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Article

Harmonic Maps and Biharmonic Maps

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Abstract: This is a survey on harmonic maps and biharmonic maps into (1) Riemannian manifolds of non-positive curvature, (2) compact Lie groups or (3) compact symmetric spaces, based mainly on my recent works on these topics.

Keywords: harmonic map; biharmonic map; Chen's conjecture; generalized Chen's conjecture; Lie group; symmetric space

1. Introduction

Harmonic maps play a central role in geometry; they are critical points of the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \, v_g$ for smooth maps φ of (M, g) into (N, h). The Euler–Lagrange equations are given by the vanishing of the tension field $\tau(\varphi)$. In 1983, J. Eells and L. Lemaire extended [1] the notion of harmonic maps to biharmonic maps, which are, by definition, critical points of the bienergy functional:

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g \tag{1}$$

After G.Y. Jiang studied [2] the first and second variation formulas of E_2, extensive studies in this area have been done (for instance, see [3–42]). Notice that harmonic maps are always biharmonic by definition.

The outline of this survey is the following:

(1) Preliminaries.
(2) Chen's conjecture and the generalized Chen's conjecture.
(3) Outline of the proofs of Theorems 3–5.
(4) Harmonic maps and biharmonic maps into compact Lie groups or compact symmetric spaces.

(5) The CR analogue of harmonic maps and biharmonic maps.
(6) Biharmonic hypersurfaces of compact symmetric spaces.

2. Preliminaries

In this section, we prepare materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map $\varphi : (M, g) \to (N, h)$, of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h), which is an extremal of the energy functional defined by:

$$E(\varphi) = \int_M e(\varphi)\, v_g$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\frac{d}{dt}\bigg|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0 \tag{2}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ, which is given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and the tension field is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally-defined frame field on (M, g), and $B(\varphi)$ is the second fundamental form of φ defined by:

$$\begin{aligned} B(\varphi)(X, Y) &= (\widetilde{\nabla} d\varphi)(X, Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y) \end{aligned} \tag{3}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, ∇ and ∇^N are connections on TM, TN of (M, g), (N, h), respectively, and $\overline{\nabla}$ and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$ and $T^*M \otimes \varphi^{-1}TN$, respectively. By Equation (2), φ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that φ is harmonic. Then,

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g \tag{4}$$

where J is an elliptic differential operator, called the Jacobi operator acting on $\Gamma(\varphi^{-1}TN)$ given by:

$$J(V) = \overline{\Delta} V - \mathcal{R}(V) \tag{5}$$

where $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i} e_i} V\}$ is the rough Laplacian and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}(V) = \sum_{i=1}^m R^N(V, d\varphi(e_i))d\varphi(e_i)$, and R^N is the curvature tensor of (N, h) given by $R^N(U, V) = \nabla^N{}_U \nabla^N{}_V - \nabla^N{}_V \nabla^N{}_U - \nabla^N{}_{[U,V]}$ for $U,\ V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire [1] proposed polyharmonic (k-harmonic) maps, and Jiang [2] studied the first and second variation formulas of biharmonic maps. Let us consider the bienergy functional defined by:

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g \tag{6}$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$.

Then, the first variation formula of the bienergy functional is given (the first variation formula) by:

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V)v_g \tag{7}$$

Here,

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathcal{R}(\tau(\varphi)) \tag{8}$$

which is called the bitension field of φ, and J is given in Equation (5).

A smooth map φ of (M, g) into (N, h) is said to be biharmonic if $\tau_2(\varphi) = 0$.

3. Chen's Conjecture and the Generalized Chen's Conjecture

Recall the famous Chen's conjecture on biharmonic submanifold of the Euclidean space:

Chen's conjecture: A biharmonic submanifold in the Euclidean space must be minimal.

One can consider biharmonic submanifolds of a Riemannian manifold of non-positive curvature, and the generalized Chen's conjecture is the following (*cf.*, R. Caddeo, S. Montaldo, P. Piu [7] and also S. Montaldo, C. Oniciuc [20], *etc.*):

The generalized Chen's conjecture: A biharmonic submanifold in a Riemannian manifold of non-positive curvature must be minimal.

Notice that the generalized Chen's conjecture was solved negatively by giving a counter example by Ou and Tang [27,41]. We first give several comments on Chen's conjecture. It should be emphasized that Chen's conjecture has been still unsolved until now.

Second, we will treat the generalized Chen's conjecture. K. Akutagawa and S. Maeta [3] gave a remarkable breakthrough to Chen's conjecture, by giving the following answer to this conjecture in the case of properly-immersed submanifolds of the Euclidean space:

If we do not assume the properness condition, N. Koiso and myself [43] gave recently a partial answer in the case of generic hypersurfaces of the Euclidean space. Namely, we obtained the following:

Theorem 1. *Let $\iota : (M^n, g) \subset E^{n+1}$ be an isometrically-immersed biharmonic hypersurface of the Euclidean space. Assume that (1) every principal curvature λ_i has multiplicity one, i.e., $\lambda_i \neq \lambda_j$ $(i \neq j)$, and (2) the principal curvature vector fields v_i $(i = 1, \cdots, n)$ along ι satisfy that $g(\nabla_{v_i} v_j, v_k) \neq 0$ for all distinct triplets of integers i, j, $k = 1, \cdots, n$. Then, it is minimal.*

For harmonic maps, it is well known that:

If a domain manifold (M, g) is complete and has non-negative Ricci curvature and the sectional curvature of a target manifold (N, h) is non-positive, then every energy finite harmonic map is a constant map (*cf.*, [44]).

See [15,45–49] for recent works on harmonic maps. Therefore, it is a natural question to consider biharmonic maps into a Riemannian manifold of non-positive curvature. In this connection, Baird, Fardoun and Ouakkas (*cf.*, [4]) showed that:

If a non-compact Riemannian manifold (M, g) is complete and has non-negative Ricci curvature and (N, h) has non-positive sectional curvature, then every bienergy finite biharmonic map of (M, g) into (N, h) is harmonic.

In our paper [21–23], we showed that:

Theorem 2. *Assume that (M, g) is complete and (N, h) has non-positive sectional curvature.*

(1) Every biharmonic map $\varphi : (M, g) \to (N, h)$ with finite energy $E(\varphi) < \infty$ and finite bienergy $E_2(\varphi) < \infty$, is harmonic.

(2) In the case $\mathrm{Vol}(M, g) = \infty$, every biharmonic map $\varphi : (M, g) \to (N, h)$ with finite bienergy $E_2(\varphi) < \infty$, is harmonic.

We do not need any assumption on the Ricci curvature of (M, g) in Theorem 3. If (M, g) is a non-compact complete Riemannian manifold whose Ricci curvature is non-negative, then $\mathrm{Vol}(M, g) = \infty$ (*cf.*, Theorem 7, p. 667, [50]). Thus, Theorem 3, Equation (2), recovers the result of Baird, Fardoun and Ouakkas. Theorem 3 is sharp, since one cannot weaken the assumptions. Indeed, the generalized Chen's conjecture does not hold if (M, g) is not complete (*cf.*, the counter examples of Ou and Tang [27]). The two assumptions of finiteness of the energy and bienergy are necessary. Indeed, there exists a biharmonic map φ, which is not harmonic, but the energy and bienergy are infinite. For example, $f(x) = r(x)^2 = \sum_{i=1}^{m} (x_i)^2, x = (x_1, \cdots, x_m) \in \mathbb{R}^m$ is biharmonic, but not harmonic, and has infinite energy and bienergy.

As the first bi-product of our method, we obtained (*cf.*, [21,22]):

Theorem 3. *Assume that (M, g) is a complete Riemannian manifold, and let $\varphi : (M, g) \to (N, h)$ be an isometric immersion; the sectional curvature of (N, h) is non-positive. If $\varphi : (M, g) \to (N, h)$ is biharmonic and $\int_M |\xi|^2 v_g < \infty$, then it is minimal. Here, ξ is the mean curvature normal vector field of the isometric immersion φ.*

Theorem 4 gave an affirmative answer to the generalized B.Y. Chen's conjecture (*cf.*, [7]) under some natural conditions.

