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Almost Contact Metric Structures on 5-Dimensional Nilpotent Lie Algebras

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Abstract: We study almost contact metric structures on 5-dimensional nilpotent Lie algebras and investigate the class of left invariant almost contact metric structures on corresponding Lie groups. We determine certain classes that a five-dimensional nilpotent Lie group can not be equipped with.

Keywords: 5-dimensional nilpotent Lie algebra; almost contact metric structure; left invariant almost contact metric structure

1. Introduction

It is well-known that every connected odd dimensional Lie group is equipped with a left invariant almost contact metric structure. These structures give rise to almost contact metric structures on corresponding Lie algebras [1]. In literature, some certain classes of such structures are studied. In [2], some general results on 5-dimensional Sasakian Lie algebras were stated, and it was proved that an odd dimensional nilpotent Lie group with a left invariant Sasakian structure is isomorphic to the real Heisenberg group. In addition, a classification of five-dimensional Sasakian Lie algebras were obtained. Then, in [3], left invariant K-contact structures on five-dimensional Lie groups were investigated. Three-dimensional homogeneous almost contact metric structures were considered in [4]. In [5], cosymplectic and α-cosymplectic Lie algebras were investigated in terms of corresponding symplectic Lie algebras and suitable derivations on them.

Our aim in this manuscript is to determine almost contact metric structures on five-dimensional nilpotent Lie algebras by direct calculation. We use the classification of five-dimensional nilpotent Lie algebras given in [6]. We consider some certain classes of almost contact metric structures, and, by this approach, we get some general results on left invariant almost contact metric structures on five-dimensional nilpotent Lie groups.

2. Preliminaries

Let $M^{2n+1}$ be a differentiable manifold of dimension $2n+1$. If there is a $(1,1)$ tensor field $\phi$, a vector field $\xi$ and a one-form $\eta$ on $M$ satisfying:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

then, $M$ is said to have an almost contact structure $(\phi, \xi, \eta)$. A manifold with an almost contact structure is called an almost contact manifold. If, in addition to an almost contact structure $(\phi, \xi, \eta)$, $M$ also admits a Riemannian metric $g$ such that

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$

where $X, Y \in \mathfrak{m}$, $\phi(X) \in \mathfrak{m}$, and $\eta(\phi(X)) = 0$, then $M$ is called a cosymplectic manifold.
for all vector fields $X, Y$, then $M$ is an almost contact metric manifold with the almost contact metric structure $(\phi, \xi, \eta, g)$. The Riemannian metric $g$ is called a compatible metric. The one-form defined by

$$\Phi(X, Y) = g(X, \phi(Y)),$$

for all $X, Y \in \mathfrak{X}(M)$, is called the fundamental two-form of the almost contact metric manifold $(M, \phi, \xi, \eta, g)$. In [7], a classification of almost contact metric manifolds was obtained via the study of the covariant derivative of the fundamental two-form. A space having the same symmetries as the covariant derivative of the fundamental two-form was written, and, then, this space was decomposed into twelve $U(n) \times 1$ irreducible components $C_1, \ldots, C_{12}$. There are $2^{12}$ invariant subspaces, each corresponding to a class of almost contact metric manifolds. For example, the trivial class for which $\nabla \Phi = 0$ [8], corresponds to the class of cosymplectic (called co-Kähler by some authors) manifolds, $C_1$ is the class of nearly-K cosymplectic manifolds, etc. [7]. For classification of almost contact metric structures (see also [9]). In this work, we focus on cosymplectic, nearly cosymplectic, $\alpha$-Sasakian, $\beta$-Kenmotsu and almost cosymplectic structures.

Let $(\phi, \xi, \eta, g)$ be an almost contact metric structure on $M$ with the fundamental two-form $\Phi$. $(\phi, \xi, \eta, g)$ is called:

- nearly cosymplectic if $\nabla_X \Phi(X, Y) = 0$,
- $a$-Sasakian ($C_a$) if $\nabla_X \phi(Y) = a(g(X,Y)\xi - \eta(Y)X)$ for a constant $a$,
- $\beta$-Kenmotsu ($C_\beta$) if $\nabla_X \phi(Y) = \beta(\Phi(X, Y)\eta(Z) - \Phi(X, Z)\eta(Y))$ for a constant $\beta$,
- semi cosymplectic ($C_1 \oplus C_2 \oplus C_3 \oplus C_5 \oplus C_6 \oplus C_{10} \oplus C_{11}$) if $\Phi = 0$ and $\delta \eta = 0$, where $\delta$ denotes the coderivative of a differential form,
- almost cosymplectic ($C_2 \oplus C_9$) if $d\Phi = 0$ and $d\eta = 0$, where $d$ denotes the exterior derivative of a differential form,

for all vector fields $X, Y, Z$ on $M$.

In literature, there are different but related definitions of cosymplectic structures. Here, we remind them and relate to the classes we use. In [5,10], an almost cosymplectic manifold is defined as a smooth manifold with a one-form $\eta$ and a two-form $\Phi$ such that $\eta \wedge \Phi^\sharp$ is a volume form. If both $\eta$ and $\Phi$ are closed, then the manifold is said to be cosymplectic. In the same context, if $d\eta = 0$ and $d\Phi = 2a\eta \wedge \phi$ for a constant $a$, then the manifold is called $a$-cosymplectic. An almost contact metric manifold $(M, \phi, \xi, \eta, g)$, where $(\eta, \Phi)$ is a $\alpha$ cosymplectic structure is called an almost co-Kähler manifold. In addition, if this manifold is normal, then it is said to be co-Kähler. An almost contact metric manifold $(M, \phi, \xi, \eta, g)$ such that $(\eta, \Phi)$ is an $\alpha$-cosymplectic structure is called an almost $\alpha$ co-Kähler manifold. A normal almost $\alpha$ co-Kähler manifold is said to be $\alpha$ co-Kähler. Refer to [5,10] and references therein. “Almost cosymplectic”, “cosymplectic” and “$\alpha$-Kenmotsu” structures in our paper correspond to “almost co-Kähler”, “co-Kähler” and “$\alpha$ co-Kähler” in [5], respectively. Throughout the paper, the definitions in and [7,8] will be followed.

The existence of metric connections on five-dimensional almost contact metric manifolds compatible with the almost contact structure was investigated in [11]. The space of torsion tensors of a metric connection splits into ten $U(2)$-irreducible subspaces $W_1, W_2, \ldots, W_{10}$. Thus, there are $2^{10}$ classes of almost contact metric structures in five-dimensions according to components of torsion tensor [11].

An almost contact metric structure $(\phi, \xi, \eta, g)$ on a connected Lie group $G$ is said to be left invariant if $g$ is left invariant and if the left multiplication map $L_a : G \rightarrow G$, $L_a(x) = a.x$ has properties

$$\phi \circ L_a = L_a \circ \phi, \quad L_a(\xi) = \xi,$$

for all $a \in G$. 


