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The Almost Periodic Rigidity of Crystallographic Bar-Joint Frameworks

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Abstract: A crystallographic bar-joint framework, $\mathcal{C}$ in $\mathbb{R}^d$, is shown to be almost periodically infinitesimally rigid if and only if it is strictly periodically infinitesimally rigid and the rigid unit mode (RUM) spectrum, $\Omega(\mathcal{C})$, is a singleton. Moreover, the almost periodic infinitesimal flexes of $\mathcal{C}$ are characterised in terms of a matrix-valued function, $\Phi_{\mathcal{C}}(z)$, on the d-torus, $\mathbb{T}^d$, determined by a full rank translation symmetry group and an associated motif of joints and bars.

Keywords: crystal framework; infinitesimal rigidity; almost periodic functions

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1. Introduction

The rigidity of a crystallographic bar-joint framework, $\mathcal{C}$, in the Euclidean spaces, $\mathbb{R}^d$, with respect to periodic first order flexes is determined by a finite matrix, the associated periodic rigidity matrix. For essentially generic frameworks of this type in two dimensions, there is a deeper combinatorial characterisation, which is a counterpart of Laman's characterisation of the infinitesimal rigidity of generic placements of finite graphs in the plane. See Ross [1]. For related results and characterisations of other forms of periodic infinitesimal rigidity, see Borcea and Streinu [2,3], Connelly, Shen and Smith [4], Malestein and Theran [5], Owen and Power [6], Power [7,8] and Ross, Schulze and Whiteley [9].

There is also extensive literature in condensed matter physics concerning the nature and multiplicity of low-energy oscillations and rigid unit modes (RUMs) for material crystals in three dimensions. In this

case, Bloch's theorem applies, and the excitation modes are periodic modulo a phase factor. The set of phase factors, or, equivalently, the set of reduced wave vectors for the modes, provides what may be viewed as the RUM spectrum of the crystal. See Dove *et al.* [10], Giddy *et al.* [11] and Wegner [12], for example. In Owen and Power [6] and Power [8], the RUM spectrum was formalised in mathematical terms as a subset, $\Omega(\mathcal{C})$, of the d-torus, $\mathbb{T}^d$, or as an equivalent subset of $[0, 1)^d$, which arises from a choice of translation group $\mathcal{T}$. This set of multi-phases is determined by a matrix-valued function, $\Phi_{\mathcal{C}}(z)$, on $\mathbb{T}^d$ with the value at $z = \hat{1} = (1, 1, \ldots, 1)$ providing the corresponding periodic rigidity matrix for $\mathcal{T}$.

In the present article, we move beyond periodicity and consider infinitesimal flexes of a crystallographic bar-joint framework, which are almost periodic in the classical sense of Bohr. Such flexes are independent of any choice of translation group and so are intrinsic to $\mathcal{C}$ as an infinite bar-joint framework. It is shown that $\mathcal{C}$ is almost periodically infinitesimally rigid if and only if for some choice of translation group, it is periodically infinitesimally rigid, and the corresponding RUM spectrum is the minimal set $\{\hat{1}\}$. More generally, we show how the almost periodic infinitesimal flexes of $\mathcal{C}$ are determined in terms of the matrix function, $\Phi_{\mathcal{C}}(z)$.

An ongoing interest in the analysis of low-energy modes in material science is to quantify the implications of symmetry and local geometry for the set of RUM wave vectors. See, for example, Kapko *et al.* [13], where the phenomenon of extensible flexibility is related to the maximal symmetry and minimal density forms of an idealised zeolite crystal framework. Here, the term, extensive flexibility, corresponds to a maximal rigid unit mode spectrum $\Omega(\mathcal{C}) = \mathbb{T}^3$. We show that for crystal frameworks whose RUM spectrum decomposes into a finite union of linear components, there is a corresponding vector space decomposition of the almost periodic flex space. The flexes in these subspaces are periodic in specific directions associated with certain symmetries of the crystallographic point group.