For the second bi-product, we can apply Theorem 3 to a horizontally-conformal submersion (*cf.*, [51,52]). Then, we obtain:

Theorem 4. *Let (M^m, g) be a non-compact complete Riemannian manifold $(m > 2)$ and (N^2, h), a Riemannian surface with non-positive curvature. Let λ be a positive function on M belonging to $C^\infty(M) \cap L^2(M)$, and $\varphi : (M, g) \to (N^2, h)$, a horizontally-conformal submersion with a dilation λ. If φ is biharmonic and $\lambda\,|\hat{\mathbf{H}}|_g \in L^2(M)$, then φ is a harmonic morphism. Here, $\hat{\mathbf{H}}$ is trace of the second fundamental form of each fiber of φ.*

4. Outline of the Proofs of Theorems 3–5

In this section, we first give a sketch of the proof of Theorem 3.

The first step: For a fixed point $x_0 \in M$, and for every $0 < r < \infty$, we first take a cut-off C^∞ function η on M satisfying that:

$$\begin{cases} 0 \le \eta(x) \le 1 \quad (x \in M), \quad \eta(x) = 1 \quad (x \in B_r(x_0)) \\ \eta(x) = 0 \quad (x \not\in B_{2r}(x_0)), \quad |\nabla \eta| \le \dfrac{2}{r} \quad (x \in M) \end{cases} \tag{9}$$

For a biharmonic map $\varphi : (M, g) \to (N, h)$, the bitension field is given as:

$$\tau_2(\varphi) = \overline{\Delta}(\tau(\varphi)) - \sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) = 0 \tag{10}$$

so we have:

$$\begin{aligned}\int_M \langle \overline{\Delta}(\tau(\varphi)), \eta^2 \, \tau(\varphi)\rangle \, v_g &= \int_M \eta^2 \sum_{i=1}^{m} \langle R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i), \tau(\varphi)\rangle \, v_g \\ &\leq 0 \end{aligned}\tag{11}$$

since the sectional curvature of (N, h) is non-positive.

The second step: Therefore, by Equation (11) and noticing that $\overline{\Delta} = \overline{\nabla}^* \, \overline{\nabla}$, we obtain:

$$\begin{aligned} 0 &\geq \int_M \langle \overline{\Delta}(\tau(\varphi)), \eta^2 \, \tau(\varphi)\rangle \, v_g \\ &= \int_M \langle \overline{\nabla}\tau(\varphi), \overline{\nabla}(\eta^2 \, \tau(\varphi))\rangle \, v_g \\ &= \int_M \sum_{i=1}^{m} \langle \overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i}(\eta^2 \, \tau(\varphi))\rangle \, v_g \\ &= \int_M \sum_{i=1}^{m} \left\{ \eta^2 \, \langle \overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i}\tau(\varphi)\rangle + e_i(\eta^2) \, \langle \overline{\nabla}_{e_i}\tau(\varphi), \tau(\varphi)\rangle \right\} v_g \\ &= \int_M \eta^2 \sum_{i=1}^{m} \left| \overline{\nabla}_{e_i}\tau(\varphi) \right|^2 v_g + 2 \int_M \sum_{i=1}^{m} \langle \eta \overline{\nabla}_{e_i}\tau(\varphi), e_i(\eta) \, \tau(\varphi)\rangle \, v_g \end{aligned}\tag{12}$$

where we used $e_i(\eta^2) = 2\eta \, e_i(\eta)$ at the last equality. By moving the second term in the last equality of Equation (12) to the left-hand side, we have:

$$\begin{aligned}\int_M \eta^2 \sum_{i=1}^{m} |\overline{\nabla}_{e_i}\tau(\varphi)|^2 \, v_g &\leq -2 \int_M \sum_{i=1}^{m} \langle \eta \, \overline{\nabla}_{e_i}\tau(\varphi), e_i(\eta) \, \tau(\varphi)\rangle \, v_g \\ &= -2 \int_M \sum_{i=1}^{m} \langle V_i, W_i\rangle \, v_g \end{aligned}\tag{13}$$

where we put $V_i := \eta \, \overline{\nabla}_{e_i}\tau(\varphi)$ and $W_i := e_i(\eta) \, \tau(\varphi)$ $(i = 1 \, \cdots, m)$.

Now, recall the following Cauchy–Schwartz inequality:

$$\pm \, 2 \, \langle V_i, W_i\rangle \leq \epsilon |V_i|^2 + \frac{1}{\epsilon} |W_i|^2 \tag{14}$$

for all positive $\epsilon > 0$ because of the inequality $0 \leq |\sqrt{\epsilon} \, V_i \pm \frac{1}{\sqrt{\epsilon}} \, W_i|^2$. Therefore, for Equation (14), we obtain:

$$-2 \int_M \sum_{i=1}^{m} \langle V_i, W_i\rangle \, v_g \leq \epsilon \int_M \sum_{i=1}^{m} |V_i|^2 \, v_g + \frac{1}{\epsilon} \int_M \sum_{i=1}^{m} |W_i|^2 \, v_g \tag{15}$$

If we put $\epsilon = \frac{1}{2}$, we obtain, by Equations (13) and (15),

$$\int_M \eta^2 \sum_{i=1}^m |\overline{\nabla}_{e_i}\tau(\varphi)|^2 \, v_g \leq \frac{1}{2}\int_M \sum_{i=1}^m \eta^2 \, |\overline{\nabla}_{e_i}\tau(\varphi)|^2 \, v_g + 2\int_M \sum_{i=1}^m e_i(\eta)^2 \, |\tau(\varphi)|^2 \, v_g \tag{16}$$

Thus, by Equations (12) and (16), we obtain:

$$\int_M \eta^2 \sum_{i=1}^m |\overline{\nabla}_{e_i}\tau(\varphi)|^2 \, v_g \leq 4\int_M |\nabla\eta|^2 \, |\tau(\varphi)|^2 \, v_g \leq \frac{16}{r^2}\int_M |\tau(\varphi)|^2 \, v_g \tag{17}$$

The third step: Since (M, g) is complete and non-compact, we can tend r to infinity. By the assumption $E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g < \infty$, the right-hand side goes to zero. Furthermore, if $r \to \infty$, the left-hand side of Equation (17) goes to $\int_M \sum_{i=1}^m |\overline{\nabla}_{e_i}\tau(\varphi)|^2 \, v_g$ since $\eta = 1$ on $B_r(x_0)$. Thus, we obtain:

$$\int_M \sum_{i=1}^m |\overline{\nabla}_{e_i}\tau(\varphi)|^2 \, v_g = 0 \tag{18}$$

Therefore, we obtain, for every vector field X in M,

$$\overline{\nabla}_X \tau(\varphi) = 0 \tag{19}$$

Then, we have, in particular, $|\tau(\varphi)|$ is constant, say c. Because, for every vector field X on M, at each point in M,

$$X \, |\tau(\varphi)|^2 = 2\langle \overline{\nabla}_X \tau(\varphi), \tau(\varphi)\rangle = 0 \tag{20}$$

Therefore, if $\mathrm{Vol}(M, g) = \infty$ and $c \neq 0$, then:

$$\tau_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g = \frac{c^2}{2}\,\mathrm{Vol}(M, g) = \infty \tag{21}$$

which yields a contradiction. Thus, we have $|\tau(\varphi)| = c = 0$, *i.e.*, φ is harmonic. We have Equation (2).

The fourth step: For Equation (1), assume both $E(\varphi) < \infty$ and $E_2(\varphi) < \infty$. Then, let us consider a one-form α on M defined by:

$$\alpha(X) := \langle d\varphi(X), \tau(\varphi)\rangle, \qquad (X \in \mathfrak{X}(M)) \tag{22}$$

Note here that:

$$\int_M |\alpha| \, v_g = \int_M \left(\sum_{i=1}^m |\alpha(e_i)|^2\right)^{1/2} v_g \leq \int_M |d\varphi| \, |\tau(\varphi)| \, v_g \leq \left(\int_M |d\varphi|^2 \, v_g\right)^{1/2} \left(\int_M |\tau(\varphi)|^2 \, v_g\right)^{1/2} = 2\sqrt{E(\varphi)\, E_2(\varphi)} < \infty \tag{23}$$

Moreover, the divergent $\delta\alpha := -\sum_{i=1}^m (\nabla_{e_i}\alpha)(e_i) \in C^\infty(M)$ turns out (*cf.*, [1], p. 9) as:

$$-\delta\alpha = |\tau(\varphi)|^2 + \langle d\varphi, \overline{\nabla}\tau(\varphi)\rangle = |\tau(\varphi)|^2 \tag{24}$$

Indeed, we have:

$$\begin{aligned}
-\delta\alpha &= \sum_{i=1}^m e_i \langle d\varphi(e_i), \tau(\varphi)\rangle - \sum_{i=1}^m \langle d\varphi(\nabla_{e_i} e_i), \tau(\varphi)\rangle \\
&= \langle \sum_{i=1}^m \left(\overline{\nabla}_{e_i}(d\varphi(e_i)) - d\varphi(\nabla_{e_i} e_i)\right), \tau(\varphi)\rangle + \sum_{i=1}^m \langle d\varphi(e_i), \overline{\nabla}_{e_i}\tau(\varphi)\rangle \\
&= \langle \tau(\varphi), \tau(\varphi)\rangle + \langle d\varphi, \overline{\nabla}\tau(\varphi)\rangle
\end{aligned}$$

which is equal to $|\tau(\varphi)|$ since $\overline{\nabla}\tau(\varphi) = 0$.