Let \( g \) be an odd dimensional Lie algebra. An almost contact metric structure on \( g \) is a quadruple \((\phi, \xi, \eta, g)\), where \( \eta \) is a one-form, \( \phi \) is an endomorphism of \( g \), \( \xi \in g \) such that

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y),
\]

for all vector fields \( X, Y \) and \( g \) is a positive definite compatible inner product on \( g \). It is also convenient to use defining relations for the structures on Lie algebras. For instance, an almost contact metric structure \((\phi, \xi, \eta, g)\) on a Lie algebra \( g \) is said to be nearly cosymplectic if \( \nabla_X \phi(X, Y) = 0 \) for any \( X, Y \) in \( g \), etc.

Let \( G \) be a connected Lie group endowed with a left invariant almost contact metric structure \((\phi, \xi, \eta, g)\) and \( g \cong T_e G \) be the corresponding Lie algebra of \( G \). Then, this structure uniquely yields an almost contact metric structure \((\phi, \xi, \eta, g)\) on \( g \).

In this work, we study almost contact metric structures on five-dimensional nilpotent Lie algebras. The classification of nilpotent Lie algebras of dimension \( \leq 5 \) was obtained in [6] (see also [12,13]). Indeed, \( g_i \) are five-dimensional nilpotent algebras with the corresponding basis \( \{e_1, \ldots, e_5\} \) and non-zero brackets as follows:

\[
\begin{align*}
g_1 : \quad & [e_1, e_2] = e_5, [e_3, e_4] = e_5, \\
g_2 : \quad & [e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5, \\
g_3 : \quad & [e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_3] = e_5, \\
g_4 : \quad & [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_4] = e_5, \\
g_5 : \quad & [e_1, e_2] = e_4, [e_1, e_3] = e_5, \\
g_6 : \quad & [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5.
\end{align*}
\]

The rest of the classes \( g_7, g_8, g_9 \) are abelian.

3. Almost Contact Metric Structures on \( g_i \)

Let \( G \) be a connected Lie group and \((\phi, \xi, \eta, g)\) a left invariant a.c.m.s. (almost contact metric structure) on \( G \). Denote the corresponding a.c.m.s. on \( g \) by the same symbols. Choose the basis \( \{e_1, \ldots, e_5\} \) such that basis elements are \( g \)-orthonormal.

First, we investigate the existence of some classes of almost contact metric structures on each \( g_i \).

The algebra \( g_1 \): By Koszul’s formula, the covariant derivatives of the basis elements are as follows:

\[
\begin{align*}
\nabla_{e_1} e_2 &= \frac{1}{2} e_5, \quad \nabla_{e_1} e_3 = -\frac{1}{2} e_2, \quad \nabla_{e_2} e_1 = -\frac{1}{2} e_3, \quad \nabla_{e_2} e_3 = \frac{1}{2} e_1, \\
\nabla_{e_1} e_4 &= \frac{1}{2} e_5, \quad \nabla_{e_3} e_5 = -\frac{1}{2} e_4, \quad \nabla_{e_3} e_4 = -\frac{1}{2} e_5, \quad \nabla_{e_4} e_3 = \frac{1}{2} e_5, \\
\nabla_{e_1} e_5 &= -\frac{1}{2} e_2, \quad \nabla_{e_2} e_4 = \frac{1}{2} e_3, \quad \nabla_{e_3} e_2 = -\frac{1}{2} e_4, \quad \nabla_{e_4} e_2 = \frac{1}{2} e_3.
\end{align*}
\]

- There exists no cosymplectic structure on \( g_1 \).

To see this, assume \( \Phi = \sum b_{ij} e^i \) is a two-form on \( g_1 \) such that \( \nabla \Phi = 0 \). Then, for any elements \( e_i, e_j, e_k \) of the basis:

\[
\begin{align*}
(\nabla_{e_i} \Phi)(e_j, e_k) &= e_i[\Phi(e_j, e_k)] - \Phi(\nabla_{e_i} e_j, e_k) - \Phi(e_j, \nabla_{e_i} e_k) \\
&= -\Phi(\nabla_{e_i} e_j, e_k) - \Phi(e_j, \nabla_{e_i} e_k) = 0.
\end{align*}
\]

It is easy to see that \( \nabla \Phi = 0 \) if and only if \( b_{ij} = 0 \) for any \( i, j \). Thus, \((\phi, \xi, \eta, g)\) is not cosymplectic.
• There is no nearly cosymplectic structure (i.e., \((\nabla_X \Phi)(X, Y) = 0\)). Let \(\Phi = \sum b_{ij} e^{ij}\), by direct calculation, we obtain,

\[
(\nabla_{e_i} \Phi)(e_i, e_j) = 0 \iff b_{13} b_{23} + b_{14} b_{24} = b_{13} b_{14} + b_{23} b_{24} = 0,
\]

where \(b_{14}^2 = b_{13}^2 + b_{24}^2\) and the remaining coefficients are zero. Thus \(\Phi = b_{13} e^{13} + b_{14} e^{14} + b_{23} e^{23} + b_{24} e^{24}\). By polarizing the equation \((\nabla_X \Phi)(X, Y) = 0\), we get

\[
\nabla_X \Phi(Y, Z) + \nabla_Y \Phi(X, Z) = 0.
\]

Then for \(X = e_2\), \(Y = e_3\) and \(Z = e_5\) in the equation (2), we obtain \(b_{13} = -b_{24}\). In addition, replacing \(e_3\), \(e_5\) and \(e_2\) for \(X\), \(Y\), \(Z\) respectively in the equation (2), we get \(b_{13} = 2b_{24}\). Thus, \(b_{13} = b_{24} = 0\). On the other hand, we get \(b_{14} = b_{23}\) and \(2b_{14} = -b_{14}\) for \(X = e_3\), \(Y = e_1\) and \(Z = e_5\) and \(X = e_1\), \(Y = e_5\) and \(Z = e_3\) respectively in the equation (2), which implies \(b_{14} = b_{23} = 0\).

• There is no non-zero parallel vector field on \(g_1\). Let \(\xi = \sum a_i e_i\) be a parallel vector field. Then, for any \(e_i, e_j\), we have \(g(\nabla_{e_i} \xi, e_j) = -g(\nabla_{e_j} \xi, e_i)\). Then,

\[
g(\nabla_{e_i} \xi, e_j) = -\frac{1}{2} a_i \text{ and } g(\nabla_{e_j} \xi, e_i) = -\frac{1}{2} a_i
\]

yields \(a_i = 0\). Similarly, since \(g(\nabla_{e_i} \xi, e_5) = -g(\nabla_{e_5} \xi, e_i)\), we have \(a_2 = 0\) and \(g(\nabla_{e_i} \xi, e_5) = -g(\nabla_{e_5} \xi, e_i)\) gives \(a_3 = 0\). In addition, \(g(\nabla_{e_5} \xi, e_5) = -g(\nabla_{e_5} \xi, e_5)\) implies \(a_4 = 0\). As a result, \(\xi = a e_5\).

• There exists 1/2-Sasakian structure on \(g_1\), where the fundamental two-form is \(\Phi = -\varepsilon^{12} - \varepsilon^{34}\) and \(\xi = e_5\). Note that this structure is given in [2] as a Sasakian structure because of the coefficient 2 in the defining relation of a Sasakian structure.

• There is no \(\beta\)-Kenmotsu structure.

