symmetry

MDPI

Article

Evaporation and Antievaporation Instabilities

Andrea Addazi *,† and Antonino Marciano *,†

Center for Field Theory and Particle Physics & Department of Physics, Fudan University, Shanghai 200433, China
* Correspondence: andrea.addazi@lngs.infn.it (A.A.); marciano@fudan.edu.cn (A.M.)
† These authors contributed equally to this work.

Received: 14 September 2017; Accepted: 20 October 2017; Published: 26 October 2017

Abstract: We review (anti)evaporation phenomena within the context of quantum gravity and extended theories of gravity. The (anti)evaporation effect is an instability of the black hole horizon discovered in many different scenarios: quantum dilaton-gravity, $f(R)$-gravity, $f(T)$-gravity, string-inspired black holes, and brane-world cosmology. Evaporating and antievaporating black holes seem to have completely different thermodynamical features compared to standard semiclassical black holes. The purpose of this review is to provide an introduction to conceptual and technical aspects of (anti)evaporation effects, while discussing problems that are still open.

Keywords: alternative gravity; black holes; gravitational instabilities.

1. Introduction

The long-standing idea to extend the standard model of Einsteinian gravity—general relativity (GR)—is strongly motivated by several open issues in cosmology and quantum gravity. Despite several known successful applications of GR to astrophysics and cosmology, its UV completion and some cosmological and astrophysical instantiations—including the inflationary paradigm and the comprehension of the nature of dark energy and dark matter—remain puzzling. The most popular extension of GR remains $f(R)$-gravity, including $(R + \zeta R^2)$ Starobinsky's model for inflation [1–5]. This theory can be conformally mapped onto scalar-tensor theories or dilaton-gravity theories [1–4] in regular unambiguous space-time backgrounds. There are many alternatives that have been hitherto suggested, such as $f(T)$-gravity [6], mimetic gravity [7–11], string-inspired black holes, and brane-world cosmologies [12–15]—see [16,17] for reviews on brane-world cosmological scenarios—to mention just a few of them.

We review aspects of the instabilities of a class of black hole solutions, which appear universally in these aforementioned classes of extended theories of gravity, and are dubbed (anti)evaporation instabilities. (Anti)evaporation phenomena consist of the exponentially (growing) decreasing radius of the black hole horizon. These were first discovered by *Bousso* and *Hawking* within the context of quantum dilaton-gravity (e.g., Ref. [18]), and then elaborated in Refs. [19–21]. *Nojiri* and *Odintsov* rediscovered the same effect in $f(R)$-gravity at the classical level in Ref. [22,23]—see also Ref. [24] for technical improvements. The two phenomena were further studied in several other contexts, such as Gauss–Bonnet gravity [24], $f(T)$-gravity [25], mimetic gravity [26–29], Bigravity [30], string-inspired black hole solutions [31], brane-world cosmology [32–35] and Bardeen–de Sitter black holes [36]. In all these theories, the two metric solutions—which turned out to be unstable—are Nariai, a degenerate Schwarzschild–de Sitter black hole, and extremal Reissner–Nördstrom solutions, in which two horizons coincide. In Ref. [37], through the analysis of the Raychaduri equation, describing the dynamics of Black Holes (BH) closed trapped Cauchy's surfaces (similar techniques were used in general relativity in Refs. [38–40]), it was argued that classical (anti)evaporation instabilities switch off the emission of Bekenstein–Hawking radiation [41–44]. Very recently, the (anti)evaporation was also discussed in relation to energy conditions in extended theories of gravity [45].

Among all the possible scenarios, it is worth mentioning that there are many realistic extensions of general relativity which are compatible with cosmological and astrophysical limits and which predict antievaporation phenomena. Certainly the minimal and more appealing scenarios seem to be the ones provided by $f(R)$-gravity models. For example, among all possible $f(R)$-gravity extensions, some simple models already proposed in literature—and well compatible with cosmological constraints—such as the Hu-Sawicki model, exponential $f(R)$-gravity, and higher-derivative polynomial extensions beyond Starobinsky's gravity universally exhibit the (anti)evaporation phenomena (see Ref. [24] for a detailed discussion of these aspects).

The plan of the paper is the following. In Section 2, we briefly introduce the concept of evaporation and antievaporation instabilities. In Section 3, we review the (anti)evaporation in quantum dilaton-gravity; in Section 4, we review the classical (anti)evaporation in $f(R)$-gravity; in Section 5, we review (anti)evaporation in $f(T)$-gravity; in Section 6, we review the classical (anti)evaporation phenomena in the context of string-inspired black holes; in Section 7, we review either Bekenstein-Hawking radiation in (anti)evaporating black holes; in Section 8 we review classical (anti)evaporation of Friedman-Robertson-Walker (FRW) brane-worlds sourced by (anti)evaporating instabilities of the higher-dimensional black hole in the bulk. In Section 9, we show our conclusions and remarks.

2. What Is (Anti)evaporation?

Evaporation and antievaporation are related to a dynamical decreasing and increasing of the black hole horizon radius in time. These instabilities may be provoked by several different dynamical origins. Their possible sources can be classified into two kinds: (i) quantum anomalies; (ii) classical instabilities sourced by extensions of general relativity. In the next sections, we will review many possible models with (anti)evaporation instabilities, lying in (i,ii) classes.

3. (Anti)evaporation in Quantum Dilaton-Gravity

In this section we review studies and results obtained on antievaporation within the context of quantum dilaton-gravity [18–21].

We start considering the four-dimensional action of N scalars fields coupled to gravity, which are included in the theory in order to allow the description of black hole radiation. The action then reads:

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g^{(4)}}\left[R^{(4)} - 2\lambda - \frac{1}{2}\sum_{1}^{N}(\nabla^{(4)}\Phi_i)^2\right], \tag{1}$$

where G_N is the Newton constant, Φ_i are N-scalar fields, and $g^{(4)}$, $R^{(4)}$, and $\nabla^{(4)}\Phi_i$ are respectively the four-dimensional metric determinant, the covariant derivative with respect to the four-dimensional metric, and the Ricci scalar.

We consider the spherically symmetric background *ansatz*:

$$ds^2 = e^{2\rho(x,t)}(dx^2 - dt^2) + e^{-2\phi(x,t)}d\Omega^2, \tag{2}$$

in which $\phi(x,t), \rho(x,t)$ are functions of space-time coordinates and $d\Omega^2$ is the two-dimensional angular line-element. In the background Equation (2), the integration of the angular modes can be performed. The 4D action reduces to a two-dimensional one, which reads:

$$S = \frac{1}{16\pi}\int d^2x\sqrt{-g}e^{-2\phi}[R + 2(\nabla\phi)^2 + 2e^{2\phi} - 2\lambda - \sum_{i=1}^{N}(\nabla\Phi_i)^2]. \tag{3}$$

It was shown in [46] that the amount of black hole radiation at infinity is proportional to the trace anomaly. The trace of the energy-momentum tensor is classically vanishing, but if we consider the quantum nature of fields, a non-vanishing expectation value of the trace can be recovered on a curved background. The inclusion of the trace anomaly in the dynamics of the system under scrutiny accommodates the analysis of the back reaction of the evaporation on the geometry. This is equivalent to considering the one-loop effective action of the matter field.

Following the same strategy as in [47], two-dimensional conformal scalar fields with exponential dilation coupling yield the the trace anomaly:

$$T = \frac{1}{24\pi}\left[R - 6(\nabla\phi)^2 - 2\partial^2\phi\right]. \tag{4}$$

The trace anomaly can be obtained from using the zeta function approach and general proprieties of the trace anomaly [47].

Equivalently, from Equation (3) the the scale-dependent part of the one-loop effective action for dilaton-coupled scalars reads:

$$S_1 = -\frac{1}{48\pi}\int d^2x\sqrt{-g}[[\frac{1}{2}R\frac{1}{\partial^2}R] - 6(\nabla\phi)^2\frac{1}{\partial^2}R - 2\phi R]. \tag{5}$$

As shown in [48], the action Equation (5) can be recast as local by introducing an auxiliary scalar field A that mimics the trace anomaly term—in other words, the trace anomaly derived from the effective action Equation (5).

As shown in [48], the action Equation (5) can be rewritten in the following form:

$$S = \frac{1}{16\pi}\int d^2x\sqrt{-g}\left[\left(e^{-2\phi} + \frac{\kappa}{2}(A + w\phi)\right)R - \frac{\kappa}{4}(\nabla A)^2 + 2 + 2e^{-2\phi}(\nabla\phi)^2 - 2e^{-2\phi}\lambda\right], \tag{6}$$

where $\kappa = 2N/3$ and w is a numerical factor. In the large N-limit, the quantum fluctuations of the metric are dominated by the quantum fluctuations of the N scalars; thus, $\kappa \gg 1$. Such a formal rewriting is possible in the framework of the scalar auxiliary field method [48].

We can now derive the effective dynamics of the system. Variations of the effective action with respect to ρ, ϕ and A lead to:

$$-\left(1 - \frac{w\kappa}{4}e^{2\phi}\right)\partial^2\phi + 2(\partial\phi)^2 + \frac{\kappa}{4}e^{2\phi}\partial^2 A + e^{2\rho+2\phi}(\lambda e^{-2\phi} - 1) = 0, \tag{7}$$

$$\left(1 - \frac{w\kappa}{4}e^{2\phi}\right)\partial^2\rho - \partial^2\phi + (\partial\phi)^2 + \lambda e^{2\rho} = 0, \tag{8}$$

$$\partial^2 A - 2\partial^2\rho = 0, \tag{9}$$

with two additional constraints to be considered; i.e.,

$$\left(1 - \frac{w\kappa}{4}e^{2\phi}\right)(\delta^2\phi - 2\delta\phi\delta\rho) - (\delta\phi)^2 = \frac{\kappa}{8}e^{2\phi}[(\delta A)^2 + 2\delta^2 A - 4\delta A\delta\rho], \tag{10}$$

$$\left(1 - \frac{w\kappa}{4}e^{2\phi}\right)(\dot\phi' - \dot\rho\phi' - \rho'\dot\phi) - \rho'\dot\phi = \frac{\kappa}{8}e^{2\phi}\left[\dot A A' + 2\dot A' - 2(\dot\rho A' + \rho'\dot A)\right], \tag{11}$$

having used the conventions:

$$\partial A\partial B = -\dot A\dot B + f'g', \;\; \partial^2 g = -\ddot A + B'',$$

$$\delta A\delta B = \dot A\dot B + A'B', \;\; \delta^2 A = \ddot A + A''.$$

From Equation (9), one obtains:

$$A = 2\rho + \eta, \tag{12}$$

with η any harmonic function of x and t. Relation Equation (12) eliminates the dependence by A in the other equation of motions (EoMs).

In such a formalism, we can study perturbations around the Nariai solution (see Appendix A). The Nariai solution—which corresponds to Equation (2) with $e^{-\phi} = \text{const}$—is a solution of the dilaton-gravity theory that reads:

$$e^{2\rho} = \frac{1}{\Lambda_1} \frac{1}{\cos^2 t}, \quad e^{2\phi} = \Lambda_2 , \tag{13}$$

where:

$$\frac{1}{\Lambda_1} = \frac{1}{8\Lambda} \left[4 - (w+2)b + \sqrt{16 - 8(w-2) + (w+2)^2 b^2} \right] , \tag{14}$$

$$\Lambda_2 = \frac{1}{2w\kappa} \left[4 + (w+2)b - \sqrt{16 - 8(w-2)b + (w+2)^2 b^2} \right] . \tag{15}$$

In these latter, we have defined $b = \kappa\Lambda$, and assumed $b \ll 1$ for $\kappa \gg 1$.

We may perturb this solution around the Nariai background, and obtain:

$$e^{2\phi} = \Lambda_2 [1 + 2\epsilon\sigma(t) \cos x] , \tag{16}$$

where $\epsilon \ll 1$. We might also perturb $e^{2\rho}$, but contributions that would arise from $e^{2\rho}$ would not enter the equation of motion for σ at the first order of the ϵ-expansion.

Let us now consider the condition for a black hole horizon $(\nabla\phi)^2 = 0$. Substituting in this latter relation Equation (16), we obtain a simple system of differential equations; i.e.,

$$\frac{\partial\phi}{\partial t} = \epsilon\dot{\sigma}\cos x , \quad \frac{\partial\phi}{\partial x} = -\epsilon\sigma \sin x . \tag{17}$$

At the first order in the ϵ-expansion, the black hole radius casts:

$$r_b(t)^{-2} = e^{2\phi} = \lambda_2 [1 + 2\epsilon\delta(t)] , \tag{18}$$

$$\delta \equiv \cos x_b = \sigma \left(1 + \frac{\dot{\sigma}^2}{\sigma^2} \right)^{-1/2} . \tag{19}$$

Consequently, the black hole horizon is controlled by the equation of motion for σ:

$$\frac{\ddot{\sigma}}{\sigma} = \frac{a}{\cos^2 t} - 1 , \tag{20}$$

where:

$$a = \frac{2\sqrt{16 - 8(w-2)b + (w+2)^2 b^2}}{4 - wb} . \tag{21}$$

The classical limit is obtained when we send $\kappa \to 0$. In this limit, the equation of motion is exactly solvable, and reduces to:

$$\dot{\sigma} = \sigma \tan t , \tag{22}$$

which yields the solution:

$$\sigma(t) = \frac{\sigma_0}{\cos t} \tag{23}$$

for the initial condition $\dot{\sigma}_0(t = 0) = 0$. This leads to a perturbation $\delta(t) = \sigma_0 = \text{const}$, which ensures that the Nariai solution is static at the classical level. Nonetheless, at the quantum level, for $\kappa > 0$, we obtain an approximated solution for the perturbations:

$$\delta(t) = \sigma_0 \left[1 - \frac{1}{2}(a-1)(a-2)t^2 + O(t^4) \right] , \quad \sigma_0 > 0 , \quad \dot{\sigma}(t = 0) = 0 . \tag{24}$$

As a remarkable consequence, the black hole size increases; i.e., the maximal Schwarzschild–de Sitter black hole has an antievaporation instability.

4. (Anti)evaporation in $f(R)$-Gravity

In this section, we review some basic aspects of evaporation and antievaporation in $f(R)$-gravity [22–24], taking into account the Nariai metric and extremal Reissner–Nodrström black holes. Let us first recall the theoretical framework.

In $f(R)$-gravity, the action reads [1–5]:

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}f(R) + S_m\,, \tag{25}$$

written in units $G_N = c = 1$. Varying the action Equation (25) with respect to the metric tensor, we obtain the equation of motions (EoMs) of the theory:

$$\frac{1}{2}g_{\mu\nu}f(R) - f'(R)R_{\mu\nu} + \nabla_\mu\nabla_\nu f'(R) = -8\pi T^m_{\mu\nu}\,. \tag{26}$$

Whenever the matter content is vanishing—namely $T^m_{\mu\nu} = 0$—and the Ricci tensor constant (i.e., $R_{\mu\nu} \sim g_{\mu\nu}$), the EoM is reduced to a more manageable form:

$$f(R) - \frac{1}{2}Rf'(R) = 0\,. \tag{27}$$

4.1. The Case of the Nariai Black Hole in $f(R)$-Gravity

The Nariai space-time is a solution of Equation (27). It can be recast (for details, see Appendix A) as:

$$ds^2 = \frac{1}{\Lambda^2}\left(\frac{1}{\cos h^2 x}(dx^2 - dt^2) + d\Omega^2\right)\,, \tag{28}$$

where Λ has one mass dimension, and again $d\Omega^2$ denotes the solid angle on a 2-sphere, i.e., $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, with $\theta \in [0,\pi)$ and $\phi \in [0, 2\pi)$. Notice also that the Ricci scalar of the Nariai space-time is $R_0 = 4\Lambda^2 = \text{const}$.