By Equation (24) and $E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g < \infty$, the function $-\delta\alpha$ is also integrable over M. Thus, together with Equation (23), we can apply Gaffney's theorem (see Theorem 6, below) for the one-form α. By integrating Equation (24) over M and by Gaffney's theorem, we have:

$$0 = \int_M (-\delta\alpha)\, v_g = \int_M |\tau(\varphi)|^2 \, v_g \tag{25}$$

which yields that $\tau(\varphi) = 0$. We have Theorem 3. $\square$

Theorem 5. *(Gaffney [53]) Let (M, g) be a complete Riemannian manifold. If a C^1 1-form α satisfies that $\int_M |\alpha| \, v_g < \infty$ and $\int_M (\delta\alpha)\, v_g < \infty$ or, equivalently, a C^1 vector field X defined by $\alpha(Y) = \langle X, Y\rangle$ $(\forall\ Y \in \mathfrak{X}(M))$ satisfies that $\int_M |X| \, v_g < \infty$ and $\int_M \mathrm{div}(X)\, v_g < \infty$, then:*

$$\int_M (-\delta\alpha)\, v_g = \int_M \mathrm{div}(X)\, v_g = 0 \tag{26}$$

Our method can be applied to an isometric immersion $\varphi : (M, g) \to (N, h)$. In this case, the one-form α defined by Equation (22) in the proof of Theorem 3 vanishes automatically without using Gaffney's theorem, since $\tau(\varphi) = m\,\xi$ belongs to the normal component of $T_{\varphi(x)}N$ $(x \in M)$, where ξ is the mean curvature normal vector field and $m = \dim(M)$. Thus, Equation (24) turns out as:

$$0 = -\delta\alpha = |\tau(\varphi)|^2 + \langle d\varphi, \overline{\nabla}\tau(\varphi)\rangle = |\tau(\varphi)|^2 \tag{27}$$

which implies that $\tau(\varphi) = m\,\xi = 0$, *i.e.*, φ is minimal. Thus, we obtain Theorem 4. $\square$

We also apply Theorem 3 to a horizontally-conformal submersion $\varphi : (M^m, g) \to (N^n, h)$ $(m > n \geq 2)$ (*cf.*, [52,54]). In the case that a Riemannian submersion from a space form of constant sectional curvature into a Riemann surface (N^2, h), Wang and Ou (*cf.*, [19,28]) showed that it is biharmonic if and only if it is harmonic. We treat with a submersion from a higher dimensional Riemannian manifold (M, g) (*cf.*, [51]). Namely, let $\varphi : M \to N$ be a submersion, and each tangent space T_xM $(x \in M)$ is decomposed into the orthogonal direct sum of the vertical space $\mathcal{V}_x = \mathrm{Ker}(d\varphi_x)$ and the horizontal space $\mathcal{H}_x$:

$$T_xM = \mathcal{V}_x \oplus \mathcal{H}_x \tag{28}$$

and we assume that there exists a positive C^∞ function λ on M, called the dilation, such that, for each $x \in M$,

$$h(d\varphi_x(X), d\varphi_x(Y)) = \lambda^2(x)\, g(X,Y), \quad (X,Y \in \mathcal{H}_x) \tag{29}$$

The map φ is said to be horizontally homothetic if the dilation λ is constant along horizontally curves in M.

If $\varphi : (M^m, g) \to (N^n, h)$ $(m > n \geq 2)$ is a horizontally-conformal submersion, then, the tension field $\tau(\varphi)$ is given (*cf.*, [51,52]) by:

$$\tau(\varphi) = \frac{n-2}{2}\, \lambda^2\, d\varphi\big(\mathrm{grad}_{\mathcal{H}}\big(\frac{1}{\lambda^2}\big)\big) - (m-n) d\varphi\big(\hat{\mathbf{H}}\big) \tag{30}$$

where $\mathrm{grad}_{\mathcal{H}}\big(\frac{1}{\lambda^2}\big)$ is the $\mathcal{H}$-component of the decomposition according to Equation (28) of $\mathrm{grad}\big(\frac{1}{\lambda^2}\big)$ and $\hat{\mathbf{H}}$ is the trace of the second fundamental form of each fiber, which is given by $\hat{\mathbf{H}} = \frac{1}{m-n}\sum_{k=n+1}^m \mathcal{H}(\nabla_{e_k} e_k)$, where a local orthonormal frame field $\{e_i\}_{i=1}^m$ on M is taken in such a way that $\{e_{ix} | i = 1, \cdots, n\}$ belong to $\mathcal{H}_x$ and $\{e_{jx} | j = n+1, \cdots, m\}$ belong to $\mathcal{V}_x$ where x is in a neighborhood in M. Then, due to Theorems 3 and Equation (29), we have immediately:

Theorem 6. *Let (M^m, g) be a complete non-compact Riemannian manifold and (N^n, h) a Riemannian manifold with the non-positive sectional curvature $(m > n \geq 2)$. Let $\varphi : (M, g) \to (N, h)$ be a horizontally-conformal submersion with the dilation λ satisfying that:*

$$\int_M \lambda^2 \left| \frac{n-2}{2}\, \lambda^2\, \mathrm{grad}_{\mathcal{H}}\big(\frac{1}{\lambda^2}\big) - (m-n)\, \hat{\mathbf{H}} \right|_g^2 v_g < \infty \tag{31}$$

Assume that, either $\int_M \lambda^2\, v_g < \infty$ or $\mathrm{Vol}(M, g) = \int_M v_g = \infty$. Then, if $\varphi : (M, g) \to (N, h)$ is biharmonic, then it is a harmonic morphism.

Due to Theorem 7, we have:

Corollary 7. *Let (M^m, g) be a complete non-compact Riemannian manifold and (N^2, h) a Riemannian surface with the non-positive sectional curvature $(m > n = 2)$. Let $\varphi : (M, g) \to (N, h)$ be a horizontally-conformal submersion with the dilation λ satisfying that:*

$$\int_M \lambda^2 \left|\hat{\mathbf{H}}\right|_g^2 v_g < \infty \tag{32}$$

Assume that either $\int_M \lambda^2\, v_g < \infty$ or $\mathrm{Vol}(M, g) = \int_M v_g = \infty$. Then, if $\varphi : (M, g) \to (N, h)$ is biharmonic, then it is a harmonic morphism.

Corollary 8 implies Theorem 5, immediately. $\square$

Remark 1. *(1) Notice that in Theorem 5, there is no restriction to the dilation λ because of* $\dim N = 2$. *This implies that for every positive C^∞ function λ in $C^\infty(M) \cap L^2(M)$ satisfying Equations (31) or (32), we have a harmonic morphism $\varphi : (M^m, g) \to (N^2, h)$.*

(2) For a biharmonic map of (M,g) into (N,h), the non-positivity of (N,h) implies that:

$$\langle \tau(\varphi), \overline{\Delta}\tau(\varphi)\rangle = \sum_{i=1}^{m} \langle R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i), \tau(\varphi)\rangle \leq 0 \tag{33}$$

which is stronger than the Bochner-type formula $|\tau(\varphi)|\,\Delta|\tau(\varphi)| \geq 0$. However, we can prove Theorem 3 in an alternative way by using the latter one. Here, $\Delta = \sum_{i=1}^m ({e_i}^2 - \nabla_{e_i}e_i)$ denotes the negative Laplace operator acting on $C^\infty(M)$.

5. Harmonic Maps and Biharmonic Maps into Compact Lie Groups or Symmetric Spaces

In this section, we treat with harmonic maps and biharmonic maps into compact Lie groups or symmetric spaces of the compact type.

