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# On Charge Conjugation, Chirality and Helicity of the Dirac and Majorana Equation for Massive Leptons

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**Abstract:** We revisit the charge-conjugation operation for the Dirac equation in its chiral representation. A new decomposition of the Dirac spinor field is suggested and achieved by means of projection operators based on charge conjugation, which is discussed here in a non-standard way. Thus, two separate two-component Majorana-type field equations for the eigenfields of the charge-conjugation operator are obtained. The corresponding free fields are entirely separated without a gauge field, but remain mixed and coupled together through an electromagnetic field term. For fermions that are charged and, thus, subjected to the gauge field of electrodynamics, these two Majorana fields can be reassembled into a doublet, which is equivalent to a standard four-component Dirac spinor field. In this way, the Dirac equation is retained in a new guise, which is fully equivalent to that equation in its chiral form.

**Keywords:** charge conjugation; Majorana fields; chiral symmetry

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## 1. Introduction

According to the canonical standard model of elementary particle physics, leptons and quarks come in three flavors, are massless and, thus, obey chiral symmetry, but then, they acquire mass through the Higgs [1,2] mechanism (see, e.g., the text book [3] for details). The Dirac equation [4] is fundamental in all of this and well understood; however, the nature of the neutrinos involved remains less clear. Are they Dirac fermions or massive Majorana [5,6] particles? In the past, neutrinos were often described by the massless Weyl [7] equations involving only two-component Pauli [8] spinors. However, since convincing empirical evidence [9] for the finite neutrino masses and the associated neutrino

oscillations [10] had been found in the past few decades, massive neutrinos have been discussed, and furthermore, another very massive neutrino species (or a sterile one) has been considered in four-neutrino models [11], for example to explain the masses of the light neutrinos by the see-saw mechanism [10].

Clearly, any realistic extension of the standard model (SM) will have to consider finite neutrinos' masses. Therefore, the Majorana equation with various mass terms [12,13] has gained strong attention, in particular in its complex two-component version (see [14–18] and the recent review by Dreiner *et al.* [19] on two-component spinor techniques), and been used in modern quantum field theory for the description of massive neutrinos. The physical state of affairs in this field and its research perspectives (as of 2006) were described comprehensively in a review by Mohapatra and Smirnov [20].

The purpose of the present paper is to show that the Dirac equation for a massive and charged fermion can be rewritten in terms of two two-component Majorana-like equations, which respectively govern two independent Pauli-spinor fields. In the case of charged leptons, they become coupled in the presence of an electromagnetic gauge field. The related field equations are developed on the basis of the chiral Dirac equation. Their derivation employs projection operator techniques related to the charge conjugation operator, which is considered here in a new way following the recent work by Marsch [16,17].

The paper is organized as follows. We first discuss the relevant aspects of the Dirac equation in chiral representation and address chirality, helicity and charge conjugation  $\mathcal{C}$ , as well as the properties of their associated projection operators. Then, the eigenfields of  $\mathcal{C}$  are derived and shown to be expressible in terms of two-component spinor fields, which obey Majorana-like equations, including a mass term. Finally, we show that a massive charged fermion (electron and positron) can be arranged in a doublet governed by what we may call the Dirac–Majorana equation. With a short conclusion section, we close the paper.

## 2. The Dirac Equation and Chiral Symmetry

### 2.1. Weyl Representation

In this tutorial introductory subsection, we first consider the Dirac equation in its chiral or Weyl [7] representation. The subsequent paragraphs provide the necessary material for the discussion in the following sections. We use standard [3] symbols, notations and definitions and conventional units of  $\hbar = c = 1$ , with the covariant four-momentum operator denoted  $P_\mu = (E, -\mathbf{p}) = i\partial_\mu = i(\partial/\partial t, \partial/\partial \mathbf{x})$ , which acts on the spinor wave function  $\psi(\mathbf{x}, t)$ . The particle mass is  $m$ . We may also sometimes abbreviate the contravariant space-time location vector  $x^\mu = (t, \mathbf{x})$  simply as  $x$ . The Dirac [4] equation in its standard form reads:

$$i\gamma^\mu \partial_\mu \psi = m\psi \quad (1)$$

The four-vector  $\gamma^\mu$  consists of the four Dirac gamma matrices that come in various representations [3]. We use here the chiral representation in which the gamma matrices may be written as follows:

$$\gamma^\mu = (\beta_{12}, \gamma\boldsymbol{\sigma}) = \left( \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) 1_2, \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \boldsymbol{\sigma} \right) \quad (2)$$

Here,  $1_2$  stands for the  $2 \times 2$  unit matrix, and the two-dimensional matrices  $\beta$  and  $\gamma$  are defined by Equation (2) implicitly. The three associated Pauli [8] matrices have their standard form given by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3)$$

Together with the unit matrix, the Pauli matrices may be combined in the four-vector form,  $\sigma_{\pm}^{\mu} = (1_2, \pm\sigma)$ , which is used later.