In Section 4, we give a small gallery of crystal frameworks, which display a variety of periodic and almost periodic flexibility properties.

2. Crystal Frameworks and the RUM Spectrum

A bar-joint framework in the Euclidean space, $\mathbb{R}^d$, is a pair consisting of a simple undirected graph $G = (V, E)$ and an injective map $p : V \to \mathbb{R}^d$. A (real) infinitesimal flex of (G, p) is a field of velocities, or velocity vectors, $u(v)$, assigned to the joints, $p(v)$, such that, for every edge, $vw \in E$,

$$(p(v) - p(w)) \cdot (u(v) - u(w)) = 0$$

If the above condition holds for all pairs $v, w \in V$, then u is a trivial infinitesimal flex of (G, p). For convenience, we let $p(E)$ denote the set of open line segments $p_e = (p(v), p(w))$ with $e = vw \in E$.

Definition 1. *A crystal framework, $\mathcal{C}$, is a bar-joint framework (G, p) for which there exist finite subsets, $F_v \subseteq p(V)$ and $F_e \subseteq p(E)$, and a full rank translation group, $\mathcal{T}$, such that:*

$$p(V) = \{T(p_v) : p_v \in F_v, T \in \mathcal{T}\}$$

$$p(E) = \{T(p_e) : p_e \in F_e, T \in \mathcal{T}\}$$

The pair (F_v, F_e) is called a motif for $\mathcal{C}$. The elements of F_v are called motif vertices, and the elements of F_e are called motif edges. The translation group, $\mathcal{T}$, is necessarily of the form:

$$\mathcal{T} = \{\sum_{j=1}^{d} k_j a_j : k_j \in \mathbb{Z}\}$$

where $a_1, a_2, \ldots, a_d$ are linearly independent vectors in $\mathbb{R}^d$. The translation $x \mapsto x + \sum_{j=1}^{d} k_j a_j$ is denoted T^k for each $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. For each motif vertex, $p(v) \in F_v$, and each $k \in \mathbb{Z}^d$, we denote by (v, k) the unique vertex for which $p(v, k) = T^k(p(v))$. For each motif edge, $p_e \in F_e$, with $p_e = (p(v, l), p(w, m))$, we let (e, k) denote the unique edge for which $p_{(e,k)} = (p(v, l+k), p(w, m+k))$. In Section 4, we provide a number of illustrative examples of crystal frameworks with natural choices of the motif and translation group.

In the consideration of infinitesimal rigidity relative to general periodic flexes, or almost periodic flexes, it is convenient and natural to consider complex velocity vectors $u : F_v \times \mathbb{Z}^d \to \mathbb{C}^d$. Indeed, such vectors are infinitesimal flexes if and only if their real and imaginary parts are infinitesimal flexes. A velocity vector $u : F_v \times \mathbb{Z}^d \to \mathbb{C}^d$ for $\mathcal{C}$ is said to be

1. local if $u(v, k) \neq 0$ for at least one and at most finitely many $(v, k) \in F_v \times \mathbb{Z}^d$;

2. strictly periodic if $u(v, k) = u(v, 0)$ for all $(v, k) \in F_v \times \mathbb{Z}^d$;

3. supercell periodic if $u(v, k) = u(v, 0)$ for each motif vertex $v \in F_v$ and for all k in a full rank subgroup of $\mathbb{Z}^d$ of the form $m_1 \mathbb{Z} \times \cdots \times m_d \mathbb{Z}$.

2.1. The Symbol Function and Rigidity Matrix

We now define the symbol function, $\Phi_{\mathcal{C}}(z)$, of a crystal framework and the rigidity matrix, $R(\mathcal{C})$, from which it is derived. The d-dimensional torus is $\mathbb{T}^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1\}$. For $k \in \mathbb{Z}^d$, the associated monomial function, $f : \mathbb{T}^d \to \mathbb{C}$, given by $f(z_1, \ldots, z_d) = z_1^{k_1} \ldots z_d^{k_d}$, is written simply as z^k.