Assume \((\phi, \zeta, \eta, g)\) is a \(\beta\)-Kenmotsu structure with fundamental two-form \(\Phi = \sum b_{ij} e^{ij}\). Then, \(g(\nabla_{e_i} \xi, e_j) = g(\nabla_{e_j} \xi, e_i)\) for any basis elements \(e_i, e_j\), which implies that \(\zeta = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4\) and \(\eta = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4\). On the other hand for \(X = e_1\), \(Y = e_1\), \(Z = e_2\) and for \(X = e_1\), \(Y = e_5\), \(Z = e_5\) in the defining relation of a \(\beta\)-Kenmotsu structure, we obtain \(b_{15} = 2b_1 b_{12}\) and \(b_{12} = 2b_1 b_{15}\), respectively. Thus, \(b_{12} = b_{15} = 0\). Similar arguments work if \(X, Y, Z\) are replaced by other basis elements. We get \(b_{ij} = 0\) for all \(i, j\). As a result the structure is not \(\beta\)-Kenmotsu.

• There is no almost cosymplectic structure.

Let \(\eta = \sum b_i e_i\) and \(\Phi = \sum b_{ij} e^{ij}\). Then, since \(de^5 = -e^{12} - e^{34}\) and \(de^i = 0\) for \(i = 1, 2, 3, 4\), we get \(d\eta = -b_5 (e^{12} + e^{34})\). This yields \(d\eta = 0\) iff \(b_5 = 0\). On the other hand, we have \(d\Phi = b_{15} e^{13} + 2b_{23} e^{24} + b_{35} e^{12} - b_{45} e^{123} + b_{25} e^{124}\), which is zero iff \(b_{15} = b_5 = b_{15} = 0\). In this case, \(\Phi \wedge \Phi = 2(b_{12} b_{34} + b_{14} b_{23} - b_{13} b_{24})e^{1234}\) and \(\eta \wedge \Phi^2 = 0\), which contradicts with the assumption that \((\phi, \zeta, \eta, g)\) is an almost contact metric structure.

• There are semi cosymplectic structures on \(g_1\).

For any vector \(X = \sum x_i e_i\) on \(g_1\), we have \(\delta \Phi(X) = x_5 (b_{12} + b_{34})\). Thus, \(\delta \Phi = 0\) for all \(X\) iff \(b_{12} = -b_{34}\). In addition, \(\delta \eta = 0\) for any one-form \(\eta\). Choose, for example, the a.c.m.s. \((\phi, \zeta, \eta, g)\), where \(\zeta = e_5\), \(\eta = e^5\) and \(\Phi = e^{12} - e^{34}\). This structure is semi cosymplectic.

• Consider the a.c.m.s. \((\phi, \zeta, \eta, g)\), where \(\phi(e_1) = -e_4\), \(\phi(e_2) = -e_3\), \(\phi(e_3) = e_2\), \(\phi(e_4) = e_1\) and \(\zeta = e_5\), \(\eta = e^5\) on \(g_1\). We show that there is a metric connection \(\nabla^c\) compatible with this structure.
Assume that $\nabla^c$ is a metric connection of $g$. Then, $\nabla^c = \nabla + A$, where $A$ is a skew-symmetric $(2, 1)$ tensor field. Since $\nabla^c$ is compatible with $\xi = e_5$, we have $\nabla^c_i e_5 = 0$ for all basis elements $e_i$. We obtain

$$A(e_1, e_5) = \frac{1}{2} e_2, \quad A(e_2, e_5) = -\frac{1}{2} e_1, \quad A(e_3, e_5) = \frac{1}{2} e_4, \quad A(e_4, e_5) = -\frac{1}{2} e_3.$$ 

Metric compatibility of $\nabla^c$ yields

$$0 = e_1 [g(e_1, e_2)] = g(\nabla^c e_1, e_2) + g(e_1, \nabla^c e_2),$$

and thus $g(e_1, \nabla^c e_2) = 0$. Note that $\nabla^c e_1 = \nabla e_1 + A(e_1, e_1) = 0$. Similarly, $g(e_2, \nabla^c e_2) = g(e_5, \nabla^c e_2) = 0$. Hence, $\nabla^c_i e_2 = a_3 e_3 + a_4 e_4$ for some constants $a_3, a_4$ and $A(e_1, e_2) = a_3 e_3 + a_4 e_4 - \frac{1}{2} e_5$.

Since $\nabla^c$ is also compatible with $\phi$, that is, $\nabla^c \phi = 0$, we have

$$0 = (\nabla^c_i \phi)(e_2) = \nabla^c_i (\phi(e_2)) - \phi(\nabla^c_i e_2) = -\nabla^c_i e_3 - \phi(a_3 e_3 + a_4 e_4).$$

Thus,

$$\nabla^c_i e_3 = -a_3 e_2 - a_4 e_1 = A(e_1, e_3).$$

In addition, $(\nabla^c_i \phi)(e_4) = 0$ implies $\phi(\nabla^c_i e_4) = 0$. By the identity $\phi^2 = -I + \eta \otimes \xi$, we get

$$0 = \phi^2(\nabla^c_i e_4) = -\nabla^c_i e_4 + g(\nabla^c_i e_4, e_5) e_5,$$

which gives $\nabla^c_i e_4 = A(e_1, e_4) = 0$. Note that $g(\nabla^c_i e_4, e_5) = 0$ since $\nabla^c$ is a metric connection. Similarly, $\nabla^c_i e_5 = A(e_2, e_5) = 0$. By direct calculation, we get

$$\nabla^c_i e_4 = a_3 e_3 + a_4 e_4, \quad A(e_3, e_4) = a_3 e_3 + a_4 e_4 - \frac{1}{2} e_5.$$ 

To sum up,

$$A = a_1 \otimes \{ a_4 (e_3^{13} + e_2^{24}) - \frac{1}{2} e_5^{35} \} + e_2 \otimes \{ a_3 (e_3^{13} + e_2^{24}) + \frac{1}{2} e_5^{15} \}$$

$$+ e_3 \otimes \{ a_3 e_2^{12} - \frac{1}{2} e_5^{45} \} + e_4 \otimes \{ a_4 e_2^{12} + \frac{1}{2} e_5^{35} \} + e_5 \otimes \{ - \frac{1}{2} e_2^{12} - \frac{1}{2} e_5^{34} \}.$$ 

Since $(\phi, \xi, \eta, g)$ has a totally skew-symmetric metric connection, by Proposition 4.1 in [11], we conclude that $(\phi, \xi, \eta, g)$ is in the class $\mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6$ with respect to the classification of Puhle in [11].

Similar observations can be made for existing structures on each $g_i$.

