On Center, Periphery and Average Eccentricity for the Convex Polytopes

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Abstract: A vertex \( v \) is a peripheral vertex in \( G \) if its eccentricity is equal to its diameter, and periphery \( P(G) \) is a subgraph of \( G \) induced by its peripheral vertices. Further, a vertex \( v \) in \( G \) is a central vertex if \( e(v) = \text{rad}(G) \), and the subgraph of \( G \) induced by its central vertices is called center \( C(G) \) of \( G \). Average eccentricity is the sum of eccentricities of all of the vertices in a graph divided by the total number of vertices, i.e., \( \text{avec}(G) = \frac{1}{n} \sum_{u \in V(G)} e_G(u) \). If every vertex in \( G \) is central vertex, then \( C(G) = G \), and hence, \( G \) is self-centered. In this report, we find the center, periphery and average eccentricity for the convex polytopes.

Keywords: eccentricity; center; periphery; average eccentricity

1. Introduction

In the facility location problem, we select a site according to some standard judgment. For example, if we want to find out the exact location for an emergency facility, such as a fire station or a hospital, we reduce the distance between that facility and the area where the emergency happens, and if we are to decide the position for a service facility, like a post office, power station or employment office, we try to reduce the traveling time of all people who have been living in that district. In the construction of a railway line, a pipeline and a superhighway, we will reduce the distance of the constructing unit for the people living in that area. All of these situations illustrate the concept of centrality but each of these three examples deals with different types of centers. Nowadays, centrality questions are being studied with the help of distance and graphs. We shall observe that many kinds of centers are helpful in facility location problems.

The most important and fundamental concept that extends to the whole of graph theory is distance. The distance is applicable in many fields, such as graph operation, extremal problems on connectivity, diameter and isomorphism testing. The theme of distance is used to check the symmetry of graphs. It also provides a base for many useful graph parameters, like radius, diameter, metric dimension, eccentricity, center and periphery, etc.

The eccentricity of the vertices in \( G \) has a fundamental importance. Recently, many indices related to eccentricity have been derived, i.e., eccentric connectivity index, adjacent eccentric sum index, Wiener index and eccentric distance sum [1]. The center and periphery is also based on minimum and
maximum eccentricity, respectively. W. Goddard and O. R. Oellermann in [2] have shown that if \( G \) is an undirected graph, then,

\[
rad(G) \leq diam(G) \leq 2rad(G)
\]

They also examined the radius and diameter of certain families of graphs in the same paper, as follows:

1. \( rad(K_n) = diam(K_n) = 1 \) for \( n \geq 2 \),
2. \( rad(C_n) = diam(C_n) = \frac{n}{2} \),
3. \( rad(K_{m,n}) = diam(K_{m,n}) = 2 \) if \( m \) and \( n \) is at least two,
4. \( rad(P_n) = \frac{n-1}{2} \), \( diam(P_n) = n - 1 \).

This implies that complete graphs \( K_n \) for \( n \geq 2 \), complete bipartite graphs \( K_{m,n} \) where \( m, n \geq 2 \) and all cycles are self-centered. Jordan [3] determined the diameter of a tree. Bela Bollobas [4] discussed the diameter of random graphs. The radius and diameter of a bridge graph are determined by Martin Farber in [5]. More general results were presented by V. Klee and D. Larman [6] and Bela Bollobas [4]. B. Hedman determined the sharp bounds for the diameter of the clique graph \( K(G) \) in terms of the diameter of \( G \). The idea of self-centered graphs is presented and elaborated by Ando, Akiyama and Avis individually [7]. These self-centered graphs are extensively studied in [7–11].

Definition 1. For a connected graph \( G \), the eccentricity \( e(v) \) of a vertex \( v \) is its distance to a vertex farthest from \( v \). Thus,

\[
e(v) = \max\{d(u,v) : u \in V(G)\}.
\]

Definition 2. The radius \( rad(G) \) of \( G \) is the minimum eccentricity among all vertices of \( G \).

Definition 3. The diameter \( diam(G) \) of \( G \) is the maximum eccentricity among all vertices of \( G \).

Definition 4. Average eccentricity is the sum of eccentricities of all of the vertices in a graph divided by the total number of vertices, i.e.,

\[
avec(G) = \frac{1}{n} \sum_{u \in V(G)} e_G(u).
\]

Definition 5. A vertex \( u \) is eccentric to a vertex \( v \) if \( d(u,v) = e(v) \).

Definition 6. A vertex \( v \) is a peripheral vertex in \( G \) if its eccentricity is equal to its diameter, and periphery \( P(G) \) is a subgraph of \( G \) induced by its peripheral vertices. Further, a vertex \( v \) in \( G \) is a central vertex if \( e(v) = rad(G) \), and the subgraph of \( G \) induced by its central vertices is called center \( C(G) \) of \( G \). If every vertex in \( G \) is a central vertex, then \( C(G) = G \), and hence, \( G \) is self-centered.
In the present report, we discuss the center, periphery and average eccentricity for families of convex polytope graphs, \(A_n, S_n\) and \(T_n\).

2. The Center and Periphery for Convex Polytope \(A_n\)

In this section, we determine the center and periphery for convex polytope \(A_n\).

**Definition 7.** The graph of convex polytope (double antiprism) \(A_n\) can be obtained from the graph of convex polytope \(R_n\) by adding new edges \(b_{i+1}c_i\), i.e.,

\[V(A_n) = V(R_n) \text{ and } E(A_n) = E(R_n) \cup \{b_{i+1}c_i : 1 \leq i \leq n\}.\]

**Theorem 1.** For the family of convex polytope \(A_n, n = 2k\), \(Cen(A_n)\) and \(Per(A_n)\) are subgraphs induced by the vertices \((b_1, b_2, ..., b_{2k})\) and \(\{a_i \cup c_i : 1 \leq i \leq 2k\}\), respectively.

**Proof.** For all even values of \(n\), select a vertex \(a_1\) on the cycle \((a_1a_2a_3...a_i...a_{2k})\). Then:

\[d(a_1, a_i) = i - 1, \quad 1 \leq i \leq k + 1\]  

when \(i = k + 2, d(a_1, a_i) = k - 1\) and for \(i = 2k, d(a_1, a_i) = 1\).

In addition, for every value of \(i\) within \(k + 2\) to \(2k\), \(d(a_1, a_i)\) must lie between \(k - 1\) and one, i.e.,

\[d(a_1, a_i) = 2k + 1 - i; \quad k + 2 \leq i \leq 2k.\]

Thus, to find the vertices farthest from \(a_1\) in \(A_n\), consider only \(1 \leq i \leq k + 1\).

As each \(a_i\) is adjacent to \(b_j, b_{j-1}\) and each \(b_i\) adjacent to \(c_j, c_{j-1}\), therefore, (1) implies,

\[d(a_1, b_i) = i, \quad 1 \leq i \leq k\]

\[d(a_1, c_i) = i + 1, \quad 1 \leq i \leq k\]

For \(k + 1 \leq i \leq 2k\), consider the cycle \((b_1b_2b_{k+1}...b_i...b_{2k})\). In this cycle, the distance between \(b_i\) and \(b_{2k}\) is \(2k - i\), and \(b_{2k}\) is adjacent to \(a_1\), therefore, the distance between \(a_1\) and \(b_{2k}\) is \(2k - i + 1\).

\[d(a_1, b_i) = 2k - i + 1, \quad k + 1 \leq i \leq 2k.\]

Now, consider the cycle \((c_1c_2...c_{k+1}...c_i...c_{2k})\). The distance between \(c_i\) and \(c_{2k-1}\) is \(2k - 1 - i\) and the vertex \(c_{2k-1}\) is adjacent to \(b_{2k}\) and \(b_{2k}\) adjacent to \(a_1\). It shows,

\[d(a_1, c_i) = 2k - i + 1, \quad k + 1 \leq i \leq 2k - 1.\]

For \(i = 2k, d(a_1, c_i) = 2\).

Hence, \(c_k\) is a vertex farthest from \(a_1\).

\[e(a_1) = k + 1\]  

Thus, the eccentricity of each vertex on inner cycle \((a_1a_2a_3...a_i...a_{2k})\) is \(k + 1\).