5.1. Biharmonic Maps into Compact Lie Groups

We first treat with harmonic maps and biharmonic maps into compact Lie groups. Let θ be the Maurer–Cartan form on G, *i.e.*, a $\mathfrak{g}$-valued left invariant one-form on G, which is defined by $\theta_y(Z_y) = Z$, ($y \in G$, $Z \in \mathfrak{g}$). For every C^∞ map ψ of (M,g) into (G,h), let us consider a $\mathfrak{g}$-valued one-form α on M given by $\alpha = \psi^*\theta$. Then, it is well known (see for example, [55]) that:

Lemma 8. *For every C^∞ map $\psi : (M,g) \to (G,h)$,*

$$\theta(\tau(\psi)) = -\delta\alpha \tag{34}$$

where $\alpha = \psi^*\theta$ and θ is the Maurer–Cartan form of G.

Thus, $\psi : (M,g) \to (G,h)$ is harmonic if and only if $\delta\alpha = 0$.

Furthermore, let $\{X_s\}_{s=1}^n$ be an orthonormal basis of $\mathfrak{g}$ with respect to the inner product $\langle\ ,\ \rangle$. Then, for every $V \in \Gamma(\psi^{-1}TG)$,

$$\begin{aligned} V(x) &= \sum_{i=1}^{n} h_{\psi(x)}(V(x), X_{s\,\psi(x)})\, X_{s\,\psi(x)} \in T_{\psi(x)}G \\ \theta(V)(x) &= \sum_{s=1}^{n} h_{\psi(x)}(V(x), X_{s\,\psi(x)})\, X_s \in \mathfrak{g} \end{aligned} \tag{35}$$

for all $x \in M$. Then, for every $X \in \mathfrak{X}(M)$,

$$\begin{aligned} \theta(\overline{\nabla}_X V) &= \sum_{s=1}^{n} h(\overline{\nabla}_X V, X_s)\, X_s = \sum_{s=1}^{n} \{X\, h(V, X_s) - h(V, \overline{\nabla}_X X_s)\} X_s \\ &= X(\theta(V)) - \sum_{s=1}^{n} h(V, \overline{\nabla}_X X_s) X_s \end{aligned} \tag{36}$$

where we regarded a vector field $Y \in \mathfrak{X}(G)$ by $Y(x) = Y(\psi(x))\,(x \in M)$ to be an element in the space $\Gamma^{-1}(TG)$ of C^∞ sections of $\psi^{-1}TG$.

Here, let us recall that the Levi–Civita connection ∇^h of (G, h) is given (*cf.*, [56,57] Volume II, p. 201, Theorem 3.3) by:

$$\nabla^h_{X_t} X_s = \frac{1}{2}\,[X_t, X_s] = \frac{1}{2}\sum_{\ell=1}^{n} C^{\ell}_{ts}\, X_\ell \tag{37}$$

where the structure constant C^{ℓ}_{ts} of $\mathfrak{g}$ is defined by $[X_t, X_s] = \sum_{\ell=1}^{n} C^{\ell}_{ts}\, X_\ell$, and satisfies that:

$$C^{\ell}_{ts} = \langle [X_t, X_s], X_\ell\rangle = -\langle X_s, [X_t, X_\ell]\rangle = -C^{s}_{t\ell} \tag{38}$$

Thus, we have by Equations (37) and (38),

$$\begin{aligned}
\sum_{s=1}^{n} h(V, \overline{\nabla}_X X_s) X_s &= \frac{1}{2}\sum_{s,t=1}^{n} h\left(V, \sum_{\ell=1}^{n} h(\psi_* X, X_t) C^{\ell}_{ts}\, X_\ell\right) X_s \\
&= -\frac{1}{2}\sum_{s,t,\ell=1}^{n} h(V, X_\ell)\, h(\psi_* X, X_t)\, C^{s}_{t\ell} X_s \\
&= -\frac{1}{2}\sum_{t,\ell=1}^{n} h(V, X_\ell)\, h(\psi_* X, X_t)\, [X_t, X_\ell] \\
&= -\frac{1}{2}\left[\sum_{t=1}^{n} h(\psi_* X, X_t)\, X_t, \sum_{\ell=1}^{n} h(V, X_\ell)\, X_\ell\right] \\
&= -\frac{1}{2}\,[\alpha(X), \theta(V)]
\end{aligned} \tag{39}$$

because we have:

$$\alpha(X) = \theta(\psi_* X) = \sum_{t=1}^{n} h(\psi_* X, X_t) X_t \tag{40}$$

and

$$\theta(V) = \sum_{\ell=1}^{n} h(V, X_\ell)\, \theta(X_\ell) = \sum_{\ell=1}^{n} h(V, X_\ell)\, X_\ell \tag{41}$$

Substituting Equations (40) and (41) into the above, we have Equation (39).

Therefore, we obtain the following together with Equations (36) and (39).

Lemma 9. *For every C^∞ map $\psi : (M, g) \to (G, h)$,*

$$\theta(\overline{\nabla}_X V) = X(\theta(V)) + \frac{1}{2}\,[\alpha(X), \theta(V)] \tag{42}$$

where $V \in \Gamma(\psi^{-1}TG)$ and $X \in \mathfrak{X}(M)$.

Then, we see immediately due to this lemma:

Theorem 10. *For every $\psi \in C^\infty(M, G)$, we have:*

$$\begin{aligned}
\theta(\tau_2(\psi)) &= \theta(J(\tau(\psi))) \\
&= -\delta\, d\, \delta\alpha - \mathrm{Trace}_g([\alpha, d\,\delta\alpha])
\end{aligned} \tag{43}$$

where $\alpha = \psi^*\theta$.

Here, let us recall the definition:

Definition 11. *For two $\mathfrak{g}$-valued one-forms α and β on M, we define $\mathfrak{g}$-valued symmetric two-tensor $[\alpha, \beta]$ on M by:*

$$[\alpha, \beta](X, Y) := \frac{1}{2}\{[\alpha(X), \beta(Y)] + [\alpha(Y), \beta(X)]\}, \quad (X, Y \in \mathfrak{X}(M)) \tag{44}$$

and its trace $\mathrm{Trace}_g([\alpha, \beta])$ *by:*

$$\mathrm{Trace}_g([\alpha, \beta]) := \sum_{i=1}^{m} [\alpha, \beta](e_i, e_i) \tag{45}$$

We also use a $\mathfrak{g}$-valued two-form $[\alpha \wedge \beta]$ on M by:

$$[\alpha \wedge \beta](X, Y) := \frac{1}{2}\{[\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]\}, \quad (X, Y \in \mathfrak{X}(M)) \tag{46}$$

Then, we have immediately by Theorem 11:

Corollary 12. *For every $\psi \in C^\infty(M, G)$, we have:*
(1) $\psi : (M, g) \to (G, h)$ is harmonic if and only if:

$$\delta\alpha = 0 \tag{47}$$

(2) $\psi : (M, g) \to (G, h)$ is biharmonic if and only if:

$$\delta\, d\, \delta\alpha + \mathrm{Trace}_g([\alpha, d\, \delta\alpha]) = 0 \tag{48}$$

5.2. Biharmonic Maps into Compact Symmetric Spaces

Now, let θ be the Maurer–Cartan form on G, *i.e.*, a $\mathfrak{g}$-valued left invariant one-form on G, which is defined by $\theta_y(Z_y) = Z$ ($y \in G$, $Z \in \mathfrak{g}$). For every C^∞ map φ of (M, g) into $(G/K, h)$ with a lift $\psi : M \to G$, let us consider a $\mathfrak{g}$-valued one-form α on M given by $\alpha = \psi^*\theta$ and the decomposition:

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}} \tag{49}$$

corresponding to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then, it is well known (see, for example, [55]) that:

Lemma 13. *For every C^∞ map $\varphi : (M, g) \to (G/K, h)$,*

$$t_{\psi(x)^{-1}*}\tau(\varphi) = -\delta(\alpha_{\mathfrak{m}} + \sum_{i=1}^{m} [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \quad (x \in M) \tag{50}$$

where $\alpha = \varphi^*\theta$, θ is the Maurer–Cartan form of G and $\delta(\alpha_{\mathfrak{m}})$ is the co-differentiation of the $\mathfrak{m}$-valued one-form $\alpha_{\mathfrak{m}}$ on (M, g).

Thus, $\varphi : (M, g) \to (G/K, h)$ is harmonic if and only if:

$$-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m} [\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] = 0 \tag{51}$$

Furthermore, we obtain:

Theorem 14. *We have:*

$$t_{\psi(x)^{-1}*}\tau_2(\varphi) = \Delta_g\left(-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)]\right)$$
$$+ \sum_{s=1}^{m}\left[\left[-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s)\right], \alpha_{\mathfrak{m}}(e_s)\right] \tag{52}$$

where Δ_g is the (positive) Laplacian of (M, g) acting on C^∞ functions on M, and $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on (M, g).