**The algebra $g_2$:** By Koszul’s formula, the covariant derivatives of the basis elements are as follows:

$$\nabla_{e_1} e_2 = \frac{1}{2} e_3, \quad \nabla_{e_1} e_3 = -\frac{1}{2} e_2 + \frac{1}{2} e_5, \quad \nabla_{e_1} e_5 = -\frac{1}{2} e_3, \quad \nabla_{e_2} e_1 = -\frac{1}{2} e_3,$$

$$\nabla_{e_2} e_3 = \frac{1}{2} e_1, \quad \nabla_{e_2} e_4 = \frac{1}{2} e_5, \quad \nabla_{e_2} e_5 = -\frac{1}{2} e_4, \quad \nabla_{e_3} e_1 = -\frac{1}{2} e_2 - \frac{1}{2} e_5,$$

$$\nabla_{e_3} e_2 = \frac{1}{2} e_1, \quad \nabla_{e_3} e_5 = \frac{1}{2} e_1, \quad \nabla_{e_4} e_2 = -\frac{1}{2} e_5, \quad \nabla_{e_4} e_5 = \frac{1}{2} e_2,$$

$$\nabla_{e_5} e_1 = -\frac{1}{2} e_3, \quad \nabla_{e_5} e_2 = -\frac{1}{2} e_4, \quad \nabla_{e_5} e_3 = \frac{1}{2} e_1, \quad \nabla_{e_5} e_4 = \frac{1}{2} e_2.$$
• There exists no cosymplectic structure.
The proof is similar to that of $g_1$.

• There exists no nearly cosymplectic structure.
Assume that there exists a nearly cosymplectic structure $(\phi, \xi, \eta, g)$ with the fundamental two-form $\Phi = \sum b_{ij} e^i e^j$. Then, for any basis elements $e_i, e_j$, we have $(\nabla e_i, \Phi)(e_i, e_j) = -\Phi(e_i, \nabla e_j) = 0$. Thus, we get:

$$(\nabla e_i, \Phi)(e_1, e_2) = 0 \Rightarrow b_{13} = 0, (\nabla e_i, \Phi)(e_1, e_3) = 0 \Rightarrow b_{15} = b_{13}, (\nabla e_2, \Phi)(e_1, e_3) = 0 \Rightarrow b_{23} = 0,$$

$$(\nabla e_2, \Phi)(e_2, e_3) = 0 \Rightarrow b_{12} = 0, (\nabla e_2, \Phi)(e_2, e_4) = 0 \Rightarrow b_{25} = 0, (\nabla e_2, \Phi)(e_2, e_5) = 0 \Rightarrow b_{24} = 0,$$

$$(\nabla e_3, e_1) = 0 \Rightarrow b_{35} = 0 \text{ and } (\nabla e_3, e_2) = 0 \Rightarrow b_{45} = 0.$$

Thus, the fundamental two-form is of type $\Phi = b_{14} e^{14} + b_{34} e^{34}$. From the equation $\Phi(X, Y) = g(X, \phi(Y))$, the endomorphism $\Phi$ is defined by $\phi(e_1) = -b_{14} e_4$, $\phi(e_2) = 0$, $\phi(e_3) = -b_{34} e_4$, $\phi(e_4) = b_{14} e_1 + b_{34} e_3$, $\phi(e_5) = 0$. Let $\xi = \sum a_i e_i$ and $\eta = \sum b_i e_i$. Then, $\phi^2(e_2) = 0 = -e_2 + \eta(e_2) \xi \Rightarrow b_2 a_2 = 1, b_2 a_5 = 0 \Rightarrow a_5 = 0$.

On the other hand,

$\phi^2(e_5) = 0 = -e_5 + \eta(e_5) \Rightarrow b_5 a_5 = 1 \Rightarrow a_5 \neq 0$. Therefore, the condition $\phi^2 = -I + \eta \otimes \xi$ does not hold. Thus, the structure is not nearly cosymplectic.

• There is no non-zero parallel vector field on $g_2$.
If a non-zero vector field $\xi = \sum a_i e_i$ is parallel ($\nabla \xi = 0$), by calculating $g(\nabla e_i, e_j)$ for basis elements, we get $a_i = 0$, for $i = 1, \cdots , 5$. It also shows that $\nabla \eta \neq 0$ for any almost contact metric structure $(\phi, \xi, \eta, g)$ on $g_2$. In particular, $(\phi, \xi, \eta, g)$ is neither $C_1$ (nearly-K-cosymplectic), nor $C_2$.

• A vector field $\xi$ on $g_2$ is Killing if and only if $\xi \in \langle e_5 \rangle$.
Assume $\xi = \sum a_i e_i$ is a Killing vector field. Then, for any $e_i, e_j$, we have $g(\nabla e_i, e_j) = -g(\nabla e_j, e_i)$.

Thus,

$g(\nabla e_i, e_3) = -\frac{1}{2} a_1, g(\nabla e_3, e_2) = -\frac{1}{2} a_1 \Rightarrow a_1 = 0,$

and similarly,

$g(\nabla e_i, e_5) = -g(\nabla e_5, e_4) \Rightarrow a_2 = 0,$

$g(\nabla e_1, e_5) = -g(\nabla e_5, e_1) \Rightarrow a_3 = 0,$

$g(\nabla e_2, e_5) = -g(\nabla e_5, e_2) \Rightarrow a_4 = 0.$

• There is no $\alpha$-Sasakian structure. Assume that a structure $(\phi, \xi, \eta, g)$ on $g_2$ is $\alpha$-Sasakian. Then, $\xi \in \langle e_5 \rangle$, since it is a Killing vector field. On the other hand, by considering the relation $\nabla_X \xi = -a(\Phi(X)$, we get the endomorphism:

$$\phi(e_1) = \frac{a_5}{2a} e_3, \phi(e_2) = \frac{a_5}{2a} e_4, \phi(e_3) = -\frac{a_5}{2a} e_1, \phi(e_4) = \frac{a_5}{2a} e_2.$$

In addition, the structure must satisfy the defining relation of the class of $\alpha$-Sasakian structures:

$$(\nabla_X \Phi)(Y) = a(g(X,Y) \xi - \eta(Y) X).$$

However, it is easy to see that this relation is not satisfied for $X = Y = e_1$. Hence, the structure is not $\alpha$-Sasakian.

Let $(\phi, \xi, \eta, g)$ be a $\beta$-Kenmotsu structure with fundamental two-form $\Phi = \sum b_{ij} e^i e^j$.

Then, $g(\nabla e_i, e_j) = g(\nabla e_j, e_i)$ for any basis elements $e_i, e_j$. Since $g(\nabla e_i, e_2) = -\frac{a_2}{2} g(\nabla e_2, e_1) = \frac{a_2}{2} a_3 = 0$ and $g(\nabla e_i, e_4) = -\frac{a_5}{2} g(\nabla e_4, e_2) = \frac{a_5}{2} a_4 = 0$, we have $\xi = a_1 e_1 + a_2 e_2 + a_4 e_4$ and $\eta = b_1 e^1 + b_2 e^2 + b_4 e^4$. On the other hand, for $X = e_1$, $Y = e_3, Z = e_5$ and for $X = e_3$, $Y = e_3, Z = e_5$ in the defining relation of a $\beta$-Kenmotsu structure, we obtain $b_{25} = 0$ and $b_{13} = 0,$
There is no non-zero parallel vector field on \( \Phi \) we see that this equation holds if and only if \( (\gamma, \xi, \eta, g) \) is a semi cosymplectic structure on \( g_2 \).

There exists an almost cosymplectic structure.

The almost contact metric structure \( (\phi, \xi, \eta, g) \), where \( \xi = e_2, \eta = e^2 \) and \( \Phi = e^{15} + e^{34} \) is almost cosymplectic, that is \( d\Phi = d\eta = 0 \).