In the same way, take cycle \((b_1b_2b_i...b_{2k})\); the distance between \(b_1\) and \(b_i\) in this cycle is,

\[d(b_1, b_i) = i - 1; \quad 1 \leq i \leq k + 1\]  

Each \(b_i\) is adjacent to \(a_i\) and \(a_{i+1}\). Therefore,

\[d(b_1, a_1) = 1, \quad (b_1, a_2) = 1\]

For \(3 \leq i \leq k + 1\), consider the path \(b_1 \rightarrow a_2 \rightarrow a_3 \rightarrow ... \rightarrow a_i\). Then, the distance between \(a_i\) and \(a_2\) is \(i - 2\). \(a_2\) is also adjacent to \(b_1\). Therefore, the distance between \(b_1\) and \(a_i\) is as follows,
\[ d(b_1, a_i) = i - 1, \quad 3 \leq i \leq k + 1 \]

For \( k + 2 \leq i \leq 2k \), consider the cycle \( (a_1a_2...a_{k+2}...a_1a_2k) \). The distance between \( a_i \) and \( a_{2k} \) is \( 2k - i \). Further, \( a_{2k} \) is adjacent to \( a_1 \) and \( a_1 \) adjacent to \( b_1 \). Therefore, the distance between \( b_1 \) and \( a_i \) is \( 2k - i + 2 \).

\[ d(b_1, a_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k \]

Further, \( b_i \) is also adjacent to \( c_i \) and \( c_{i-1} \); it follows from (3):

\[ d(b_1, c_i) = i, \quad 1 \leq i \leq k \]

For \( k + 1 \leq i \leq 2k \), consider the cycle \( (c_1c_2...c_{k+1}...c_1c_{2k}) \). The distance between \( c_i \) and \( c_{2k} \) is \( 2k - i \), where \( c_{2k} \) is also adjacent to \( b_1 \). Therefore,

\[ d(b_1, c_i) = 2k - i + 1, \quad k + 1 \leq i \leq 2k \]

Hence, \( b_{k+1}, a_{k+1} \) and \( c_k \) are the vertices farthest from \( b_1 \). Therefore:

\[ e(b_1) = k \tag{4} \]

Hence, each vertex on the middle cycle \( (b_1b_2...b_{i-1}...b_2b_1) \) has eccentricity \( k \).

Further, to find out the eccentricity of the vertices on the outer cycle \( (c_1c_2...c_{i-1}...c_{2k}) \), choose a vertex \( c_1 \) on this cycle. The distance between \( c_1 \) and \( c_i \) is \( i - 1 \).

\[ d(c_1, c_i) = i - 1, \tag{5} \]

Each \( c_i \) is adjacent to \( b_i \) and \( b_{i+1} \), i.e.,

\[ d(c_1, b_i) = 1, \tag{6} \]

For \( 3 \leq i \leq k + 1 \), consider the path \( c_1 \to b_2 \to b_3 \to ... \to b_i \). The distance between \( b_2 \) and \( b_i \) is \( i - 2 \). As \( b_2 \) is adjacent to \( c_1 \), therefore, \( c_1 \) and \( b_i \) has the following distance,

\[ d(c_1, b_i) = i - 1, \quad 3 \leq i \leq k + 1 \]

For \( k + 2 \leq i \leq 2k \), consider the cycle \( (b_1b_2b_{k+1}...b_1b_{2k}) \). The distance between \( b_i \) and \( b_{2k} \) is \( 2k - i \). As \( b_{2k} \) is adjacent to \( b_1 \) and \( b_1 \) adjacent to \( c_1 \), therefore, the distance between \( c_1 \) and \( b_i \) is \( 2k - i + 2 \).

\[ d(c_1, b_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k \]

Each \( b_i \) is also adjacent to \( a_i \) and \( a_{i+1} \), using the result of (6),

\[ d(c_1, a_1) = 2, \quad d(c_1, a_2) = 2 \]

For \( 3 \leq i \leq k + 2 \), consider the path \( c_1 \to b_2 \to b_3 \to ... \to b_{i-1} \to a_i \). In this path, the distance between \( b_2 \) and \( b_{i-1} \) is \( i - 3 \). \( b_{i-1} \) is adjacent to \( a_i \) and \( b_2 \) adjacent to \( c_1 \). Therefore, the distance between \( c_1 \) and \( a_i \) is \( i - 1 \).

\[ d(c_1, a_i) = i - 1, \quad 3 \leq i \leq k + 2 \]

For \( k + 3 \leq i \leq 2k \), consider the cycle \( (a_1a_2...a_{k+2}...a_1a_{2k}) \). The distance between \( a_i \) and \( a_{2k} \) is \( 2k - i \). \( a_{2k} \) is adjacent to \( a_1 \); \( a_1 \) adjacent to \( b_1 \) and \( b_1 \) adjacent to \( c_1 \). Therefore, the distance between \( c_1 \) and \( a_i \) is \( 2k - i + 3 \).

\[ d(c_1, a_i) = 2k - i + 3, \quad k + 3 \leq i \leq 2k \]

This shows that \( a_{k+1} \) is a vertex farthest from \( c_1 \). Therefore:
\[ e(c_1) = k + 1. \] (7)

Therefore, (2), (4) and (7) imply,

\[ \text{diam}(A_n) = k + 1 = \frac{n}{2} + 1. \]

and:

\[ \text{rad}(A_n) = k = \frac{n}{2}. \]

Consequently, \( \text{Cen}(A_n) \) is a subgraph induced by vertices \( (b_1, b_2, ..., b_{2k}) \), while the set of vertices \( \{a_1, a_2, ..., a_{2k}, c_1, c_2, ..., c_{2k}\} \) is the peripheral vertices. Therefore, the periphery of \( A_n \) is the subgraph induced by all of these vertices. \( \square \)

**Theorem 2.** For the family of convex polytope \( A_n \), \( n \) is odd.

\[ \text{Cen}(A_n) = \text{Per}(A_n) = A_n. \]

**Proof.** Consider, \( n = 2k + 1 \ k \geq 2 \). Select vertex \( a_1 \) on the cycle \( (a_1 a_2 a_3 ... a_{2k+1}) \). By using this,

\[ d(a_1, a_i) = i - 1, \ 1 \leq i \leq k + 1 \] (8)

while \( i \) increases from \( k + 2 \) to \( 2k + 1 \), \( d(a_1, a_i) \) reduces from \( k \) to one.

\[ d(a_1, a_i) = 2k + 2 - i, \ k + 2 \leq i \leq 2k + 1 \]

Thus, to find the vertices farthest from \( a_1 \) in \( A_n \), we have to take only those values of \( i \) that lie between one and \( k + 1 \).

As each \( a_i \) is adjacent to \( b_i, b_{i-1} \) and each \( b_i \) adjacent to \( c_i, c_{i-1} \), therefore, (8) implies,

\[ d(a_1, b_i) = i, \ 1 \leq i \leq k + 1 \]

\[ d(a_1, c_i) = 1 + i, \ 1 \leq i \leq k \]

For \( k + 2 \leq i \leq 2k + 1 \), consider the cycle \( (b_1 b_2 ... b_{k+1} ... b_{2k+1}) \). In this cycle, the distance between \( b_1 \) and \( b_{2k+1} \) is \( 2k + 1 - i \), and \( b_{2k+1} \) is adjacent to \( a_1 \). Therefore, the distance between \( a_1 \) and \( b_i \) is \( 2k - i + 2 \).

\[ d(a_1, b_i) = 2k - i + 2, \ k + 2 \leq i \leq 2k + 1. \]

Now, consider the cycle \( (c_1 c_2 ... c_{k+1} ... c_{2k+1}) \). The distance between \( c_i \) and \( c_{2k} \) is \( 2k - i \). The vertex \( c_{2k} \) is adjacent to \( b_{2k+1} \) and \( b_{2k+1} \) adjacent to \( a_1 \). It shows that the distance between \( a_1 \) and \( c_i \) is \( 2k - i + 2 \).

\[ d(a_1, c_i) = 2k - i + 2, \ k + 1 \leq i \leq 2k. \]

For \( i = 2k + 1 \), \( d(a_1, c_i) = 2 \).

Hence, \( c_k \) and \( b_{k+1} \) are the vertices farthest from \( a_1 \). Therefore:

\[ e(a_1) = k + 1 \] (9)

Thus, the eccentricity of each vertex on inner cycle \( (a_1 a_2 a_3 ... a_{2k+1}) \) is \( k + 1 \).