Now, we can also introduce the chiral matrix  $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ , which in chiral representation, takes the form:

$$\gamma^5 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \quad (4)$$

and obeys  $(\gamma^5)^2 = 1_4$ , where  $1_4$  stands for the  $4 \times 4$  unit matrix. By use of  $\gamma^5$ , the well-known projection operators can be defined as  $P_{R,L} = \frac{1}{2}(1_4 \pm \gamma^5)$ , which are idempotent and represent a decomposition of the identity operator. In matrix form, we obtain:

$$P_L = \begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1_2 \end{pmatrix} \quad (5)$$

With their help, any Dirac spinor field can be decomposed into its right- and left-chiral component,  $\psi = P_R\psi + P_L\psi = \psi_R + \psi_L$ .

Finally, we may introduce a gauge field, which is obtained by the minimal substitution. We first consider the Abelian gauge field  $A_{\mu}(x)$  of electrodynamics. Conventionally, this is inserted into the Dirac field Equation (1) according to the minimal coupling principle, *i.e.*, by replacing the time-space derivative  $\partial_{\mu}$  by the covariant derivative:

$$D_{\mu} = \partial_{\mu} + iqA_{\mu} \quad (6)$$

The particle charge is denoted by  $q$ . The resulting Dirac equation reads:

$$i\gamma^{\mu}(\partial_{\mu} + iqA_{\mu})\psi = m\psi \quad (7)$$

By its definition,  $\gamma^5$  anticommutes with all gamma matrices, which means  $\{\gamma^5, \gamma^{\mu}\} = 0$ , where the curly brackets denote the anticommutator. Consequently, we have  $\gamma^{\mu}P_{R,L} = P_{L,R}\gamma^{\mu}$ , and by using this, we obtain the coupled Dirac equations for the right- and left-chiral field,

$$i\gamma^{\mu}(\partial_{\mu} + iqA_{\mu})\psi_{R,L} = m\psi_{L,R} \quad (8)$$

The interchanged indices for the mass term indicate that it breaks chiral symmetry. Yet, note that the gauge field coupling term has no effect on the chiral decomposition of the Dirac field.

## 2.2. Eigenfunctions

We here derive and present the eigenfunctions of the Dirac equation in an unconventional, but convenient way using the chiral representation, in which the Dirac equation may be written as follows:

$$i \left( \beta 1_2 \frac{\partial}{\partial t} + \gamma \boldsymbol{\sigma} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \psi = m\psi \quad (9)$$

To derive its eigenfunctions, we make the usual plane-wave ansatz for the particles (negative frequency):

$$\psi_P(\mathbf{x}, t) = v(\mathbf{p}) \exp(-iEt + i\mathbf{p} \cdot \mathbf{x}) \quad (10)$$

and antiparticles (positive frequency):

$$\psi_A(\mathbf{x}, t) = w(\mathbf{p}) \exp(iEt - i\mathbf{p} \cdot \mathbf{x}) \quad (11)$$

The spatial differentiation in Equation (9) yields a term involving the spin-related helicity operator, and therefore, it is convenient to use its eigenfunctions. They obey the eigenvalue equation of the helicity operator in Fourier space:

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{p}})u_{\pm}(\hat{\mathbf{p}}) = \pm u_{\pm}(\hat{\mathbf{p}}) \quad (12)$$

These two eigenvectors depend only on the momentum unit vector  $\hat{\mathbf{p}} = \mathbf{p}/p = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$  and can be written as:

$$u_+(\hat{\mathbf{p}}) = \begin{pmatrix} \cos \frac{\theta}{2} e^{-\frac{i}{2}\phi} \\ \sin \frac{\theta}{2} e^{\frac{i}{2}\phi} \end{pmatrix}, \quad u_-(\hat{\mathbf{p}}) = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-\frac{i}{2}\phi} \\ \cos \frac{\theta}{2} e^{\frac{i}{2}\phi} \end{pmatrix} \quad (13)$$