The Nariai metric can be obtained from the more general expression:

$$ds^2 = e^{2\rho(x,t)}(dx^2 - dt^2) + e^{-2\phi(x,t)}d\Omega^2\,, \tag{29}$$

where $\phi(x,t), \rho(x,t)$ are functions of space-time coordinates governed by the following EoMs:

$$\begin{aligned} 0 = &-\frac{e^{2\rho}}{2}f(R) - (-\ddot{\rho} + 2\ddot{\phi} + \rho'' - 2\dot{\phi}^2 - 2\rho'\phi' - 2\dot{\rho}\dot{\phi})f'(R) + \frac{\partial^2 f'}{\partial t^2} - \dot{\rho}\frac{\partial f'}{\partial t} - \rho'\frac{\partial f'}{\partial x} \\ &+ e^{2\phi}\left\{-\frac{\partial}{\partial t}\left(e^{-2\phi}\frac{\partial f'}{\partial t}\right) + \frac{\partial}{\partial x}\left(e^{-2\phi}\frac{\partial f'}{\partial x}\right)\right\}\,, \end{aligned} \tag{30}$$

$$\begin{aligned} 0 = &\frac{e^{2\rho}}{2}f - (\ddot{\rho} + 2\phi'' - \rho'' - 2\phi'^2 - 2\rho'\phi' - 2\dot{\rho}\dot{\phi})f' + \frac{\partial^2 f'}{\partial x^2} \\ &-\dot{\rho}\frac{\partial f'}{\partial t} - \rho'\frac{\partial f'}{\partial x} - e^{2\phi}\left\{-\frac{\partial}{\partial t}\left(e^{-2\phi}\frac{\partial f'}{\partial t}\right) + \frac{\partial}{\partial x}\left(e^{-2\phi}\frac{\partial f'}{\partial x}\right)\right\}\,, \end{aligned} \tag{31}$$

$$0 = -(2\dot{\phi}' - 2\phi'\dot{\phi} - 2\rho'\dot{\phi} - 2\dot{\rho}\phi')f' + \frac{\partial^2 f'}{\partial t\partial x} - \dot{\rho}\frac{\partial f'}{\partial x} - \rho'\frac{\partial f'}{\partial t}\,, \tag{32}$$

$$\begin{aligned} 0 = &\frac{e^{-2\phi}}{2}f - e^{-2(\rho+\phi)}(-\ddot{\phi} + \phi'' - 2\phi'^2 + 2\dot{\phi}^2)f' - f' + e^{-2(\rho+\phi)}\left(\dot{\phi}\frac{\partial f'}{\partial t} - \phi'\frac{\partial f'}{\partial x}\right) \\ &-e^{-2\rho}\left\{-\frac{\partial}{\partial t}\left(e^{-2\phi}\frac{\partial f'}{\partial t}\right) + \frac{\partial}{\partial}\left(e^{-2\phi}\frac{\partial f'}{\partial x}\right)\right\}\,. \end{aligned} \tag{33}$$

From EoMs in the metric Equation (74), one can study the evolution of the perturbations around the Nariai background:

$$\rho = -\ln(\Lambda\cosh x) + \delta\rho\,, \tag{34}$$

$$\phi = \ln \Lambda + \delta\phi\,. \tag{35}$$

Substituting these expressions into EoMs, one obtains a set of four equations in $\delta\rho, \delta\phi$, namely:

$$\begin{aligned} 0 &= \tfrac{-f'(R_0)+2\Lambda^2 f''(R_0)}{2\Lambda^2\cosh^2 x}\delta R - \tfrac{f(R_0)}{\Lambda^2\cosh^2 x}\delta\rho \\ &-f'(R_0)(-\delta\ddot{\rho} + 2\delta\ddot{\phi} + \delta\rho'' + 2\tanh x\delta\phi') + \tanh x f''(R_0)\delta R' + f''(R_0)\delta R''\,, \end{aligned} \tag{36}$$

$$\begin{aligned} 0 &= -\tfrac{-f'(R_0)+2\Lambda^2 f''(R_0)}{2\Lambda^2\cosh^2 x}\delta R + \tfrac{f(R_0)}{\Lambda^2\cosh^2 x}\delta\rho \\ &-f'(R_0)(\delta\ddot{\rho} + 2\delta\phi'' - \delta\rho'' + 2\tanh x\delta\phi') + f''(R_0)\delta\ddot{R} + \tanh x f''(R_0)\delta R'\,, \end{aligned} \tag{37}$$

$$0 = -2(\delta\dot{\phi}' + \tanh x\delta\dot{\phi}) + \frac{f''(R_0)}{f'(R_0)}(\delta\dot{R}' + \tanh x\delta\dot{R})\,, \tag{38}$$

$$0 = -\tfrac{-f'(R_0)+2\Lambda^2 f''(R_0)}{2\Lambda^2}\delta R - \tfrac{f(R_0)}{\Lambda^2}\delta\phi - \cosh^2 x f'(R_0)(-\delta\ddot{\phi} + \delta\phi'') - \cosh^2 x f''(R_0)(-\delta\ddot{R} + \delta R'')\,, \tag{39}$$

where:

$$\delta R = 4\Lambda^2(-\delta\rho + \delta\phi) + \Lambda^2\cosh^2 x(2\delta\ddot{\rho} - 2\delta\rho'' - 4\delta\ddot{\phi} + \delta\phi'')\,. \tag{40}$$

(see Appendix C for more technical details).

The third equation can be integrated, leading to:

$$-2\delta\phi + \frac{f''(R_0)}{f'(R_0)}\delta R = c_x(x) + \frac{c_t(t)}{\cosh x}\,, \tag{41}$$

where $c_x(x), c_t(t)$ are arbitrary integration functions of x, t respectively. From a linear combination of the first, second, and fourth equations, one can obtain the equations:

$$0 = \frac{-f'(R_0) + 2\Lambda^2 f''(R_0)}{2\Lambda^2\cosh^2 x}\delta R - f'(R_0)\partial^2\left(\delta\rho - \delta\phi - \frac{f''(R_0)}{f'(R_0)}\delta R\right)\,, \tag{42}$$

$$0 = \frac{2\Lambda^2}{\cosh^2 x}\delta\phi + \partial^2\left(\delta\rho + \frac{f''(R_0)}{2f'(R_0)}\delta R\right)\,. \tag{43}$$

Once combined with Equation (41), Equations (42) and (43) allow to find the differential equation in ϕ:

$$0 = \frac{1}{\alpha\cosh^2 x}\left(2(2\alpha - 1)\delta\phi + (\alpha - 1)\left(c_x(x) + \frac{c_t(t)}{\cosh x}\right)\right) + \partial^2\left(3\delta\phi + c_x(x) + \frac{c_t(t)}{\cosh x}\right)\,, \tag{44}$$

where:

$$\alpha \equiv \frac{2\Lambda^2 f''(R_0)}{f'(R_0)}\,. \tag{45}$$

We emphasize that Equation (44) can have unstable modes in specific subregions of the parameter space. Since in homogeneous and isotropic backgrounds $\delta\phi(t, x) \equiv \phi(t)$, Equation (44) reduces to:

$$\frac{d^2\delta\phi}{dt^2} + \tanh t\frac{d\delta\phi}{\delta t} - m^2\delta\phi = 0\,, \tag{46}$$

where the effective mass of the mode is expressed by:

$$m^2 = \frac{2(2\alpha - 1)}{3\alpha}\,, \tag{47}$$

having assumed the initial conditions $c_x = c_t = 0$ in Equation (44). Such an equation has tachyon-like modes for $m^2 > 0$ ($\alpha < 0$ and $\alpha > 1/2$) and for $1 + 4m^2 \geq 0$ ($\alpha < 0$ and $\alpha > 8/19$).

The horizon is located in correspondence of the condition:

$$\nabla\delta\phi \times \nabla\delta\phi = 0\,, \tag{48}$$

which specifies the requirement that the gradient of the two-sphere size is equal to zero. This means that for a black hole located in x_0, the radius is:

$$r_0(t)^{-2} = e^{2\phi(t,x_0)}\,.$$

Consequently, either an increase or a decrease of ϕ correspond to a dynamical displacement of the horizon.

4.2. Extremal Reissner–Nordström Black Holes

In this section, we will review evaporation and antievaporation of the extremal Reissner–Nörstrom (RN) black holes in $f(R)$-gravity [23].

The extremal RN solution is recovered in the limit in which the two possible RN radii coincide. The extremal RN-black hole metric can then be recast as (see Appendix B for further details):

$$ds^2 = \frac{r_0^2}{\left(1 - \frac{r_0^2 R_0}{2}\right)\cosh^2 x}(d\tau^2 - dx^2) + r_0^2 d\Omega^2\,. \tag{49}$$

This expression shares several similarities with the aforementioned Nariai metric. Indeed, the extremal RN solution can also be reshuffled as:

$$ds^2 = \frac{e^{2\rho(x,\tau)}}{\Lambda^2}(d\tau^2 - dx^2) + \frac{e^{-2\phi(x,\tau)}}{\Lambda'^2}d\Omega^2\,. \tag{50}$$

The form of $\rho(x,\tau)$ finally induces the explicit formula:

$$ds^2 = \frac{1}{\Lambda^2\cosh^2 x}(d\tau^2 - dx^2) + \frac{e^{-2\phi}}{\Lambda'^2}d\Omega^2\,, \tag{51}$$

$$\Lambda = \frac{\sqrt{1 - \frac{r_0^2 R_0}{2}}}{r_0}, \quad \Lambda' = \frac{1}{r_0}\,. \tag{52}$$

Then, using the same ansatz on the metric we deployed while tackling the Nariai metric, the EoM, written in components (τ,τ), (x,x), (τ,x) and (θ,θ) $((\psi,\psi))$, cast:

$$\begin{aligned} 0 = \tfrac{e^{2\rho}}{2\Lambda^2}f(R) - \left(-\ddot\rho + 2\ddot\phi + \rho'' - 2\dot\phi^2 - 2\rho'\phi' - 2\dot\rho\dot\phi\right)f'(R) + \tfrac{\partial^2 f'(R)}{\partial\tau^2} - \dot\rho\tfrac{\partial f'}{\partial\tau} - \rho'\tfrac{\partial f'}{\partial x} \\ + e^{2\phi}\left\{-\tfrac{\partial}{\partial\tau}\left(e^{-2\phi}\tfrac{\partial f'}{\partial\tau}\right) + \tfrac{\partial}{\partial x}\left(e^{-2\phi}\tfrac{\partial F'}{\partial x}\right)\right\}\,, \end{aligned} \tag{53}$$

$$\begin{aligned} 0 = \tfrac{e^{2\rho}}{2\Lambda^2}f(R) - (\ddot\rho + 2\phi'' - \rho'' - 2\phi'^2 - 2\rho'\phi' - 2\dot\rho\dot\phi)f' + \tfrac{\partial^2 f'}{\partial x^2} - \dot\rho\tfrac{\partial f'}{\partial\tau} - \rho'\tfrac{\partial f'}{\partial x} \\ - e^{2\phi}\left\{-\tfrac{\partial}{\partial\tau}\left(e^{-2\phi}\tfrac{\partial f'}{\partial\tau}\right) + \tfrac{\partial}{\partial x}\left(e^{-2\phi}\tfrac{\partial f'}{\partial x}\right)\right\}\,, \end{aligned} \tag{54}$$

$$0 = -(2\dot\phi' - 2\phi'\dot\phi - 2\rho'\dot\phi - 2\dot\rho\phi')f' + \frac{\partial^2}{\partial\tau\partial x} - \dot\rho\frac{\partial f'}{\partial x} - \rho'\frac{\partial f'}{\partial\tau}\,, \tag{55}$$

$$\begin{aligned} 0 = -\tfrac{e^{-2\phi}}{2\Lambda'^2}f - \tfrac{\Lambda^2}{\Lambda'^2}e^{-2(\rho+\phi)}\left(-\ddot\phi + \phi'' - 2\phi'^2 + 2\dot\phi^2\right)f' + f' \\ + \tfrac{\Lambda^2}{\Lambda'^2}e^{-2(\rho+\phi)}\left(\dot\phi\tfrac{\partial f'}{\partial\tau} - \phi'\tfrac{\partial f'}{\partial x}\right) - \tfrac{\Lambda^2}{\Lambda'^2}e^{-2\rho}\left\{-\tfrac{\partial}{\partial\tau}\left(e^{-2\phi}\tfrac{\partial f'}{\partial\tau}\right) + \tfrac{\partial}{\partial x}\left(e^{-2\phi}\tfrac{\partial f'}{\partial x}\right)\right\}\,. \end{aligned} \tag{56}$$

(see Appendix D for more technical details).

Perturbations with respect to the extremal RN background can be considered following the same strategy as in the previous sections. We then add a generic perturbation to the expressions:

$$\rho = -\log\cosh x + \delta\rho, \;\; \phi = \delta\phi \tag{57}$$

and then recover:

$$0 = f''(R_0)\left\{-\frac{1}{\cosh^2 x}\delta R + \tanh x\,\delta R' + \delta R''\right\}, \tag{58}$$

$$0 = f''(R_0)\left\{\frac{1}{\cosh^2 x}\delta R + \tanh x\,\delta R' + \delta\ddot{R}\right\}, \tag{59}$$

$$0 = f''(R_0)\left\{\delta\dot{R}' + \tanh x\delta\dot{R}\right\}, \tag{60}$$

$$0 = f''(R_0)\left\{\delta R - \cosh^2 x(-\delta\ddot{R} + \delta R'')\right\}, \tag{61}$$

where:

$$\delta R = -4\Lambda^2\delta\rho + 4\Lambda'^2\delta\phi - \Lambda^2\cosh^2 x\{2(\delta\ddot{\rho} - 2\delta\rho'') - 4(\delta\ddot{\phi} - \delta\phi'')\}. \tag{62}$$

To study the instabilities of the system, we can adopt the parametrization:

$$\delta\phi = \phi_0\cosh\omega\tau\cosh^\beta x, \qquad \delta\rho = \rho_0\cosh\omega\tau\cosh^\beta x, \tag{63}$$

where $\rho_0, \phi_0, \omega, \beta$ are constant parameters. Using the definition of the horizon $g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = 0$, we then end up recovering the solutions:

$$\delta\phi \equiv \delta\phi_H = \phi_0\cosh^2\beta t, \tag{64}$$

$$r_H = \frac{1}{\Lambda}e^{-\delta\phi_H} = \frac{e^{-\phi_0\cosh^2\beta\tau}}{\Lambda}. \tag{65}$$

What is remarkable in this case is that the instabilities seem to be independent by the particular kind of $f(R)$-gravity under scrutiny.

5. (Anti)evaporation in $f(T)$-Gravity

In this section we move to the discussion of the evaporation and antievaporation phenomena within the context of $f(T)$-gravity [25]. Once again, we start reviewing the theoretical framework of these models.

In $f(T)$-gravity, the action reads:

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}f(T) + S_m, \tag{66}$$

in which we again use units $G_N = c = 1$. We then introduce internal indices in the description of the gravitational field, and represent the gravitational degrees of freedom in terms of a frame field that constitutes the tetrad matrix. The line element then recasts:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ij}\theta^i\theta^j, \tag{67}$$

$$dx^\mu = e_i^\mu\theta^i, \quad \theta^i = e^i_\mu dx^\mu, \tag{68}$$

where $e_i^\mu e^i_\nu = \delta^\mu_\nu$, $\eta_{ij} = \mathrm{diag}(-1,1,1,1)$, $\sqrt{-g} = e = \det[e^i_\mu]$.

The Weitzenböck connection deployed in the construction of the $f(T)$ theory is purely torsional. Its relation to the torsion tensor can be straightforwardly determined to be:

$$T^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{j}(\partial_{\mu} e^{i}_{\nu} - \partial_{\nu} e^{i}_{\mu})\,. \tag{69}$$

The Euler–Lagrange equations of the theory are then recovered by variation of the action with respect to the tetrad field e^{i}_{μ}; namely:

$$S^{\nu\rho}_{\mu}\partial_{\rho} T \frac{d^2 f}{dT^2} + e^{-1} e^{i}_{\mu}\partial_{\rho}[e S^{\nu\rho}_{\alpha} e^{\alpha}_{i} + T^{\alpha}_{\mu\sigma} S^{\nu\sigma}_{\alpha}]\frac{df}{dT} + \frac{1}{2}\delta^{\nu}_{\mu} f = 4\pi T^{(m)}_{\mu\nu}\,. \tag{70}$$

In Equation (70), $T^{(m)}_{\mu\nu}$ denotes the energy-momentum tensor, while $S^{\nu\rho}_{\mu}$ is expressed by the relation:

$$S^{\mu\nu}_{\alpha} = \frac{1}{2}(\delta^{\mu}_{\alpha} T^{\nu\beta}_{\beta} - \delta^{\mu}_{\beta} T^{\nu\beta}_{\alpha} + K^{\mu\nu}_{\alpha})\,, \tag{71}$$

$K^{\mu\nu}_{\alpha}$ standing for the co-torsion. Finally, the scalar torsion reads:

$$T = T^{\alpha}_{\mu\nu} S^{\mu\nu}_{\alpha}\,. \tag{72}$$

General relativity with a cosmological constant can be recovered in the limit $\frac{d^2 f}{dT^2} \to 0$; i.e., $f(T) = a + bT$.

5.1. The Case of Nariai Black Hole in Diagonal Tetrads Gauge

For the $f(T)$ theory, the Nariai space-time acquires the form:

$$ds^2 = \frac{1}{\Lambda}\left[-\frac{1}{\cos^2\tau}(dx^2 - d\tau^2) + d\Omega^2\right]\,, \tag{73}$$

where Λ is the cosmological constant, once again $d\Omega^2$ stands for the solid angle on a 2-sphere $d\Omega^2 = d\theta^2 + \sin^2\theta d\psi^2$, and $0 < \tau < \pi/2$, $0 < t < \infty$, with the mutual relation $\cosh t = 1/\cos\tau$. Notice that also in this case the Ricci scalar of the Nariai space-time is constant, since $R = 4\Lambda$.

The Nariai space-time is a solution of Equation (70) in the diagonal tetrad ansatz:

$$ds^2 = e^{2\rho(x,t)}(-dx^2 + d\tau^2) + e^{-2\phi(x,t)} d\Omega^2\,, \tag{74}$$

$$e^{a}_{\mu} = [e^{\rho}, e^{\rho}, e^{-\phi}, e^{-\phi}\sin\theta]\,. \tag{75}$$

The dynamical aspects of the Nariai solutions can be studied resorting to the methods of perturbation theory. We can consider arbitrary variations of the functions:

$$\rho = -ln[\sqrt{\Lambda}\cos\tau] + \delta\rho(\tau, x)\,, \tag{76}$$

$$\phi = \ln\sqrt{\Lambda} + \delta\phi(\tau, x)\,, \tag{77}$$

and then find the relation:

$$\delta T = -2\Lambda\sin(2\tau)\delta\dot{\phi}\,. \tag{78}$$

Inserting Equations (77) and (78) in Equation (70), we may recover:

$$\delta\phi(x,\tau) = k_1 \sin(x - \bar{x})\sec\tau + k_2\,, \tag{79}$$

where $\bar{x}$ is the fixed initial condition and $k_{1,2,}$ are two integration constants.

Consider now that the horizon is defined through the condition:

$$\left(\frac{\partial\delta\phi}{\partial\tau}\right)^2 = \left(\frac{\partial\delta\phi}{\partial x}\right)^2 . \tag{80}$$

From this, we obtain:

$$x_h = \bar{x} - \tau + m\pi - \frac{\pi}{2} , \tag{81}$$

where $m = 0, 1, ...$, and correspondingly we recover:

$$\delta\phi(\tau, x_h) = k_1(-1)^{n+1} + k_2 , \tag{82}$$

$$r_h(\tau)^{-2} = 1 + \delta\phi(\tau, x_h) . \tag{83}$$

We can interpret this result by saying that the black hole radius is fixed; i.e., no evaporation or antievaporation instabilities occur.

It is worth noting that the choice of diagonal tetrads should be handled carefully in the case of spherically symmetric solutions. This issue was extensively studied in Refs. [49–52]. In Ref. [52], it was shown that the rigorous way to implement the choice of tetrads consists of also taking the connection into account. These arguments highly motivate to relax the diagonal tetrads choice, as discussed in the following section.