Similarly as above, the vertices \( b_1 \) and \( b_i \) on cycle \( (b_1 b_2 ... b_i ... b_{2k+1}) \) have the distance as,

\[ d(b_1, b_i) = i - 1, \ 1 \leq i \leq k + 1 \] (10)
Each \( b_i \) is adjacent to \( a_i \) and \( a_{i+1} \). Therefore,

\[
d(b_1, a_1) = 1, \quad (b_1, a_2) = 1
\]

For \( 3 \leq i \leq k + 2 \), consider the path \( b_1 \rightarrow a_2 \rightarrow a_3 \rightarrow ... \rightarrow a_i \). Then, the distance between \( a_i \) and \( a_2 \) is \( i - 2 \). \( a_2 \) is also adjacent to \( b_1 \). Therefore, the distance between \( b_1 \) and \( a_i \) is \( i - 1 \), i.e.,

\[
d(b_1, a_i) = i - 1, \quad 3 \leq i \leq k + 2
\]

For \( k + 3 \leq i \leq 2k + 1 \), consider the cycle \((a_1a_2...a_{k+3}...a_i...a_{2k+1})\). The distance between \( a_i \) and \( a_{2k+1} \) is \( 2k - i + 1 \). Further, \( a_{2k+1} \) is adjacent to \( a_1 \), and \( a_1 \) is adjacent to \( b_1 \). Therefore, the distance between \( b_1 \) and \( a_i \) is \( 2k - i + 3 \).

\[
d(b_1, a_i) = 2k - i + 3, \quad k + 3 \leq i \leq 2k + 1
\]

Further, \( b_i \) is also adjacent to \( c_i \) and \( c_{i-1} \); it follows from (10):

\[
d(b_1, c_i) = i, \quad 1 \leq i \leq k + 1
\]

For \( k + 2 \leq i \leq 2k + 1 \), consider the cycle \((c_1c_2...c_{k+2}...c_i...c_{2k+1})\). The distance between \( c_i \) and \( c_{2k+1} \) is \( 2k + 1 - i \). \( c_{2k+1} \) is also adjacent to \( b_1 \). Therefore,

\[
d(b_1, c_i) = 2k + 2 - i, \quad k + 2 \leq i \leq 2k + 1
\]

Hence, \( a_{k+2} \) and \( c_{k+1} \) are the vertices farthest from \( b_1 \). Therefore:

\[
e(b_1) = k + 1.
\] (11)

Hence, each vertex on the middle cycle \((b_1b_2...b_i...b_{2k+1})\) has eccentricity \( k + 1 \).

Further, to find out the eccentricity of the vertices on the outer cycle \((c_1c_2...c_i...c_{2k+1})\), choose a vertex \( c_1 \) on this cycle. The distance between \( c_1 \) and \( c_i \) is \( i - 1 \).

\[
d(c_1, c_i) = i - 1, \quad 1 \leq i \leq k + 1
\] (12)

Each \( c_i \) is adjacent to \( b_i \) and \( b_{i+1} \). Therefore,

\[
d(c_1, b_1) = 1, \quad d(c_1, b_2) = 1
\] (13)

For \( 3 \leq i \leq k + 2 \), consider the path \( c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow ... \rightarrow b_i \). The distance between \( b_2 \) and \( b_1 \) is \( i - 2 \). \( b_2 \) is adjacent to \( c_1 \), therefore, the distance between \( c_1 \) and \( b_1 \) is \( i - 1 \).

\[
d(c_1, b_i) = i - 1, \quad 3 \leq i \leq k + 2
\]

For \( k + 3 \leq i \leq 2k + 1 \), consider the cycle \((b_1b_2...b_{k+3}...b_i...b_{2k+1})\). The distance between \( b_i \) and \( b_{2k+1} \) is \( 2k - i + 1 \). \( b_{2k+1} \) is adjacent to \( b_1 \) and \( b_1 \) adjacent to \( c_1 \), therefore, the distance between \( c_1 \) and \( b_1 \) is \( 2k - i + 3 \).

\[
d(c_1, b_i) = 2k - i + 3, \quad k + 3 \leq i \leq 2k + 1
\]

Each \( b_i \) is also adjacent to \( a_i \) and \( a_{i+1} \), using the result of (13):

\[
d(c_1, a_1) = 2, \quad d(c_1, a_2) = 2
\]
For $3 \leq i \leq k + 2$, consider the path $c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow ... \rightarrow b_{i-1} \rightarrow a_i$. In this path, the distance between $b_2$ and $b_{i-1}$ is $i - 3$. $b_{i-1}$ is adjacent to $a_i$, and $b_2$ is adjacent to $c_1$. Therefore, the distance between $c_1$ and $a_i$ is $i - 1$.

$$d(c_1, a_i) = i - 1, \quad 3 \leq i \leq k + 2$$

For $k + 3 \leq i \leq 2k$, consider the cycle $(a_1a_2...a_{k+3}...a_{i-1}a_i)$. The distance between $a_i$ and $a_{2k+1}$ is $2k + 1 - i$. $a_{2k+1}$ is adjacent to $a_1$ and $a_1$ adjacent to $b_1$. In addition, $b_1$ is adjacent to $c_1$. Therefore, the distance between $c_1$ and $a_i$ is $2k - i + 4$.

$$d(c_1, a_i) = 2k - i + 4, \quad k + 3 \leq i \leq 2k + 1$$

This shows that $a_{k+2}$ and $b_{k+2}$ are the vertices farthest from $c_1$. Therefore:

$$e(c_1) = k + 1. \quad (14)$$

Consequently, (9), (11) and (14) show the smallest, In addition, the greatest eccentricity of these vertices is $k + 1$. Therefore:

$$diam(A_n) = rad(A_n) = k + 1 = \frac{n - 1}{2} + 1 = \frac{n + 1}{2}.$$ 

Implies:

$$\text{Cen}(A_n) = \text{Per}(A_n) = A_n.$$ 

Hence, the family of $A_n$ is self-centered for odd values of $n$.  \(\square\)

### 2.1. Average Eccentricity for Convex Polytope $A_n$

Here, we also are concerned with calculating the average eccentricity for the graph of convex polytope $A_n$. The average eccentricity of any graph can be calculated by dividing the sum of the eccentricities of all of the vertices to the total number of vertices ($n$). There are three circles in the graph of convex polytope $A_n$, and each circle consists of $n$ vertices. Therefore, $A_n$ has a total of $3n$ vertices; it follows,

$$avec(A_n) = \frac{1}{n} \sum_{u \in V(G)} e_G(u) \quad (15)$$

By Theorem 1:

$$avec(A_n) = \frac{1}{3 \times (\hat{n})} \left[ n \times \left\{ e(a_1) + e(b_1) + e(c_1) \right\} \right]$$

$$= \frac{1}{3 \times n} \left[ n \times (k + 1) + (k) + (k + 1) \right] = k + \frac{2}{3} = \frac{n}{2} + \frac{1}{3}.$$ 

and by Theorem 2,

$$avec(A_n) = \frac{1}{3 \times n} \left[ n \times 3(k + 1) \right]$$

$$= k + 1 = \frac{n - 1}{2} + 1 = \frac{n + 1}{2}.$$ 

Therefore, we have the following result:

$$avec(A_n) = \begin{cases} \frac{n + 1}{2}, & \text{if } n = 2k + 1; \\ \frac{n}{2} + \frac{2}{3}, & \text{if } n = 2k. \end{cases}$$
2.2. Illustration

Consider the graph of $A_8$. We have labeled each of its vertices by its eccentricities. The center and periphery are shown in Figures 1 and 2.

![Graph of $A_8$](image)

**Figure 1.** The graph of convex polytope $A_8$.

![Centrality in $A_8$](image)

**Figure 2.** Centrality in the graph of convex polytope $A_8$.

3. The Center and Periphery for Convex Polytope $S_n$

Here, we examine the center and periphery for convex polytope $S_n$.

**Definition 8.** The graph of convex polytope (double antiprism) $S_n$ can be obtained from the graph of convex polytope $Q_n$ by adding new edges $c_i c_{i+1}$, i.e.,

$$V(S_n) = V(Q_n) \quad \text{and} \quad E(S_n) = E(Q_n) \cup \{c_i c_{i+1} : 1 \leq i \leq n\}.$$ 

For our convenience, we identify the cycle induced by the vertices $(a_1, a_2, ..., a_n)$, $(b_1, b_2, ..., b_n)$, $(c_1, c_2, ..., c_n)$ and $(d_1, d_2, ..., d_n)$ as the inner cycle, interior cycle, exterior cycle and outer cycle, respectively.

**Theorem 3.** For the family of convex polytope $S_n$, when $n$ is even, we have:

$$\text{diam}(S_n) = \frac{n}{2} + 1.$$ 

$$\text{rad}(S_n) = \frac{n}{2} + 2.$$
**Proof.** Suppose, \( n = 2k, \ k \geq 2. \) Consider the cycle \((a_1a_2...a_i...a_{2k})\). Here, the eccentricity of only one vertex, i.e., \(a_1\), is determined, and due to the symmetry of the graph, all other vertices have the same eccentricity as \(a_1\) on this cycle. Using this cycle,

\[
d(a_1, a_i) = i - 1, \ 1 \leq i \leq k + 1.
\]  

(16)

For \( k + 2 \leq i \leq 2k, \) \(d(a_1, a_i)\) varies from \(k - 1\) to one, i.e.,

\[
d(a_1, a_i) = 2k - i + 1, \ k + 2 \leq i \leq 2k.
\]

Thus, to identify a vertex at maximum distance from \(a_1\) in \(S_n\), take only \(1 \leq i \leq k + 1\).