The algebra \( g_3 \): By Koszul’s formula, the covariant derivatives of the basis elements are as follows:

\[
\begin{align*}
\nabla_{e_1}e_2 &= \frac{1}{2}e_3, & \nabla_{e_1}e_3 &= -\frac{1}{2}e_2 + \frac{1}{2}e_4, & \nabla_{e_1}e_4 &= -\frac{1}{2}e_3 + \frac{1}{2}e_5, & \nabla_{e_1}e_5 &= -\frac{1}{2}e_4, \\
\nabla_{e_2}e_3 &= \frac{1}{2}e_1 + \frac{1}{2}e_5, & \nabla_{e_2}e_5 &= -\frac{1}{2}e_3, \\
\nabla_{e_3}e_1 &= -\frac{1}{2}e_2 - \frac{1}{2}e_4, & \nabla_{e_3}e_2 &= \frac{1}{2}e_1 - \frac{1}{2}e_5, & \nabla_{e_3}e_4 &= \frac{1}{2}e_1, & \nabla_{e_3}e_5 &= \frac{1}{2}e_2, \\
\nabla_{e_4}e_1 &= -\frac{1}{2}e_3, & \nabla_{e_4}e_3 &= \frac{1}{2}e_1, & \nabla_{e_4}e_5 &= \frac{1}{2}e_1, \\
\nabla_{e_5}e_1 &= -\frac{1}{2}e_4, & \nabla_{e_5}e_2 &= -\frac{1}{2}e_3, & \nabla_{e_5}e_3 &= \frac{1}{2}e_2, & \nabla_{e_5}e_4 &= \frac{1}{2}e_1.
\end{align*}
\]

• There exists no cosymplectic structure.

The proof is similar to that of \( g_1 \).

• There exists no nearly cosymplectic structure.

Let \( (\phi, \xi, \eta, g) \) be a nearly cosymplectic structure with fundamental two-form \( \Phi = \sum b_{ij}e^j \). Then, for any basis elements \( e_i, e_j \), we have \( \langle \nabla_{e_i} \Phi \rangle(e_i, e_j) = -\Phi(e_i, \nabla_{e_j} e_i) = 0 \). After some calculations, we see that this equation holds if and only if \( \Phi = b_{24}e^{24} \). However, the condition \( \eta \wedge \Phi \neq 0 \) is not satisfied since \( \Phi \wedge \Phi = 0 \).

• There is no non-zero parallel vector field on \( g_3 \).

The proof is similar to these of \( g_1 \) and \( g_2 \).

A vector field \( \xi \) on \( g_3 \) is Killing if and only if \( \xi \in \langle e_5 \rangle \).

Let \( \xi = \sum a_i e_i \) be a Killing vector field. Then, for any \( e_i, e_j \), we have \( g(\nabla_{e_i} \xi, e_j) = -g(\nabla_{e_j} \xi, e_i) \).

Now, \( g(\nabla_{e_1} \xi, e_3) = -g(\nabla_{e_3} \xi, e_1) \Rightarrow a_2 = 0 \), \( g(\nabla_{e_1} \xi, e_4) = -g(\nabla_{e_4} \xi, e_1) \Rightarrow a_3 = 0 \), \( g(\nabla_{e_1} \xi, e_5) = -g(\nabla_{e_5} \xi, e_1) \Rightarrow a_4 = 0 \), \( g(\nabla_{e_2} \xi, e_3) = -g(\nabla_{e_3} \xi, e_2) \Rightarrow a_1 = 0 \). In other words, \( \xi \) is Killing if and only if \( \xi = a_5 e_5 \).

• There is no \( \alpha \)-Sasakian structure.

Let \( (\phi, \xi, \eta, g) \) be an \( \alpha \)-Sasakian structure on \( g_3 \). Then, \( \xi \notin \langle e_5 \rangle \), since it is a Killing vector field.

On the other hand, by considering the relation \( \nabla_X \xi = -\alpha \phi(X) \), we get the endomorphism as:

\[
\phi(e_1) = \frac{a_5}{2\alpha} e_4, \quad \phi(e_2) = \frac{a_5}{2\alpha} e_3, \quad \phi(e_3) = -\frac{a_5}{2\alpha} e_2, \quad \phi(e_4) = -\frac{a_5}{2\alpha} e_1.
\]

However, for \( X = Y = e_1 \), this structure does not satisfy the defining relation \( (\nabla_X \phi)(Y) = \alpha(g(X, Y) \xi - \eta(Y) X) \).

• There is no \( \beta \)-Kenmotsu structure.

Let \( (\phi, \xi, \eta, g) \) be a \( \beta \)-Kenmotsu structure with fundamental two-form \( \Phi = \sum b_{ij}e^j \), \( \xi = \sum a_i e_i \), \( \eta = \sum b_i e^i \). Then, \( g(\nabla_{e_i} \xi, e_j) = g(\nabla_{e_j} \xi, e_i) \) for any basis elements \( e_i, e_j \), which implies that \( \xi = a_1 e_1 + a_2 e_2 \) and \( \eta = b_1 e^1 + b_2 e^2 \). However, replacing basis elements for vector fields in the
The algebra $\mathfrak{g}_4$: By Koszul’s formula, the covariant derivatives of the basis elements are as follows:

$$
\nabla_{e_1}e_2 = \frac{1}{2}e_3, \quad \nabla_{e_1}e_3 = -\frac{1}{2}e_2 + \frac{1}{2}e_4, \quad \nabla_{e_1}e_4 = -\frac{1}{2}e_3 + \frac{1}{2}e_5, \quad \nabla_{e_1}e_5 = -\frac{1}{2}e_4.
$$

$$
\nabla_{e_2}e_1 = -\frac{1}{2}e_3, \quad \nabla_{e_2}e_3 = \frac{1}{2}e_1, \\
\nabla_{e_2}e_4 = \frac{1}{2}e_2 + \frac{1}{2}e_5, \quad \nabla_{e_2}e_5 = \frac{1}{2}e_1, \\
\nabla_{e_3}e_1 = \frac{1}{2}e_2 - \frac{1}{2}e_4, \quad \nabla_{e_3}e_2 = \frac{1}{2}e_1, \quad \nabla_{e_3}e_4 = \frac{1}{2}e_1, \quad \nabla_{e_3}e_5 = \frac{1}{2}e_1, \\
\nabla_{e_4}e_1 = \frac{1}{2}e_3 - \frac{1}{2}e_5, \quad \nabla_{e_4}e_2 = \frac{1}{2}e_1, \quad \nabla_{e_4}e_3 = \frac{1}{2}e_1, \quad \nabla_{e_4}e_5 = \frac{1}{2}e_1, \\
\nabla_{e_5}e_1 = \frac{1}{2}e_4, \quad \nabla_{e_5}e_2 = \frac{1}{2}e_1, \\
\nabla_{e_5}e_3 = \frac{1}{2}e_4, \quad \nabla_{e_5}e_4 = \frac{1}{2}e_1.
$$

There exists no non-zero parallel vector field on $\mathfrak{g}_4$. Thus, there is no non-zero parallel two-form on $\mathfrak{g}_4$.