in which the half-angles of  $\theta$  and  $\phi$  appear. According to Equation (13), we have  $u_{\pm}^{\dagger}(\hat{\mathbf{p}})u_{\pm}(\hat{\mathbf{p}}) = 1$  and  $u_{\mp}^{\dagger}(\hat{\mathbf{p}})u_{\pm}(\hat{\mathbf{p}}) = 0$ . The dagger denotes, as usual, the transposed and complex conjugated vector, respectively, matrix. Thus, the eigenvectors for the same  $\hat{\mathbf{p}}$  are orthogonal to each other and normalized to unity. They further obey the relation  $u_{\pm}(-\hat{\mathbf{p}}) = iu_{\mp}(\hat{\mathbf{p}})$ . The trace matrix elements of the spin operator between these two eigenvectors read:

$$u_{\pm}^{\dagger}(\hat{\mathbf{p}})\boldsymbol{\sigma}u_{\pm}(\hat{\mathbf{p}}) = \pm \hat{\mathbf{p}} \quad (14)$$

The four-component spinors,  $v$  and  $w$ , can now be decomposed into:

$$v_s(\mathbf{p}) = \varphi_s(p)u_s(\hat{\mathbf{p}}) \quad (15)$$

$$w_s(\mathbf{p}) = \chi_s(p)u_s(\hat{\mathbf{p}}) \quad (16)$$

whereby we added the subscript  $s$ , which indicates the helicity eigenvalue that can be  $s = \pm 1$ , and thus,  $s^2 = 1$ . The above two-component spinors obey the matrix equations:

$$(\beta E - \gamma sp - m)\varphi_s(p) = 0 \quad (17)$$

$$(\beta E - \gamma sp + m)\chi_s(p) = 0 \quad (18)$$

for particles (top) and antiparticles (bottom), having opposite signs of their mass terms. Nontrivial solutions require the determinant of the above  $2 \times 2$  matrices to vanish, which yields the positive eigenvalue:

$$E(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2} \quad (19)$$

The negative root related to antiparticles must not be considered explicitly, as it is already implied by the ansatz (11). The energy does not depend on the spin index  $s$ . When solving the Equations (17) and (18) for their eigenfunctions, we obtain:

$$\varphi_s(p) = \frac{1}{\sqrt{2m(E(p) + sp)}} \begin{pmatrix} m \\ E(p) + sp \end{pmatrix} \quad (20)$$

$$\chi_s(p) = \frac{1}{\sqrt{2m(E(p) - sp)}} \begin{pmatrix} -E(p) + sp \\ m \end{pmatrix} \quad (21)$$

These spinors are both real and can be normalized in the standard Lorentz-invariant way by introducing the conjugate spinor  $\bar{\varphi} = (\beta\varphi)^T$ , where superscript  $T$  denotes the transposition. It is straightforward to show that  $\bar{\varphi}_s\varphi_s = 1$ ,  $\bar{\chi}_s\chi_s = -1$ ,  $\bar{\varphi}_s\chi_s = 0$ , and  $\bar{\chi}_s\varphi_s = 0$ , where the dispersion relation (19) has been used. Finally, we obtain the normalized orthogonal particle and antiparticle spinor fields:

$$\psi_P(\mathbf{x}, t; \mathbf{p}, s) = \varphi_s(p)u_s(\hat{\mathbf{p}}) \exp(-iE(p)t + i\mathbf{p} \cdot \mathbf{x}) \quad (22)$$

$$\psi_A(\mathbf{x}, t; \mathbf{p}, s) = \chi_s(p)u_s(\hat{\mathbf{p}}) \exp(iE(p)t - i\mathbf{p} \cdot \mathbf{x}) \quad (23)$$

For opposite helicities, the orthogonality of  $\psi_A$  and  $\psi_P$  is ensured by the orthogonality of the corresponding helicity eigenvectors. Finally, we note the following symmetry properties:

$$\gamma\chi_s = \varphi_{-s}, \quad \gamma\varphi_{-s} = -\chi_s \quad (24)$$

which are used subsequently in connection with charge conjugation.

### 3. The Dirac Equation and Charge Conjugation

#### 3.1. Projection Operators

According to standard procedures [3], the charge conjugation symmetry of the Dirac equation is given by the operator  $\mathcal{C}$ , which has the effect that  $\psi$  transforms into  $\psi^{\mathcal{C}} = \mathcal{C}\psi = -i\gamma_y\psi^*$ , where the asterisk means complex conjugation and the phase factor in front is apt convention.