As each \(a_i\) is adjacent to \(b_j\), therefore,

\[
d(a_1, b_i) = i, \ 1 \leq i \leq k + 1.
\]

(17)

For \( k + 2 \leq i \leq 2k, \) take the interior cycle \((b_1b_2...b_i...b_{2k})\). In this cycle, the vertices \(b_1\) and \(b_{2k}\) are at a distance \(2k - i\). Further, \(b_{2k}\) is adjacent to \(b_1\) and \(b_1\) adjacent to \(a_1\). Therefore, The distance between \(a_1\) and \(b_i\) is \(2k - i + 2\).

\[
d(a_1, b_i) = 2k - i + 2, \ k + 2 \leq i \leq 2k.
\]

Each \(b_i\) is adjacent to \(c_i\) and \(c_{i-1}\), by using (17),

\[
d(a_1, c_i) = i + 1, \ 1 \leq i \leq k.
\]

(18)

For \( k + 1 \leq i \leq 2k, \) consider the exterior cycle \((c_1c_2...c_i...c_{2k})\). The vertices \(c_i\) and \(c_{2k}\) are at a distance \(2k - i\). As \(c_{2k}\) is adjacent to \(b_1\) and \(b_1\) adjacent to \(a_1\), therefore, \(a_1\) and \(c_i\) are at a distance \(2k - i + 2\).

\[
d(a_1, c_i) = 2k - i + 2, \ k + 1 \leq i \leq 2k.
\]

\(c_i\) is also adjacent to \(d_i\), so (18) implies,

\[
d(a_1, d_i) = i + 2, \ 1 \leq i \leq k.
\]

(19)

For \( k + 1 \leq i \leq 2k, \) the vertices \(d_i\) and \(d_{2k}\) are at a distance \(2k - i\) in the outer cycle \((d_1d_2...d_k...d_{2k})\). As \(d_{2k}\) is adjacent to \(c_{2k}\), \(c_{2k}\) adjacent to \(b_1\) and \(b_1\) adjacent to \(a_1\), therefore, \(a_1\) and \(d_i\) are at a distance \(2k - i + 3\).

\[
d(a_1, d_i) = 2k - i + 3, \ k + 1 \leq i \leq 2k.
\]

Therefore, \(e(a_1) = k + 2\).

In the same manner as above, we calculate the eccentricity of \(b_1\) in the cycle \((b_1b_2...b_i...b_{2k})\). The distance between \(b_1\) and \(b_i\) in this cycle is,

\[
d(b_1, b_i) = i - 1, \ 1 \leq i \leq k + 1.
\]

(19)

and,

\[
d(b_1, b_i) = 2k + 1 - i, \ k + 2 \leq i \leq 2k.
\]

Therefore, we only take values of \(i\) between one and \(k + 1\). As each \(b_i\) is adjacent to \(c_i\) and \(c_{i-1}\), using (19),

\[
d(b_1, c_i) = i, \ 1 \leq i \leq k.
\]

(20)
For \( k + 1 \leq i \leq 2k \), consider the cycle \( (c_1c_2...c_{k+2}...c_i...c_{2k}) \). The distance between \( c_i \) and \( c_{2k} \) is \( 2k - i \). Since, \( c_{2k} \) is adjacent to \( b_1 \). Thus,

\[
  d(b_1, c_i) = 2k - i + 1, \quad k + 1 \leq i \leq 2k.
\]

Each \( c_i \) is also adjacent to \( d_i \). Therefore, (20) shows,

\[
  d(b_1, d_i) = i + 1, \quad 1 \leq i \leq k.
\]

For \( k + 1 \leq i \leq 2k \), consider the cycle \( (d_1d_2...d_i...d_{2k}) \). The distance between \( d_i \) and \( d_{2k} \) is \( 2k - i \). As \( d_{2k} \) is adjacent to \( c_{2k} \) and \( c_{2k} \) adjacent to \( b_1 \), therefore,

\[
  d(b_1, d_i) = 2k - i + 2, \quad k + 1 \leq i \leq 2k.
\]

\( b_i \) is also adjacent to \( a_i \), i.e.,

\[
  d(b_1, a_i) = i, \quad 1 \leq i \leq k + 1.
\]

For \( k + 2 \leq i \leq 2k \), consider the cycle \( (a_1,a_2...a_{k+2}...a_i...a_{2k}) \). The distance between the vertices \( a_i \) and \( a_{2k} \) is \( 2k - i \). As \( a_{2k} \) is adjacent to \( a_{i+1} \), therefore,

\[
  d(b_1, a_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k.
\]

As, \( d_k \) and \( a_{k+1} \) are farthest from \( b_1 \), therefore, \( e(b_1) = k + 1 \). Next, the distance between \( c_1 \) and \( c_i \) in the cycle \( (c_1c_2...c_i...c_{2k}) \) is \( i - 1 \).

\[
  d(c_1,c_i) = i - 1, \quad 1 \leq i \leq k + 1. \quad (21)
\]

Additionally, for \( k + 2 \leq i \leq 2k \),

\[
  d(c_1,c_i) = 2k - i + 1, \quad k + 2 \leq i \leq 2k
\]

Each \( c_i \) is adjacent to \( d_i \), from (21):

\[
  d(c_1,d_i) = i, \quad 1 \leq i \leq k + 1.
\]

For \( k + 2 \leq i \leq 2k \), the vertices \( d_i \) and \( d_{2k} \) are at a distance \( 2k - i \) in the cycle \( (d_1d_2...d_{k+2}...d_i...d_{2k}) \). The vertex \( d_{2k} \) is adjacent to \( d_1 \) and \( d_1 \) adjacent to \( c_1 \). Therefore,

\[
  d(c_1,d_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k.
\]

Each \( c_i \) is adjacent to \( b_i \) and \( b_{i+1} \); Equation (17) implies,

\[
  d(c_1, b_1) = 1, \quad d(c_1, b_2) = 1. \quad (22)
\]

For \( 3 \leq i \leq k + 1 \), \( b_2 \) and \( b_i \) are at a distance \( i - 2 \) in the path \( c_1 \to b_2 \to b_3 \to ... \to b_i \). Again, \( b_2 \) is adjacent to \( c_1 \); thus, we have:

\[
  d(c_1, b_i) = i - 1, \quad 3 \leq i \leq k + 1. \quad (23)
\]

For \( k + 2 \leq i \leq 2k \), consider the cycle \( (b_1b_2...b_{k+2}...b_i...b_{2k}) \). The distance between \( b_i \) and \( b_{2k} \) is \( 2k - i \) in this cycle. As, \( b_{2k} \) is adjacent to \( b_1 \), \( b_1 \) adjacent to \( c_1 \). Therefore,

\[
  d(c_1,b_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k.
\]
Since $b_i$ is adjacent to $a_i$, it follows from (22) that:

$$d(c_1, a_1) = 2, \quad d(c_1, a_2) = 2$$

For $3 \leq i \leq k + 1$, $b_i$ and $b_2$ are at a distance $i - 2$ in the path $c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow \ldots \rightarrow b_i \rightarrow a_i$. The vertex $b_i$ is adjacent to $a_i$ and $b_2$ adjacent to $c_1$. Therefore,

$$d(c_1, a_i) = i, \quad 3 \leq i \leq k + 1. \quad \ldots (23)$$

For $k + 2 \leq i \leq 2k$, the distance between $a_i$ and $a_{2k}$ is $2k - i$ in the cycle $(a_1a_2\ldots a_{k+2}\ldots a_i\ldots a_{2k})$. $a_{2k}$ is again adjacent to $a_i$, $a_1$ adjacent to $b_1$ and $b_1$ adjacent to $c_1$. For that reason,

$$d(c_1, a_i) = 2k - i + 3, \quad k + 2 \leq i \leq 2k.$$  

Consequently, $d_{k+1}$ and $a_{k+1}$ are farthest from $c_1$. Therefore, $e(c_1) = k + 1$.