Therefore, we obtain immediately the following two corollaries.

Corollary 15. *Let $(G/K, h)$ be a Riemannian symmetric space and $\varphi : (M, g) \to (G/K, h)$ a C^∞ mapping. Then, we have:*

(1) The map $\varphi : (M, g) \to (G/K, h)$ is harmonic if and only if:

$$-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] = 0 \tag{53}$$

(2) The map $\varphi : (M, g) \to (G/K, h)$ is biharmonic if and only if:

$$\Delta_g\left(-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)]\right)$$
$$+ \sum_{s=1}^{m}\left[\left[-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s)\right], \alpha_{\mathfrak{m}}(e_s)\right] = 0 \tag{54}$$

Corollary 16. *Let $(G/K, h)$ be a Riemannian symmetric space and $\varphi : (M, g) \to (G/K, h)$ a C^∞ mapping with a horizontal lift $\psi : M \to G$, i.e., $\varphi = \pi \circ \psi$ and $\psi_x(T_xM) \subset H_{\psi(x)}$, which is equivalent to $\alpha_{\mathfrak{k}} \equiv 0$.*

Then, we have:

(1) The map $\varphi : (M, g) \to (G/K, h)$ is harmonic if and only if:

$$\delta(\alpha_{\mathfrak{m}}) = 0 \tag{55}$$

(2) and the map $\varphi : (M, g) \to (G/K, h)$ is biharmonic if and only if:

$$\delta\, d\, \delta(\alpha_{\mathfrak{m}}) + \sum_{s=1}^{m}[[\delta(\alpha_{\mathfrak{m}}), \alpha_{\mathfrak{m}}(e_s)], \alpha_{\mathfrak{m}}(e_s)] = 0. \tag{56}$$

For applications and examples, see [29,30].

6. The CR Analogue of Harmonic Maps and Biharmonic Maps

In the 1970s, Chern and Moser initiated [58] the geometry and analysis of strictly convex CR manifolds, and many mathematicians have worked on CR manifolds (*cf.*, [59]). Recently, Barletta, Dragomir and Urakawa gave [60] the notion of the pseudo-harmonic map, and also Dragomir and Montaldo settled [10] the one of the pseudo-biharmonic map.

6.1. Conjecture and Results

In this part, we raise:

The CR analogue of the generalized Chen's conjecture:

Let (M, g_θ) be a complete strictly pseudoconvex CR manifold and assume that (N, h) is a Riemannian manifold of non-positive curvature.

Then, every pseudo-biharmonic isometric immersion $\varphi : (M, g_\theta) \to (N, h)$ must be pseudo-harmonic.

We will see that this conjecture holds under some L^2 condition on a complete strongly pseudoconvex CR manifold (*cf.*, Theorem 18) and will give characterization theorems on pseudo-biharmonic immersions from CR manifolds into the unit sphere or the complex projective space (*cf.*, Theorems 19 and 20). More precisely, we will see:

Theorem 17. *(cf., Theorem 21) Let φ be a pseudo-biharmonic map of a complete CR manifold (M, g_θ) into a Riemannian manifold (N, h) of non-positive curvature. If the pseudo-energy and the pseudo-bienergy of φ are finite, then φ is pseudo-harmonic.*

Then, we have:

Theorem 18. *Let φ be an isometric immersion of a CR manifold (M^{2n+1}, g_θ) into the unit sphere $S^{2n+2}(1)$ of curvature one. Assume that the pseudo-mean curvature is parallel, but not pseudo-harmonic.*

Then, φ is pseudo-biharmonic if and only if the restriction of the second fundamental form B_φ to the holomorphic subspace $H_x(M)$ of T_xM $(x \in M)$ satisfies that:

$$\| B_\varphi|_{H(M)\times H(M)} \|^2 = 2n$$

Furthermore, we have:

Theorem 19. *Let φ be an isometric immersion of a CR manifold (M^{2n+1}, g_θ) into the complex projective space $(\mathbb{P}^{n+1}(c), h, J)$ of holomorphic sectional curvature $c > 0$. Assume that the pseudo-mean curvature is parallel, but not pseudo-harmonic. Then, φ is pseudo-biharmonic if and only if one of the following holds:*

(1) $J(d\varphi(T))$ is tangent to $\varphi(M)$ and:

$$\| B_\varphi|_{H(M)\times H(M)} \|^2 = \frac{c}{4}(2n+3)$$

(2) $J(d\varphi(T))$ is normal to $\varphi(M)$ and:

$$\| B_\varphi|_{H(M)\times H(M)} \|^2 = \frac{c}{4}(2n) = \frac{n}{2}\, c$$

Here, T is the characteristic vector field of (M, g_θ), $H_x(M) \oplus \mathbb{R}T_x = T_x(M)$ and $B_\varphi|_{H(M)\times H(M)}$ is the restriction of the second fundamental form B_φ to $H_x(M)$ $(x \in M)$.

Several examples of pseudo-biharmonic immersions of (M, g_θ) into the unit sphere or complex projective space are given in [31].

6.2. Explanations of Notions and Proofs of the CR Rigidity

We explain the terminologies in the above results following Dragomir and Montaldo [10] and also Barletta, Dragomir and Urakawa [60]. We also prepare the materials on pseudo-harmonic maps and pseudo-biharmonic maps (see also [61]).

Let M be a strictly pseudoconvex CR manifold of $(2n+1)$-dimension, T the characteristic vector field on M, J the complex structure of the subspace $H_x(M)$ of $T_x(M)$ $(x \in M)$ and g_θ the Webster–Riemannian metric on M defined for $X, Y \in H(M)$ by:

$$g_\theta(X,Y) = (d\theta)(X, JY),\ \ g_\theta(X,T) = 0,\ \ g_\theta(T,T) = 1$$

Let us recall for a C^∞ map φ of (M, g_θ) into another Riemannian manifold (N, h); the pseudo-energy $E_b(\varphi)$ is defined [60] by:

$$E_b(\varphi) = \frac{1}{2}\int_M \sum_{i=1}^{2n} (\varphi^* h)(X_i, X_i)\,\theta \wedge (d\theta)^n \tag{57}$$

where $\{X_i\}_{i=1}^{2n}$ is an orthonormal frame field on $(H(M), g_\theta)$. Then, the first variational formula of $E_b(\varphi)$ is as follows [60]. For every variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\left.\frac{d}{dt}\right|_{t=0} E_b(\varphi_t) = -\int_M h(\tau_b(\varphi), V)\, d\theta \wedge (d\theta)^n = 0 \tag{58}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is defined by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$. Here, $\tau_b(\varphi)$ is the pseudo-tension field, which is given by:

$$\tau_b(\varphi) = \sum_{i=1}^{2n} B_\varphi(X_i, X_i) \tag{59}$$

where $B_\varphi(X, Y)$ $(X,\ Y \in \mathfrak{X}(M))$ is the second fundamental form of Equation (3) for a C^∞ map of (M, g_θ) into (N, h). Then, φ is pseudo-harmonic if $\tau_b(\varphi) = 0$.

The second variational formula of E_b is given as follows ([60], p.733):

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E_b(\varphi_t) = \int_M h(J_b(V), V)\,\theta \wedge (d\theta)^n \tag{60}$$

where J_b is a subelliptic operator acting on $\Gamma(\varphi^{-1}TN)$ given by:

$$J_b(V) = \Delta_b V - \mathcal{R}_b(V) \tag{61}$$

Here, for $V \in \Gamma(\varphi^{-1}TN)$),

$$\begin{cases} \Delta_b V = (\overline{\nabla}^H)^* \,\overline{\nabla}^H V = -\displaystyle\sum_{i=1}^{2n} \left\{ \overline{\nabla}_{X_i}(\overline{\nabla}_{X_i} V) - \overline{\nabla}_{\nabla_{X_i} X_i} V \right\} \\ \mathcal{R}_b(V) = \displaystyle\sum_{i=1}^{2n} R^h(V, d\varphi(X_i)) d\varphi(X_i) \end{cases} \tag{62}$$

where ∇ is the Tanaka–Webster connection, $\overline{\nabla}$ the induced connection on $\phi^{-1}TN$ induced from the Levi–Civita connection ∇^h and $\{X_i\}_{i=1}^{2n}$ a local orthonormal frame field on $(H(M), g_\theta)$, respectively. Here, $(\overline{\nabla}^H)_X V := \overline{\nabla}_{X^H} V$ $(X \in \mathfrak{X}(M),\, V \in \Gamma(\phi^{-1}TN))$, corresponding to the decomposition $X = X^H + g_\theta(X,T)\,T$ $(X^H \in H(M))$. Define $\pi_H(X) = X^H$ $(X \in T_x(M))$, and $(\overline{\nabla}^H)^*$ is the formal adjoint of $\overline{\nabla}^H$.