The charge conjugation operation can be more concisely written by the help of a new operator named  $\tau = -i\sigma_y\mathbb{C}$ , where  $\mathbb{C}$  denotes the complex conjugation operator, which transmutes a complex number  $z$  into  $z^*$ . This operator turned out to be very convenient in the treatment of the complex two-component Majorana equation [16,17]. It is antiunitary and obeys  $\tau^\dagger = -\tau = \tau^{-1}$  and  $\tau^2 = -1$ . Furthermore, it anticommutes with all Pauli matrices,  $\{\tau, \boldsymbol{\sigma}\} = \mathbf{0}$ , which means it flips the spin by interchanging the sign of the Pauli matrix three-vector. Furthermore, with the above eigenfunctions of the helicity operator, we find the important property:

$$\tau u_s(\hat{\mathbf{p}}) = s u_{-s}(\hat{\mathbf{p}}) \quad (25)$$

Using the above operator  $\tau$ , we can now define the charge conjugation [3] by  $\mathcal{C} = \delta$  in the chiral representation, in which it is given by the matrix operator:

$$\delta = \gamma\tau = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix} \quad (26)$$

It is unitary and its own inverse, and its square yields the unit matrix:  $\delta^2 = 1_4$ ,  $\delta^\dagger = \delta^{-1} = \delta$ . These properties qualify the charge conjugation operator for the construction of a projector in the following form,  $P_\pm = \frac{1}{2}(1 \pm \delta)$ , which in matrix form reads:

$$P_\pm = \frac{1}{2} \begin{pmatrix} 1 & \pm\tau \\ \mp\tau & 1 \end{pmatrix} \quad (27)$$

Using the properties of  $\tau$ , it is straightforward to validate that  $P_\pm$  is idempotent. As the operator  $\mathbb{C}$  changes  $i$  into its negative, we have  $P_\pm i = iP_\mp$ . Concerning the Dirac matrices, we find  $[P_\pm, i\gamma^\mu] = 0$ , where the square brackets denote the commutator. This is nontrivial for  $\gamma^0$ , but readily verified and also immediately obtained for the three spatial components of gamma by noting that  $[\tau, i\sigma] = 0$ . With the help of  $P_\pm$ , any Dirac spinor field can now be decomposed [13] into two orthogonal charge-conjugated components,

$$\psi = P_+\psi + P_-\psi = \psi_+ + \psi_- \quad (28)$$

obeying  $\delta\psi_\pm = \pm\psi_\pm$ . This decomposition can be made in a Lorentz-invariant way. As  $[\delta, i\gamma_\mu] = 0$ , one finds that the projection operator  $P_\pm$  commutes with the operator of the Lorentz transformation.

It is then straightforward to decompose by projection the Dirac equation including the electromagnetic gauge field into two equations for the eigenfields of  $\mathcal{C}$  or  $\delta$  with the result:

$$i\gamma^\mu(\partial_\mu\psi_\pm + iqA_\mu\psi_\mp) = m\psi_\pm \quad (29)$$

Contrary to the chiral decomposition (8), the present decomposition decouples the two fermion fields in their mass term; however, not unexpectedly, it couples the two fields (of opposite eigenvalue  $\pm 1$  of  $\delta$ ) via their common gauge four-vector potential. If a fermion field carries no electric charge, *i.e.*,  $q = 0$ , like for the neutrino, the two fields decouple entirely, and either one of them may be chosen, where the requirement  $\psi = \mathcal{C}\psi$ , *i.e.*,  $\psi_- = 0$ , seems to be the most natural [18] choice. We may rename the so-constrained Dirac field as  $\psi_0$ , indicating the zero charge by the subscript. Such a field is reduced and has only two independent components.

The above formal decomposition of the Dirac spinor field can be illustrated explicitly when use is made of the four known free eigenfields in the chiral representation, as given in Equations (22) and (23), which we here add up with equal weight to form  $\hat{\psi}$  as a superposition. For the sake of simplicity, we only keep at this point the spin index ( $s = \pm 1$ ), and then can write:

$$\hat{\psi} = \psi_P(+) + \psi_P(-) + \psi_A(+) + \psi_A(-) \quad (30)$$

for any given time, location and momentum. Using the properties of  $\tau$  and  $\gamma$  as stated in Equations (24) and (25), one finds the effect of  $\mathcal{C}$  or  $\delta$  operating on the eigenfunctions with the key result:

$$\delta\psi_P(s) = -s\psi_A(-s), \text{ and } \delta\psi_A(s) = s\psi_P(-s) \quad (31)$$

Correspondingly, one obtains:

$$\delta\hat{\psi} = -\psi_A(-) + \psi_A(+) + \psi_P(-) - \psi_P(+) \quad (32)$$

Taking the difference, respectively the sum, of Equations (30) and (32), and exploiting the definition of the projection operator, one obtains:

$$\hat{\psi}_+ = \psi_P(-) + \psi_A(+) \quad (33)$$

$$\hat{\psi}_- = \psi_P(+) + \psi_A(-) \quad (34)$$

This result shows that these two orthogonal fields consist, so to speak, of “half” of the electron and positron, in so far as they combine a particle state with a state of its antiparticle, yet of opposite helicity.