Next, take a vertex $d_1$ on the outer cycle. In this cycle $(d_1d_2\ldots d_{2k})$,

$$d(d_1, d_i) = i - 1, \quad 1 \leq i \leq k + 1. \quad (24)$$

Additionally,

$$d(d_1, d_i) = 2k - i + 1, \quad k + 2 \leq i \leq 2k.$$  

In addition, each $d_i$ is adjacent to $c_i$,

$$d(d_1, c_i) = i, \quad 1 \leq i \leq k + 1.$$  

For $k + 2 \leq i \leq 2k$, take a cycle $(c_1c_2\ldots c_{k+2}\ldots c_{2k})$. The vertices $c_i$ and $c_{2k}$ are at a distance $2k - i$ in this cycle. In addition, $c_{2k}$ is adjacent to $c_1$ and $c_1$ adjacent to $d_1$. Then,

$$d(d_1, c_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k.$$  

Each $c_i$ is adjacent to $b_i$ and $b_{i+1}$, i.e., $d(d_1, b_1) = 2$ and:

$$d(d_1, b_2) = 2. \quad (25)$$

For $3 \leq i \leq k + 1$, the vertices $b_2$ and $b_i$ are at a distance $i - 2$ in the path $d_1 \rightarrow c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow \ldots \rightarrow b_i$. $b_2$ is adjacent to $c_1$ and $c_1$ adjacent to $d_1$ in $S_n$. Therefore,

$$d(d_1, b_i) = i, \quad 3 \leq i \leq k + 1, \quad (26)$$

For $k + 2 \leq i \leq 2k$, the vertices $b_i$ and $b_{2k}$ are at distance $2k - i$ in the cycle $(b_1b_2\ldots b_{k+2}\ldots b_i\ldots b_{2k})$. $b_{2k}$ is adjacent to $b_1$, $b_1$ adjacent to $c_1$ and $c_1$ adjacent to $d_1$; for this,

$$d(d_1, b_i) = 2k - i + 3, \quad k + 2 \leq i \leq 2k.$$  

In addition, $b_i$ is adjacent to $a_i$. This implies from (25),

$$d(d_1, a_1) = 3, \quad d(d_1, a_2) = 3.$$  

For $3 \leq i \leq k + 1$, the vertices $b_2$ and $b_i$ are at a distance $i - 2$ in the path $d_1 \rightarrow c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow \ldots \rightarrow b_i \rightarrow a_i$. $b_i$ is adjacent to $a_i$, $b_2$ adjacent to $c_1$ and $c_1$ adjacent to $d_1$ in $S_n$. Therefore,

$$d(d_1, a_i) = i + 1, \quad 3 \leq i \leq k + 1.$$
For \( k + 2 \leq i \leq 2k \), consider the cycle \((a_1, a_2, ..., a_{k+2}, ..., a_{2k})\). The vertices \(a_i\) and \(a_{2k}\) are at a distance \(2k - i\) in this cycle. \(a_{2k}\) is adjacent to \(a_1\), \(a_1\) adjacent to \(b_1\), \(b_1\) adjacent to \(c_1\) and \(c_1\) adjacent to \(d_1\). As a result,
\[
d(d_1, a_i) = 2k - i + 4, \quad k + 2 \leq i \leq 2k.\]
Consequently,
\[
e(d_1) = k + 2.
\]
Thus, it is concluded that maximum eccentricity among all of the vertices of \(S_n\) is \(k + 2\), and the minimum eccentricity is \(k + 1\).

Therefore
\[
diam(S_n) = k + 2 = \frac{n}{2} + 2.
\]
\[
rad(S_n) = k + 1 = \frac{n}{2} + 1.
\]

\[\square\]

The following corollary is straightforward.

**Corollary 1.** The center and periphery for the family of convex polytope \((S_n)\), when \(n\) is even, are subgraphs induced by all of the central vertices \(\{b_1, b_2, ..., b_{k+2}, c_1, c_2, ..., c_{k+2}\}\) and peripheral vertices \(\{a_1, a_2, ..., a_{k+2}, d_1, d_2, ..., d_{k+2}\}\) of \(S_n\), respectively.

Now, we find out the radius and diameter of \(S_n\), when \(n\) is odd.

**Theorem 4.** When \(n\) is odd, the family of convex polytope \(S_n\) has the radius and diameter as,
\[
diam(S_n) = \frac{n - 1}{2} + 3,
\]
\[
rad(S_n) = \frac{n - 1}{2} + 2.
\]

**Proof.** Let \(n = 2k + 1\), \(k \geq 2\). Consider the cycle \((a_1a_2..., a_i...a_{2k+1})\), and select a vertex \(a_1\) in it. It is clear that,
\[
d(a_1, a_i) = i - 1, \quad 1 \leq i \leq k + 1
\]
\[
d(a_1, a_i) = 2k + 2 - i, \quad k + 2 \leq i \leq 2k + 1, \quad (27)
\]
Thus, the equations above lead to the proof including only \(1 \leq i \leq k + 1\) in order to find a vertex having the greatest distance from \(a_1\) in \(S_n\). Since each \(a_i\) is adjacent to \(b_i\), therefore, \((27)\) implies that:
\[
d(a_1, b_i) = i, \quad 1 \leq i \leq k + 1. \quad (28)
\]

For \(k + 2 \leq i \leq 2k + 1\), the vertices \(b_i\) and \(b_{2k+1}\) are at a distance \(2k - i + 1\) in the cycle \((b_1b_2..., b_{k+2}..., b_{2k+1})\). \(b_{2k+1}\) is adjacent to \(b_1\) and \(b_1\) adjacent to \(a_1\). Therefore, The distance between \(a_1\) and \(b_1\) is \(2k - i + 3\).
\[
d(a_1, b_i) = 2k + 3 - i, \quad k + 2 \leq i \leq 2k + 1.
\]
Again, each \(b_i\) is adjacent to \(c_i\) and \(c_{i-1}\), by using \((28)\).
\[
d(a_1, c_i) = i + 1, \quad 1 \leq i \leq k + 1. \quad (29)
\]
For $k + 2 \leq i \leq 2k + 1$, the distance between the vertices $c_i$ and $c_{2k+1}$ is $2k + 1 - i$ in the cycle $(c_1c_2...c_{k+1}...c_i...c_{2k+1})$. Since, $c_{2k+1}$ is adjacent to $b_1$ and $b_1$ adjacent to $a_1$, therefore, $a_1$ and $c_i$ are at a distance $2k - i + 3$.

$$d(a_1, c_i) = 2k - i + 3, \quad k + 2 \leq i \leq 2k + 1.$$ 

In addition, $c_i$ is adjacent to $d_i$, therefore, (29) shows,

$$d(a_1, d_i) = i + 2, \quad 1 \leq i \leq k + 1.$$

For $k + 2 \leq i \leq 2k + 1$, the vertices $d_i$ and $d_{2k+1}$ are at a distance $2k + 1 - i$ in the cycle $(d_1d_2...d_{k+1}...d_i...d_{2k+1})$. In addition, $d_{2k+1}$ is adjacent to $c_{2k+1}$, $c_{2k+1}$ adjacent to $b_1$ and $b_1$ adjacent to $a_1$. Therefore, $a_1$ and $d_i$ are at a distance $2k - i + 4$.

$$d(a_1, d_i) = 2k - i + 4, \quad k + 2 \leq i \leq 2k + 1.$$ 

As a result, $d_{k+1}$ is farthest from $a_1$; therefore, $c(a_1) = k + 3$.

In order to find out the eccentricity of the vertices on the cycle $(b_1b_2...b_{i-1}b_{2k+1})$, the distance between $b_1$ and $b_i$ in this cycle is $i - 1$.

$$d(b_1, b_i) = i - 1, \quad 1 \leq i \leq k + 1. \quad (30)$$

In addition,

$$d(b_1, b_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k + 1.$$

Further, each $b_i$ is adjacent to $c_i$ and $c_{i-1}$, therefore, (30) shows,

$$d(b_1, c_i) = i, \quad 1 \leq i \leq k + 1. \quad (31)$$

For $k + 2 \leq i \leq 2k + 1$, consider the cycle $(c_1c_2...c_{k+2}...c_i...c_{2k+1})$. The distance between $c_i$ and $c_{2k+1}$ is $2k - i + 1$. Since, $c_{2k+1}$ is adjacent to $b_1$, thus,

$$d(b_1, c_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k + 1.$$

Each $c_i$ is also adjacent to $d_i$. It is shown from (31),

$$d(b_1, d_i) = i + 1, \quad 1 \leq i \leq k + 1.$$

For $k + 2 \leq i \leq 2k + 1$, consider the cycle $(d_1d_2...d_{i-1}d_{2k+1})$. The distance between $d_i$ and $d_{2k+1}$ is $2k - i + 1$. As $d_{2k+1}$ is adjacent to $c_{2k+1}$ and $c_{2k+1}$ adjacent to $b_1$, therefore,

$$d(b_1, d_i) = 2k - i + 3, \quad k + 2 \leq i \leq 2k + 1.$$

$b_i$ is also adjacent to $a_i$, i.e.,

$$d(b_1, a_i) = i, \quad 1 \leq i \leq k + 1.$$

For $k + 2 \leq i \leq 2k + 1$, consider the cycle $(a_1a_2...a_{k+2}...a_i...a_{2k+1})$. The vertices $a_i$ and $a_{2k+1}$ is $2k + 1 - i$. As $a_{2k+1}$ is adjacent to $a_1$, $a_1$ adjacent to $b_1$, therefore,

$$d(b_1, a_i) = 2k - i + 3, \quad k + 2 \leq i \leq 2k + 1.$$ 

Since, $d_{k+1}$ is a vertex farthest from $b_1$, therefore, $c(b_1) = k + 2$.