Definition 2. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ with motif (F_v, F_e), and for $e = (v, k)(w, l) \in F_e$, let $p(e) = p(v, k) - p(w, l)$. Then, $\Phi_{\mathcal{C}}(z)$ is a matrix-valued function on $\mathbb{T}^d$, which assigns a finite $|F_e| \times d|F_v|$ matrix to each $z \in \mathbb{T}^d$. The rows of $\Phi_{\mathcal{C}}(z)$ are labelled by the edges of F_e, and the columns are labelled by the vertex-coordinate pairs in $F_v \times \{1, \ldots, d\}$. The row for an edge $e = (v, k)(w, l)$ with $v \neq w$ takes the form,*

$$e \begin{bmatrix} 0 & \cdots & 0 & \overset{v}{p(e)\overline{z}^k} & 0 & \cdots & 0 & \overset{w}{-p(e)\overline{z}^l} & 0 & \cdots & 0 \end{bmatrix}$$

while if $v = w$, it takes the form,

$$e \begin{bmatrix} 0 & \cdots & 0 & \overset{v}{p(e)(\overline{z}^k - \overline{z}^l)} & 0 & \cdots & 0 \end{bmatrix}$$

Definition 3. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ with motif (F_v, F_e), and for $e = vw \in F_e$, let $p(e) = p(v) - p(w)$. Then, $R(\mathcal{C})$ is the infinite matrix, whose rows are labelled by the edges, (e, k) in $F_e \times \mathbb{Z}^d$, and whose columns are labelled by the pairs, (v, k) in $F_v \times \mathbb{Z}^d$. The row for an edge $(e, k) = (v, l + k)(w, m + k)$, with $e = (v, l)(w, m) \in F_e$, takes the form:*

$$
\begin{array}{c}
 \qquad\qquad\qquad \overset{(v,l+k)}{} \qquad\qquad \overset{(w,m+k)}{} \\
{}_{(e,k)} \left[\begin{array}{ccccccccccc} \cdots & \cdots & 0 & p(e) & 0 & \cdots & 0 & -p(e) & 0 & \cdots & \cdots \end{array} \right]
\end{array}
$$

It follows from this definition that a velocity vector $u : F_v \times \mathbb{Z}^d \to \mathbb{R}^d$ is an infinitesimal flex for $\mathcal{C}$ if and only if $R(\mathcal{C})u = 0$.

2.2. The RUM Spectrum

Let $\omega = (w_1, \ldots, \omega_d) \in \mathbb{T}^d$. A velocity vector $u : F_v \times \mathbb{Z}^d \to \mathbb{C}^d$ is said to be ω-phase-periodic if $u(v, k) = \omega^k u(v, 0)$ for all $(v, k) \in F_v \times \mathbb{Z}^d$. Here, ω^k is the product $\omega_1^{k_1} \ldots \omega_d^{k_d}$. We also write $u = b \otimes e_\omega$ for this vector, where b is the vector $(u(v, 0))_v$ in $\mathbb{C}^{d|F_v|}$ and e_ω is the multi-sequence $(\omega^k)_{k \in \mathbb{Z}^d}$. We refer to ω as a multi-phase for u. The velocity vector u is said to be phase-periodic if it is ω-phase-periodic for some multi-phase $\omega \in \mathbb{T}^d$.

To explain this terminology, note that a multi-phase is a d-tuple of unimodular complex numbers $\omega = (e^{i\theta_1}, \ldots, e^{i\theta_d})$. The k-th entry corresponds to the phase variation of the velocity vector, u, in the direction of the k-th lattice direction. The d-tuple of real numbers $\theta = (\theta_1, \ldots, \theta_d)$ can be taken to have entries in the interval $[0, 2\pi)$, in which case, θ can be considered as the reduced wave vector of the velocity vector, u (see [8] for further remarks on this dynamical connection).