Dragomir and Montaldo [10] introduced the pseudo-bienergy given by:

$$E_{b,2}(\varphi) = \frac{1}{2}\int_M h(\tau_b(\varphi), \tau_b(\varphi))\,\theta \wedge (d\theta)^n \tag{63}$$

where $\tau_b(\varphi)$ is the pseudo-tension field of φ. They gave the first variational formula of $E_{b,2}$ as follows ([10], p. 227):

$$\left.\frac{d}{dt}\right|_{t=0} E_{b,2}(\varphi_t) = -\int_M h(\tau_{b,2}(\varphi), V)\,\theta \wedge (d\theta)^n \tag{64}$$

where $\tau_{b,2}(\varphi)$ is called the pseudo-bitension field given by:

$$\tau_{b,2}(\varphi) = \Delta_b\big(\tau_b(\varphi)\big) - \sum_{i=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_i))\,d\varphi(X_i) \tag{65}$$

Then, a smooth map φ of (M, g_θ) into (N, h) is said to be pseudo-biharmonic if $\tau_{b,2}(\varphi) = 0$. By definition, a pseudo-harmonic map is always pseudo-biharmonic.

Theorem 20. *(cf., Theorem 18) Assume that φ is a pseudo-biharmonic map of a strictly pseudoconvex complete CR manifold (M, g_θ) into another Riemannian manifold (N, h) of non-positive curvature.*

If φ has finite pseudo-bienergy $E_{b,2}(\varphi) < \infty$ and finite pseudo-energy $E_b(\varphi) < \infty$, then it is pseudo-harmonic, i.e., $\tau_b(\varphi) = 0$.

(Proof of Theorem 21) The proof is divided into several steps.

The first step: For an arbitrarily fixed point $x_0 \in M$, let $B_r(x_0) = \{x \in M : r(x) < r\}$ where $r(x)$ is a distance function on (M, g_θ), and let us take a cut-off function η on (M, g_θ), *i.e.*,

$$\begin{cases} 0 \le \eta(x) \le 1 \quad (x \in M), & \eta(x) = 1 \quad (x \in B_r(x_0)) \\ \eta(x) = 0 \quad (x \notin B_{2r}(x_0)), & |\nabla^{g_\theta}\eta| \le \dfrac{2}{r} \quad (x \in M) \end{cases} \tag{66}$$

where r is the distance function and ∇^{g_θ} is the Levi–Civita connection of (M, g_θ), respectively. Assume that $\varphi : (M, g_\theta) \to (N, h)$ is a pseudo-biharmonic map, *i.e.*,

$$\tau_{b,2}(\varphi) = J_b(\tau_b(\varphi)) = \Delta_b(\tau_b(\varphi)) - \sum_{j=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_j))\,d\varphi(X_j) = 0 \tag{67}$$

The second step: Then, we have:

$$\begin{aligned} &\int_M \langle \Delta_b(\tau_b(\varphi)), \eta^2\,\tau_b(\varphi)\rangle\,\theta \wedge (d\theta)^n \\ &= \int_M \eta^2 \sum_{j=1}^{2n} \langle R^h(\tau_b(\varphi), d\varphi(X_j))\,d\varphi(X_j),\ \tau_b(\varphi)\rangle\,\theta \wedge (d\theta)^n \le 0 \end{aligned} \tag{68}$$

since (N, h) has the non-positive sectional curvature. However, for the left-hand side of Equation (68), it holds that:

$$\begin{aligned}
&\int_M \langle \Delta_b(\tau_b(\varphi)), \eta^2\, \tau_b(\varphi)\rangle\, \theta \wedge (d\theta)^n \\
&\quad = \int_M \langle \overline{\nabla}^H\, \tau_b(\varphi), \overline{\nabla}^H\, (\,\eta^2\, \tau_b(\varphi)\,)\,\rangle\, \theta \wedge (d\theta)^n \\
&\quad = \int_M \sum_{j=1}^{2n} \langle \overline{\nabla}_{X_j} \tau_b(\varphi), \overline{\nabla}_{X_j}(\eta^2\, \tau_b(\varphi))\,\rangle\, \theta \wedge (d\theta)^n
\end{aligned} \tag{69}$$

Here, let us recall, for $V,\ W \in \Gamma(\varphi^{-1}TN))$,

$$\langle \overline{\nabla}^H V, \overline{\nabla}^H W\rangle = \sum_{\alpha=1}^{2n+1} \langle \overline{\nabla}^H_{e_\alpha} V, \overline{\nabla}^H_{e_\alpha} W\rangle = \sum_{j=1}^{2n} \langle \overline{\nabla}_{X_i} V, \overline{\nabla}_{X_i} W\rangle$$

where $\{e_\alpha\}_{\alpha=1}^{2n+1}$ is a locally-defined orthonormal frame field of (M, g_θ), $X_{j\,j=1}^{\,2n}$ is an orthonormal frame of $H(M)$ and $\overline{\nabla}^H_X W$ $(X \in \mathfrak{X}(M),\ W \in \Gamma(\varphi^{-1}TN))$ is defined by:

$$\overline{\nabla}^H_X W = \sum_j \{(X^H f_j)\, V_j + f_j \overline{\nabla}_{X^H}\, V_j\}$$

for $W = \sum_j f_i\, V_j$ $(f_j \in C^\infty(M)$ and $V_j \in \Gamma(\varphi^{-1}TN)$. Here, X^H is the $H(M)$-component of X corresponding to the decomposition of $T_x(M) = H_x(M) \oplus \mathbb{R}T_x$ $(x \in M)$, and $\overline{\nabla}$ is the induced connection of $\varphi^{-1}TN$ from the Levi–Civita connection ∇^h of (N, h).

Since

$$\overline{\nabla}_{X_j}(\eta^2\, \tau_b(\varphi)) = 2\eta\, X_j\eta\, \tau_b(\varphi) + \eta^2\, \overline{\nabla}_{X_j} \tau_b(\varphi) \tag{70}$$

the right-hand side of Equation (69) is equal to:

$$\begin{aligned}
&\int_M \eta^2 \sum_{j=1}^{2n} \big|\, \overline{\nabla}_{X_j} \tau_b(\varphi)\, \big|^2\, \theta \wedge (d\theta)^n \\
&\quad + 2 \int_M \sum_{j=1}^{2n} \langle \eta\, \overline{\nabla}_{X_j} \tau_b(\varphi),\ (X_j\eta)\, \tau_b(\varphi)\rangle\, \theta \wedge (d\theta)^n
\end{aligned} \tag{71}$$

Therefore, together with Equation (68), we have:

$$\begin{aligned}
&\int_M \eta^2 \sum_{j=1}^{2n} \big|\, \overline{\nabla}_{X_j}\, \tau_b(\varphi)\, \big|^2\, \theta \wedge (d\theta)^n \\
&\quad \leq -2 \int_M \sum_{j=1}^{2n} \langle \eta\, \overline{\nabla}_{X_j} \tau_b(\varphi),\ (X_j\eta)\, \tau_b(\varphi)\rangle\, \theta \wedge (d\theta)^n \\
&\quad =: -2 \int_M \sum_{j=1}^{2n} \langle V_j, W_j\rangle\, \theta \wedge (d\theta)^n
\end{aligned} \tag{72}$$

where we define $V_j,\ W_j \in \Gamma(\varphi^{-1}TN)$ $(j = 1, \cdots, 2n)$ by:

$$V_j := \eta\, \overline{\nabla}_{X_j}\, \tau_b(\varphi), \quad W_j := (X_j\eta)\, \tau_b(\varphi)$$