5.2. Classical Evaporation and Antievaporation in Non-Diagonal Tetrads

We can now generalize the previous result, considering a non-diagonal tetrad of the form:

$$e^0_0 = e^\rho, \; e^3_3 = e^{1,2,3}_0 = e^0_{1,2,3} = 0 , \tag{84}$$

$$e^1_1 = \cos\psi \sin\theta \, e^\rho , \; e^2_1 = \cos\psi \cos\theta \, e^{-\phi} , \; e^3_1 = -\sin\psi \sin\theta \, e^{-\phi} , \tag{85}$$

$$e^1_2 = \sin\psi \sin\theta \, e^\rho , \; e^1_3 = \cos\theta \, e^\rho , \; e^2_2 = \sin\psi \cos\theta \, e^{-\phi} , \tag{86}$$

$$e^2_3 = \cos\psi \sin\theta \, e^{-\phi}, \; e^3_2 = -\sin\theta \, e^{-\phi} . \tag{87}$$

Under this ansatz, we obtain:

$$\delta\phi = A \sec\tau \cos(x - \bar{x}) + B(\tan\tau)^{3/2} e^{\frac{1+2\cos^2\tau}{4\cos^4\tau}} , \tag{88}$$

where A, B are integration constants. This entails for the horizon the expression:

$$x_h = \bar{x} - \tau + \arcsin\left(\frac{\cos^2\tau}{A}\frac{d}{d\tau}\varphi(\tau)\right) , \tag{89}$$

where:

$$\varphi(\tau) = B(\tan\tau)^{3/2} e^{\frac{1+2\cos^2\tau}{4\cos^4\tau}} . \tag{90}$$

Notice that Equation (88) has a divergence in $\tau \to \pi/2$ — this is the extreme time-like angle excluded from the range of the Nariai solution. Depending on the integration constants, Equation (89) represents a solution either increasing or decreasing in time. The first class of instabilities corresponds to the classical antievaporation, while the second class to the classical evaporation.

6. (Anti)evaporation in String-Inspired Black Holes

We discuss dyonic black hole solutions in the case of $f(R)$-gravity coupled with a dilaton and two gauge bosons. The study of such a model is highly motivated from string theory. Our black hole solutions are extensions of the one first studied by Kallosh, Linde, Ortín, Peet, and Van Proeyen

(KLOPV) in Ref. [53]. We will show that extreme solutions are unstable. In particular, these solutions have Bousso–Hawking–Nojiri–Odintsov (anti)evaporation instabilities.

It is known that the low-energy limit of a dimensionally-reduced superstring theory dimensionally reduced to $d = 4$ is $\mathcal{N} = 4$ supergravity. There are two versions: $SO(4)$ and $SU(4)$. The first one is invariant under a (rigid) $SU(4) \times SU(1,1)$ symmetry. Black hole solutions of the reduced sector $U(1)^2$ were studied by Kallosh, Linde, Ortín, Peet, and Van Proeyen (KLOPV) in Ref. [53]. In particular, they consider $U(1)^2$ charged dilaton black holes. These solutions are Reissner–Nordström-like black holes, or more precisely dyonic black holes. In particular, the dilaton field is the real part of an initial complex scalar, while the imaginary part is an axion pseudoscalar field. They assumed the axion stabilized to a constant vacuum expectation value (VEV) . The effective bosonic action corresponds to the Einstein–Hilbert one coupled with a dilaton field and two $U(1)$ fields. Extreme limits of dyonic solutions are shown to saturate $\mathcal{N} = 4$ supersymmetry in $d = 4$. On the other hand, the presence of non-perturbative stringy effects could modify the effective action in the low-energy limit. For instance, higher derivative terms may be generated by euclidean D-brane or worldsheet instantons. In particular, the Einstein–Hilbert sector coupled to the dilaton and $U(1)$-fields can be extended from R to an analytic function $f(R)$ (see Ref. [54] for a review on this subject, see Refs. [55–58] for recent investigations of E-brane instantons in particle physics).

KLOPP solutions are particularly important in string theory. For instance, the famous derivation of the Hawking BH entropy from Bogomolnyi-Prasad-Sommerfield states (BPS) microstates shown by Strominger and Vafa is based on five-dimensional KLOPP solutions [59]. The Vafa–Strominger result has inspired the so-called fuzzball proposal, which has the ambition to solve the BH information paradox [60].

It is worth mentioning that the existence of modes' correlations inside the Hawking radiation was discussed in Ref. [61]. On the other hand, the unitarity time evolution of quantum black hole formation and evaporation processes in the framework of the Bohr-like approach was studied in Ref. [62].

In this paper, we will study black hole solutions in string-inspired $f(R)$-gravity, coupled with a dilaton field and two gauge bosons (it is conceivable that the analysis of branes in higher-dimensional $f(R)$-gravity—see Refs. [32–34]—may be connected to these issues). We assume that the asymptotic space-time is Minkowski's one. Let us clarify that we will not consider a $f(R)$-*supergravity* coupled to gauge bosons and dilatons. In fact, it was recently shown that the only $f(R)$-supergravity which is not plagued by ghosts and tachyons is Starobinsky's supergravity [63,64]. Nevertheless, one can consider the case in which higher-derivative terms are generated by exotic instantons or fluxes after a spontaneous supersymmetry breaking mechanism. In this sense, our model—which has a stable vacuum and is not plagued by ghosts and tachyons—is inspired by string theory. Clearly, it is impossible to calculate instantonic corrections from a realistic stringy model at the moment. We believe that this highly motivates our effective field theory analysis, in which coefficients inside the $f(R)$-functional parametrize our ignorance about the string theory vacua. We will show that extreme dyonic solutions have Bousso–Hawking–Nojiri–Odintsov (BHNO) (anti)evaporation instabilities. In particular, Nojiri and Odintsov have discovered (anti)evaporation instabilities in Reissner–Nordström black holes in $f(R)$-gravity [23]. A posteriori, our result is understood as a generalization of Nojiri–Odintsov calculations in Ref. [23]. On the other hand, the peculiar thermodynamical proprieties of antievaporating solutions were discussed in our recent paper [37].

Let us consider the case of a $f(R)$-gravity with two $U(1)$-gauge bosons and a dilaton. In particular, we will consider the action:

$$S = \int d^4x\sqrt{-g}[-f(R) + 2\partial^{\mu}\phi\partial_{\mu}\phi + 2\nabla_{\mu}\phi\nabla_{\nu}\phi - e^{-2\phi}(2F_{\mu\lambda}F_{\nu\delta}g^{\lambda\delta} - \frac{1}{2}g_{\mu\nu}F^2)]\,, \tag{91}$$

where:

$$F_{\mu\nu} = \partial_{\nu}A_{\mu} - \partial_{\mu}A_{\nu}, \quad \tilde{B}_{\mu\nu} = \partial_{\nu}\tilde{B}_{\mu} - \partial_{\mu}\tilde{B}_{\nu}$$

and A_μ, B_μ are gauge bosons of $U(1) \times U(1)$, we conveniently use unit $2\kappa_{(4)} = 1$, where $\kappa_{(4)}$ is the four-dimensional gravitational coupling (coming from the Kaluza–Klein reduction of the ten-dimensional gravitational coupling). The action Equation (1) comes from the $SO(4)$, $d = 4$, $\mathcal{N} = 4$ supergravity, and it is formulated in the Einstein-frame, with an opportune and understood redefinition of the dilaton field.

The equations of motion are:

$$0 = \nabla_\mu (e^{-2\phi} F^{\mu\nu}) , \tag{92}$$

$$0 = \nabla_\mu (e^{2\phi} \tilde{G}^{\mu\nu}) , \tag{93}$$

$$0 = \nabla^2 \phi - \frac{1}{2} e^{-2\phi} F^2 + \frac{1}{2} e^{2\phi} \tilde{G}^2 , \tag{94}$$

$$\begin{aligned} 0 = f_R(R) R_{\mu\nu} + \tfrac{1}{2}(R f_R - f(R)) g_{\mu\nu} - \nabla_\mu \nabla_\nu f_R(R) + g_{\mu\nu} \partial^2 f_R(R) \\ +2\nabla_\mu \phi \nabla_\nu \phi - e^{-2\phi}(2F_{\mu\lambda} F_{\nu\delta} g^{\lambda\delta} - \tfrac{1}{2} g_{\mu\nu} F^2) - e^{2\phi}(2\tilde{G}_{\mu\lambda} \tilde{G}_{\nu\delta} g^{\lambda\delta} - \tfrac{1}{2} g_{\mu\nu} \tilde{G}^2) . \end{aligned} \tag{95}$$

A solution of these equations is:

$$\begin{gathered} ds^2 = e^{2U} dt^2 - e^{-2U} dr^2 - R^2 d\Omega \\ e^{2\phi} = e^{2\phi_0} \tfrac{r+\Sigma}{r-\Sigma}, \;\; F = \tfrac{Q e^{\phi_0}}{(r-\Sigma)^2} dt \wedge dr , \\ \tilde{G} = \tfrac{P e^{-\phi_0}}{(r+\Sigma)^2} dt \wedge dr, \;\; e^{2U} = \tfrac{(r-r_+)(r-r_-)}{R^2} , \\ R^2 = r^2 - \Sigma^2, \;\; \Sigma = \tfrac{P^2 - Q^2}{2M}, \;\; r_\pm = M \pm r_0 , \\ r_0^2 = M^2 + \Sigma^2 - P^2 - Q^2 = M^2 + \Sigma^2 - e^{-2\phi_0} P_m^2 - e^{-2\phi_0} Q_{el}^2 . \end{gathered} \tag{96}$$

The solutions depend on independent parameters M, Q, P, ϕ_0. M is the BH mass, ϕ_0 is the asymptotic value of the dilaton field. $Q_{el} = e^{\phi_0 Q}$ is the F-field electric charge, while $P_m = e^{\phi_0} P$ is the G-field magnetic charge (electric charge of $\tilde{G}$).

These equations imply the relation:

$$C f_R(R_0) = q^2 \equiv \sqrt{Q^2 + P^2} = e^{-\phi_0} \sqrt{Q_{el}^2 + P_m^2}$$

where C is an integration constant.

In the case of an extremal dyonic black hole, the metric can be conveniently rewritten as [23]:

$$ds^2 = \frac{M^2}{\cosh^2 x} (d\tau^2 - dx^2) + M^2 d\Omega^2$$

This suggests the ansatz:

$$ds^2 = M^2 e^{2\rho(x,\tau)} (d\tau^2 - dx^2) + M^2 e^{-2\varphi(x,\tau)} (d\tau^2 - dx^2) d\Omega^2$$

and the gravitational EoM can be rewritten as:

$$\begin{aligned} 0 = -(-\ddot{\rho} + 2\ddot{\varphi} + \rho'' - 2\dot{\varphi}^2 - 2\rho' \varphi' - 2\dot{\rho}\dot{\varphi}) f_R + \frac{M^2}{2} e^{2\rho} f + \frac{\partial^2}{\partial \tau^2} f_R \\ -\rho' \frac{\partial}{\partial x} f_R + \dot{\rho} \frac{\partial}{\partial \tau} f_R + \frac{q^2 M^2 e^{2\rho}}{2} + e^{2\varphi} \left[-\frac{\partial}{\partial \tau} \left(e^{-2\varphi} \frac{\partial f_R}{\partial \tau} \right) + \frac{\partial}{\partial x} \left(e^{-2\varphi} \frac{\partial f_R}{\partial x} \right) \right] , \end{aligned} \tag{97}$$

$$\begin{aligned} 0 = \tfrac{-M^2}{2} e^{2\rho} f - \left(\ddot{\rho} + 2\varphi'' - \rho'' - 2\varphi'^2 - 2\rho' \varphi' - 2\dot{\rho}\dot{\varphi} \right) f_R \\ -\tfrac{q^2 M^2 e^{2\rho}}{2} + \tfrac{\partial^2}{\partial x^2} f_R - \dot{\rho} \tfrac{\partial f_R}{\partial \tau} - \rho' \tfrac{\partial f_R}{\partial x} - e^{2\varphi} \left[-\tfrac{\partial}{\partial \tau} \left(e^{-2\varphi} \tfrac{\partial f_R}{\partial \tau} \right) + \tfrac{\partial}{\partial x} \left(e^{-2\varphi} \tfrac{\partial f_R}{\partial x} \right) \right] \end{aligned} \tag{98}$$

$$0 = -(2\dot{\varphi}' - 2\varphi' \dot{\varphi} - 2\rho' \dot{\varphi} - 2\dot{\rho} \varphi') f_R + \frac{\partial^2 f_R}{\partial \tau \partial x} - \dot{\rho} \frac{\partial f_R}{\partial x} - \rho' \frac{\partial f_R}{\partial \tau} \tag{99}$$

$$
\begin{aligned}
0 = \quad & -2M^2e^{-2\varphi}f - e^{-2(\rho+\varphi)}(-\ddot{\varphi}+\varphi''+2\varphi'^2+2\dot{\varphi}^2)f_R + f_R + e^{-2(\rho+\varphi)}\left(\dot{\varphi}\frac{\partial f_R}{\partial t} - \varphi'\frac{\partial f_R}{\partial x}\right) + \frac{q^2M^2e^{2\rho}}{2} \\
& -e^{-2\rho}\left[-\frac{\partial}{\partial\tau}\left(e^{-2\varphi}\frac{\partial f_R}{\partial\tau}\right) + \frac{\partial}{\partial x}\left(e^{-2\varphi}\frac{\partial f_R}{\partial x}\right)\right]
\end{aligned} \tag{100}
$$

Now, let us consider perturbations around the background extremal solution as:

$$\rho = -\ln(\cosh x) + \delta\rho, \quad \varphi = \delta\varphi\,. \tag{101}$$

The perturbed EoM are:

$$
\begin{aligned}
0 = \frac{f_R(R_0)+2M^{-2}f_{RR}(R_0)}{2}\delta R - f_R(R_0)M^{-2}\cosh^2 x(-\delta\ddot{\rho}+2\delta\ddot{\varphi}+\delta\rho''+2\tanh x\,\delta\phi') \\
-2f_R(R_0)M^{-2}\delta\rho + f_{RR}(R_0)M^{-2}\cosh^2 x(\tanh x\,\delta R' + \delta R'')
\end{aligned} \tag{102}
$$

$$
\begin{aligned}
0 = \quad & -\frac{f_R(R_0)+2M^{-2}f_{RR}(R_0)}{2}\delta R + 2f_R(R_0)M^{-2}\delta\rho - f_R(R_0)M^{-2}\cosh^2 x\,(\delta\ddot{\rho}+2\delta\varphi''-\delta\rho''+2\tanh x\,\delta\varphi') \\
& +f_{RR}(R_0)M^{-2}\cosh^2 x\,(\tanh x\,\delta R' + \delta\ddot{R})
\end{aligned} \tag{103}
$$

$$0 = -2(\delta\dot{\varphi}' + \tanh x\,\delta\dot{\varphi}) + \frac{f_{RR}(R_0)}{f_R(R_0)}(\delta\dot{R}' + \tanh x\,\delta\dot{R}) \tag{104}$$

$$
\begin{aligned}
0 = \quad & -\frac{f_R(R_0)+2M^{-2}f_{RR}(R_0)}{2}\delta R - 2M^{-2}f_R(R_0)\delta\varphi - f_R(R_0)M^{-2}\cosh^2 x\,(-\delta\ddot{\varphi}+\delta\varphi'') \\
& -f_{RR}(R_0)M^{-2}\cosh^2 x\,(-\delta\ddot{R}+\delta R'')\,.
\end{aligned} \tag{105}
$$

A convenient parametrization of perturbations is:

$$\delta\rho = \rho_0\cosh\omega\tau\cosh^\beta x, \quad \delta\varphi = \varphi_0\cosh\omega\tau\cosh^\beta x\,, \tag{106}$$

where ρ_0, ϕ_0, β are arbitrary constants.

Solving EoM, we find conditions:

$$\omega^2 = \beta^2 \tag{107}$$

and:

$$\beta = \beta_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{3}M^2\left(\frac{f_R(R_0)}{f_{RR}(R_0)}\right)}\right] \tag{108}$$

from:

$$\partial^2\delta\varphi = [\beta^2 + \beta(\beta-1)\cosh^{-2}x - \omega^2]\delta\varphi\,. \tag{109}$$

Let us note that β always has a Real part which is positive, implying exponential instabilities. In particular, for $\phi_0 < 0$ the antievaporation phase is obtained while $\phi_0 > 0$ corresponds to the evaporation. Hence, this is not enough to demonstrate that the extremal solution is unstable. So, we show the numerical solution of the horizon radius obtained by EoM perturbed up to the second order in $\delta\rho, \delta\phi$. Finally, we claim that a similar analysis in the case of the $SU(4)$-inspired model (despite the $SO(4)$ gauge group) leads to the same kind of instabilities, as can be easily checked (we mention that some solutions in other extended theories of gravity also have geodetic instabilities [65]).

7. Evaporation, Antievaporation, and Hawking's Radiation

In this section, we will discuss the suppression of Bekenstein–Hawking radiation in $f(R)$-gravity and $f(T)$-gravity.

7.1. Path Integral Approach in $f(R)$-Gravity

In general, the path integral over all euclidean metrics and matter fields $\phi_i, \psi_j, A^\mu_k, ..$ is:

$$Z_E = \int \mathcal{D}g \mathcal{D}\phi_i \mathcal{D}\psi_j \mathcal{D}A^\mu_k e^{-I[g,\phi_i,\psi_j,A^\mu_k,...]}\,, \tag{110}$$

where g the euclidean metric tensor. In semiclassical general relativity, the leading terms in the action are:

$$I_E = -\int_\Sigma \sqrt{g} d^4x \left(\mathcal{L}_m + \frac{1}{16\pi}R\right) + \frac{1}{8\pi}\int_{\partial\Sigma} \sqrt{h} d^3x (K - K^0)\,, \tag{111}$$

where $\mathcal{L}_m$ is the matter Lagrangian:

$$\mathcal{L}_m = \frac{Y^{ii'}}{2} g_{\mu\nu} \partial\phi^{i\mu} \partial\phi^{i'\nu} + ...$$

K the trace of the curvature induced on the boundary $\partial\Sigma$ of the region Σ considered, h is the metric induced on the boundary $\partial\Sigma$, K^0 is the trace of the curvature induced imbedded in flat space. The last term is a contribution from the boundary. We consider infinitesimal perturbations of matter and metric as $\phi = \phi_0 + \delta\phi$, $A = A^0 + \delta A$, (...) and $g = g_0 + \delta g$, so that:

$$I[\phi, A, ..., g] = I[\phi_0, A_0, ..g_0] + I_2[\delta\phi, \delta A, ...\delta g] + higher\ orders$$

$$I_2[\delta\phi, \delta A, .., \delta g] = I_2[\delta\phi, \delta A, ...] + I_2[\delta g]$$

$$logZ = -I[\phi_0, A^0, ..., g_0] + log \int \mathcal{D}\delta\phi \mathcal{D}\delta A(...) \mathcal{D}\delta g e^{-I_2[\delta g, \delta\phi, \delta A, ...]} \tag{112}$$

In a euclidean Schwarzschild solution, the metric has a time dimension compactified on a circle S^1, with periodicity $i\beta$, and:

$$\beta = T^{-1} = 8\pi M$$

where T, M are BH temperature and mass. The euclidean S. metric has the form:

$$ds^2_E = \left(1 - \frac{2M}{r}\right) d^2\tau + \left(1 - \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 \tag{113}$$

A convenient change of coordinates:

$$x = 4M\sqrt{1 - \frac{2M}{r}}$$

leads to:

$$ds^2_E = \left(\frac{x}{4M}\right)^2 + \left(\frac{r^2}{4M^2}\right)^2 dx^2 + r^2 d\Omega^2\,. \tag{114}$$

Equation (114) has not more a (mathematical) singularity in $r = 2M$. The boundary $\partial\Sigma$ is $S^1 \times S^2$ with S_2 with conveniently fixed radius r_0. The path integral becomes a partition function of a (canonical) ensamble, with a euclidean time related to the temperature $T = \beta^{-1}$. The leading contribution to the path integral is:

$$Z_{ES} = e^{-\frac{\beta^2}{16\pi}}\,. \tag{115}$$

Contributions to this term are only coming from surface terms in the gravitational action; i.e., bulk geometry does not contribute to Equation (115).