The following theorem is given in [8,14].

Theorem 1. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$; let $\omega \in \mathbb{T}^d$ and let $u = b \otimes e_\omega$, with $b \in F_v \times \mathbb{C}^d$, be a ω-phase-periodic velocity vector. Then, the following conditions are equivalent:*

(i) $R(\mathcal{C})u = 0$;

(ii) $\Phi_{\mathcal{C}}(\overline{\omega})b = 0$.

The RUM spectrum, $\Omega(\mathcal{C})$, of $\mathcal{C}$ is defined to be the set of multi-phases, ω, for which there exists a nonzero phase-periodic infinitesimal flex for $\mathcal{C}$ as an infinite-bar-joint framework, or, equivalently, as the set of multi-phases for which the rank of the matrix, $\Phi_{\mathcal{C}}(\overline{\omega})$, is less than $d|F_v|$.

Note that the strictly periodic velocity vectors for $\mathcal{C}$ are, by definition, precisely the phase-periodic velocity vectors, which have multi-phase $\hat{1} = (1, \ldots, 1) \in \mathbb{T}^d$. Evidently, the RUM spectrum must always contain the point, $\hat{1} \in \mathbb{T}^d$, as every constant velocity vector, u, is a phase-periodic infinitesimal flex with multi-phase $\hat{1}$. Thus, the RUM spectrum of $\mathcal{C}$ is a singleton (*i.e.*, it contains only the single point, $\hat{1}$) if and only if every phase-periodic infinitesimal flex for $\mathcal{C}$ is strictly periodic.

Corollary 1. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ with translation group $\mathcal{T}$. Then, the following statements are equivalent:*

(i) The $\mathcal{T}$-periodic real infinitesimal flexes of $\mathcal{C}$ are trivial;

(ii) The $\mathcal{T}$-periodic complex infinitesimal flexes of $\mathcal{C}$ are trivial;

(iii) The periodic rigidity matrix $\Phi_{\mathcal{C}}(1, \ldots, 1)$ has rank equal to $d|F_v| - d$.

We may view phase-periodic infinitesimal flexes of a crystal framework as the pure flexes with (possibly incommensurate) oscillatory variation relative to the periodicity lattice. Note that finite linear combinations of such flexes need not be phase-periodic, but will in fact be almost periodic in the sense of Definition 4 below. Our main theorem may be viewed as an almost periodic flex variant of the general principle that general motions can be synthesised by combinations of purely oscillatory motions.

3. Almost Periodic Rigidity

In this section, we first outline the proof of the fundamental approximation theorem for uniformly almost periodic functions and its counterpart for almost periodic sequences. These ensure that a function (or sequence), which is almost periodic in the sense of Bohr, is approximable by trigonometric functions (or sequences) that are obtained in an explicit manner from convolution with Bochner–Fejér kernels. A convenient self-contained exposition of this fact for function approximation is given in Partington [15]. (See also [16,17]). The direct arguments there can be extended to almost periodic vector-valued functions on $\mathbb{Z}^d$, and this embraces the setting of velocity fields relevant to the almost periodic rigidity of crystal frameworks (see Definition 4). The constructive approximation theorem that we require is given in Theorem 2. This theorem together with Lemmas 1 and 2 leads to the almost periodic rigidity theorem.

3.1. Almost Periodic Sequences

First, we recall the classical theory for univariable functions on $\mathbb{R}$. The Fejér kernel functions are given by:

$$K_n(x) = \sum_{|m| \leq n+1} (1 - \frac{|m|}{n+1}) e^{imx}, \quad x \in \mathbb{R}$$

The positivity of the K_n and their approximate identity property under convolution with a continuous periodic functions feature in a standard proof that a continuous 2π-periodic function, $f(x)$, on the real line is uniformly approximable by the explicit trigonometric functions:

$$g_n(x) = \int_0^{2\pi} f(s) K_n(s - x) \frac{ds}{2\pi}$$

Almost periodic functions on the real line in the sense of Bohr are similarly uniformly approximable by an explicit sequence of trigonometric polynomials that are determined by convolution with certain Bochner–Fejér kernels.