Let us continue by discussing some further properties of the charge conjugation and chirality operators. First, by their definitions and according to Equations (4) and (26), these two Hermitian operators do not commute, but we have:

$$\{\gamma^5, \delta\} = 0 \quad (35)$$

and thus, they cannot have common eigenfunctions. Of course, by definition, the left-chiral and right-chiral Weyl fields obey:

$$\gamma^5\psi_{R,L} = \pm\psi_{R,L} \quad (36)$$

and similarly, the charge-conjugated fields obey:

$$\delta\psi_{\pm} = \pm\psi_{\pm} \quad (37)$$

Moreover, because of Equation (35), we find that  $\delta P_{R,L} = P_{L,R}\delta$  and  $\gamma^5 P_{\pm} = P_{\mp}\gamma^5$ . Thus, chirality and charge conjugation are intimately linked.

We may generally write any Dirac four-component spinor field in terms of two two-component Pauli fields,

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad (38)$$

Then, the Weyl fields (eigenfields of  $\gamma^5$ ) and the eigenfields of  $\mathcal{C}$  can be written as:

$$\psi_L = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R = \begin{pmatrix} 0 \\ \chi \end{pmatrix} \quad (39)$$

$$\psi_{\pm} = \frac{1}{2} \begin{pmatrix} \phi \pm \tau\chi \\ \chi \mp \tau\phi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ \mp\tau \end{pmatrix} (\phi \pm \tau\chi) \quad (40)$$

Note that the two chiral fields are reduced by two degrees of freedom as compared to the full Dirac field, which is obvious from Equation (39). The charge-conjugated fields also have two degrees of freedom less than  $\psi$ , as they consist of the linear combination of its independent upper and lower components, as given in Equation (40), and only their sum gives the full  $\psi$ .

### 3.2. Links between the Equations as Obtained by Projection through Chirality- or Charge-Conjugation

Let us now consider again the Dirac equation, including an electromagnetic gauge field, and write it in the chiral decomposition according to Equation (39). The coupled set of two-component chiral spinor fields reads:

$$\sigma_+^\mu (i\partial_\mu - qA_\mu) \chi = m\phi \quad (41)$$

$$\sigma_-^\mu (i\partial_\mu - qA_\mu) \phi = m\chi \quad (42)$$

The sign in front of the charge is the same in both equations. As a consequence of the previous discussion, we now use the positive and negative eigenfields (40) of  $\mathcal{C}$  or  $\delta$  and insert them into the Dirac Equation (29), which will result in two coupled fields for linear combinations of  $\phi$  and  $\chi$ . We then obtain four two-component equations, which must be satisfied together. To be explicit for the sake of clarity, we quote the full result (omitting the common factor of 1/2, which cancels), which is:

$$\sigma_+^\mu (i\partial_\mu(\chi - \tau\phi) - qA_\mu(\chi + \tau\phi)) = m(\phi + \tau\chi) \quad (43)$$

$$\sigma_-^\mu (i\partial_\mu(\phi + \tau\chi) - qA_\mu(\phi - \tau\chi)) = m(\chi - \tau\phi) \quad (44)$$

$$\sigma_+^\mu (i\partial_\mu(\chi + \tau\phi) - qA_\mu(\chi - \tau\phi)) = m(\phi - \tau\chi) \quad (45)$$

$$\sigma_-^\mu (i\partial_\mu(\phi - \tau\chi) - qA_\mu(\phi + \tau\chi)) = m(\chi + \tau\phi) \quad (46)$$

When operating with  $\tau$  on Equations (44) and (46), we find these equations to be identical with Equations (43) and (45). Therefore, out of four, only two equations are independent. If we take the sum of Equations (43) and (45) and of Equations (44) and (46), we retain the chiral Equations (41) and (42). If we take the corresponding differences, operating from left with  $\tau$  on both equations, then we obtain again the two chiral Equations (41) and (42). Hereby, use was made of  $\tau^\dagger = -\tau$ , *i.e.*, that  $\tau$  is an anti-Hermitian operator, and that  $\tau^2 = -1_2$ .