Next, the distance between $c_1$ and $c_i$ in the cycle $(c_1c_2...c_i...c_{2k+1})$ is,

$$d(c_1, c_i) = i - 1, \quad 1 \leq i \leq k + 1. \quad (32)$$
Additionally,
\[ d(c_1, c_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k + 1. \]

Each \( c_i \) is adjacent to \( d_i \), from (32):
\[ d(c_1, d_i) = i, \quad 1 \leq i \leq k + 1. \]

For \( k + 2 \leq i \leq 2k + 1 \), the vertices \( d_i \) and \( d_{2k+1} \) are at a distance \( 2k + 1 - i \) in the cycle \( (d_1d_2...d_{k+2}...d_i...d_{2k+1}) \). The vertex \( d_{2k+1} \) is adjacent to \( d_1 \) and \( d_1 \) adjacent to \( c_1 \). Therefore,
\[ d(c_1, d_i) = 2k - i + 3, \quad k + 2 \leq i \leq 2k + 1. \]

Each \( c_i \) is adjacent to \( b_i \) and \( b_{i+1} \); Equation (28) implies,
\[ d(c_1, b_1) = 1, \quad d(c_1, b_2) = 1. \] (33)

For \( 3 \leq i \leq k + 2 \), \( b_2 \) and \( b_1 \) are at a distance \( i - 2 \) in the path \( c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow ... \rightarrow b_i \). Again, \( b_2 \) is adjacent to \( c_1 \); thus, we have:
\[ d(c_1, b_i) = i - 1, \quad 3 \leq i \leq k + 2. \] (34)

For \( k + 3 \leq i \leq 2k + 1 \), consider the cycle \( (b_1b_2...b_{k+1}...b_i...b_{2k+1}) \). The distance between \( b_i \) and \( b_{2k+1} \) is \( 2k + 1 - i \) in this cycle. As \( b_{2k+1} \) is adjacent to \( b_1 \) and \( b_1 \) adjacent to \( c_1 \), therefore,
\[ d(c_1, b_i) = 2k - i + 3, \quad k + 3 \leq i \leq 2k + 1. \]

Since \( b_i \) is adjacent to \( a_i \), it follows from (33) that:
\[ d(c_1, a_1) = 2, \quad d(c_1, a_2) = 2 \]

For \( 3 \leq i \leq k + 2 \), \( b_1 \) and \( b_2 \) are at a distance \( i - 2 \) in the path \( c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow ... \rightarrow b_i \). The vertex \( b_i \) is adjacent to \( a_i \), \( b_2 \) adjacent to \( c_1 \). Therefore,
\[ d(c_1, a_i) = i, \quad 3 \leq i \leq k + 2. \]

For \( k + 3 \leq i \leq 2k + 1 \), the distance between \( a_i \) and \( a_{2k+1} \) in the cycle \( (a_1a_2...a_{k+2}...a_i...a_{2k+1}) \). \( a_{2k+1} \) is again adjacent to \( a_1 \), \( a_1 \) adjacent to \( b_1 \) and \( b_1 \) adjacent to \( c_1 \). For that reason,
\[ d(c_1, a_i) = 2k - i + 4, \quad k + 3 \leq i \leq 2k + 1. \]

Consequently, \( a_{k+2} \) is a vertex farthest from \( c_1 \). Therefore, \( e(c_1) = k + 2 \). Next, take a vertex \( d_1 \) on the outer cycle. In this cycle, \( (d_1d_2...d_i...d_{2k+1}) \),
\[ d(d_1, d_i) = i - 1, \quad 1 \leq i \leq k + 1. \] (35)

\( d(d_1, d_i) \) starts to decrease for \( k + 2 \leq i \leq 2k + 1 \) as,
\[ d(d_1, d_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k + 1. \]

Each \( d_i \) is adjacent to \( c_i \),
\[ d(d_1, c_i) = i, \quad 1 \leq i \leq k + 1. \]
For $k + 2 \leq i \leq 2k + 1$, take a cycle $(c_1c_2\ldots c_{k+2}\ldots c_i\ldots c_{2k+1})$. The vertices $c_i$ and $c_{2k+1}$ are at a distance $2k + 1 - i$ in this cycle. In addition, $c_{2k+1}$ is adjacent to $c_1$ and $c_1$ adjacent to $d_1$. Then,

\[ d(d_1, c_i) = 2k - i + 3, \quad k + 2 \leq i \leq 2k + 1. \]

Each $c_i$ is adjacent to $b_i$ and $b_{i+1}$, i.e., $d(d_1, b_1) = 2$ and

\[ d(d_1, b_i) = 2, \quad 3 \leq i \leq k + 2. \quad (36) \]

For $3 \leq i \leq k + 2$, the vertices $b_2$ and $b_i$ are at a distance $i - 2$ in the path $d_1 \rightarrow c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow \ldots \rightarrow b_i$. $b_2$ is adjacent to $c_1$ and $c_1$ adjacent to $d_1$ in $S_n$. Therefore,

\[ d(d_1, b_i) = i, \quad 3 \leq i \leq k + 2. \quad (37) \]

For $k + 3 \leq i \leq 2k + 1$, consider the cycle $(b_1b_2\ldots b_{k+2}\ldots b_i\ldots b_{2k+1})$. The vertices $b_i$ and $b_{2k+1}$ are $2k + 1 - i$. $b_{2k+1}$ is adjacent to $b_1$, $b_1$ adjacent to $c_1$ and $c_1$ adjacent to $d_1$; for this,

\[ d(d_1, b_i) = 2k - i + 4, \quad k + 3 \leq i \leq 2k + 1. \]

In addition, $b_i$ is adjacent to $a_i$. Therefore, (36) implies,

\[ d(d_1, a_1) = 3, \quad d(d_1, a_2) = 3. \]

For $3 \leq i \leq k + 2$, the vertices $b_2$ and $b_i$ are at a distance $i - 2$ in the path $d_1 \rightarrow c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow \ldots \rightarrow b_i \rightarrow a_i$. $b_i$ is adjacent to $a_i$, $b_2$ adjacent to $c_1$ and $c_1$ adjacent to $d_1$ in $S_n$. Therefore,

\[ d(d_1, a_i) = i + 1, \quad 3 \leq i \leq k + 2. \]

For $k + 3 \leq i \leq 2k + 1$, consider the cycle $(a_1a_2\ldots a_{k+2}\ldots a_i\ldots a_{2k+1})$. The vertices $a_i$ and $a_{2k+1}$ are $2k + 1 - i$. $a_{2k+1}$ is adjacent to $a_1$, $a_1$ adjacent to $b_1$, $b_1$ adjacent to $c_1$ and $c_1$ adjacent to $d_1$. As a result,

\[ d(d_1, a_i) = 2k - i + 5, \quad k + 3 \leq i \leq 2k + 1. \]

This means,

\[ e(d_1) = k + 3. \]

It shows that the maximum and minimum eccentricity among all of the vertices of $S_n$ are $k + 3$ and $k + 2$, respectively. Therefore:

\[ \text{diam}(S_n) = k + 3 = \frac{n - 1}{2} + 3. \]

\[ \text{rad}(S_n) = k + 2 = \frac{n - 1}{2} + 2. \]

\[ \square \]

Thus, we can summarize the above results as,

**Corollary 2.** The center for the family of convex polytope $S(n)$ is a subgraph induced by all of the vertices of the interior and exterior cycles, and the periphery is the subgraphs induced by all of the peripheral vertices \{${a_1, a_2, \ldots, a_i, \ldots, a_{2k}, d_1, d_2, \ldots, d_i, \ldots, d_{2k}}$\} of $S_n$, respectively.
3.1. Average Eccentricity for Convex Polytopes $S_n$

Here, the average eccentricity for the family of $S_n$ is being determined. The graph of $S_n$ consists of four major circles, and there are $n$ vertices in each circle. Therefore, the total number of vertices in $S_n$ (i.e., $\hat{n}$) is $4n$; it follows,

$$avec(S_n) = \frac{1}{\hat{n}} \sum_{u \in V(G)} e_G u$$

By Theorem 3:

$$avec(S_n) = \frac{1}{4} \times \frac{n}{\hat{n}} \left[ n \times \{ e(a_1) + e(b_1) + e(c_1) + e(d_1) \} \right]$$

$$= \frac{1}{4 \times n} \left[ n \times \{ (k+2) + (k+1) + (k+1) + (k+2) \} \right]$$

$$= \frac{1}{4 \times n} \left[ 2n \times \{ (k+2) + (k+1) \} \right]$$

$$= \frac{1}{2} \left[ 2k + 3 \right]$$

$$= k + \frac{3}{2}$$

$$= \frac{n + 3}{2}.$$  

and by Theorem 4,

$$avec(S_n) = \frac{1}{4} \times \frac{n}{\hat{n}} \left[ n \times \{ (k+3) + (k+2) + (k+2) + (k+3) \} \right]$$

$$= \frac{1}{4 \times n} \left[ 2n \times \{ (k+3) + (k+2) \} \right]$$

$$= \frac{1}{2} \left[ 2k + 5 \right]$$

$$= k + \frac{5}{2}$$

$$= \frac{n - 1}{2} + \frac{5}{2}$$

$$= \frac{n + 4}{2}.$$  

Therefore, we have the following result:

$$avec(S_n) = \begin{cases} 
\frac{n + 3}{2}, & \text{for all even values of } n \\
\frac{n + 4}{2}, & \text{for all odd values of } n.
\end{cases}$$

3.2. Illustration

Consider the graph of $S_6$. Its center and periphery are shown in Figures 3 and 4.
4. The Center and Periphery for Convex Polytopes $T_n$

Here, we established the center and periphery for $T_n$ and show that $T_n$ is not self-centered.