The average energy (or internal energy) is:

$$\langle E \rangle = -\frac{d}{d\beta}(logZ) = \frac{\beta}{8\pi}\,. \tag{116}$$

On the other hand, the free energy F is related to Z as:

$$F = -T\log Z\,. \tag{117}$$

Finally, the entropy is:

$$S = \beta(F - \langle E\rangle)\,. \tag{118}$$

As a consequence, Bekeinstein–Hawking radiation can be related to the partition function as follows:

$$S = \beta(logZ - \frac{d}{d\beta}(logZ)) = \frac{\beta^2}{16\pi} = \frac{1}{4}A \tag{119}$$

In $f(R)$-gravity, we can reformulate a euclidean approach. Through a conformal transformation, we can be more conveniently remapped $f(R)$-gravity to a scalar-tensor theory. The new relevant action in the semiclassical regime has the form:

$$I = -\frac{1}{16\pi}\int_{\Sigma} d^4x\sqrt{g}\left(f(\phi) + f'(\phi)(R-\phi)\right) - \frac{1}{8\pi}\int_{\partial\Sigma} d^3x\sqrt{h}f'(\phi)(K-K_0) \tag{120}$$

that can be remapped to the corresponding $f(R)$-gravity action as:

$$I = -\frac{1}{16\pi}\int d^4x\sqrt{-g}f(R) - \frac{1}{8\pi}\int d^4x\sqrt{h}f'(R)(K-K_0)\,. \tag{121}$$

Let us assume a generic spherical symmetric static solution for $f(R)$-gravity with a euclidean periodic time $\tau \to \tau + \beta$ where $\beta = 8\pi M$,

$$ds_E^2 = J(r)d\tau^2 + J(r)^{-1}dr^2 + r^2 d\Omega^2\,. \tag{122}$$

As in GR, the leading contribution is zero from the bulk geometry. However, the boundary term has a non-zero contribution. One can evaluate the boundary integral considering suitable surface $\partial\Sigma$. In this case, the obvious choice is a $S_2 \times S_1$ surface with with radius r of S_2. We obtain:

$$\int_{\partial\Sigma} d^3x\sqrt{h}f'(R)(K-K_0) = f'(R_0)\int_{\partial\Sigma} d^3x\sqrt{h}(K-K_0) = 8\pi\beta r - 12\pi\beta M - 8\pi\beta r\sqrt{1-\frac{r_S}{r}}\,, \tag{123}$$

where $r_S = 2M$ and R_0 is the scalar curvature of the classical black hole background. In the limit of $r \to \infty$, the resulting action, partition function, and entropy are:

$$I = f'(R_0)\beta^2, \quad Z_E = e^{-f'(R_0)\beta^2}, \quad S = 16\pi f'(R_0)\frac{A}{4}\,. \tag{124}$$

The same result was also found in [66]. This result seems in antithesis with our statements in the Introduction: Equation (124) leads to a Bekeinstein-Hawking like radiation. In fact, as mentioned, a Nariai solution is nothing but a Schwarzschild–de Sitter one with $J(r) = 1 - J(r)_{Schwarzschild} - \frac{\Lambda}{3}r^2$, with a black hole radius $r \simeq H^{-1}$ (limit of BH mass $M \to \frac{1}{3}\Lambda^{-1/2}$), with mass scale $\mathcal{M} = \Lambda$. However, result Equation (124) is based on a strong assumption on the metric Equation (122): it is assumed that the gravitational action will not lead to a dynamical evolution. For example, in Nariai solution obtained by Nojiri and Odintsov in $f(R)$-gravity, $J(r,t)$ is also a function of time: the mass parameter is a function of time $r_S(t)$. As a consequence, the analysis performed here is not valid.

As a consequence, the result obtained in this section must be considered with caution: Equation (124) can be applied if and only if one has a spherically symmetric stationary and static solution of $f(R)$-gravity.

Let us also comment that the same entropy in Equation (124) can be obtained by the Wald entropy charge integral. The Wald entropy is:

$$S_W = -2\pi \int_{S^2} d^2x\sqrt{-h^{(2)}}\left(\frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}\right)_{S^2}\hat{\epsilon}_{\mu\nu}\hat{\epsilon}_{\rho\sigma} = \frac{A}{4G_{eff}}, \quad (125)$$

where $\hat{\epsilon}$ is the antisymmetric binormal vector to the surface S^2 and:

$$(2\pi G_{eff})^{-1} = -\left(\frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}\right)_{S^2}\hat{\epsilon}_{\mu\nu}\hat{\epsilon}_{\rho\sigma}, \quad (126)$$

leading to $G_{eff} = G/f'(R_0)$ [67].

However, again, this result can be applied if and only if the spherical symmetric solution is static. As argument in Section 2, this is not the case of Nariai BHs in $f(R)$-gravity.

Let us argue on the non-applicability of these results in dynamical cases. The euclidean path integral approach is supposing a euclidean black hole inside an ideal box, in thermal equilibrium with it. However, thermodynamical limit can be applied only for systems in equilibrium, so a statistical mechanics approach can be reasonably considered. However, a dynamical space-time inside a box is generally an out-of-equilibrium system. In fact, in the next section, we will show a simple argument leading to the conclusion that Bekenstein–Hawking evaporation is suppressed by the increasing of Nariai's horizon in $f(R)$-gravity. A thermal equilibrium at $T_{B.H.}$ in an external ideal box will never be approached by a dynamical Nariai black hole.

7.2. Bekenstein–Hawking Radiation is Turned Off

Let us consider a Bekenstein–Hawking pair in a dynamical horizon. These are created nearby BH horizon, and they become real in the external gravitational background. Now, one of this pair can pass the horizon as a quantum tunnel effect, with a certain rate Γ_{bh}. However, the horizon is displacing outward the previous radius because of antievaporation effect. As a consequence, the Bekenstein–Hawking pair will be trapped in the black hole interior, in a space-like surface $\mathcal{A}_{space-like}$. From such a space-like surface, a tunnel effect of one particle is impossible. As a consequence, the only way to escape is if $\Gamma_{bh}^{-1} < \Delta t$, where Δt is the minimal effective time scale (from an external observer in a rest frame) from a $\mathcal{A}_{time-like} \rightarrow \mathcal{A}_{space-like}$ transition—from a surface on the Black Hole horizon $\mathcal{A}_{time-like}$ to a surface inside the Black Hole horizon $\mathcal{A}_{space-like}$. However, Δt can also be infinitesimal, on the order of λ, where λ is the effective separation scale between the Bekestein-Hawking pair. In fact, defining Δr as the radius increasing with Δt, it is sufficient that $\Delta r > \lambda$ in order to "eat" the Bekenstein–Hawking pair in the space-like interior. However, for black holes with a radius $r_S >> l_{Pl}$, the tunneling time is expected to be $\Gamma_{bh}^{-1} >>> \Delta t$. As a consequence, a realistic Bekenstein–Hawking emission is impossible for non-Planckian black holes. The same argument can be iteratively applied during all the evolution time and the external horizon. That Bekenstein–Hawking radiation cannot be emitted by a space-like surface was rigorously proven in [38–40], with tunneling approach, eikonal approach, and Hawking's original derivation with Bogoliubov coefficients.

Let us consider this situation from the energy conservation point of view. In stationary black holes (as in Schwarzschild in GR), the BH horizon is necessary a Killing bifurcation surface. In fact, one can define two Killing vector fields for the interior and the exterior of the BH. In the exterior region, the Killing vector ζ^μ is time-like, while in the interior it is space-like. This aspect is crucially connected with particles' energies: the energy of a particle is $E = -p_\mu\zeta^\mu$, where p^μ is the 4-momentum of the particle. As a consequence, energy is always $E > 0$ outside the horizon, while it is $E < 0$ inside the horizon. In the Killing horizon, a real particle creation is energetically possible. On the other hand, in the dynamical case, it is not possible to define a conserved energy of a particle E for a dynamical space-time; i.e., it is not possible to define a Killing vector field for time translation in a dynamical space-time. As discussed above, the Bekenstein–Hawking particle–antiparticle pair will be displaced inside the horizon in a space-like region. The creation of a real particle from a space-like region is a violation of causality. In fact, it is an acausal exchange of energy (i.e., of classical information). In fact, a particle inside the horizon is inside a light-cone with a space-like axis.

As shown in [38], one can distinguish marginally outer trapped 3-surface (we will remind at the end of this section the definition of a null trapped surface, as well as those of marginally outer and marginally inner trapped surfaces) emitting Hawking's pair (timelike surface), from the outer non-emitting one (space-like). Let us consider the null or optics Raychaduri equation for null geodesic congruences:

$$\dot{\hat{\theta}} = -\hat{\theta}^2 - 2\hat{\sigma}_{ab}\sigma^{ab} + \hat{\omega}_{cd}\hat{\omega}^{cd} - R_{\mu\nu}k^{\mu}k^{\nu}\,, \tag{127}$$

where the hats indicate that the expansion, shear, twist, and vorticity are defined for the transverse directions. The Ricci tensor encodes the dynamical proprieties of $f(R)$-gravity EoM. Let us also specify that $\dot{\hat{\theta}} = \frac{\partial}{\partial\lambda}\hat{\theta}$, where λ is the affine parameter, while k^a is $k^a = \frac{dx^a}{d\lambda}$, with $k^2 = 0$, and $\hat{\theta} = k^a_{;a}$ also defined as the relative variation of the cross sectional are:

$$\hat{\theta} = 2\frac{1}{A}\frac{dA}{d\lambda}$$

From Equation (138) one can define an emitting marginally outer 2-surface $\mathcal{A}_{time-like}$ and the non-emitting inner 2-surface $\mathcal{A}_{space-like}$. Let us call the divergence of the outgoing null geodesics $\hat{\theta}_+$ in a S^2-surface. With the increasing of the black hole gravitational field, $\hat{\theta}_+$ is decreasing (light is more bended). On the other hand, the divergence of ingoing null geodesics is $\hat{\theta}_- < 0$ everywhere, while $\hat{\theta}_+ > 0$ for $r > 2m$ in Schwarzschild. The marginally outer trapped 2-surface $\mathcal{A}^{2d}_{space-like}$ is rigorously defined as a space-like 2-sphere with:

$$\hat{\theta}_+(\mathcal{A}^{2d}_{space-like}) = 0\,. \tag{128}$$

As mentioned above, in a Schwarzschild BH the radius of the S^2-sphere $\mathcal{A}^{2d}_{space-like}$ is exactly equal to the Schwarzschild radius. As a consequence, S^2-spheres with radii smaller than $r_S = 2M$ will be trapped surfaces (TSs) with $\theta(\mathcal{A}^{2d}_{TS}) < 0$.

From the 2D definition, one can construct a generalized definition for 3D surfaces. The dynamical horizon is a marginally outer trapped 3-Surface. It is foliated by marginally trapped 2D surfaces. In particular, a dynamical horizon if it can be foliated by a chosen family of S^2 with $\theta_{(n)}$ of one null normal m_a vanishing while $\theta_{n\neq m} < 0$ for each S^2. In particular, one can distinguish among an emitting marginally outer trapped 3-surface $\mathcal{A}^{3d}_{time-like}$ and a non-emitting one $\mathcal{A}^{3d}_{time-like}$ by their derivative of $\hat{\theta}_m$ with respect to an ingoing null tangent vector n_a.

$$\hat{\theta}_m(\mathcal{A}^{3d}_{time-like}) = 0, \quad \partial\hat{\theta}_m(\mathcal{A}^{3d}_{time-like})/\partial n^a > 0\,, \tag{129}$$

while the non-emitting one is defined as:

$$\hat{\theta}_m(\mathcal{A}^{3d}_{space-like}) = 0, \quad \partial\hat{\theta}_m(\mathcal{A}^{3d}_{space-like})/\partial n^a < 0\,. \tag{130}$$

Now, armed with these definitions, let us demonstrate that the antievaporation will displace the emitting marginally trapped 3-surface to a non-emitting space-like 3-surface. We can consider the Raychaudhuri equation associated to our problem. Let us suppose an initial condition $\theta(\bar{\lambda}) > 0$ with $\bar{\lambda}$ an initial value of the affine parameter λ. In the antievaporation phenomena, the null Raychauduri equation is bounded as:

$$\frac{d\hat{\theta}}{d\lambda} < -R_{ab}k^a k^b\,. \tag{131}$$

Let us consider such an equation for an infinitesimal Δt, so that we can expand the Schwarzschild radius:

$$r_S = \frac{1}{\mathcal{M}}e^{-\phi_0} - \frac{1}{\mathcal{M}}\beta^2 e^{-\phi_0}\phi_0 t^2 + \frac{1}{6\mathcal{M}}\beta^4 e^{-\phi_0}\phi_0(-2+3\phi_0)t^4 + O(t^5)$$

and we can consider only the first 0th leading term. For any $\lambda > \bar{\lambda}$, $R_{ab}k^a k^b > C > 0$, where C is a constant associated to the 0th leading order of $R_{ab}k^a k^b$ with time. As a consequence, $\hat{\theta}$ is bounded as:

$$\hat{\theta}(\lambda) < \hat{\theta}(\lambda) + C(\lambda - \bar{\lambda}) , \tag{132}$$

leading to $\hat{\theta}(\lambda) < 0$ for $\lambda > \lambda_1 + \hat{\theta}_1/C$, where $\lambda_1, \hat{\theta}_1$ are defined in a characteristic time t_1. As a consequence, even for a small Δt, a constant 0th contribution coming from antievaporation will cause an extra effective focusing term in the Raychauduri equation. On the other hand, the dependence of the extra focusing term on time is exponentially growing. This formalizes the argument given above. As a consequence, an emitting marginally trapped 3-surface will exponentially evolve to a non-emitting marginally one. Bekenstein–Hawking emission are completely suppressed by this dynamical evolution because of space-like surface cannot emit thermal Bekenstein–Hawking radiation, mixed states (solutions of Raychauduri equations are strictly related to energy conditions; in $f(R)$-gravity, energy conditions like null energy condition are generically not satisfied [68,69]).

Now let us consider the Raychaudhuri equation in $f(T)$-gravity [6]:

$$\dot{\hat{\theta}} = -\frac{1}{3}\hat{\theta}^2 - 2\hat{\sigma}_{\mu\nu}\sigma^{\mu\nu} + \hat{\omega}_{\mu\nu}\hat{\omega}^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu - \tilde{\nabla}\tilde{a} - 2U^\nu T^\sigma_{\mu\nu}\left(\frac{1}{3}h^\mu_\sigma\tilde{\theta} + \tilde{\sigma}^\mu_\sigma + \tilde{\omega}^\mu_\sigma - U_\sigma\tilde{a}^\mu\right) , \tag{133}$$

$\hat{\theta}, \hat{\sigma}, \hat{\omega}$ are the expansion, shear, twist, vorticity, and acceleration in $f(T)$-gravity. In general, $\hat{\theta}, \hat{\sigma}, \hat{\omega}$ will be corrected by the torsion as:

$$\tilde{\theta} = \theta_{(GR)} - 2T^\rho U_\rho , \tag{134}$$

$$\tilde{\sigma}_{\mu\nu} = \sigma_{(GR)\mu\nu} + 2h^\rho_\mu h^\sigma_\nu K^\lambda_{(\rho\sigma)}U_\lambda , \tag{135}$$

$$\tilde{\omega}_{\mu\nu} = \omega_{(GR)\mu\nu} + 2h^\rho_\mu h^\sigma_\nu K^\lambda_{[\rho\sigma]}U , \tag{136}$$

$$\tilde{a}_\rho = a_{\rho(GR)} + U^\mu K^\sigma_{\mu\rho}U_\sigma , \tag{137}$$

where U^μ is the four velocity and:

$$\tilde{\nabla}_\mu U_\nu = \tilde{\sigma}_{\mu\nu} + \frac{1}{3}h_{\mu\nu}\tilde{\theta} + \tilde{\omega}_{\mu\nu} - U_\mu\tilde{a}_\nu$$

$\dot{\hat{\theta}} = \frac{\partial}{\partial\lambda}\hat{\theta}$, where λ is the affine parameter in the the optical null case, and $U^a = k^a$ is $k^a = \frac{dx^a}{d\lambda}$, with $k^2 = 0$, and:

$$\hat{\theta} = k^a_{;a} = 2\frac{1}{\Sigma}\frac{d\Sigma}{d\lambda} .$$

We can define an emitting marginally outer 2-surface $\Sigma_{time-like}$ and the non-emitting inner 2-surface $\Sigma_{space-like}$.

The marginally outer trapped 2-surface $\Sigma^{2d}_{space-like}$ has a topology of space-like 2-sphere with the condition:

$$\hat{\theta}_+(\Sigma^{2d}_{space-like}) = 0 \tag{138}$$

where $\hat{\theta}_+$ in a S^2-surface is the divergence of the outgoing null geodesics.