We may now define, based on the two chiral fields, two new charge-conjugation-related fields that we name  $\chi_\pm = \phi \pm \tau\chi$  and  $\phi_\pm = \chi \mp \tau\phi$ , corresponding to the combinations that appear in the above four equations, and in Equation (40). We thus obtain two sets of still coupled (via the gauge field) Majorana-type (because of the  $\tau$  in the mass term) equations, which read as follows:

$$\sigma_-^\mu (i\partial_\mu\chi_\pm - qA_\mu\chi_\mp) = \mp m\tau\chi_\pm \quad (47)$$

$$\sigma_+^\mu (i\partial_\mu\phi_\pm - qA_\mu\phi_\mp) = \pm m\tau\phi_\pm \quad (48)$$

Note that the respective plus and minus fields are not decoupled owing to the gauge field term, but certainly are so for zero charge. Moreover, the equations for  $\phi_\pm$  and  $\chi_\pm$  are equivalent, since, by definition,  $\phi_\pm = \mp\tau\chi_\pm$ . Thus, by operating with  $\mp\tau$  on Equation (47), we get Equation (48), and *vice versa*. Therefore, we only need consider one of the two sets, corresponding to the left-chiral version for the phi-fields and right-chiral form for the chi-fields. This illustrates that the sign of the spin  $\sigma$ , *i.e.*, helicity and chirality, and charge conjugation (apparently meaning here the replacement of  $\sigma_+^\mu$  by  $\sigma_-^\mu$  by operation of  $\tau$ ) are intimately linked.

### 3.3. Majorana Equations

With the two Equations (47) or (48), we decomposed the single four-component Dirac field (which includes the electron and positron) into two two-component fields, which are not linked by their mass terms for which they yet have opposite signs and which also are, like in the original Equation (29), linked by the common gauge-field term. Therefore, solving either the set of coupled (via the mass term) chiral Equations (41) and (42) or the above set of coupled (via the gauge field) Majorana-type equations gives the complete solution of the Dirac equation of a charged massive lepton in an electromagnetic field.

The original and charge-conjugated Dirac fields  $\psi$  and  $\psi_{\pm}$  are in terms of the fields  $\chi_{\pm}$  and  $\phi_{\pm}$  according to Equations (38) and (40) given by either:

$$\psi_{\pm} = \begin{pmatrix} \pm\tau\phi_{\pm} \\ \phi_{\pm} \end{pmatrix}, \text{ or } \psi_{\pm} = \begin{pmatrix} \chi_{\pm} \\ \mp\tau\chi_{\pm} \end{pmatrix} \quad (49)$$

whereby, by definition, the field obeys  $\mathcal{C}\psi_{\pm} = \pm\psi_{\pm}$ . This provides a general decomposition of the Dirac fermion field  $\psi$  and differs from the usual Majorana field obtained by imposing the condition that  $\psi_{-} = 0$  and, thus,  $\psi = \psi_{+}$ . Furthermore, once we know the field  $\psi_{\pm}$ , we can readily obtain either  $\chi_{\pm}$  from its upper component, or  $\phi_{\pm}$  from its lower component, simply by reading them off from of the above Equation (49).

To give an important example, when considering in the chiral representation the four free eigenfields (22) and (23) and exploiting the relations in Equation (31), we find for their charge-conjugated versions the results:

$$\psi_{P\pm}(s) = \frac{1}{2} (\psi_P(s) \mp s\psi_A(-s)) \quad (50)$$

$$\psi_{A\pm}(s) = \frac{1}{2} (\psi_A(s) \pm s\psi_P(-s)) \quad (51)$$

Referring to the general definition of  $\psi_{\pm}$  in Equation (49) and inserting the solutions (22) and (23) into the above equations, we find the two-component spinor solutions as follows:

$$\chi_{P\pm}(s) = \frac{m u_s \exp(-ixp) \pm (E + sp) s u_{-s} \exp(ixp)}{\sqrt{2E(E + sp)}} \quad (52)$$

$$\chi_{A\pm}(s) = \frac{\pm m s u_{-s} \exp(-ixp) - (E - sp) u_s \exp(ixp)}{\sqrt{2E(E - sp)}} \quad (53)$$

Here, we use  $xp = E(p)t - \mathbf{p} \cdot \mathbf{x}$  as the abbreviation and, for short,  $E = E(p)$  and  $u_s = u_s(\hat{\mathbf{p}})$ . Furthermore, we renormalized the spinor, such that when calculating the product  $\chi_{J\pm}^{\dagger}(s)\chi_{J\pm}(s)$ , one finds it to be unity, so that the spinor  $\chi_{J\pm}(s)$  is normalized; however,  $\chi_{J+}(s)$  is not orthogonal to  $\chi_{J-}(s)$ , where J stands for the subscript A or P. Finally, we remind the reader that  $\tau u_s = s u_{-s}$ .