**Definition 9.** The graph of convex polytope $T_n$ can be obtained from the graph of convex polytope $Q_n$ by adding new edges. It consist of three-sided faces, five-sided faces and $n$-sided faces. \( V(T_n) = V(Q_n) \) and \( E(T_n) = E(Q_n) \cup \{a_i+1b_i : 1 \leq i \leq n\} \).

This section begins with the following theorem on $T_n$.

**Theorem 5.** The diameter for the family of convex polytope $T_n$ is,

\[
\text{diam}(T_n) = \begin{cases} 
\frac{n}{2} + 2, & \text{for } n = 2k; \\
\frac{n-1}{2} + 2, & \text{for } n = 2k + 1.
\end{cases}
\]

In addition, its radius,

\[
\text{rad}(T_n) = \begin{cases} 
\frac{n}{2} + 1, & \text{for } n \text{ to be even}; \\
\frac{n+1}{2}, & \text{for } n \text{ to be odd}.
\end{cases}
\]

**Proof.** Consider, $n = 2k, k \geq 2$. Choose take cycle \((a_1a_2...a_i...a_{2k})\). In this cycle:

\[d(a_1,a_i) = i - 1, \ 1 \leq i \leq k + 1.\]
For $k + 2 \leq i \leq 2k$, the distance between $a_1$ and $a_i$ decreases from $k - 1$ to one, i.e.,
\[
d(a_1, a_i) = 2k - i + 1, \quad k + 2 \leq i \leq 2k.
\]

Therefore, we must considered $1 \leq i \leq k + 1$ in order to find the distance of a vertex $a_1$ from a vertex farthest from it in $T_n$.

In the graph of $T_n$, each $a_i$ is adjacent to $b_i$ and $b_i-1$; thus, (38) implies,
\[
d(a_1, b_i) = i, \quad 1 \leq i \leq k.
\]

For $k + 1 \leq i \leq 2k$, the vertices $b_i$ and $b_{2k}$ are at a distance $2k - i$ in the cycle $(b_1 b_2 \ldots b_{k+1} \ldots b_{2k})$. In addition, $b_{2k}$ is adjacent to $a_1$. Therefore, The distance between $a_1$ and $b_i$ is $2k - i + 1$.
\[
d(a_1, b_i) = 2k - i + 1, \quad k + 1 \leq i \leq 2k.
\]

Further, each $b_i$ is adjacent to $c_i$ and $c_{i-1}$, using (39).
\[
d(a_1, c_1) = 2, d(a_1, c_{2k}) = 2
\]

for $2 \leq i \leq k$, consider path $a_1 \to b_1 \to b_2 \to \ldots \to b_i \to c_i$. The distance between $b_1$ and $b_i$ is $i - 1$. Each $b_i$ is adjacent to $c_i$ and $b_1$ adjacent to $a_1$. Therefore,
\[
d(a_1, c_i) = i + 1, \quad 2 \leq i \leq k.
\]

Next, for $k + 1 \leq i \leq 2k - 1$, consider the cycle $(b_1 b_2 \ldots b_{i+1} \ldots b_{2k})$. The vertices $b_{2k}$ and $b_{i+1}$ are at a distance $2k - i - 1$. Further, $b_{2k}$ is adjacent to $a_1$ and $b_{i+1}$ adjacent to $c_i$. Therefore,
\[
d(a_1, c_i) = 2k - i + 1, \quad k + 1 \leq i \leq 2k - 1.
\]

$c_i$ is also adjacent to $d_i$. Therefore, (40) implies
\[
d(a_1, d_i) = i + 2, \quad 1 \leq i \leq k.
\]

For $k + 1 \leq i \leq 2k$, the vertices $d_i$ and $d_{2k}$ are at a distance $2k - i$ in the cycle $(d_1 d_2 \ldots d_i \ldots d_{2k})$. In addition, each $d_{2k}$ is adjacent to $c_{2k}$, $c_{2k}$ adjacent to $b_1$ and $b_1$ adjacent to $a_1$; therefore,
\[
d(a_1, d_i) = 2k + 3 - i, \quad k + 1 \leq i \leq 2k.
\]

Hence, $d_k$ is a vertex at the largest distance from $a_1$. Therefore, $e(a_1) = k + 2$.

Next, continue this for cycle $(b_1 b_2 \ldots b_i \ldots b_{2k})$; we choose a vertex $b_1$, such that,
\[
d(b_1, b_i) = i - 1, \quad 1 \leq i \leq k + 1.
\]

The distance between $b_1$ and $b_i$ decreases from $k - 1$ to one, when $i$ increases from $k + 2$ to $2k$.
\[
d(b_1, b_i) = 2k - i + 1, \quad k + 2 \leq i \leq 2k.
\]

In addition, each $b_i$ is adjacent to $a_i$ and $a_{i+1}$.
\[
d(b_1, a_1) = 1, \quad d(b_1, a_2) = 1
\]
and when $3 \leq i \leq k + 1$, consider the path $b_1 \rightarrow a_2 \rightarrow a_3 \rightarrow ... \rightarrow a_i$. $a_2$ and $a_i$ are at a distance $i - 2$ in this path, and $a_2$ is adjacent to $b_1$; therefore, 

$$d(b_1, a_i) = i - 1, \quad 3 \leq i \leq k + 1.$$ 

For $k + 2 \leq i \leq 2k$, consider the cycle $(a_1a_2...a_{k+2}...a_{2k})$. The distance between $a_i$ and $a_{2k}$ is $2k - i$. As $a_{2k}$ is adjacent to $a_1$ and $a_1$ adjacent to $b_1$, therefore, 

$$d(b_1, a_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k.$$ 

In addition, $b_i$ is adjacent to $c_i$ and $c_{i-1}$; using (41), we have:

$$d(b_1, c_i) = i, \quad 1 \leq i \leq k. \quad (42)$$

For $k + 1 \leq i \leq 2k$, consider the path $b_1 \rightarrow b_{2k} \rightarrow b_{2k-1} \rightarrow ... \rightarrow b_{i+1} \rightarrow c_i$. The distance between $b_{2k}$ and $b_{i+1}$ is $2k - i - 1$. Further, $b_{2k}$ is adjacent to $b_1$. In addition, $b_{i+1}$ is adjacent to $c_i$. Therefore, the distance between $b_1$ and $c_i$ is $2k - i + 1$.

$$d(b_1, c_i) = 2k - i + 1, \quad k + 1 \leq i \leq 2k.$$ 

Further, $c_i$ is adjacent to $d_i$; hence, (42) shows, 

$$d(b_1, d_i) = i + 1, \quad 1 \leq i \leq k.$$ 

For $k + 1 \leq i \leq 2k$, the vertices $d_i$ and $d_{2k}$ are at a distance $2k - i$ in the cycle $(d_1d_2...d_{k+2}...d_i...d_{2k})$. The vertex $d_{2k}$ is adjacent to $c_{2k}$ and $c_{2k}$ adjacent to $b_1$. Therefore, 

$$d(b_1, d_i) = 2k - i + 2, \quad k + 1 \leq i \leq 2k.$$ 

Hence, $d_k$ is a vertex farthest from $b_1$. Therefore, $e(b_1) = k + 1$

Next, to find out the eccentricity of the vertices $\{c_i, 1 \leq i \leq 2k\}$, take a vertex $c_1$ among all $c_i$'s, and each $c_i$ is adjacent to $b_i$, $b_{i+1}$, i.e., 

$$d(c_1, b_1) = 1, \quad d(c_1, b_2) = 1.$$

and when $3 \leq i \leq k + 1$, consider the path $c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow ... \rightarrow b_i$. $b_2$ and $b_i$ are at distance $i - 2$, and again, $b_2$ is adjacent to $c_1$; thus, we have:

$$d(c_1, b_i) = i - 1, \quad 3 \leq i \leq k + 1. \quad (43)$$

For $k + 2 \leq i \leq 2k$, consider the cycle $(b_1b_2...b_{k+2}...b_i...b_{2k})$. The distance between $b_i$ and $b_{2k}$ is $2k - i$ in this cycle. As $b_{2k}$ is adjacent to $b_1$ and $b_1$ adjacent to $c_1$, therefore, 