There is no nearly cosymplectic structure. If a non-zero vector field $\xi$ is Killing if and only if $g(\nabla_{\xi}\Phi, e_i) = 0$. We have $g(\nabla_{e_1}\Phi, e_i) = 0$ for any basis elements $e_1, e_2, e_3$. Then $\xi = e_3$ is parallel.

Let $\Phi = \sum b_{ij}e^j$ be the two-form of a nearly cosymplectic a.c.m.s. Replacing $X$ and $Y$ by basis elements, we get $a_i = 0$ for $i = 1, \cdots, 5$. It also shows that $\nabla\eta \neq 0$ for any almost contact metric structure $(\phi, \xi, \eta, g)$ on $\mathfrak{g}_4$.

A vector field $\xi$ on $\mathfrak{g}_4$ is Killing if and only if $\xi \in \langle e_5 \rangle$. Let $\xi = \sum_{i} \alpha_i e_i$ be a non-zero Killing vector field. Then, for any $e_i, e_j$, we have $g(\nabla_{\xi}e_j, e_i) = g(\nabla_{e_i}\xi, e_j)$. Thus,

$$
g(\nabla_{e_1}\xi, e_3) = -g(\nabla_{e_3}\xi, e_1) \Rightarrow a_2 = 0,
$$

$$
g(\nabla_{e_1}\xi, e_4) = -g(\nabla_{e_4}\xi, e_1) \Rightarrow a_3 = 0,
$$

$$
g(\nabla_{e_1}\xi, e_5) = -g(\nabla_{e_5}\xi, e_1) \Rightarrow a_4 = 0,
$$

$$
g(\nabla_{e_2}\xi, e_3) = -g(\nabla_{e_3}\xi, e_2) \Rightarrow a_1 = 0.
$$

No condition is obtained for $a_5$. In other words, $\xi$ is Killing if and only if $\xi = a_5 e_5$.

There is no $\alpha$-Sasakian structure. Let $(\phi, \xi, \eta, g)$ be an $\alpha$-Sasakian structure on $\mathfrak{g}_4$. Then, $\xi = e_5$, since it is a unit Killing vector field. On the other hand, by considering the relation $\nabla_X\xi = -a\phi(X)$, we get $\phi(e_5) = -\frac{1}{2\alpha} \nabla e_5 = 0$.

However, in this case, $g(\phi(e_2), \phi(e_2)) \neq g(e_2, e_2) - \eta(e_2)\eta(e_2)$.

There is no $\beta$-Kenmotsu structure. Let $(\phi, \xi, \eta, g)$ be a $\beta$-Kenmotsu structure with fundamental two-form $\Phi = \sum b_{ij}e^j$, $\xi = \sum a_i e_i$.
\( \eta = \sum b_i e^i \). Then, \( g(\nabla e_i \xi, e_j) = g(\nabla e_j \xi, e_i) \) for any basis elements \( e_i, e_j \), which implies that \( \xi = a_1 e_1 + a_2 e_3 \) and \( \eta = b_1 e^1 + b_2 e^2 \). However, after an easy calculation on the defining relation of a \( \beta \)-Kenmotsu structure, we get \( b_{ij} = 0 \), for any \( i, j \).

- There exists a semi cosymplectic structure.
  For any two-form \( \Phi = \sum b_i e^i \) any \( X = \sum x_i e_i \in g_4 \),
  \[ \delta \Phi(X) = -\sum (\nabla e_i \Phi)(e_i, X) = -\{x_3 b_{12} + x_4 b_{13} + x_5 b_{14}\} \]
  Thus \( \delta \Phi(X) = 0 \) for any \( X \) iff \( b_{12} = b_{13} = b_{14} = 0 \). In addition, for any one-form \( \eta = \sum b_i e^i \), we have
  \[ \delta \eta = -\sum (\nabla e_i \eta)(e_i) = -\sum g(\nabla e_i \xi, e_i) = 0. \]
  Thus for example, the a.c.m.s. for which \( \xi = e_1, \eta = e^1 \) and \( \Phi = e^{23} + e^{45} \) is semi cosymplectic.

- There exists no almost cosymplectic structure.
  Since \( d\eta(X, Y) = \frac{1}{2} \{ (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) \} \), \( d\eta(X, Y) = 0 \) iff \( (\nabla_X \eta)(Y) = (\nabla_Y \eta)(X) \), or equivalently, \( g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X) \) for all \( X, Y \) in \( g_4 \). Substituting basis elements for \( X \) and \( Y \) implies that \( d\eta = 0 \) iff \( \xi = a_1 e_1 + a_2 e_2 \). Any almost cosymplectic structure is almost-K-contact, thus for the fundamental form \( \Phi = \sum b_i e^i \) of an almost cosymplectic structure, we have \( \nabla_\xi \Phi = 0 \), where \( \xi = a_1 e_1 + a_2 e_2 \). \( \nabla_\xi \Phi(e_i, e_j) = 0 \) yields \( \Phi = 0 \).

**The algebra \( g_5 \):** By Koszul’s formula; the covariant derivatives of the basis elements are as follows:
\[
\begin{align*}
\nabla e_1 e_2 &= \frac{1}{2} e_4, & \nabla e_1 e_3 &= \frac{1}{2} e_5, & \nabla e_1 e_4 &= -\frac{1}{2} e_2, & \nabla e_1 e_5 &= -\frac{1}{2} e_3, \\
\nabla e_2 e_1 &= -\frac{1}{2} e_4, & \nabla e_2 e_3 &= \frac{1}{2} e_1, & \nabla e_2 e_4 &= \frac{1}{2} e_5, & \nabla e_2 e_5 &= \frac{1}{2} e_1, \\
\nabla e_3 e_1 &= -\frac{1}{2} e_2, & \nabla e_3 e_4 &= \frac{1}{2} e_1, & \nabla e_3 e_5 &= -\frac{1}{2} e_3, & \nabla e_3 e_2 &= \frac{1}{2} e_1.
\end{align*}
\]

- There exists no cosymplectic structure.
  The proof is similar to that in other algebras.

- There is no nearly cosymplectic structure on \( g_5 \).
  Let \( \Phi = \sum b_i e^i \) be the two-form of a nearly cosymplectic a.c.m.s. Replacing \( X \) and \( Y \) by basis elements, we have \( (\nabla e_i \Phi)(e_i, e_j) = 0 \), which shows that \( b_{ij} = 0 \), except for \( b_{23}, b_{35}, b_{34} \) and \( b_{45} \). Then \( \Phi = b_{23} e^{23} + b_{25} e^{25} + b_{34} e^{34} + b_{45} e^{45} \). We obtain \( b_{23} = b_{25} = b_{34} = b_{45} = 0 \) for \( X = e_1, Y = e_2, Z = e_3; X = e_1, Y = e_2, Z = e_3; X = e_3, Y = e_1, Z = e_2 \) and \( X = e_4, Y = e_1, Z = e_3 \) respectively in the equation (2).

- There is no non-zero parallel vector field on \( g_5 \).
  The proof is similar to other cases. In particular, \( (\phi, \xi, \eta, g) \) is neither \( C_1 \) (nearly-K-cosymplectic), nor \( C_2 \).