Close inspection of Equations (52) and (53) shows that, what looks like eight, actually are only four independent equations, since one finds the symmetry relations:

$$\chi_{P\pm}(s) = (\mp s)\chi_{A\pm}(-s), \text{ and } \chi_{A\pm}(s) = (\pm s)\chi_{P\pm}(-s) \quad (54)$$

Therefore, there is no cross-coupling between the  $\pm$  indices, which correspond to the eigenvalues of the charge-conjugation  $\mathcal{C}$  or its operator  $\delta$  in the chiral representation of Dirac's equation. Consequently,

for both helicity eigenvalues  $s = \pm 1$ , we can associate the plus-minus solutions either with particles or antiparticles,

$$\chi_{\pm}(s) = \chi_{P\pm}(s), \text{ or } \chi_{\pm}(s) = \chi_{A\pm}(s) \quad (55)$$

This interpretation is consistent with Equation (47), in which the two-component free spinor fields  $\chi_+$  and  $\chi_-$  are fully decoupled from each other for a vanishing electromagnetic field. Finally, we quote again the two sets of possible solutions explicitly as:

$$\chi_{\pm}(\mathbf{x}, t; \mathbf{p}, s) = \frac{m u_s(\hat{\mathbf{p}}) \exp(-ixp) \pm (E(p) + sp) s u_{-s}(\hat{\mathbf{p}}) \exp(ixp)}{\sqrt{2E(p)(E(p) + sp)}} \quad (56)$$

or alternatively:

$$\chi_{\pm}(\mathbf{x}, t; \mathbf{p}, s) = \frac{\pm m s u_{-s}(\hat{\mathbf{p}}) \exp(-ixp) - (E(p) - sp) u_s(\hat{\mathbf{p}}) \exp(ixp)}{\sqrt{2E(p)(E(p) - sp)}} \quad (57)$$

These two-component spinor fields are normalized to unity. Direct insertion shows that both pairs solve the two independent complex Majorana equations:

$$\sigma_-^{\mu} i \partial_{\mu} \chi_{\pm} = \mp m \tau \chi_{\pm} \quad (58)$$

Upon insertion of these solutions into  $\psi_{\pm}$ , as given in Equation (49), we are provided with the two charge-conjugated solutions of the Dirac equation in its Weyl or chiral representation. Similar calculations can be done for the  $\phi_{\pm}$  fields and deliver equivalent results. These solutions correspond to a superposition of half of the electron and positron, yet with opposite helicity, and are valid for the two spin orientations or helicities given by  $s = \pm 1$ .

Now, let us consider an uncharged particle, like the massive neutrino. With  $q = 0$ , we get from Equations (47) and (48) the same equation for  $\chi_+$  and  $\chi_-$ , with the exception of the opposite sign of the mass term. Taking one of the fields to be zero, namely  $\chi_-$ , and the other non-zero, namely  $\chi_+$ , whereby we name that field  $\chi_0$ , indicating zero charge, we obtain the two-component complex Majorana equation, which for the massive neutrino, reads:

$$\sigma_+^{\mu} i \partial_{\mu} \chi_0 = -m \tau \chi_0 \quad (59)$$

Equivalently, we obtain:

$$\sigma_+^{\mu} i \partial_{\mu} \phi_0 = m \tau \phi_0 \quad (60)$$

The solutions are connected by the relation  $\phi_0 = -\tau \chi_0$ . Note the advantage of the two-component Pauli spinor-field description as given by the three Equations (47), (48) and (59) over the four-component description using the Dirac matrices, for which the projection constraints are implicit, but still have to be obeyed, whereas in the two-component theory, they have already been incorporated by the reduction of the spinor fields from four to two components.

### 3.4. Electromagnetic Gauge

As discussed in the previous section, we can describe the coupled charge-conjugated Dirac fields in terms of two-component Majorana-like fields, which obey the separate equations for the two free eigenfields of the charge conjugation operator, but become coupled through the electromagnetic gauge field. This doublet is associated with the  $\sigma_x$ -like  $2 \times 2$  charge matrix:

$$\varepsilon = q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (61)$$

Inspection of the mass term suggest introducing another  $\sigma_z$ -like  $2 \times 2$  mass matrix:

$$\mu = m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (62)$$

The two fields  $\chi_+$  and  $\chi_-$  (and  $\phi_+$  and  $\phi_-$ ) can now be assembled into a doublet spinor field  $\Xi$  (and  $\Phi$ ), which reads:

$$\Xi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}, \quad \text{and} \quad \Phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \quad (63)$$

and which obeys the Dirac–Majorana equation:

$$\sigma_-^\mu (i\partial_\mu - \varepsilon A_\mu) \Xi = -\mu\tau\Xi \quad (64)$$

$$\sigma_+^\mu (i\partial_\mu - \varepsilon A_\mu) \Phi = \mu\tau\Phi \quad (65)$$