$$d(c_1, b_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k.$$ 

Moreover, $b_i$ is adjacent to $a_i$ and $a_{i+1}$; it follows from (43) that:

$$d(c_1, a_1) = 2, \quad d(c_1, a_2) = 2, \quad d(c_1, a_3) = 2.$$ 

For $4 \leq i \leq k + 2$, $a_i$ and $a_2$ are at a distance $i - 3$ in the path $c_1 \rightarrow b_2 \rightarrow a_3 \rightarrow ... \rightarrow a_i$. Furthermore, $a_2$ is adjacent to $b_2$ and $b_2$ adjacent to $c_1$. Thus, 

$$d(c_1, a_i) = i - 1, \quad 4 \leq i \leq k + 2. \quad (44)$$
For $k + 3 \leq i \leq 2k$, the distance between $a_i$ and $a_{2k}$ in the cycle $(a_1a_2...a_{k+3}...a_i...a_{2k})$ is $2k - i$, and $a_{2k}$ is adjacent to $a_1$, $a_1$ adjacent to $b_1$ and $b_1$ adjacent to $c_1$. For that reason,

$$d(c_1, a_i) = 2k - i + 3, \quad k + 3 \leq i \leq 2k.$$ 

Again, $c_i$ is adjacent to $d_i$. Hence,

$$d(c_1, d_i) = i, \quad 1 \leq i \leq k + 1$$

For $k + 2 \leq i \leq 2k$, the vertices $d_{2k}$ and $d_i$ are at a distance $2k - i$ in the cycle $(d_1d_2...d_{k+2}...d_i...d_{2k})$, and $d_{2k}$ is adjacent to $d_1$ and $d_1$ adjacent to $c_1$ in $T_n$. Therefore,

$$d(c_1, d_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k$$

In order to find the distance between $c_1$ and $c_i$, $1 \leq i \leq k + 1$, consider the path $c_1 \rightarrow b_2 \rightarrow b_3... \rightarrow b_i \rightarrow c_i$. The distance between $b_2$ and $b_i$ is $i-2$, and $b_i$ is adjacent to $c_i$ and $b_2$ adjacent to $c_1$. Therefore,

$$d(c_1, c_i) = i, \quad 1 \leq i \leq k + 1.$$ 

For more values of $i$, $d(c_1, c_i)$ begins to reduce as,

$$d(c_1, c_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k$$

This means that $a_{k+2}$, $c_{k+1}$ and $d_{k+1}$ are the vertices farthest from $c_1$. Therefore, $e(c_1) = k + 1$.

Now, we find the eccentricities of the vertices on the cycle $(d_1d_2...d_i...d_{2k})$. In this cycle,

$$d(d_1, d_i) = i - 1, \quad 1 \leq i \leq k + 1.$$ 

For $k + 2 \leq i \leq 2k$, the distance between $d_1$ and $d_i$ decreases from $k - 1$ to one.

$$d(d_1, d_i) = 2k + 1 - i, \quad k + 2 \leq i \leq 2k.$$ 

As $d_i$ adjacent to $c_i$:

$$d(d_1, c_i) = i, \quad 1 \leq i \leq k + 1.$$ 

(45)

When $i$ increases from $k + 2$ to $2k$, the distance between $d_1$ and $d_{2k}$ is $2k - i$ in the cycle $(d_1d_2...d_{k+2}...d_i...d_{2k})$. In addition, $d_{2k}$ is adjacent to $d_1$ and $d_1$ adjacent to $c_i$. Thus,

$$d(d_1, c_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k.$$ 

As each $c_i$ is adjacent to $b_i$, $b_{i+1}$.

$$d(d_1, b_1) = 2, \quad d(d_1, b_2) = 2,$$ 

(46)

For $3 \leq i \leq k + 1$, consider a path $d_1 \rightarrow c_1 \rightarrow b_2 \rightarrow b_3... \rightarrow b_i$. $b_2$ and $b_i$ are at a distance $i - 2$, and $b_2$ is adjacent to $c_1$ and $c_1$ adjacent to $d_1$ in $T_n$. Therefore,

$$d(d_1, b_i) = i, \quad 3 \leq i \leq k + 1$$

For $k + 2 \leq i \leq 2k$, consider the cycle $(b_1b_2...b_{k+2}...b_i...b_{2k})$. The distance between the vertices $b_i$ and $b_{2k}$ is $2k - i$. $b_{2k}$ is adjacent to $b_1$, $b_1$ adjacent to $c_1$ and $c_1$ adjacent to $d_1$; for that reason,

$$d(d_1, b_i) = 2k - i + 3, \quad k + 2 \leq i \leq 2k.$$
In addition, $b_i$ is adjacent to $a_i$ and $a_{i+1}$. Therefore, (46) implies,

$$d(d_1, a_1) = 3, \quad d(d_1, a_2) = 3, \quad d(d_1, a_3) = 3$$

For $4 \leq i \leq k + 2$, the vertices $a_3$ and $a_i$ are at a distance $i - 3$ in the path $d_1 \rightarrow c_1 \rightarrow b_1 \rightarrow a_3 \rightarrow a_4 \rightarrow \ldots \rightarrow a_i$. Further, $a_3$ is adjacent to $b_2$, $b_2$ adjacent to $c_1$ and $c_1$ adjacent to $d_1$ in $T_n$. Therefore,

$$d(d_1, a_i) = i, \quad 4 \leq i \leq k + 2.$$ 

For $k + 3 \leq i \leq 2k$, consider the cycle $(a_1a_2\ldots a_{k+3}\ldots a_i\ldots a_{2k})$. The distance between the vertices $a_i$ and $a_{2i}$ is $2k - i$. $a_{2i}$ is adjacent to $a_1$ and $a_1$ adjacent to $b_1$. Further, $b_1$ adjacent to $c_1$ and $c_1$ adjacent to $d_1$. As a result,

$$d(d_1, a_i) = 2k - i + 4, \quad k + 3 \leq i \leq 2k.$$ 

This shows that $a_{k+2}$ is at the highest distance from $d_1$. Therefore, $e(d_1) = k + 2$.

Thus, it is concluded that the maximum eccentricity among all of the vertices of $T_n$ is $k + 2$, and $k + 1$ is the minimum eccentricity. Therefore, diam$(T_n) = k + 2 = \frac{n}{2} + 2$.

For odd $n$, the proof is analogous to the case discussed above and omitted. $\square$

**Corollary 3.** The center of $T_n$, when $n$ is even, is the subgraph induced by the central vertices $\{b_1 \cup c_1 : 1 \leq i \leq n\}$, while the periphery is the subgraph induced by the vertices of inner and outer cycles.

### 4.1. Average Eccentricity for Convex Polytopes $T_n$

There are four circles in the graph of $T_n$, and each circle has $n$ vertices. The average eccentricity for the graph of convex polytope $T_n$ can be found out by dividing sum of eccentricities of all vertices on each circle to its total number of vertices. Therefore,

$$avec(T_n) = \frac{1}{n} \sum_{u \in V(G)} e_G(u)$$

By Theorem 5:

$$avec(T_n) = \frac{1}{4 \times n} \left[ n \times \{(k + 2) + (k + 1) + (k + 1) + (k + 2)\} \right]$$

$$= \frac{1}{4 \times n} \left[ 2n \times \{(k + 2) + (k + 1)\} \right] = \frac{1}{2} \left[ 2k + 3 \right] = k + \frac{3}{2} = \frac{n + 3}{2}.$$ 

and by Theorem 5,

$$avec(T_n) = \frac{1}{4 \times n} \left[ n \times \{(k + 2) + (k + 2) + (k + 1) + (k + 2)\} \right]$$

$$= \frac{1}{4 \times n} \left[ 3n \times (k + 2) + n \times (k + 1)\right] = \frac{1}{4} \left[ 4k + 7 \right] = \frac{1}{4} \left[ 4 \left( \frac{n - 1}{2} \right) + 7 \right] = \frac{n + 5}{4}.$$ 

Therefore, we get the following immediate result:

$$avec(T_n) = \begin{cases} \frac{n + 5}{4}, & \text{if } n = 2k + 1; \\ \frac{n + 3}{2}, & \text{if } n = 2k. \end{cases}$$

### 4.2. Illustration

Consider the graph $T_6$. The center and periphery for $T_6$ are shown in Figures 5 and 6.
5. Concluding Remarks

In summary, we have studied the center and periphery of three types of families of convex polytopes via finding a subgraph induced by central and peripheral vertices. The predetermined facts about the eccentricity, radius and diameter of graphs play an important role in order to find the center and periphery for specific families of graphs; the average eccentricity of the above families of graphs has also been demonstrated.

6. Open Problems

This paper consist of the center and periphery for families of convex polytope graphs. This is an open problem for new researchers to find the center and periphery for others families of graphs, such as the corona product, composition product and lexicographic product of families of graphs.

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