Let us remember that $\hat{\theta}_+$ decrease with the increasing of the gravitational field. $\hat{\theta}_+ > 0$ for $r > 2M$ in the Schwarzschild case. The opposite variable is the divergence of ingoing null geodesics $\hat{\theta}_-$, $\hat{\theta}_- < 0$ everywhere.

The radius of the S^2-sphere $\Sigma^{2d}_{space-like}$ coincides with the Schwarzschild radius. S^2-spheres with radii smaller than $r_S = 2M$ will be trapped surfaces (TSs) (a trapped null surface is a set of points individuating a closed surface on which future-oriented light rays are converging. In this respect, the light rays are actually moving inwards. For any compact, orientable, and space-like surface, a null trapped surface can be recovered by first finding its outward pointing normal vectors, and then by studying whether the light rays directed along these latter are converging or diverging. We will say that, given a null congruence orthogonal to a space-like two-surface that has a negative expansion rate, there exists a surface that is "trapped". For these peculiar features, trapped null surfaces are often deployed in the definition of apparent horizon surrounding black holes); i.e., $\theta(\Sigma^{2d}_{TS}) < 0$.

We can generalize these topological definitions for 3D surfaces.

The dynamical horizon is a marginally outer trapped 3D surface. It is foliated by marginally trapped 2D surfaces. In particular, a dynamical horizon can be foliated by a chosen family of S^2 with $\theta_{(n)}$ of a null normal vector m_a vanishing while $\theta_{n\neq m} < 0$, for each S^2. In particular, one can distinguish among an emitting marginally outer trapped 3D surface $\Sigma^{3d}_{time-like}$ and a non-emitting one $\Sigma^{3d}_{time-like}$ by their derivative of $\hat{\theta}_m$ with respect to an ingoing null tangent vector n_a.

$$\hat{\theta}_m(\Sigma^{3d}_{time-like}) = 0, \quad \frac{\partial\hat{\theta}_m(\Sigma^{3d}_{time-like})}{\partial n^a} > 0 \tag{139}$$

and the non-emitting one is defined as:

$$\hat{\theta}_m(\Sigma^{3d}_{space-like}) = 0, \quad \frac{\partial\hat{\theta}_m(\Sigma^{3d}_{space-like})}{\partial n^a} < 0 \tag{140}$$

Now, adopting these definitions, we demonstrate that the antievaporation will transmute the emitting marginally trapped 3D surface to a non-emitting space-like 3D surface. We can consider the Raychaudhuri–Landau equation associated to our problem. Let us suppose an initial condition $\theta(\bar{\lambda}) > 0$ with $\bar{\lambda}$ an initial value of the affine parameter λ. In the antievaporation phenomena, the null Raychauduri–Landau equation is bounded as:

$$\frac{d\hat{\theta}}{d\lambda} < -\mathcal{R}_{ab}k^a k^b \tag{141}$$

where $\mathcal{R}_{ab}k^a k^b$ is the effective contraction of the Ricci tensor with null 4-vectors, corrected by torsion contributions:

$$\begin{aligned}\mathcal{R}_{\mu\nu}k^\mu k^\nu = {} & R_{\mu\nu}k^\mu k^\nu + \tfrac{2}{3}T^\rho k_\rho - 2h^\rho_\mu h^\sigma_\nu K^\lambda_{(\rho\sigma)}k_\lambda - 2h^\rho_\mu h^\sigma_\nu K^\lambda_{[\rho\sigma]}k_\lambda + k^\mu K^\sigma_{\mu\rho}k_\sigma k^\rho \\ & +2k^\nu T^\sigma_{\mu\nu}\left(-\tfrac{2}{3}h^\mu_\sigma T^\rho k_\rho + 2h^\rho_\mu h^\sigma_\nu K^\lambda_{(\rho\sigma)}k_\lambda + 2h^\rho_\mu h^\sigma_\nu K^\lambda_{[\rho\sigma]}k_\lambda - k_\sigma k^\mu K^\sigma_{\mu\rho}k^\rho\right).\end{aligned} \tag{142}$$

Let us consider the antievaporation case: for $\lambda > \bar{\lambda}$, it is $\mathcal{R}_{ab}k^a k^b > K > 0$, where K is the 0-th leading order of the scalar function $\mathcal{R}_{ab}k^a k^b(t)$. So that

$$\hat{\theta}(\lambda) < \hat{\theta}(\lambda) - K(\lambda - \bar{\lambda}), \tag{143}$$

leading to $\hat{\theta}(\lambda) < 0$ for $\lambda > \lambda_0 + \hat{\theta}_0/K$, where $\lambda_0, \hat{\theta}_0$ are defined at a characteristic time t_0. For a small δt, a constant 0th contribution sourced by the torsion will cause an effective focusing term in the Raychauduri equation. This phenomena is exponentially growing in time. So, an emitting marginally trapped 3D surface will exponentially evolve to a non-emitting marginally trapped one.

Now let us consider a Bekenstein–Hawking pair in an antievaporating solution. They are imagined to be created in the black hole horizon as a virtual pair. Then, the external gravitational field can promote them to be real particles. Then, a particle of this pair can quantum tunnel outside the black hole horizon with a certain characteristic time scale τ_{bh}. With an understood correction to the black hole entropy formula, this conclusion seems compatible with Nariai solutions in diagonal tetrad choice. Bekenstein–Hawking's calculations are performed in the limit of a static horizon and a black hole in thermal equilibrium with the environment. This approximation cannot work for antievaporating black holes. In fact, the horizon is displacing outward the previous radius. The Bekenstein–Hawking pair will be trapped in the black hole interior, foliated in space-like surfaces $\Sigma_{space-like}$. However, from a space-like surface, the tunneling effect of a particle is impossible—otherwise, causality will be violated. As a consequence, Bekenstein–Hawking radiation requests $\tau_{bh} < \delta t$, where δt is the minimal effective time scale in the external rest frame for a $\Sigma_{time-like} \to \Sigma_{space-like}$ transition. The Bekenstein–Hawking radiation is exponentially turned off with time. In fact, Bekenstein–Hawking radiation cannot be emitted from a space-like surface in all possible approaches, as proven in [38–40].

7.3. A New Radiation in Non-Diagonal Evaporating Solutions

Now, let us comment on what happens in the opposite case: evaporating solutions. In this case, $f(T)$-gravity will source an extra anti-focalizing term in the null Raychauduri equation. This will cause exactly the opposite transition: a null-like horizon is pushed out the black hole radius, and it will become time-like. Defining δt as the transition time $\Sigma_{space-like} \to \Sigma_{time-like}$, Bekenstein–Hawking effect will happen if $\tau_{bh} << \delta t$. However, with $\delta t < \tau_{bh}$, the Bekenstein–Hawking pair is pushed-off from the black hole horizon. In other words, they will *both* be emitted from the black hole. They can annihilate outside the black hole, producing radiation. Contrary to Bekenstein–Hawking radiation, unitarity is not violated in black hole formation during the gravitational collapse. In fact, the firewall paradox is exactly coming by from the entanglement of the two pairs combined by the fact that one is falling inside the interior while its twin tunnels out. In our case, both are emitted outwards because of evaporation effects. In the Bekenstein–Hawking case, outgoing information is exactly copied with the interior information. In our case, there is no entanglement among black hole interior and the external environment. This radiation does not introduce any new information paradoxes.

8. Brane-Worlds Instabilities

In this section, we will study the presence of evaporation and antievaporation instabilities in brane-world scenarios [35]. Let us consider the $F(R)$-gravity theory in five dimensions:

$$S = \frac{1}{2\kappa_5^2} \int \sqrt{-g} \left[F^{(5)}(R) + S_m \right] , \tag{144}$$

where κ_5 is the five-dimensional gravitational constant and S_m is the action of the matter. The equations of motion in the vacuum are given by:

$$F_R^{(5)}(R) \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = \frac{1}{2} g_{\mu\nu} \left[F^{(5)}(R) - R F_R^{(5)}(R) \right] + \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \partial^2 \right] F_R^{(5)}(R) , \tag{145}$$

where $F_R^{(5)} = dF^{(5)}/dR$. Especially if we assume that the metric is covariantly constant (that is, $R_{\mu\nu} = K g_{\mu\nu}$ with a constant K), we find:

$$0 = R F_R^{(5)}(R) - \frac{5}{2} F^{(5)}(R) . \tag{146}$$

We denote the solution of Equation (146) as $R = R_0$ and define the length parameter l by $R_0 = 20/l^2$.

We should note that the metric of the Schwarzschild–de Sitter solution is covariantly constant and given by,

$$ds^2_{SdS,(5)} = \frac{1}{h(a)} da^2 - h(a) dt^2 + a^2 d\Omega^2_{(3)} , \quad h(a) = 1 - \frac{a^2}{l^2} - \frac{16\pi G_{(5)} M}{3a^2} . \tag{147}$$

Here M corresponds to the mass of the black hole and $G_{(5)}$ is defined by $8\pi G_{(5)} = \kappa_5^2$. The space-time expressed by the metric Equation (147) has two horizons at:

$$a^2 = a_\pm^2 = \frac{l^2}{2} \left\{ 1 \pm \sqrt{1 - \frac{64\pi G_{(5)} M}{3 l^2}} \right\} . \tag{148}$$

The two horizons degenerate in the limit,

$$\frac{64\pi G_{(5)} M}{3l^2} \to 1\,, \tag{149}$$

and we obtain the degenerate Schwarzschild–de Sitter (Nariai) solution. The metric in the Nariai space-time is given by:

$$ds^2 = \frac{1}{\Lambda}\left(- \sin^2\chi d\psi^2 + d\chi^2 + d\Omega_{(3)}^2 \right)\,, \tag{150}$$

where there are the horizons at $\chi = 0$, π, and $\Lambda = \frac{2}{l^2}$. Let us perform the coordinate transformation $\chi = \arccos\zeta$,

$$ds^2 = -\frac{1}{\Lambda}\left(1-\zeta^2\right) d\psi^2 + \frac{d\zeta^2}{\Lambda\left(1-\zeta^2\right)} + \frac{1}{\Lambda} d\Omega_{(3)}^2\,, \tag{151}$$

which is singular at $\zeta = \pm 1$. By changing the coordinate $\zeta = \tanh\xi$, the metric can be rewritten as,

$$ds^2 = \frac{1}{\Lambda\cosh^2\xi}\left(-d\psi^2 + d\xi^2\right) + \frac{1}{\Lambda} d\Omega_{(3)}^2\,. \tag{152}$$

We often analytically continue the coordinates by:

$$\psi = ix\,, \quad \zeta = i\tau\,, \tag{153}$$

and we obtain the following metric:

$$ds^2 = -\frac{1}{\Lambda\cos^2\tau}\left(-d\tau^2 + dx^2\right) + \frac{1}{\Lambda} d\Omega_{(3)}^2\,. \tag{154}$$

Of course, after the analytic continuation, the obtained space is a solution of the equations although the topology is changed. This expression of the metric was used in [18].

In order to consider the perturbation, we now consider the general metric in the following form,

$$ds^2 = \mathrm{e}^{2\rho(x,\tau)}\left(-d\tau^2 + dx^2\right) + \mathrm{e}^{-2\phi(x,\tau)} d\Omega_{(3)}^2\,, \tag{155}$$

which generalizes the Nariai metric in Equation (154) with generic functions $\rho(x,\tau), \phi(x,\tau)$.

Then the equation of motion can be decomposed in components as:

$$\begin{aligned} 0 = & -\tfrac{\mathrm{e}^{2\rho}}{2} F^{(5)} - \left(-\ddot\rho + 3\ddot\phi + \rho'' - 3\dot\phi^2 - 3\dot\rho\dot\phi - 3\rho'\phi'\right) F_R^{(5)} + \ddot F_R^{(5)} \\ & -\dot\rho \dot F_R^{(5)} - \rho'\left(F_R^{(5)}\right)' + \mathrm{e}^{2\phi}\left[-\tfrac{\partial}{\partial\tau}\left(\mathrm{e}^{-2\phi}\dot F_R^{(5)}\right) + \left(\mathrm{e}^{-2\phi}(F_R^{(5)})'\right)'\right]\,, \end{aligned} \tag{156}$$

$$\begin{aligned} 0 = & \tfrac{\mathrm{e}^{2\rho}}{2} F^{(5)} - \left(-\rho'' + 3\phi'' + \ddot\rho - 3\phi'^2 - 3\rho'\phi' - 3\dot\rho\dot\phi\right) F_R^{(5)} + {F_R^{(5)}}'' - \dot\rho\dot F_R^{(5)} - \rho'\left(F_R^{(5)}\right)' \\ & -\mathrm{e}^{2\phi}\left[-\tfrac{\partial}{\partial\tau}\left(\mathrm{e}^{-2\phi}\dot F_R^{(5)}\right) + \left(\mathrm{e}^{-2\phi}\left(F_R^{(5)}\right)'\right)'\right]\,, \end{aligned} \tag{157}$$

$$0 = -\left(3\dot\phi' - 3\phi'\dot\phi - 3\rho'\dot\phi - 3\dot\rho\phi'\right) F_R^{(5)} + \frac{\partial^2 F_R^{(5)}}{\partial x\partial\tau} - \dot\rho\left(F_R^{(5)}\right)' - \rho'\dot F_R^{(5)}\,, \tag{158}$$

$$0 = \frac{e^{-2\phi}}{2} F^{(5)} - e^{-2(\rho+\phi)} \left(-\ddot{\phi} + \phi'' + 3\dot{\phi}^2 - 3\phi'^2\right) F_R^{(5)} - F_R^{(5)} + e^{-2(\rho+\phi)} \left(\dot{\phi} \dot{F}_R^{(5)} - \phi' F_{RR}^{(5)\prime}\right) \\ -e^{-2\rho} \left[-\frac{\partial}{\partial\tau}\left(e^{-2\phi} \dot{F}_R^{(5)}\right) + \left(e^{-2\phi} F_{RR}^{(5)\prime}\right)'\right], \tag{159}$$

(see Appendix E for more technical details), where $F' = \frac{\partial F}{\partial x}$ and $\dot{F} = \frac{\partial F}{\partial \tau}$ and we have used the expressions of the curvatures Equation (A33) in Appendix A.

We consider the perturbations at the first order around the Nariai background Equation (154) with $R_0 = \frac{20}{l^2}$,

$$0 = \frac{-F_R^{(5)}(R_0) + 2\Lambda F_{RR}^{(5)}(R_0)}{2\Lambda \cos^2\tau} \delta R - \frac{F^{(5)}(R_0)}{\Lambda \cos^2\tau} \delta\rho - F_R^{(5)}(R_0)\left(-\delta\ddot{\rho} + 3\delta\ddot{\phi} + \delta\rho'' - 3\tan\tau\delta\dot{\phi}\right) \\ - \tan\tau F_{RR}^{(5)}(R_0)\delta\dot{R} + F_{RR}^{(5)}(R_0)\delta R'', \tag{160}$$

$$0 = -\frac{-F_R^{(5)}(R_0) + 2\Lambda F_{RR}^{(5)}(R_0)}{2\Lambda \cos^2\tau} \delta R + \frac{F^{(5)}(R_0)}{\Lambda \cos^2\tau} \delta\rho - F_R^{(5)}(R_0)\left(\delta\ddot{\rho} + 3\delta\phi'' - \delta\rho'' - 3\tan\tau\delta\dot{\phi}\right) \\ - \tan\tau F_{RR}^{(5)}(R_0)\delta\dot{R} + F_{RR}^{(5)}(R_0)\delta R'', \tag{161}$$

$$0 = -3F_R^{(5)}(R_0)\left(\delta\dot{\phi}' - \tan\tau\delta\phi'\right) + F_{RR}^{(5)}(R_0)\left(\delta\dot{R}' - \tan\tau\delta R'\right), \tag{162}$$

$$0 = -\frac{-F_R^{(5)}(R_0) + 2\Lambda F_{RR}^{(5)}(R_0)}{2\Lambda \cos^2\tau} \delta R - \frac{F^{(5)}(R_0)}{\Lambda \cos^2\tau} \delta\phi - F_R^{(5)}(R_0)\left(-\delta\ddot{\phi} + \delta\phi''\right) \\ -F_{RR}^{(5)}(R_0)(-\delta\ddot{R} + \delta R''). \tag{163}$$

The perturbation of the scalar curvature δR is given in terms of $\delta\rho$ and $\delta\phi$ as follows,

$$\delta R = 4\Lambda(-\delta\rho + \delta\phi) + \Lambda\cos^2\tau(2\delta\ddot{\rho} - 2\delta\rho'' - 6\delta\ddot{\phi} + 6\delta\phi''). \tag{164}$$

Therefore, the four equations of motion include only two $\delta\phi$ and $\delta\rho$, which tell that only two equations in the four equations should be independent.

One can find that Equation (162) can be easily integrated:

$$\delta R = 3\frac{F_R^{(5)}(R_0)}{F_{RR}^{(5)}(R_0)}\delta\phi + \frac{c_1(x)}{\cos\tau} + c_2(\tau). \tag{165}$$

Here $c_1(x)$ and $c_2(\tau)$ are arbitrary functions, but because δR should vanish when both of $\delta\rho$ and $\delta\phi$ vanish as seen from Equation (164), we can put $c_1(x) = c_2(\tau) = 0$.