Essentially, these equations are just other forms of the Dirac equation coming in a new guise, with the main difference (and perhaps complication) being that the operator  $\tau$  appears at the mass term as in the complex Majorana equation. For zero mass and  $A_\mu = 0$ , Equations (64) and (65) then are equivalent to a doublet of two Weyl fields having the opposite chirality. We recall that  $\tau$  and  $i$  anticommute, and also,  $[\varepsilon, \mu] = 0$ . Therefore, we have  $[i\varepsilon, \tau\mu] = 0$ . Although it seems less obvious, we can readily establish gauge invariance by noting that Equations (64) and (65) are invariant under a phase transformation of, for example, the spinor field  $\Phi$  of the kind:

$$\Phi \rightarrow \Phi' = \exp(i\varepsilon\lambda(x))\Phi \quad (66)$$

with some real gauge function  $\lambda(x)$  ( $x$  stands as the abbreviation for  $x^\mu$ ). We may write the phase factor out explicitly, which is in fact a matrix that reads:

$$\exp(i\varepsilon\lambda(x)) = 1_2 \cos(\lambda(x)) + i\varepsilon \sin(\lambda(x)) \quad (67)$$

This complex phase factor slips through the mass term and, then, can finally be canceled from Equation (65) if the gauge field is redefined, as usual, by  $A'_\mu = A_\mu + \partial_\mu\lambda(x)$ . Finally, we note that Equation (65) is, as it should be, invariant against charge conjugation, which here means  $\varepsilon$  is replaced by  $-\varepsilon$ . If we introduce the anti-diagonal  $\sigma_y$ -like matrix:

$$\kappa = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (68)$$

this is found to obey  $\kappa = \kappa^{-1}$  and  $\kappa^2 = 1_2$ , and one can readily see that it anticommutes with  $\varepsilon$  and  $\mu$ , but commutes with the mass term, as  $\tau$  also anticommutes with  $\kappa$ . Therefore, if  $\Phi(\varepsilon)$  solves Equation (65), then  $\Phi(-\varepsilon) = \kappa\Phi(\varepsilon)$  solves the charge-conjugated equation.

#### 4. Conclusions

Starting from the chiral Dirac equation, we have shown that its spinor field can be decomposed into the two eigenfields  $\psi_{\pm}$  of the charge-conjugation operator  $\mathcal{C}$ , for which appropriate projectors have been defined in Equation (27). These eigenfields are constrained Dirac spinor fields and are fully determined by the two complex two-component Majorana fields, as they are given in the left-chiral version by the solution of Equation (47) or the right-chiral version by Equation (48). The two equations are separated for the plus and minus sign and, unlike the chiral Weyl fields, are not linked by their respective mass terms, which have opposite signs. However, they become closely linked via the electromagnetic gauge field. The resulting Dirac–Majorana equations in the form of either Equations (64) or (65) have, to our knowledge of the literature, not been derived before.

Thus, with the solutions of the two basic Equations (47) and (48), we can by means of Equation (49) decompose the single four-component Dirac field into two two-component Majorana fields. Therefore, solving either the set of coupled (via the mass term) chiral Weyl Equations (41) and (42) or the above set of coupled (via the gauge field) Majorana equations provides us with a complete solution of the Dirac equation of a charged massive lepton in an electromagnetic field. The Majorana equations are by their mass terms characterized by the operator  $\tau$ , which is at the heart of charge conjugation. Yet, its explicit appearance is unavoidable, and  $\tau$  may somewhat complicate the involved algebraic calculations as compared to those that are usually carried out in the standard or chiral representation of the Dirac equation.

However, this new decomposition (involving complex conjugation) is not too complicated, even though we went into considerable algebraic detail in the previous illustrative calculations. Adding and subtracting the original and charge-conjugated Dirac fields [13] is exactly what is implied by the decomposition (28) employing the projection operator (27). However, this procedure is not standard, and in textbooks and the special literature, the eigenfunction  $\psi_{-}$  with a negative eigenvalue of  $\mathcal{C}$  is usually not considered, but only  $\psi_{+}$  with a positive eigenvalue, corresponding to the “reality condition” [18] imposed on a Dirac spinor field within the real Majorana representation of the Dirac matrices. The general use of projection operators for charge conjugation permits more algebraic transparency when proceeding from the standard Dirac equation to the Majorana equation including an electromagnetic field.

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## Conflicts of Interest

The author declares no conflict of interest.

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