Then, since it holds that $0 \leq \left|\sqrt{\epsilon}\, V_i \pm \frac{1}{\sqrt{\epsilon}}\, W_i\right|^2$ for every $\epsilon > 0$, we have,

$$\text{RHS of Equation (72)} \leq \epsilon \int_M \sum_{j=1}^{2n} \left|\, V_j \,\right|^2 \theta \wedge (d\theta)^n + \frac{1}{\epsilon} \int_M \sum_{j=1}^{2n} \left|\, W_j \,\right|^2 \theta \wedge (d\theta)^n \tag{73}$$

for every $\epsilon > 0$. By taking $\epsilon = \dfrac{1}{2}$, we obtain:

$$\begin{aligned}
&\int_M \eta^2 \sum_{j=1}^{2n} \left|\, \overline{\nabla}_{X_j} \tau_b(\varphi) \,\right|^2 \theta \wedge (d\theta)^n \\
&\leq \frac{1}{2} \int_M \sum_{j=1}^{2n} \eta^2 \left| \overline{\nabla}_{X_j} \tau_b(\varphi) \right|^2 \theta \wedge (d\theta)^n + 2 \int_M \sum_{j=1}^{2n} \left| X_j \eta \right|^2 \left| \tau_b(\varphi) \,\right|^2 \theta \wedge (d\theta)^n
\end{aligned} \tag{74}$$

Therefore, we obtain, due to the properties that $\eta = 1$ on $B_r(x_0)$ and $\sum_{j=1}^{2n} |X_j \eta|^2 \leq |\nabla^{g_\theta} \eta|^2 \leq \left(\frac{2}{r}\right)^2$,

$$\begin{aligned}
\int_{B_r(x_0)} \sum_{j=1}^{2n} \left| \overline{\nabla} \tau_b(\varphi) \right|^2 \theta \wedge (d\theta)^n &\leq \int_M \eta^2 \sum_{j=1}^{2n} \left| \overline{\nabla}_{X_j} \tau_b(\varphi) \right|^2 \theta \wedge (d\theta)^n \\
&\leq 4 \int_M \sum_{j=1}^{2n} \left| X_j \eta \right|^2 \left| \tau_b(\varphi) \right|^2 \theta \wedge (d\theta)^n \leq \frac{16}{r^2} \int_M |\tau_b(\varphi)|^2 \, \theta \wedge (d\theta)^n
\end{aligned} \tag{75}$$

The third step: If we let $r \to \infty$, then $B_r(x_0)$ goes to M, and the right-hand side of Equation (75) goes to zero due to our assumptions that $E_{b,2}(\varphi) = \frac{1}{2} \int_M |\tau_b(\varphi)|^2 \, \theta \wedge (d\theta)^n < \infty$ and (M, g_θ) is complete. Thus, we have:

$$\int_M \sum_{j=1}^{2n} \left|\, \overline{\nabla}_{X_j} \tau_b(\varphi) \,\right|^2 \theta \wedge (d\theta)^n = 0 \tag{76}$$

This implies that:

$$\overline{\nabla}_X \tau_b(\varphi) = 0 \qquad\qquad \text{(for all } X \in H(M)\text{)} \tag{77}$$

The fourth step: Let us take a one form α on M defined by:

$$\alpha(X) = \begin{cases} \langle d\varphi(X), \tau_b(\varphi) \rangle, & (X \in H(M)) \\ 0 & (X = T) \end{cases}$$

Then, we have:

$$\begin{aligned}
&\int_M |\alpha| \, \theta \wedge (d\theta)^n = \int_M \left(\sum_{j=1}^{2n} \alpha(X_j)|^2 \right)^{\frac{1}{2}} \theta \wedge (d\theta)^n \\
&\leq \left(\int_M |d_b \varphi|^2 \, \theta \wedge (d\theta)^n \right)^{\frac{1}{2}} \left(\int_M |\tau_b(\varphi)|^2 \, \theta \wedge (d\theta)^n \right)^{\frac{1}{2}} \\
&= 2 \sqrt{E_b(\varphi) \, E_{b,2}(\varphi)} < \infty
\end{aligned} \tag{78}$$

where we put $d_b\varphi := \sum_{i=1}^{2n} d\varphi(X_i) \otimes X_i$,

$$|d_b\varphi|^2 = \sum_{i,j=1}^{2n} g_\theta(X_i, X_j)\, h(d\varphi(X_i), d\varphi(X_j)) = \sum_{i=1}^{2n} h(d\varphi(X_i), d\varphi(X_i))$$

and

$$E_b(\varphi) = \frac{1}{2}\int_M |d_b\varphi|^2\; \theta \wedge (d\theta)^n \tag{79}$$

Furthermore, let us define a C^∞ function $\delta_b\alpha$ on M by:

$$\delta_b\alpha = -\sum_{j=1}^{2n} (\nabla_{X_j}\alpha)(X_j) = -\sum_{j=1}^{2n} \left\{X_j(\alpha(X_j)) - \alpha(\nabla_{X_j}X_j)\right\} \tag{80}$$

where ∇ is the Tanaka–Webster connection. Notice that:

$$\begin{aligned}
\text{div}(\alpha) &= \sum_{j=1}^{2n} (\nabla^{g_\theta}_{X_j}\alpha)(X_j) + (\nabla^{g_\theta}_T\alpha)(T) \\
&= \sum_{j=1}^{2n} \left\{X_j(\alpha \circ \pi_H(X_j)) - \alpha \circ \pi_H(\nabla^{g_\theta}_{X_j}X_j)\right\} \\
&\quad + T(\alpha \circ \pi_H(T)) - \alpha \circ \pi_H(\nabla^{g_\theta}_T T) \\
&= \sum_{j=1}^{2n} \left\{X_j(\alpha(X_j)) - \alpha(\pi_H(\nabla^{g_\theta}_{X_j}X_j))\right\} = \sum_{j=1}^{2n} \left\{X_j(\alpha(X_j)) - \alpha(\nabla_{X_j}X_j)\right\} \\
&= -\delta_b\alpha
\end{aligned} \tag{81}$$

where $\pi_H : T_x(M) \to H_x(M)$ is the natural projection. We used the facts that $\nabla^{g_\theta}_T T = 0$ and $\pi_H(\nabla^{g_\theta}_X Y) = \nabla_X Y$ $(X, Y \in H(M))$ ([61], p. 37). Here, recall again that ∇^{g_θ} is the Levi–Civita connection of g_θ and ∇ is the Tanaka–Webster connection. Then, we have, for Equation (80),

$$\begin{aligned}
\delta_b\alpha &= -\sum_{j=1}^{2n} \left\{X_j \left\langle d\varphi(X_j), \tau_b(\varphi)\right\rangle - \left\langle d\varphi(\nabla_{X_j}X_j), \tau_b(\varphi)\right\rangle\right\} \\
&= -\sum_{j=1}^{2n} \left\{ \begin{array}{l} \langle \overline{\nabla}_{X_j}(d\varphi(X_j)), \tau_b(\varphi)\rangle + \langle\, d\varphi(X_j), \overline{\nabla}_{X_j}\tau_b(\varphi)\rangle \\ -\,\langle\, d\varphi(\nabla_{X_j}X_j), \tau_b(\varphi)\rangle \end{array} \right\} \\
&= -\left\langle \sum_{j=1}^{2n} \left\{\overline{\nabla}_{X_j}(d\varphi(X_j)) - d\varphi(\nabla_{X_j}X_j)\right\}, \tau_b(\varphi) \right\rangle = -|\tau_b(\varphi)|^2
\end{aligned} \tag{82}$$

We used Equation (77) to derive the last second equality of Equation (82). Then, due to Equation (82), we have for $E_{b,2}(\varphi)$,

$$\begin{aligned}
E_{b,2}(\varphi) &= \frac{1}{2}\int_M |\tau_b(\varphi)|^2\, \theta \wedge (d\theta)^n = -\frac{1}{2}\int_M \delta_b\alpha\;\theta \wedge (d\theta)^n \\
&= \frac{1}{2}\int_M \text{div}(\alpha)\, \theta \wedge (d\theta)^n = 0
\end{aligned} \tag{83}$$

In the last equality, we used Gaffney's theorem (*cf.*, Theorem 6, [23], p. 271, or [53]).