- A vector field \( \xi \) on \( g_5 \) is Killing if and only if \( \xi \in \langle e_4, e_5 \rangle \).
  Let \( \xi = \sum a_i e_i \) be a non-zero Killing vector field. Then, for any \( e_i, e_j \), we have
  \[ g(\nabla e_i \xi, e_j) = -g(\nabla e_j \xi, e_i). \]
  Thus,
  \[
  g(\nabla e_i \xi, e_4) = -g(\nabla e_4 \xi, e_i) \Rightarrow a_2 = 0,
  \]
  \[
  g(\nabla e_i \xi, e_5) = -g(\nabla e_5 \xi, e_i) \Rightarrow a_3 = 0,
  \]
  \[
  g(\nabla e_2 \xi, e_4) = -g(\nabla e_4 \xi, e_2) \Rightarrow a_1 = 0.
  \]
  No condition is obtained for \( a_4 \) and \( a_5 \). Thus, \( \xi \) is Killing if and only if \( \xi = a_4 e_4 + a_5 e_5 \).

- There is no \( \alpha \)-Sasakian structure.
  Let \( (\phi, \xi, \eta, g) \) be an \( \alpha \)-Sasakian structure on \( g_5 \). Then, \( \xi = a_4 e_4 + a_5 e_5 \), where \( a_4^2 + a_5^2 = 1 \) and \( \eta = b_4 e_4 + b_5 e_5 \). By the relation \( \nabla_X \xi = -a \phi(X) \), we get \( \phi(e_2) = -\frac{a_4}{a_5} e_1 \) and \( \phi(e_3) = -\frac{a_5}{a_4} e_1 \).
Since \( g(\phi(e_2), \phi(e_3)) = g(e_2, e_3) - \eta(e_2)\eta(e_3) \), we have \( a_4, a_5 = 0 \). This implies \( \phi(e_2) = 0 \), or \( \phi(e_3) = 0 \). Assume without loss of generality that \( \phi(e_2) = 0 \). Then, \( g(\phi(e_2), \phi(e_2)) \neq g(e_2, e_2) - \eta(e_2)\eta(e_2) \).

- There is no non-zero parallel vector field on \( g \).
- There exists no cosymplectic structure on \( g \).
- There exists a semi cosymplectic structure.

Thus, \( \xi = a_1e_1 + a_2e_2 + a_3e_3 \) and \( \eta = b_1e_1^2 + b_2e_2^2 + b_3e_3^3 \). Replacing basis elements for \( X, Y, Z \) in the defining relation of \( \beta \)-Kenmotsu structures results in \( \Phi = 0 \). Thus, there does not exist a \( \beta \)-Kenmotsu structure.

- There exists a semi cosymplectic structure.

For any two-form \( \Phi = \sum b_i e_i^i \) any \( X = \sum x_i e_i \in g_5 \),
\[
\delta \Phi(X) = -\sum (\nabla_{e_i} \Phi)(e_i, X) = -\{x_4b_{12} + x_5b_{13}\}
\]
Thus, \( \delta \Phi(X) = 0 \) for any \( X \) iff \( b_{12} = b_{13} = 0 \). In addition, for any one-form \( \eta = \sum b_i e_i \), we have
\[
\delta \eta = -\sum (\nabla_{e_i} \eta)(e_i) = -\sum g(\nabla_{e_i} \xi, e_i) = 0.
\]
Thus, for example, the a.c.m.s. for which \( \xi = e_5, \eta = e^5 \) and \( \Phi = e^{14} + e^{23} \) is semi cosymplectic.

- There exists an almost cosymplectic structure.

Consider, for instance, the a.c.m.s. given by \( \xi = e_1, \eta = e^1 \) and \( \Phi = e^{25} + e^{34} \).

**The algebra \( g_5 \):** By Kozsul’s formula, the covariant derivatives of the basis elements are as follows:
\[
\nabla_{e_1} e_2 = \frac{1}{2} e_3, \quad \nabla_{e_1} e_3 = -\frac{1}{2} e_2 + \frac{1}{2} e_4, \quad \nabla_{e_1} e_4 = -\frac{1}{2} e_3, \\
\nabla_{e_2} e_1 = -\frac{1}{2} e_3, \quad \nabla_{e_2} e_3 = \frac{1}{2} e_1 + \frac{1}{2} e_5, \quad \nabla_{e_2} e_5 = -\frac{1}{2} e_3, \\
\nabla_{e_3} e_1 = -\frac{1}{2} e_2 - \frac{1}{2} e_4, \quad \nabla_{e_3} e_2 = \frac{1}{2} e_1 - \frac{1}{2} e_5, \quad \nabla_{e_3} e_4 = \frac{1}{2} e_1, \quad \nabla_{e_3} e_5 = \frac{1}{2} e_2, \\
\nabla_{e_4} e_1 = -\frac{1}{2} e_3, \quad \nabla_{e_4} e_3 = \frac{1}{2} e_1, \quad \nabla_{e_4} e_5 = -\frac{1}{2} e_3, \quad \nabla_{e_4} e_5 = \frac{1}{2} e_2.
\]

- There exists no cosymplectic structure on \( g_6 \).

It is easy to see that \( \nabla \Phi = 0 \) if and only if \( \Phi = 0 \), where \( \Phi \) is a two-form.

- There is no nearly cosymplectic structure.

Let \( \Phi = \sum b_i e_i^i \) be a two-form with the property that \( \nabla_X \Phi(X, Y) = 0 \). Then, we obtain
\[ \Phi = b_{15}e^{15} + b_{24}e^{24} + b_{45}e^{45} \]. By considering \( \Phi \) as the fundamental two-form of an almost contact metric structure \( (\phi, \xi, \eta, g) \), from the condition \( \phi^2 = -I + \eta \otimes \xi \), we get \( b_{15}^2 = 1 \) and \( b_{45} = 0 \). We get \( b_{15} = b_{24} = 0 \) for \( X = e_5, Y = e_2, Z = e_3 \) and \( X = e_1, Y = e_3, Z = e_4 \) respectively in the equation (2).

- There is no non-zero parallel vector field on \( g_6 \).

The proof is the same as before.

- A vector field \( \xi \) on \( g_6 \) is Killing if and only if \( \xi \in \langle e_4, e_5 \rangle \).

Let \( \xi = \sum a_i e_i \) be a Killing vector field. Then, for any \( e_i, e_j \), we have \( g(\nabla_{e_i} \xi, e_j) = -g(\nabla_{e_j} \xi, e_i) \). Thus,
\[
\quad g(\nabla_{e_2} \xi, e_3) = -g(\nabla_{e_3} \xi, e_2) \Rightarrow a_1 = 0, \\
\quad g(\nabla_{e_3} \xi, e_3) = -g(\nabla_{e_2} \xi, e_2) \Rightarrow a_2 = 0, \\
\quad g(\nabla_{e_1} \xi, e_4) = -g(\nabla_{e_4} \xi, e_1) \Rightarrow a_3 = 0.
\]

No conditions are obtained for \( a_4 \) and \( a_5 \).
Theorem 1. An almost contact metric structure on a five-dimensional nilpotent Lie algebra \( g \) is cosymplectic if

Let \( g \) be one of \( g_6 \). Then, \( \xi \) has the form \( \xi = a_4 e_4 + a_5 e_5 \) and satisfies the equation \( \nabla_X \xi = -\alpha \xi \). Thus, the endomorphism can be expressed with:

\[
\phi(e_1) = \frac{a_4}{2a} e_3, \quad \phi(e_2) = \frac{a_5}{2a} e_3, \quad \phi(e_3) = -\frac{a_5}{2a} e_2, \quad \phi(e_4) = 0, \quad \phi(e_5) = 0.
\]

From the condition \( \phi^2 = -I + \eta \otimes \xi \), we have

\[
\phi^2(e_4) = 0 = (a_4^2 - 1)e_4 + a_4 a_5 e_5 \Rightarrow a_4^2 = 1, \quad a_4 a_5 = 0
\]

and

\[
\phi^2(e_5) = 0 = a_5 a_4 e_4 + (a_5^2 - 1)e_5 \Rightarrow a_5^2 = 1, \quad a_4 a_5 = 0.
\]

However, since \( a_4^2 = a_5^2 = 1 \), the number \( a_4 a_5 \) is non-zero.