Then, one can directly consider Equation (164): Substituting in it $\delta R(\delta\phi)$ obtained in Equation (165), we find a simple equation:

$$\left(\partial^2 + \frac{M^2}{\cos^2\tau}\right)\delta\phi = 0, \quad \partial^2 \equiv -\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial x^2}. \tag{166}$$

Here:

$$M^2 = \frac{1}{2}\frac{4\alpha - 1}{\alpha}, \quad \alpha = \frac{4\Lambda F_{RR}^{(5)}(R_0)}{F_R^{(5)}(R_0)} = \frac{F(R_0)F_{RR}(R_0)}{[F_R(R_0)]^2}. \tag{167}$$

Equation (166) is nothing but a time-dependent Klein–Gordon equation for the $\delta\phi$ mode, with an effective oscillating mass term in time. An explicit solution of Equation (166) is given by:

$$\delta\phi = \phi_0 \cos(\beta x)\cos^\beta\tau. \tag{168}$$

Here β is given by solving the equation $M^2 = \beta(\beta - 1)$. The antievaporation corresponds to the increasing of the radius of the apparent horizon, which is defined by the condition:

$$\nabla\delta\phi \cdot \nabla\delta\phi = 0\,. \tag{169}$$

In other words, it is imposed that the (flat) gradient of the two-sphere size is null. By using the solution in Equation (168), we find $\tan\beta x = \tan\tau$; that is, $\beta x = \tau$. Therefore on the apparent horizon, we find:

$$\delta\phi = \phi_0 \cos^{\beta+1}\tau\,. \tag{170}$$

Because the horizon radius r_H is given by $r_H = \mathrm{e}^{-\phi}$, we find:

$$r_H = \frac{\mathrm{e}^{-\phi_0 \cos^{\beta+1}\tau}}{\sqrt{\Lambda}}\,. \tag{171}$$

Then, if $\beta < -1$, the horizon grows up, which corresponds to the antievaporation depending on the sign of ϕ_0. The sign could be determined by the initial condition of the perturbation. On the other hand, the case in which β, ω are complex parameters is also possible. In this case, solutions of perturbed equations read:

$$\delta\phi = \mathrm{Re}\left\{(C_1\mathrm{e}^{\beta t} + C_2\mathrm{e}^{-\beta t})\mathrm{e}^{\beta x}\right\}\,, \tag{172}$$

where $C_{1,2}$ are complex numbers. $\delta\phi$ always increase in time for $C_1 \neq 0$ because $\mathrm{Re}\beta > 0$. This means that the Nariai solution is also unstable in this region of parameters. A particular class among possible complex parameter solutions is:

$$\delta\phi = \phi_0\left\{\mathrm{e}^{\frac{-t+x}{2}}\left(\cos\frac{\gamma(t-x)}{2} + \frac{1}{\gamma}\sin\frac{\gamma(t-x)}{2}\right) + \mathrm{e}^{\frac{t+x}{2}}\left(\cos\frac{\gamma(t+x)}{2} - \frac{1}{\gamma}\sin\frac{\gamma(t+x)}{2}\right)\right\}\,, \tag{173}$$

where $\beta \equiv \frac{1}{2}(1 + i\gamma)$ and $\gamma \equiv \pm\sqrt{\frac{2-9\alpha}{\alpha}}$.

On the horizon, the fluctuations must satisfy the condition $\frac{\phi_0^2}{2}\gamma^2\mathrm{e}^x\sin\frac{\gamma(t-x)}{2}\sin\frac{\gamma(t+x)}{2} = 0$, which corresponds to two classes of solutions with $x = \mp t + \frac{2n\pi}{\gamma}$,

$$\delta\phi = \phi_0(-1)^n\left\{\mathrm{e}^{\frac{n\pi}{\gamma}} + \mathrm{e}^{\mp t + \frac{n\pi}{\gamma}}\left(\cos\gamma t \mp \frac{1}{\gamma}\sin\gamma t\right)\right\}\,, \tag{174}$$

which implies an oscillating horizon radius.

Let us consider a class of $F^{(5)}(R)$ models:

$$F^{(5)}(R) = \frac{R}{2\kappa^2} + f_2R^2 + f_0\mathcal{M}^{5-2n}R^n\,. \tag{175}$$

Here f_2 and $\mathcal{M}$ are constants with a mass dimension and f_0 is a dimensionless constant. In this case, α is given by:

$$\alpha = \frac{4\Lambda\left(2f_2 + n(n-1)f_0\mathcal{M}^{5-2n}R_0^{n-2}\right)}{1/2\kappa^2 + 2f_2R_0 + nf_0M^{5-2n}R_0^{n-1}}\,. \tag{176}$$

Then β is given by:

$$\beta^2 - \beta = \frac{1}{2\alpha}(4\alpha - 1)\,; \tag{177}$$

that is,

$$\beta_\pm = \frac{1}{2}\left(1 \pm \sqrt{\frac{9\alpha - 2}{\alpha}}\right)\,. \tag{178}$$

Then, the condition of the antievaporation $\beta < -1$ (for $\phi_0 < 0$) can be satisfied only by β_- and for $\alpha < 0$. On the other hand, for β as a complex parameter in Equation (174), the oscillation instabilities are obtained for $0 < \alpha < 2/9$. In this case, evaporation and antievaporation phases are iterated.

Brane Dynamics in the Bulk

We now consider the $F^{(d+1)}(R)$ gravity in the $d+1$ dimensional space-time M with d dimensional boundary B, whose action is given by:

$$S = \frac{1}{2\kappa^2}\int_M d^{d+1}x\sqrt{-g}F^{(d+1)}(R)\,, \tag{179}$$

which can be rewritten in the scalar-tensor form. We begin by rewriting the action Equation (179) by introducing the auxiliary field A as follows:

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}\left\{F^{(d+1)\prime}(A)\left(R-A\right)+F^{(d+1)}(A)\right\}\,. \tag{180}$$

By the variation of the action with respect to A, we obtain the equation $A = R$ and by substituting the obtained expression $A = R$ into the action Equation (180), we find that the action in Equation (179) is reproduced. If we rescale the metric by conformal transformation,

$$g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}\,, \quad \sigma = -\ln F^{(d+1)\prime}(A)\,, \tag{181}$$

we obtain the action in the Einstein frame,

$$\begin{aligned} S_E &= \tfrac{1}{2\kappa^2}\textstyle\int_M d^{d+1}x\sqrt{-g}\left(R-(d-1)\partial^2\sigma - \tfrac{(d-2)(d-1)}{4}\partial^\mu\sigma\partial_\mu\sigma - V(\sigma)\right) \\ &= \tfrac{1}{2\kappa^2}\textstyle\int_M d^{d+1}x\sqrt{-g}\left(R - \tfrac{(d-2)(d-1)}{4}\partial^\mu\sigma\partial_\mu\sigma - V(\sigma)\right) + (d-1)\int_B d^d x\sqrt{-\hat{g}}n^\mu\partial_\mu\sigma\,, \\ V(\sigma) &= e^{\sigma}g\left(e^{-\sigma}\right) - e^{2\sigma}f\left(g\left(e^{-\sigma}\right)\right) = \tfrac{A}{F^{(d+1)\prime}(A)} - \tfrac{F^{(d+1)}(A)}{F^{(d+1)\prime}(A)^2}\,. \end{aligned} \tag{182}$$

Here $g\left(e^{-\sigma}\right)$ is given by solving the equation $\sigma = -\ln F^{(d+1)\prime}(A)$ as $A = g\left(e^{-\sigma}\right)$. By the integration of the term $\partial^2\sigma$, there appears the boundary term, where n^μ is the unit vector perpendicular to the boundary and the direction of the vector is inside. Furthermore, $\hat{g}_{\mu\nu}$ is the metric induced on the boundary, $\hat{g}_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. The existence of the boundary makes the variational principle with respect to σ ill-defined; we cancel the term by introducing the boundary action:

$$S_B = -(d-1)\int_B d^d x\sqrt{-\hat{g}}n^\mu\partial_\mu\sigma\,. \tag{183}$$

Then, one may forget the boundary term,

$$S_E \to S_E + S_B = \frac{1}{2\kappa^2}\int_M d^{d+1}x\sqrt{-g}\left(R - \frac{(d-2)(d-1)}{4}\partial^\mu\sigma\partial_\mu\sigma - V(\sigma)\right)\,. \tag{184}$$

As is well-known, because the scalar curvature R includes the second derivative term, the variational principle is still ill-defined in the space-time with boundary [70] (see also Refs. [32,33,71,72]). Because the variation of the scalar curvature with respect to the metric is given by:

$$R = -\delta g_{\mu\nu}R^{\mu\nu} + g^{\sigma\nu}\left(\nabla_\mu\delta\Gamma^\mu_{\sigma\nu} - \nabla_\sigma\delta\Gamma^\mu_{\mu\nu}\right)\,, \tag{185}$$

the variation of the action with respect to the metric is given by:

$$\delta S_{\mathrm{E}} = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} Q^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{2\kappa^2} \int_B d^d x \sqrt{-\hat{g}} g^{\sigma\nu} \left(-n_\mu \delta \Gamma^\mu_{\sigma\nu} + n_\sigma \delta \Gamma^\mu_{\mu\nu} \right) . \tag{186}$$

Here the Einstein equation in the bulk is given by $Q_{\mu\nu} = 0$. Then, the variational principle becomes well-defined if we add the following boundary term:

$$\tilde{S}_b = -\frac{1}{2\kappa^2} \int_B d^d x \sqrt{-\hat{g}} g^{\sigma\nu} \left(-n_\mu \Gamma^\mu_{\sigma\nu} + n_\sigma \Gamma^\mu_{\mu\nu} \right) . \tag{187}$$

Although the above boundary term Equation (187) is not invariant under the reparametrization, because:

$$\nabla_\mu n_\nu = \partial_\mu n_\nu - \Gamma^\lambda_{\mu\nu} n_\lambda \, , \quad \nabla_\mu n^\nu = \partial_\mu n^\nu + \Gamma^\nu_{\mu\lambda} n^\lambda \, , \tag{188}$$

we find:

$$g^{\sigma\nu} \left(-n_\mu \Gamma^\mu_{\sigma\nu} + n_\sigma \Gamma^\mu_{\mu\nu} \right) = -\partial_\mu n^\mu - 2 g^{\delta\rho} \partial_\delta n_\rho \nabla_\mu n^\mu \, , \tag{189}$$

which is just equal to $\nabla_\mu n^\mu$ on the boundary [32,33,70–72] . Therefore we can replace the boundary term Equation (187) by the Gibbons–Hawking boundary term,

$$S_{GH} = \frac{1}{\kappa^2} \int_B d^d x \sqrt{-\hat{g}} \nabla_\mu n^\mu \, . \tag{190}$$

Let the boundary be defined by a function $f(x^\mu)$ as $f(x^\mu) = 0$. Then, by the analogy of the relation between the electric field and the electric potential in the electromagnetism, we find that the vector $\left(\partial_\mu f\left(x^\mu\right)\right)$ is perpendicular to the boundary because $dx^\mu \partial_\mu f\left(x^\mu\right) = 0$ on the boundary, which gives an expression for n_μ as:

$$n_\mu = \frac{\partial_\mu f}{\sqrt{g^{\rho\sigma} \partial_\rho f \partial_\sigma f}} \, . \tag{191}$$

Then, with respect to the variation of the metric, the variation of n^μ is given by:

$$\delta n_\mu = \frac{1}{2} \frac{\partial_\mu f}{\left(g^{\rho\sigma} \partial_\rho f \partial_\sigma f \right)^{\frac{3}{2}}} \partial^\tau f \partial^\eta f \delta g_{\tau\eta} = \frac{1}{2} n_\mu n^\rho n^\sigma \delta g_{\rho\sigma} \, . \tag{192}$$

By using the expression in Equation (192), one finds the variation of $\nabla_\mu n^\mu$ with respect to the metric,

$$\delta \left(2 \nabla_\mu n^\mu \right) = -2 \delta g_{\mu\nu} n^\mu n^\nu - n^\mu \nabla^\nu \delta g_{\mu\nu} - g^{\mu\nu} n_\rho \delta \Gamma^\rho_{\mu\nu} + n^\nu \delta \Gamma^\mu_{\mu\nu} \, . \tag{193}$$

The last two terms in Equation (193) are necessary to make the variational principle well-defined, but the second term $n^\mu \nabla^\nu \delta g_{\mu\nu}$ may also violate the variational principle. However, by using the reparametrization invariance, we can choose the gauge condition so that $\nabla^\nu \delta g_{\mu\nu} = 0$.

We may also add the following boundary term:

$$S_{\mathrm{BD}} = \int_B d^d x \sqrt{-\hat{g}} \mathcal{L}_{\mathrm{B}} \, . \tag{194}$$

The variation of the total action,

$$S_{\mathrm{total}} = S_{\mathrm{E}} + S_B + S_{\mathrm{GH}} + S_{\mathrm{BD}} \, , \tag{195}$$

is given by:

$$\delta S_{\mathrm{total}} = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} Q^{\mu\nu} \delta g_{\mu\nu} + \int_B d^d x \sqrt{-\hat{g}} \left[\frac{1}{2\kappa^2} \left(\frac{1}{2} \mathcal{K} \hat{g}^{\mu\nu} - \mathcal{K}^{\mu\nu} \right) + \frac{1}{2} T^{\mu\nu}_{\mathrm{B}} \right] \delta g_{\mu\nu} \, . \tag{196}$$

Here we have defined the extrinsic curvature by $\mathcal{K}_{\mu\nu} \equiv \nabla_\mu n_\nu$ and $\mathcal{K} \equiv g^{\mu\nu}\mathcal{K}_{\mu\nu}$. We also wrote the variation of $S_{\rm BD}$ as:

$$\delta S_{\rm BD} = \frac{1}{2}\int_B d^d x\sqrt{-\hat{g}}T_{\rm B}^{\mu\nu}\delta g_{\mu\nu}\,. \tag{197}$$

Then, on the boundary, we obtain the following equation:

$$0 = \frac{1}{2}\mathcal{K}\hat{g}^{\mu\nu} - \mathcal{K}^{\mu\nu} + \kappa^2 T_{\rm B}^{\mu\nu}\,, \tag{198}$$

which may be called the brane equation. Especially if the boundary action $S_{\rm BD}$ consists of only the brane tension $\tilde{\kappa}$,

$$S_{\rm B} = \frac{\tilde{\kappa}}{\kappa^2}\int_B d^d x\sqrt{-\hat{g}}\,, \tag{199}$$

we find:

$$0 = \frac{1}{2}\mathcal{K}\hat{g}^{\mu\nu} - \mathcal{K}^{\mu\nu} + \tilde{\kappa}g^{\mu\nu}\,, \tag{200}$$

which can be rewritten as,

$$0 = \frac{2}{d-2}\tilde{\kappa}\hat{g}^{\mu\nu} - \mathcal{K}^{\mu\nu}\,. \tag{201}$$

If we consider the model which is given by gluing two space-time as in the Randall–Sundrum model [13,14], the contribution from the bulk doubles and therefore the Gibbons–Hawking term also doubles:

$$0 = \frac{2}{d-2}\tilde{\kappa}\hat{g}^{\mu\nu} - 2\mathcal{K}^{\mu\nu}\,. \tag{202}$$

Let us consider the following five-dimensional geometry:

$$ds_5^2 = g_{\mu\nu}dx^\mu dx^\nu = -\mathrm{e}^{2\rho}dt^2 + \mathrm{e}^{-2\rho}da^2 + a^2 d\Omega_3^2\,. \tag{203}$$

Here $d\Omega_3^2 = \tilde{g}_{ij}dx^i dx^j$ expresses the metric of the unit sphere in two dimensions. We now introduce a new time variable τ so that the following condition is satisfied,

$$-\mathrm{e}^{2\rho}\left(\frac{\partial t}{\partial\tau}\right)^2 + \mathrm{e}^{-2\rho}\left(\frac{\partial a}{\partial\tau}\right)^2 = -1\,. \tag{204}$$

Then, we obtain the following FRW metric:

$$ds_4^2 = \tilde{g}_{\rm ij}dx^i dx^j = -d\tau^2 + a^2 d\Omega_3^2\,. \tag{205}$$

Then,

$$n^\mu = \left(-\mathrm{e}^{-2\rho}\frac{\partial a}{\partial\tau}, -\mathrm{e}^{2\rho}\frac{\partial t}{\partial\tau}, 0, 0, 0\right)\,, \tag{206}$$

because:

$$\mathcal{K}_{ij} = \frac{\kappa}{2}\mathrm{e}^{4\rho}a\tilde{g}_{ij}\frac{dt}{d\tau}\,. \tag{207}$$

From Equation (200), we obtain:

$$\mathrm{e}^{2\rho}\frac{dt}{d\tau} = -\frac{\tilde{\kappa}}{2}a\,. \tag{208}$$

Using Equation (204) and defining the Hubble rate by $H = \frac{1}{a}\frac{da}{d\tau}$, one finds the following FRW equation for the brane:

$$H^2 = -\frac{\mathrm{e}^{2\rho(a)}}{a^2} + \frac{\tilde{\kappa}^2}{4}\,. \tag{209}$$

Then, in the case of the Schwarzschild–de Sitter black hole,

$$e^{2\rho} = \frac{1}{a^2}\left(-\mu + a^2 - \frac{a^4}{l_{\mathrm{dS}}^2}\right), \tag{210}$$

we obtain:

$$H^2 = \frac{1}{l_{\mathrm{dS}}^2} - \frac{1}{a^2} + \frac{\mu}{a^4} + \frac{\kappa^2}{4}. \tag{211}$$

Here l_{dS} is the curvature radius of the de Sitter space-time and μ is the black hole mass. On the other hand, in the Schwarzschild–(anti-)de Sitter (AdS) black hole,

$$e^{2\rho} = \frac{1}{a^2}\left(-\mu + a^2 + \frac{a^4}{l_{\mathrm{AdS}}^2}\right), \tag{212}$$

we obtain,

$$H^2 = -\frac{1}{l_{\mathrm{AdS}}^2} - \frac{1}{a^2} + \frac{\mu}{a^4} + \frac{\kappa^2}{4}. \tag{213}$$

In the Jordan frame, the metric is given by:

$$ds_{\mathrm{J}4}^2 = F^{(5)\prime}(R)ds_4^2 = \left(-d\tau^2 + a^2 d\Omega_3^2\right). \tag{214}$$

Because the scalar curvature is a constant in the Schwarzschild–(anti-)de Sitter space-time, $F^{(5)\prime}(R)$ can be absorbed into the redefinition of τ and a:

$$d\tilde{\tau} \equiv dt\sqrt{F^{(5)\prime}(R)}, \quad \tilde{a} \equiv a\sqrt{F^{(5)\prime}(R)}. \tag{215}$$

Then, the qualitative properties are not changed in the Jordan frame compared with the Einstein frame. We should also note that the motion of the brane does not depend on the detailed structure of $F^{(5)}(R)$.

In the Nariai space, the radius a is a constant and therefore $H = 0$. Furthermore, in the Nariai space, we find $e^{2\rho(a)} = 0$, and therefore Equation (209) shows that the brane tension $\tilde{\kappa}$ should vanish. That is, if and only if the tension vanished, the brane can exist. The non-vanishing tension might be cancelled with the contribution from the trace anomaly by tuning the brane tension. We should note, however, that there should not be any (FRW) dynamics of the brane in the Nariai space.