Therefore, we obtain $\tau_b(\varphi) \equiv 0$, *i.e.*, φ is pseudo-harmonic. $\square$

7. Biharmonic Hypersurfaces of Compact Symmetric Spaces

7.1. Characterization of Biharmonic Maps

In the first part of this section, we show a characterization theorem for an isometric immersion φ of an m-dimensional Riemannian manifold (M, g) into a Riemannian manifold (N, h) whose tension field $\tau(\varphi)$ satisfies that $\overline{\nabla}^{\perp}_X\tau(\varphi) = 0$ $(X \in \mathfrak{X}(M))$ is biharmonic. Let us recall the following theorem due to [2]:

Theorem 21. *Let $\varphi : (M^m, g) \to (N^n, h)$ be an isometric immersion. Assume that $\overline{\nabla}^{\perp}_X\tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$. Then, φ is biharmonic if and only if the following holds:*

$$\begin{aligned}
&-\sum_{j,k=1}^{m} h\big(\tau(\varphi), R^h(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k)\big)\, d\varphi(e_j)\\
&+\sum_{j,k=1}^{m} h(\tau(\varphi), B_\varphi(e_j, e_k))\, B_\varphi(e_j, e_k) - \sum_{j=1}^{m} R^h(\tau(\varphi), d\varphi(e_j))\, d\varphi(e_j) = 0
\end{aligned} \tag{84}$$

where R^h is the curvature tensor field of (N, h) given by $R^h(U, V)W = \nabla^h_U(\nabla^h_V W) - \nabla^h_V(\nabla^h_U W) - \nabla^h_{[U,V]}W$, $(U, V, W \in \mathfrak{X}(N))$, $B_\varphi(X, Y)$ $(X, Y \in \mathfrak{X}(M))$ is the second fundamental form of the immersion φ given by $B_\varphi(X, Y) = \nabla^h_{d\varphi(X)}d\varphi(Y) - d\varphi(\nabla^g_X Y)$ and $\{e_j\}$ is a locally-defined orthonormal frame field on (M, g).

We obtain:

Theorem 22. *Let $\varphi : (M^m, g) \to (N^n, h)$ be an isometric immersion. Assume that $\overline{\nabla}^{\perp}_X\tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$. Then, φ is biharmonic if and only if the following equations hold:*

(1) The tangential part:

$$\left(\sum_{k=1}^{m} R^h(\tau(\varphi), d\varphi(e_k))d\varphi(e_k)\right)^{\top} = 0 \tag{85}$$

(2) The normal part:

$$\left(\sum_{k=1}^{m} R^h(\tau(\varphi), d\varphi(e_k))\, d\varphi(e_k)\right)^{\perp} = \sum_{j,k=1}^{m} h(\tau(\varphi), B_\varphi(e_j, e_k))\, B_\varphi(e_j, e_k) \tag{86}$$

As a corollary of Theorem 2, we obtain:

Corollary 23. *Assume that the sectional curvature of the target space (N^n, h) is non-positive. Let $\varphi : (M^m, g) \to (N^n, h)$ be an isometric immersion whose tension field satisfies $\overline{\nabla}^{\perp}_X\tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$. Then, if φ is biharmonic, then it is harmonic.*

Remark 2. *Corollary 24 gives partial evidence to the generalized B.-Y. Chen's conjecture: every biharmonic isometric immersion into a non-positive curvature manifold must be harmonic. On the other hand, notice that the generalized B.-Y. Chen's conjecture was given by a counter example due to Y. Ou and L. Tang [27].*

7.2. Biharmonic Submanifolds in Einstein Manifolds

In the second part of this section, we apply Theorem 23 to an isometric immersion into an Einstein manifold (N^n, h) whose Ricci transform is denoted by ρ^h, by definition, $\rho^h(u) := \sum_{i=1}^n R^h(u, e'_i)e'_i$ $(u \in T_yN,\ y \in N)$, where $\{e'_i\}_{i=1}^n$ is a locally-defined orthonormal frame field on (N^n, h). Then, we obtain:

Theorem 24. *Assume that $\varphi : (M^m, g) \to (N^n, h)$ is an isometric immersion whose tension field $\tau(\varphi)$ that satisfies that $\overline{\nabla}^{\perp}_X \tau(\varphi) = 0$, and the target space (N, h) is an Einstein, i.e., the Ricci transform ρ^h of (N, h) satisfies $\rho^h = c\,\mathrm{Id}$ for some constant c. Then, φ is biharmonic if and only if the following holds:*

$$c\,\tau(\varphi) - \sum_{i=1}^p R^h(\tau(\varphi), \xi_i)\xi_i = \sum_{j,k=1}^m h(\tau(\varphi), B_\varphi(e_j, e_k))\, B_\varphi(e_j, e_k) \tag{87}$$

where $\{\xi_i\}_{i=1}^p$ is a local orthonormal frame field of the normal bundle corresponding to the immersion $\varphi : M \to N$.

In the following, we treat with a hypersurface $\varphi : (M^m, g) \to (N^n, h)$, *i.e.*, $p = 1$, and $m = \dim M = \dim N - 1 = n - 1$. In this case, we obtain the following theorem:

Theorem 25. *Assume that $\varphi : (M^m, g) \to (N^n, h)$ is an isometric immersion whose tension field $\overline{\nabla}^{\perp}_X \tau(\varphi) = 0$ $(\forall\ X \in \mathfrak{X}(M))$ and φ is a hypersurface, i.e., $m = n - 1$.*

(1) If φ is not harmonic, then φ is biharmonic if and only if:

$$\rho^h(\xi) = \|B_\varphi\|^2\, \xi, \tag{88}$$

where ρ^h is the Ricci transform of (N, h) and ξ is a unit normal vector field along φ.

(2) In particular, if (N, h) is an Einstein manifold, i.e., $\rho^h = c\,\mathrm{Id}$ and φ is not harmonic, then φ is biharmonic if and only if $\|B_\varphi\|^2 = c$.

Furthermore, we have:

Theorem 26. *Assume that $\varphi : (M, g) \to (N, h)$ is an isometric immersion into a Riemannian manifold (N, h) whose Ricci curvature is non-positive, $\dim M = \dim N - 1$, and $\overline{\nabla}^{\perp}_X \tau(\varphi) = 0$ for all C^∞ vector field X on M. Then, if φ is biharmonic, it is harmonic.*

Finally in this section, we give a criterion for which the condition $\overline{\nabla}^{\perp}_X \tau(\varphi) = 0$ $(\forall\ X \in \mathfrak{X}(M))$ holds:

Proposition 27. *Assume that $\varphi : (M^m, g) \to (N^n, h)$ is an isometric immersion with $m = \dim M = \dim N - 1 = n - 1$. Then, the following equivalence holds: The condition that $\overline{\nabla}^{\perp}_X \tau(\varphi) = 0$ $(\forall\ X \in \mathfrak{X}(M)$ holds if and only if the mean curvature $\mathbf{H} = \frac{1}{m}\sum_{i=1}^m H_{ii}$ is constant on M. Here, $B_\varphi(e_i, e_j) = H_{ij}\,\xi$, and ξ is a unit normal vector field along φ.*

Summarizing Theorems 26 and 27 and Proposition 28, we obtain:

Corollary 28. *Let $\varphi : (M^m, g) \to (N^n, h)$ be an isometric immersion. Assume that $m = \dim M = \dim N - 1 = n - 1$ and the mean curvature of φ, $\mathbf{H} = \frac{1}{m}\sum_{i=1}^m H_{ii} = \frac{1}{m}\tau(\varphi)$, is constant. Then, the following hold:*

(1) Assume that $\mathbf{H} \neq 0$, i.e., φ is not harmonic. Then, it holds that φ is biharmonic if and only if $\rho^h(\xi) = \|B_\varphi\|^2\,\xi$, where ρ^h is the Ricci transform of (N, h), ξ is a unit normal vector field along φ and B_φ is the second fundamental form of φ.

(2) Assume that $\mathbf{H} \neq 0$ and (N, h) is Einstein, i.e., $\rho^h = c\,\mathrm{Id}$ for some constant c. Then, φ is biharmonic if and only if $\|B_\varphi\|^2 = c$.

(3) Assume that $\mathbf{H} \neq 0$ and the Ricci curvature of (N, h) is non-positive. Then, if φ biharmonic, it is harmonic.

Due to the above, we have a classification of homogeneous hypersurfaces in compact symmetric spaces. Refer to our recent paper [32]. Finally, we raise a problem related to this topic (*cf.*, [32]):

Problem 29. *Let (N^n, h) be an Einstein manifold with the Ricci transform $\rho^h = c\,\mathrm{Id}$ for some constant $c > 0$ and admitting a low cohomogeneity action of some compact Lie group H. Determine all of the H-orbits, which are harmonic or biharmonic.*

8. Conclusions

We have given a survey on recent progresses on Chen's conjecture and the generalized Chen's conjecture, the outlines of the proofs on the L^2 rigidity theorems of biharmonic maps due to N. Nakauchi and myself, and also its CR analogue. Then, we have given a survey on constructions and classification of harmonic maps and biharmonic maps into compact Lie groups or compact symmetric spaces. Finally, we have explained how to construct biharmonic hypersurfaces of compact symmetric spaces. We hope for young geometers to read this survey, and to give new results by extending our results, and to attack and obtain answers of the above problem 29.

Conflicts of Interest

The author declares no conflict of interest.

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