There is no \( \alpha \)-Sasakian structure.

Let \((\phi, \xi, \eta, g)\) be an \( \alpha \)-Sasakian structure on \( g_6 \). Then, \( \xi \) has the form \( \xi = a_4 e_4 + a_5 e_5 \) and satisfies the equation \( \nabla_X \xi = -\alpha \xi \). Thus, the endomorphism can be expressed with:

\[
\phi(e_1) = \frac{a_4}{2a} e_3, \quad \phi(e_2) = \frac{a_5}{2a} e_3, \quad \phi(e_3) = -\frac{a_5}{2a} e_2, \quad \phi(e_4) = 0, \quad \phi(e_5) = 0.
\]

From the condition \( \phi^2 = -I + \eta \otimes \xi \), we have

\[
\phi^2(e_4) = 0 = (a_4^2 - 1)e_4 + a_4 a_5 e_5 \Rightarrow a_4^2 = 1, \quad a_4 a_5 = 0
\]

and

\[
\phi^2(e_5) = 0 = a_5 a_4 e_4 + (a_5^2 - 1)e_5 \Rightarrow a_5^2 = 1, \quad a_4 a_5 = 0.
\]

However, since \( a_4^2 = a_5^2 = 1 \), the number \( a_4 a_5 \) is non-zero.

There is no \( \beta \)-Kenmotsu structure.

Let \((\phi, \xi, \eta, g)\) be a \( \beta \)-Kenmotsu structure with fundamental two-form \( \Phi = \sum b_i e_i \), \( \xi = \sum a_i e_i \).

Then, \( g(\nabla_{e_i} \xi, e_j) = g(\nabla_{e_i} \xi, e_j) \) for any basis elements \( e_i, e_j \), which implies that \( \xi = a_1 e_1 + a_2 e_2 \).

However, after calculations on the defining relation, we get \( \Phi = b_{12} e^{12} \). However, in this case, \( \Phi \wedge \Phi = 0 \).

There exists a semi cosymplectic structure.

The a.c.m.s. \((\phi, \xi, \eta, g)\) with \( \xi = e_3, \eta = e^3 \) and \( \Phi = e^{14} + e^{25} \) is semi cosymplectic.

There is no almost cosymplectic structure.

Obviously, \( de_1 = 0, de_2 = 0, de_3 = -e^{12}, de_4 = -e^{13}, de_5 = -e^{23} \). Thus, for a one-form \( \eta = \sum b_i e_i \), we have \( d\eta = 0 \iff b_3 = b_4 = b_5 = 0 \), and for a two-form \( \Phi = \sum b_i e_i \), we get \( d\Phi = 0 \iff b_{15} = b_{24}, b_{34} = b_{35} = b_{45} = 0 \). So, if \((\phi, \xi, \eta, g)\) with the fundamental two-form \( \Phi = \sum b_i e_i \) is an almost cosymplectic structure on \( g_6 \), then, \( \Phi \) and \( \eta \) have the forms \( \Phi = b_{12} e^{12} + b_{13} e^{13} + b_{14} e^{14} + b_{15} e^{15} + b_{23} e^{23} + b_{15} e^{24} + b_{25} e^{25} \) and \( \eta = b_1 e^1 + b_2 e^2 \). However, it is easy to see that \( \eta \wedge \Phi \wedge \Phi \) vanishes. Thus, the structure is not almost cosymplectic.

In summary, we state the following.

**Theorem 1.** An almost contact metric structure on a five-dimensional nilpotent Lie algebra \( g \) is cosymplectic if and only if \( g \) is abelian.

The existence of cosymplectic structures on Lie groups and on their compact quotients by uniform discrete subgroups was studied in [14]. We state Theorem 1 by direct calculation.

In the sequel, we deduce

**Corollary 2.** There is no cosymplectic left invariant almost contact metric structure on a five-dimensional connected Lie group whose corresponding Lie algebra is nilpotent.

**Theorem 3.** There is no nearly cosymplectic structure on any five-dimensional nilpotent Lie algebra.

**Corollary 4.** There is no nearly cosymplectic left invariant almost contact metric structure on a five-dimensional connected Lie group whose corresponding Lie algebra is nilpotent.

**Theorem 5.** There exists no non-zero parallel vector field on any five-dimensional nilpotent Lie algebra.

There are non-zero Killing vector fields on \( g_i \) for \( i \in \{1, 2, 3, 4, 5, 6\} \).

**Theorem 6.** Let \( g \) be one of \( g_1, g_2, g_3 \) or \( g_4 \). A vector field \( \xi \) on \( g \) is Killing if and only if \( \xi \in < e_5 > \).

In addition, if \( g \) is \( g_3 \) or \( g_6 \), then \( \xi \) is Killing if \( \xi \in < e_4, e_5 > \).
Theorem 7. If \( g \) has an \( \alpha \)-Sasakian structure, then \( g \) is isomorphic to \( g_1 \).

Theorem 8. There is no \( \beta \)-Kenmotsu a.c.m.s. on any five-dimensional nilpotent Lie algebra.

We may conclude

Corollary 9. There is no \( \beta \)-Kenmotsu left invariant almost contact metric structure on a five-dimensional connected Lie group whose corresponding Lie algebra is nilpotent.

Theorem 10. There exist semi cosymplectic a.c.m. structures on each \( g_i \).

Theorem 11. An a.c.m.s. on \( g \) is almost cosymplectic iff \( g \) is isomorphic to one of \( g_2, g_3 \) or \( g_5 \).

Let \( G \) be a simply-connected nilpotent Lie group with Lie algebra \( g \). It is known that there exists a co-compact discrete subgroup \( \Gamma \) of \( G \) such that \( G/\Gamma \) is a compact nilmanifold \[15\]. Giving examples of discrete subgroups \( \Gamma \) for simply-connected nilpotent Lie group \( G_i \) with Lie algebra \( g_i \) is an ongoing study.

4. Conclusions

In this paper, we examined almost contact metric structures on five dimensional nilpotent Lie algebras by direct calculation and obtained some results about the relations between the classes of almost contact metric structures and five dimensional nilpotent Lie algebras. In addition, we got some general results on left invariant almost contact metric structures on five dimensional nilpotent Lie groups by studying their corresponding Lie algebras.

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References


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