However, the antievaporation may induce the dynamics of the brane. For the metric Equation (155), one gets the expressions of the connection in Equation (A32). We introduce a new time coordinate $\tilde{t}$ in the metric Equation (155) as follows:

$$d\tilde{t}^2 \equiv e^{2\rho}\left(d\tau^2 - dx^2\right). \tag{216}$$

Then, the metric Equation (155) reduces to the form of the FRW-like metric,

$$ds^2 = -d\tau^2 + e^{-2\phi(x,\tau)}d\Omega_{(3)}^2, \tag{217}$$

if we identify $e^{-\phi(x,\tau)}$ with the scale factor a, $a = e^{-\phi(x,\tau)}$. Then, the unit vector perpendicular to the brane is given by:

$$n^\mu = \left(-e^{-2\rho}\frac{\partial x}{\partial \tilde{t}}, -e^{-2\rho}\frac{\partial \tau}{\partial \tilde{t}}, 0, 0, 0\right), \tag{218}$$

and the (i,j) $(i,j=1,2,3)$ components Equation (200) give:

$$-\mathrm{e}^{-2\rho}\frac{\partial\phi}{\partial\tau}\frac{\partial x}{\partial\tilde{t}}-\mathrm{e}^{-2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\tau}{\partial\tilde{t}}=\tilde{\kappa}\,. \tag{219}$$

As we discussed, in order that the brane exists in the Nariai space-time, we find $\tilde{\kappa}=0$. By using the solution in Equation (168), and analytically recontinuing the coordinates $x\to -i\tau$, $\tau\to -ix$, if we assume:

$$\phi=\ln\Lambda+\phi_0\cosh\omega\tau\cosh^{\beta}x\,, \tag{220}$$

with $\omega^2=\beta^2$, we find:

$$-\omega\sinh\omega\tau\cosh^{\beta}x\frac{\partial x}{\partial\tilde{t}}-\beta\cosh\omega\tau\cosh^{\beta-1}x\sinh x\frac{\partial\tau}{\partial\tilde{t}}=0\,; \tag{221}$$

that is,

$$\frac{\partial x}{\partial\tilde{t}}=-\frac{\beta\tanh x}{\omega\tanh\omega\tau}\frac{\partial\tau}{\partial\tilde{t}}\,. \tag{222}$$

Assuming that x and τ only depend on $\tilde{t}$ on the brane,

$$0=\frac{1}{\beta\tanh x}\frac{dx}{d\tilde{t}}+\frac{1}{\omega\tanh\omega\tau}\frac{d\tau}{d\tilde{t}}=\frac{d}{d\tilde{t}}\left(\frac{1}{\beta}\ln\sinh x+\ln\tanh\omega\tau\right)\,; \tag{223}$$

that is, $\frac{1}{\omega}\ln\sinh x+\ln\sinh\omega\tau$ is a constant, which gives the trajectory of the brane,

$$\sinh x=\frac{C}{\sinh^{\beta}\omega\tau}\,. \tag{224}$$

Here C is a constant. Of course, the expression in Equation (224) is valid as long as the perturbation $\delta\phi=\phi_0\cosh\omega\tau\cosh^{\beta}x$ is small enough. We should also note that because $F^{(5)\prime}(R)$ is not a constant due to the perturbation, Equation (215) also gives another source of the dynamics of the brane. However, Equation (215) gives only a small correction to Equation (224).

9. Discussions and Open Problems

In this review we have discussed the evaporation and antievaporation phenomena within the framework of extended theories of gravity. In particular, we have identified two particular metrics—the Nariai and the extremal Reissner–Nordström black hole solutions—that are unstable at the first order of metric perturbations. Explicit analyses with the cases of dilaton-gravity, $f(R)$-gravity, $f(T)$-gravity, mimetic gravity, and string-inspired gravity show up the emerging of the evaporation and antievaporation instabilities. We have seen how these instabilities completely change the thermodynamical behavior of black holes. The most surprising result is the suppression of the Bekenstein–Hawking radiation.

Several further questions may naturally arise. First of all, since (anti)evaporation instabilities seem to be so ubiquitous, we may ask whether any fundamental principle could be found, common to extended theories of gravity, that could motivate the emergence of such a phenomena. Second, since evaporation and antievaporation turn off Bekenstein–Hawking radiation, we may ask whether these phenomena can be relevant for the black hole information paradox.

Another unclear point remains the sensitivity of the evaporation/antievaporation transition on integration constants that seems undetermined by the initial conditions of the problem. Is there any principle to establish them? To use Hawking's words, is there a loss of predictability behind such a problem?

The last point is also crucially related to a possible cosmological problem. The production of primordial black holes, described by Nariai metrics, can lead to a disastrous cosmological instability. In fact, the antievaporation, turning off Bekenstein–Hawking emission, can lead to a catastrophic exponential expansion of primordial black holes. This is an issue that still needs to be better understood in the literature. We emphasize indeed that the implications of these phenomena on the information loss paradox and on the holographic principle have not yet been discussed in the literature. In other words, the interpretation of such instabilities of the black hole in the bulk has not a clear interpretation on the boundary theory.

(Anti)evaporation may be related to a way out from the the Firewall paradox [73,74]. The Firewall paradox is originated from the holographic entanglement among the black hole interior and the emitted Bekenstein–Hawking radiation [73,74]. This leads to a paradoxical violation of unitarity in quantum mechanics as well as the equivalence principle of general relativity. However, the (anti)evaporation instability radically changes the black hole emission, leading to a suppression of the Bekenstein–Hawking radiation, as mentioned above. So, it is conceivable that the (anti)evaporation carries deep consequences in our understanding of the black hole information paradox.

On the other hand, it is still unknown if exotic black hole solutions with multi-event horizons of alternative theories of gravity in the presence of a non-linear electrodynamical field—like the one recently found in Ref. [75]—can have (anti)evaporation in some regions of the parameter space.

In the era of gravitational waves discovery from the LIGO collaboration [76–78], crucial information on the (anti)evaporation phenomena can be provided from information on the gravitational waves signal, as it was redundant , searching for deviations from general relativity predictions [79].

We can then conclude that evaporation and antievaporation instabilities are interesting new phenomena that cannot be found in standard general relativity, but are common in many of its extensions. Several deep issues could still be hidden behind them, and for the time being it seems that we are still far from a complete understanding of their implications.

Acknowledgments: Andrea Addazi thanks Salvatore Capozziello and Sergei Odintsov for many discussions on these subjects.

Author Contributions: Both the authors contribute equally to the paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Nariai Metric

The Nariai metric can be obtained as a particular limit of the Schwarzschild–de Sitter (SdS) black hole. Let us start from the SdS solution in four space-time dimensions:

$$ds^2 = -\Phi(R)dt^2 + \frac{dr^2}{\Phi(r)} + r^2 d\Omega^2 \,, \tag{A1}$$

where $d\Omega^2 = \sin\psi^2 d\theta^2 + d\psi^2$ and:

$$\Phi(r) = 1 - \frac{2M}{r} - \frac{\lambda}{3} r^2 \,, \tag{A2}$$

where M is the black hole mass and λ is a cosmological constant.

Now, in order to smoothly perform the limit to the extremal SdS black hole, where $9M^2\lambda \to 1$, let us introduce two new coordinates ψ', χ, as follows:

$$t = \frac{1}{\epsilon\Lambda}\psi', \quad r = \frac{1}{\Lambda}\left[1 - \epsilon\cos\chi - \frac{1}{6\epsilon^2}\right] , \tag{A3}$$

where $9M^2\lambda = 1 - 3\epsilon^2$, $0 \le \epsilon << 1$, $\lambda = \sqrt{\Lambda}$. The new metric in this coordinate system is:

$$ds^2 = -\frac{1}{\Lambda^2}\left(1 + \frac{2}{3}\epsilon\cos\chi\right)\sin^2\chi d\psi'^2 + \frac{1}{\Lambda^2}\left(1 - \frac{2}{3}\epsilon\cos\chi\right)d\chi^2 + \frac{1}{\Lambda^2}(1 - 2\epsilon\cos\chi)d\Omega^2 \,. \tag{A4}$$

which, in the limit of $\epsilon \to 0$, smoothly converges to the so-called Nariai space-time:

$$ds^2 = \frac{1}{\Lambda^2}(-\sin^2\chi d\psi'^2 + d\chi^2) + \frac{1}{\Lambda^2}d\Omega^2\,. \tag{A5}$$

At this point, one can introduce a series of coordinate changes. First of all, we introduce $\chi = -\arcsin z$, such that:

$$ds^2 = -\frac{1}{\Lambda^2}(1-z^2)d\psi'^2 + \frac{dz^2}{\Lambda^2(1-z^2)} + \frac{1}{\Lambda^2}d\Omega^2\,. \tag{A6}$$

Then, we can write the cosmological time variable and the comoving coordinate x as:

$$t = \psi' + \frac{1}{2}\log(1-z^2), \quad x = \frac{z}{(1-z)^{1/2}}e^{\pm t}\,, \tag{A7}$$

leading to the metric:

$$ds^2 = \frac{1}{\Lambda^2}(-dt^2 + \cosh^2 t dx^2 + d\Omega^2) \tag{A8}$$

as a linear combination of metrics:

$$ds^2 = \frac{1}{\Lambda^2}(-dt^2 + e^{\pm 2t}dx^2) + \frac{1}{\Lambda^2}d\Omega^2\,. \tag{A9}$$

Finally, let us remark that with an analytic continuation of the time-coordinate, one can transform the dependences on the $\cosh^2 t$ to $\cos^2 t$, as done previously in the literature.

Appendix B. Extremal Reissner–Nördstrom Metric

In this section we will review the extremal Reissner–Nordström metric. Let us start from the Reissner–Nordström metric:

$$ds^2 = -\Phi(r)dt^2 + \Phi(r)^{-1}dr^2 + r^2 d\Omega^2\,, \tag{A10}$$

where:

$$\Phi(r) = 1 - \frac{R_0 r^2}{12} - \frac{M}{r} + \frac{Q}{r^2}\,. \tag{A11}$$

We can rewrite the mass and the charge in terms of the two radii:

$$Q = r_0 r_1\left(1 - \frac{R_0(r_0^2 + r_1^2 + r_0 r_1)}{12}\right) \tag{A12}$$

and:

$$\Phi(r) = \left(1 - \frac{r_0}{r}\right)\left(1 - \frac{r_1}{r}\right)\left\{1 - \frac{R_0[(r+r_0)(r+r_1) + r_0^2 + r_1^2]}{12}\right\}. \tag{A13}$$

Now, we can smoothly perform the limit of $r_0 \to r_1$; i.e., the extremal limit. We can consider the following coordinate change:

$$r_1 = r_0 + \epsilon, \quad r = r_0 + \frac{\epsilon}{2}(1 + \tanh x)\,. \tag{A14}$$

For $\epsilon \to 0$,

$$\Phi \to -\frac{\epsilon^2}{4r_0^2}\left(1 - \frac{r_0^2 R_0}{2}\right)\cosh^{2x}\,. \tag{A15}$$

By redefining t as:

$$t = \frac{2r_0^2}{\epsilon\left(1 - \frac{r_0^2 R_0}{2}\right)}\tau\,, \tag{A16}$$

one obtains:

$$ds^2 = \frac{r_0^2}{\left(1 - \frac{r_0^2 R_0}{2}\right)\cosh^2 x}(d\tau^2 - dx^2) + r_0^2 d\Omega^2\,. \tag{A17}$$

Appendix C. Components of the Ricci Tensors and Ricci Scalar of 4D Nariai Black Holes in $f(R)$-Gravity

$$\Gamma^{\tau}_{\tau\tau} = \Gamma^{\tau}_{xx} = \Gamma^{x}_{\tau x} = \dot{\rho}\,, \quad \Gamma^{x}_{xx} = \Gamma^{x}_{\tau\tau} = \Gamma^{\tau}_{x\tau} = \rho'\,, \tag{A18}$$

$$\Gamma^{\tau}_{\psi\psi} = \Gamma^{\tau}_{\theta\theta}\sin^2\theta = -\dot{\phi}e^{-(\rho+\phi)}\sin^2\theta\,, \Gamma^{x}_{\psi\psi} = \Gamma^{x}_{\theta\theta}\sin^2\theta = \phi' e^{-(\rho+\phi)}\sin^2\theta\,, \tag{A19}$$

$$\begin{gathered}\Gamma^{\theta}_{\tau\theta} = \Gamma^{\tau}_{\theta\tau} = \Gamma^{\psi}_{\psi\tau} = -\dot{\phi}, \quad \Gamma^{\theta}_{x\theta} = \Gamma^{\theta}_{\theta x} = \Gamma^{\psi}_{x\psi} = \Gamma^{\psi}_{\psi x} = -\phi', \\ \Gamma^{\theta}_{\psi\psi} = -\sin\theta\cos\theta, \quad \Gamma^{\psi}_{\psi\theta} = \Gamma^{\psi}_{\theta\psi} = \cot\theta\,,\end{gathered} \tag{A20}$$

$$R_{\tau\tau} = -\ddot{\rho} + 2\ddot{\phi} + \rho'' - 2\dot{\phi}^2 - 2\dot{\rho}\dot{\phi} - 2\rho'\phi', \quad R_{xx} = -\rho'' + \ddot{\rho} + 2\phi'' - 2\phi'^2 - 2\dot{\rho}\dot{\phi} - 2\rho'\phi'\,, \tag{A21}$$

$$\begin{gathered}R_{x\tau} = R_{\tau x} = 2\dot{\phi}' - 2\phi'\dot{\phi} - 2\rho'\dot{\phi} - 2\dot{\rho}\phi', \\ R_{\phi\phi} = R_{\theta\theta}\sin^2\theta = \left\{1 + e^{-2(\rho+\phi)}\left(-\ddot{\phi} + \phi'' + 2\dot{\phi}^2 - 2\phi'^2\right)\right\}\sin^2\theta\end{gathered} \tag{A22}$$

$$R = (2\ddot{\rho} - 2\rho'' - 4\ddot{\phi} + 4\phi'' + 6\dot{\phi}^2 - 6\phi'^2)e^{-2\rho} + 2e^{2\phi}\,. \tag{A23}$$

Appendix D. Components of the Ricci Tensors and Ricci Scalar in Extremal Reissner-Nördstrom BH

$$\Gamma^{\tau}_{\tau\tau} = \Gamma^{\tau}_{xx} = \Gamma^{\tau}_{\tau x} = \Gamma^{\tau}_{x\tau} = \dot{\rho}, \quad \Gamma^{x}_{xx} = \Gamma^{x}_{\tau\tau} = \Gamma^{\tau}_{\tau x} = \Gamma^{\tau}_{x\tau} = \rho'\,, \tag{A24}$$

$$\Gamma^{\tau}_{\psi\psi} = \Gamma^{\tau}_{\theta\theta}\sin^2\theta = -\frac{\Lambda^2}{\Lambda'^2}\dot{\phi}e^{-(\rho+\phi)}\sin^2\theta, \quad \Gamma^{x}_{\phi\phi} = \Gamma^{x}_{\theta\theta}\sin^2\theta = \frac{\Lambda^2}{\Lambda'^2}\phi' e^{-(\rho+\phi)}\sin^2\theta\,, \tag{A25}$$

$$\Gamma^{\theta}_{\tau\theta} = \Gamma^{\theta}_{\theta\tau} = \Gamma^{\psi}_{\tau\psi} = \Gamma^{\psi}_{\psi\tau} = -\dot{\phi}, \quad \Gamma^{\theta}_{x\theta} = \Gamma^{\theta}_{\theta x} = \Gamma^{\psi}_{x\psi} = \Gamma^{\psi}_{\psi x} = -\phi'\,, \tag{A26}$$

$$\Gamma^{\theta}_{\psi\psi} = -\sin\theta\cos\theta, \quad \Gamma^{\psi}_{\psi\theta} = \Gamma^{\psi}_{\theta\psi} = \cot\theta\,, \tag{A27}$$

$$R_{\tau\tau} = -\ddot{\rho} + 2\ddot{\phi} + \rho'' - 2\dot{\phi}^2 - 2\dot{\rho}\dot{\phi} - 2\rho'\phi', \quad R_{xx} = -\rho'' + \ddot{\rho} + 2\phi'' - 2\phi'^2 - 2\dot{\rho}\dot{\phi} - 2\rho'\phi'\,, \tag{A28}$$

$$\begin{gathered}R_{\tau x} = R_{x\tau} = 2\dot{\phi}' - 2\phi'\dot{\phi} - 2\rho'\dot{\phi} - 2\dot{\rho}\phi', \\ R_{\psi\psi} = R_{\theta\theta}\sin^2 = \left\{1 + \tfrac{\Lambda^2}{\Lambda'^2}e^{-2(\rho+\phi)}(-\ddot{\phi} + \phi'' + 2\dot{\phi}^2 - 2\phi'^2)\right\}\sin^2\theta\,,\end{gathered} \tag{A29}$$

$$R = \Lambda^2(2\ddot{\rho} - 2\rho'' - 4\ddot{\phi} + 4\phi'' + 6\dot{\phi}^2 - 6\phi'^2)e^{-2\rho} + \Lambda'^2 e^{2\phi}\,. \tag{A30}$$

Appendix E. Components of the Ricci Tensors and Ricci Scalar in Five-Dimensional Nariai Black Holes

For the metric Equation (155), we write $\eta_{ab} = \mathrm{diag}(-1, 0)$, $(a, b = \tau, x)$. For the metric $d\Omega^2_{(3)}$ of three-dimensional unit sphere, we also write as:

$$d\Omega^2_{(3)} = \hat{g}_{ij}dx^i dx^j\,, \quad (i, j = 1, 2, 3)\,. \tag{A31}$$

Then, we obtain $\hat{R}_{ij} = 2\hat{g}_{ij}$. Here $\hat{R}_{ij}$ is the Ricci curvature given by $\hat{g}_{ij}$.

Then, we find the following expression of the connections:

$$\Gamma^a_{bc} = \delta^a{}_b\rho_{,c} + \delta^a{}_c\rho_{,b} - \eta_{bc}\rho^{,a}\,, \quad \Gamma^a_{ij} = e^{-2(\rho+\phi)}\hat{g}_{ij}\phi^{,a}\,, \quad \Gamma^i_{aj} = \Gamma^i_{ja} = -\delta^i{}_j\phi_{,a}\,, \quad \Gamma^i_{jk} = \hat{\Gamma}^i_{jk}\,. \tag{A32}$$

Here $\hat{\Gamma}^i_{jk}$ is the connection given by $\hat{g}_{ij}$. By using the expressions in Equation (A32), the curvatures are given by:

$$\begin{aligned} R_{ab} &= 3\phi_{,ab} - \eta_{ab}\partial^2\rho - 3\left(\phi_{,a}\rho_{,b} + \phi_{,b}\rho_{,a}\right) + 3\eta_{ab}\phi_{,c}\rho^{,c} - 3\phi_{,a}\phi_{,b}\,, \\ R_{ij} &= \hat{R}_{ij} + \hat{g}_{ij}e^{-2(\rho+\phi)}\left(\partial^2\phi - 3\phi_{,a}\phi^{,a}\right)\,, \quad R_{ia} = R_{ai} = 0\,, \\ R &= e^{2\phi}\hat{R} + \mathrm{e}^{-2\rho}\left(6\partial^2\phi - 2\partial^2\rho - 12\phi_{,a}\phi^{,a}\right)\,. \end{aligned} \tag{A33}$$

We should note $\hat{R} = 6$ because $\hat{R}_{ij} = 2\hat{g}_{ij}$.

References

1. Nojiri, S.; Odintsov, S.D. Introduction to modified gravity and gravitational alternative for dark energy. *Int. J. Geom. Methods Mod. Phys.* **2006**, *4*, 115–145, doi:10.1142/S0219887807001928.
2. Njiri, S.; Odintsov, S.D. Unified cosmic history in modified gravity: From F(R) theory to Lorentz non-invariant models. *Phys. Rep.* **2011**, *505*, 59–144, doi:10.1016/j.physrep.2011.04.001.
3. Clifton, T.; Ferreira, P.G.; Padilla, A.; Skordis, C. Modified Gravity and Cosmology. *Phys. Rep.* **2012**, *513*, 1–189, doi:10.1016/j.physrep.2012.01.001.
4. Capozziello, S.; De Laurentis, M. Extended Theories of Gravity. *Phys. Rep.* **2011**, *509*, 167–321, doi:10.1016/j.physrep.2011.09.003.
5. Nojiri, S.; Odintsov, S.D.; Oikonomou, V.K. Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution. *arXiv*, 2017, arXiv:1705.11098.
6. Cai, Y.F.; Capozziello, S.; De Laurentis, M.; Saridakis, E.N. f(T) teleparallel gravity and cosmology. *Rep. Prog. Phys.* **2016**, *79*, doi:10.1088/0034-4885/79/10/106901.
7. Chamseddine, A.H.; Mukhanov, V.; Vikman, A. Cosmology with Mimetic Matter. *J. Cosmol. Astropart. Phys.* **2014**, *1406*, doi:10.1088/1475-7516/2014/06/017.
8. Sebastiani, L.; Vagnozzi, S.; Myrzakulov, R. Mimetic gravity: A review of recent developments and applications to cosmology and astrophysics. *Adv. High Energy Phys.* **2017**, *2017*, doi:10.1155/2017/3156915.
9. Ijjas, A.; Ripley, J.; Steinhardt, P.J. NEC violation in mimetic cosmology revisited. *Phys. Lett. B* **2016**, *760*, 132–138, doi:10.1016/j.physletb.2016.06.052.
10. Rabochaya, Y.; Zerbini, S. A note on a mimetic scalar-tensor cosmological model. *Eur. Phys. J. C* **2016**, *76*, 85, doi:10.1140/epjc/s10052-016-3926-y.
11. Nojiri, S.; Odintsov, S.D. Mimetic $F(R)$ gravity: Inflation, dark energy and bounce. *Mod. Phys. Lett. A* **2014**, *29*, doi:10.1142/S0217732314502113.
12. Arkani-Hamed, N.; Dimopoulos, S.; Dvali, G.R.; Kaloper, N. Infinitely large new dimensions. *Phys. Rev. Lett.* **2000**, *84*, 586–589, doi:10.1103/PhysRevLett.84.586 [hep-th/9907209].
13. Randall, L.; Sundrum, R. A Large mass hierarchy from a small extra dimension. *Phys. Rev. Lett.* **1999**, *83*, 3370–3373, doi:10.1103/PhysRevLett.83.3370.
14. Randall, L.; Sundrum, R. An Alternative to compactification. *Phys. Rev. Lett.* **1999**, *83*, 4690–4693, doi:10.1103/PhysRevLett.83.4690.
15. Dvali, G.R.; Gabadadze, G.; Porrati, M. 4-D gravity on a brane in 5-D Minkowski space. *Phys. Lett. B* **2000**, *485*, 208–214, doi:10.1016/S0370-2693(00)00669-9.
16. Maartens, R.; Koyama, K. Brane-World Gravity. *Living Rev. Relativ.* **2010**, *13*, 5, doi:10.12942/lrr-2010-5.
17. Brax, P.; van de Bruck, C. Cosmology and brane worlds: A Review. *Class. Quantum Gravity* **2003**, *20*, R201, doi:10.1088/0264-9381/20/9/202.
18. Bousso, R.; Hawking, S.W. (Anti)evaporation of Schwarzschild-de Sitter black holes. *Phys. Rev. D* **1998**, *57*, 2436–2442.
19. Nojiri, S.; Odintsov, S.D. Quantum evolution of Schwarzschild-de Sitter (Nariai) black holes. *Phys. Rev. D* **1999**, *59*, doi:10.1103/PhysRevD.59.044026.
20. Nojiri, S.; Odintsov, S.D. Effective action for conformal scalars and anti-evaporation of black holes. *Int. J. Mod. Phys. A* **1999**, *14*, 1293–1304, doi:10.1142/S0217751X9900066X.

21. Elizalde, E.; Nojiri, S.; Odintsov, S.D. Possible quantum instability of primordial black holes. *Phys. Rev. D* **1999**, *59*, doi:10.1103/PhysRevD.59.061501.
22. Nojiri, S.; Odintsov, S.D. Anti-Evaporation of Schwarzschild-de Sitter Black Holes in $F^{(d+1)}(R)$ gravity. *Class. Quantum Gravity* **2013**, *30*, 125003.
23. Nojiri, S.; Odintsov, S.D. Instabilities and anti-evaporation of Reissner-Nordström black holes in modified $F^{(d+1)}(R)$ gravity. *Phys. Lett. B* **2014**, *735*, 376–382.
24. Sebastiani, L.; Momeni, D.; Myrzakulov, R.; Odintsov, S.D. Instabilities and (anti)-evaporation of Schwarzschild—De Sitter black holes in modified gravity. *Phys. Rev. D* **2013**, *88*, doi:10.1103/PhysRevD.88.104022.
25. Houndjo, M.J.S.; Momeni, D.; Myrzakulov, R.; Rodrigues, M.E. Evaporation phenomena in $f(T)$ gravity. *Can. J. Phys.* **2015**, *93*, 377–383, doi:10.1139/cjp-2014-0281.
26. Matsumoto, J.; Odintsov, S.D.; Sushkov, S.V. Cosmological perturbations in a mimetic matter model. *Phys. Rev. D* **2015**, *91*, doi:10.1103/PhysRevD.91.064062.
27. Astashenok, A.V.; Odintsov, S.D.; Oikonomou, V.K. Modified Gauss—Bonnet gravity with the Lagrange multiplier constraint as mimetic theory. *Class. Quantum Gravity* **2015**, *32*, doi:10.1088/0264-9381/32/18/185007.
28. Oikonomou, V.K. A note on Schwarzschild-de Sitter black holes in mimetic $F^{(d+1)}(R)$ gravity. *Int. J. Mod. Phys. D* **2016**, *25*, doi:10.1142/S0218271816500784.
29. Oikonomou, V.K. Reissner-Nordström Anti-de Sitter Black Holes in Mimetic $F(R)$ Gravity. *Universe* **2016**, *2*, 10, doi:10.3390/universe2020010.
30. Katsuragawa, T.; Nojiri, S. Stability and antievaporation of the Schwarzschild?de Sitter black holes in bigravity. *Phys. Rev. D* **2015**, *91*, doi:10.1103/PhysRevD.91.084001.
31. Addazi, A. (Anti)evaporation of Dyonic Black Holes in string-inspired dilaton $f(R)$-gravity. *Int. J. Mod. Phys. A* **2017**, *32*, doi:10.1142/S0217751X17501020.
32. Chakraborty, S.; SenGupta, S. Spherically symmetric brane spacetime with bulk $f(\mathcal{R})$ gravity. *Eur. Phys. J. C* **2015**, *75*, 11, doi:10.1140/epjc/s10052-014-3234-3.
33. Chakraborty, S.; SenGupta, S. Effective gravitational field equations on m-brane embedded in n-dimensional bulk of Einstein and $f(\mathcal{R})$ gravity. *Eur. Phys. J. C* **2015**, *75*, 538, doi:10.1140/epjc/s10052-015-3768-z.
34. Chakraborty, S.; SenGupta, S. Solving higher curvature gravity theories. *Eur. Phys. J. C* **2016**, *76*, 552, doi:10.1140/epjc/s10052-016-4394-0 .
35. Addazi, A.; Nojiri, S.; Odintsov, S. Evaporation and antievaporation instability of a Schwarzschild-de Sitter braneworld: The case of five-dimensional $F(R)$ gravity. *Phys. Rev. D* **2017**, *95*, doi:10.1103/PhysRevD.95.124020.
36. Singh, D.V.; Singh, N.K. Anti-Evaporation of Bardeen de-Sitter Black Holes. *arXiv*, 2017, arXiv:1704.01831.
37. Addazi, A.; Capozziello, S. The fate of Schwarzschild-de Sitter Black Holes in $F(R)$ gravity. *Mod. Phys. Lett. A* **2016**, *31*, doi:10.1142/S0217732316500541.
38. Ellis, G.F.R. Astrophysical black holes may radiate, but they do not evaporate. *arXiv*, 2013, arXiv:1310.4771.
39. Firouzjaee, J.T.; Ellis, G.F.R. Cosmic Matter Flux May Turn Hawking Radiation off. *Gen. Relativ. Gravit.* **2015**, *47*, 6.
40. Firouzjaee, J.T.; Ellis, G.F.R. Particle creation from the quantum stress tensor. *Phys. Rev. D* **2015**, *91*, doi:10.1103/PhysRevD.91.103002.
41. Bekenstein, J.D. Black holes and entropy. *Phys. Rev. D* **1973**, *7*, doi:10.1103/PhysRevD.7.2333.
42. Hawking, S.W. Particle Creation by Black Holes. *Commun. Math. Phys.* **1975**, *43*, 199–220, doi:10.1007/BF02345020.
43. Hawking, S.W. Black holes and Thermodynamics. *Phys. Rev. D* **1976**, *13*, doi:10.1103/PhysRevD.13.191.
44. Hawking, S.W. Breakdown of Predictability in Gravitational Collapse. *Phys. Rev. D* **1976**, *14*, 2460–2473.
45. Addazi, A. Evaporation/Antievaporation and energy conditions in alternative gravity. *arXiv*, 2017, arXiv:1707.05724.
46. Christensen, S.M.; Fulling, S.A. Trace anomalies and the Hawking effect. *Phys. Rev. D* **1977**, *15*, 2088–2104.
47. Bousso, R.; Hawking, S.W. Trace anomaly of dilaton coupled scalars in two-dimensions. *Phys. Rev. D* **1997**, *56*, 7788–7791, doi:10.1103/PhysRevD.56.7788.
48. Hayward, J.D. Entropy in the RST model. *Phys. Rev. D* **1995**, *52*, doi:10.1103/PhysRevD.52.2239.
49. Gonzalez, P.A.; Saridakis, E.N.; Vasquez, Y. Circularly symmetric solutions in three-dimensional Teleparallel, f(T) and Maxwell-f(T) gravity. *J. High Energy Phys.* **2012**, *2012*, doi:10.1007/JHEP07(2012)053.
50. Capozziello, S.; Gonzalez, P.A.; Saridakis, E.N.; Vasquez, Y. Exact charged black-hole solutions in D-dimensional f(T) gravity: Torsion vs. curvature analysis. *J. High Energy Phys.* **2013**, *1302*, doi:10.1007/JHEP02(2013)039.

51. Iorio, L.; Saridakis, E.N. Solar system constraints on f(T) gravity. *Mon. Not. R. Astron. Soc.* **2012**, *427*, 1555–1561, doi:10.1111/j.1365-2966.2012.21995.x.
52. Krššák, M.; Saridakis, E.N. The covariant formulation of f(T) gravity. *Class. Quantum Gravity* **2016**, *33*, doi:10.1088/0264-9381/33/11/115009.
53. Kallosh, R.; Linde, A.D.; Ortin, T.; Peet, A.W.; Van Proeyen, A. Supersymmetry as a cosmic censor. *Phys. Rev. D* **1992**, *46*, 5278–5302, doi:10.1103/PhysRevD.46.5278.
54. Bianchi, M.; Samsonyan, M. Notes on unoriented D-brane instantons. *Int. J. Mod. Phys. A* **2009**, *24*, 5737–5763.
55. Addazi, A. Direct generation of a Majorana mass for the Neutron from Exotic Instantons. *Phys. Lett. B* **2016**, *757*, 462–467, doi:10.1016/j.physletb.2016.04.018.
56. Addazi, A.; Bianchi, M.; Ricciardi, G. Exotic see-saw mechanism for neutrinos and leptogenesis in a Pati-Salam model. *J. High Energy Phys.* **2016**, *1602*, doi:10.1007/JHEP02(2016)035.
57. Addazi, A.; Khlopov, M. Way-out to the gravitino problem in intersecting *D*-brane Pati-Salam models. *Mod. Phys. Lett. A* **2016**, *31*, 1650111, doi:10.1142/S021773231650111X.
58. Addazi, A.; Valle, J.W.F.; Vaquera-Araujo, C.A. String completion of an $\mathrm{SU}(3)_c \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ electroweak model. *Phys. Lett. B* **2016**, *759*, 471–478, doi:10.1016/j.physletb.2016.06.015.
59. Strominger, A.; Vafa, C. Microscopic origin of the Bekenstein-Hawking entropy. *Phys. Lett. B* **1996**, *379*, 99–104, doi:10.1016/0370-2693(96)00345-0.
60. Mathur, S.D. The Information paradox: A Pedagogical introduction. *Class. Quantum Gravity* **2009**, *26*, doi:10.1088/0264-9381/26/22/224001.
61. Zhang, B.; Cai, Q.Y.; Zhan, M.S.; You, L. Information conservation is fundamental: Recovering the lost information in Hawking radiation. *Int. J. Mod. Phys. D* **2013**, *22*, doi:10.1142/S0218271813410149.
62. Corda, C. Time dependent Schrödinger equation for black hole evaporation: no information loss. *Ann. Phys.* **2015**, *353*, 71–82, doi:10.1016/j.aop.2014.11.002.
63. Ketov, S.V.; Terada, T. Old-minimal supergravity models of inflation. *J. High Energy Phys.* **2013**, *1312*, 40, doi:10.1007/JHEP12(2013)040.
64. Ferrara, S.; Kehagias, A.; Porrati, M. Vacuum structure in a chiral $R + R^n$ modification of pure supergravity. *Phys. Lett. B* **2013**, *727*, 314–318, doi:10.1016/j.physletb.2013.10.027.
65. Addazi, A.; Capozziello, S. External stability for Spherically Symmetric Solutions in Lorentz Breaking Massive Gravity. *Int. J. Theor. Phys.* **2015**, *54*, 1818–1829, doi:10.1007/s10773-014-2387-z.
66. Dyer, E.; Hinterbichler, K. Boundary Terms, Variational Principles and Higher Derivative Modified Gravity. *Phys. Rev. D* **2009**, *79*, doi:10.1103/PhysRevD.79.024028.
67. Briscese, F.; Elizalde, E. Black hole entropy in modified gravity models. *Phys. Rev. D* **2008**, *77*, doi:10.1103/PhysRevD.77.044009.
68. Albareti, F.D.; Cembranos, J.A.R.; de la Cruz-Dombriz, A.; Dobado, A. On the non-attractive character of gravity in f(R) theories. *J. Cosmol. Astropart. Phys.* **2013**, *1307*, doi:10.1088/1475-7516/2013/07/009.
69. Mimoso, J.P.; Lobo, F.S.N.; Capozziello, S. Extended Theories of Gravity with Generalized Energy Conditions. *J. Phys. Conf. Ser.* **2015**, *600*, doi:10.1088/1742-6596/600/1/012047.
70. Gibbons, G.W.; Hawking, S.W. Action Integrals and Partition Functions in Quantum Gravity. *Phys. Rev. D* **1977**, *15*, doi:10.1103/PhysRevD.15.2752.
71. Nojiri, S.; Odintsov, S.D. Is brane cosmology predictable? *Gen. Relativ. Gravit.* **2005**, *37*, 1419–1425, doi:10.1007/s10714-005-0126-8.
72. Deruelle, N.; Sasaki, M.; Sendouda, Y. Junction conditions in F(R) theories of gravity. *Prog. Theor. Phys.* **2008**, *119*, 237–251, doi:10.1143/PTP.119.237. [arXiv:0711.1150.
73. Almheiri, A.; Marolf, D.; Polchinski, J.; Sully, J. Black Holes: Complementarity or Firewalls? *J. High Energy Phys.* **2013**, *1302*, 62.
74. Braunstein, S.L.; Pirandola, S.; yczkowski, K. Better Late than Never: Information Retrieval from Black Holes. *Phys. Rev. Lett.* **2013**, *110*, doi:10.1103/PhysRevLett.110.101301.
75. Nojiri, S.; Odintsov, S.D. Regular Multi-Horizon Black Holes in Modified Gravity with Non-Linear Electrodynamics. *arXiv*, 2017, arXiv:1708.05226.
76. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Abernathy, M.R.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* **2016**, *116*, doi:10.1103/PhysRevLett.116.061102.

77. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Abernathy, M.R.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; et al. GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence. *Phys. Rev. Lett.* **2016**, *116*, doi:10.1103/PhysRevLett.116.241103.
78. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; Adya, V.B.; et al. GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2. *Phys. Rev. Lett.* **2017**, *118*, doi:10.1103/PhysRevLett.118.221101.
79. Corda, C. Interferometric detection of gravitational waves: The definitive test for General Relativity. *Int. J. Mod. Phys. D* **2009**, *18*, 2275–2282, doi:10.1142/S0218271809015904.