Note first that it is elementary that the functions, $g_n(x)$, has the form $g_n(x) = [f, R_x K_n]$, where $[\cdot, \cdot]$ is the mean inner product:

$$[f_1, f_2] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_1(s) \overline{f_2(s)} ds$$

and $R_x K_n(s) = K_n(s - x)$. It is classical that for a function, $f(x)$ in $AP(\mathbb{R}, \mathbb{C})$, there is a sequence of Bochner–Fejér kernels K_n', which provide, by the same formula, a uniformly approximating sequence

of trigonometric polynomials, $g_n(x)$. In this case, the frequencies, λ, in the nonzero terms, $ae^{i\lambda x}$, of these approximants appear in a countable set derived (by rational division) from the spectrum, $\Lambda(f)$, of f defined by:

$$\Lambda(f) = \{\lambda \in \mathbb{R} : [f(x), e^{i\lambda x}] \neq 0\}$$

It follows from a Parceval inequality for almost periodic functions that this spectrum, which we refer to as the Bohr spectrum, is a well-defined finite or countable set.

Similar considerations apply to the space, $AP(\mathbb{Z}, \mathbb{C})$, of almost periodic sequences. The approximants are general trigonometric sequences, that is, sequences, $(h(k))_{k\in\mathbb{Z}}$, that have a finite sum form:

$$h(k) = \sum_{\omega = e^{i\lambda} : \lambda \in F} a_\lambda \omega^k$$

so that, in our earlier notation,

$$h = \sum_{\omega = e^{i\lambda} : \lambda \in F} a_\lambda e_\omega$$

where F is a finite subset of $\mathbb{R}$. The Bohr spectrum of a sequence, h in $AP(\mathbb{Z}, \mathbb{C})$, is defined to be the set:

$$\Lambda(h) = \{\lambda \in \mathbb{R} : [h, e_\omega]_{\mathbb{Z}} \neq 0, \text{ for } \omega = e^{i\lambda}\}$$

where

$$[h_1, h_2]_{\mathbb{Z}} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|k|\leq N} h_1(k)\overline{h_2(k)}$$

This spectrum of ω values is now a subset of $\mathbb{T}$. Partial counterparts of the Fejér kernel functions, $K_n(x)$, are given by the Fejér sequences:

$$K_{(n,\lambda)} = \sum_{|m|\leq n+1} (1 - \frac{|m|}{n+1})e_{\omega^m}$$

associated with a single frequency $\omega = e^{i\lambda}$. In particular:

$$K_{(n,\lambda)}(k) = \sum_{|m|\leq n+1} (1 - \frac{|m|}{n+1})\omega^{mk}$$

The following fundamental approximation theorem indicates the explicit construction of the Bochner–Fejér kernels K'_n for h as coordinate-wise products of appropriate Fejér sequences.

Theorem 2. *Let h be a sequence in $AP(\mathbb{Z}, \mathbb{C})$; let $\alpha_1, \alpha_2, \ldots$ be a maximal subset of the Bohr spectrum $\Lambda(h)$, which is independent over $\mathbb{Q}$, and for $n = 1, 2, \ldots$ let:*

$$K'_n = \prod_{k=1}^{n} K_{(n.n!-1, \alpha_k/n!)}$$

Then, h is the uniform limit of the sequence $g_1, g_2, \ldots$ of trigonometric sequences in $AP(\mathbb{Z}, \mathbb{C})$ given by:

$$g_n(k) = [h, R_k K'_n]_{\mathbb{Z}}, \quad k \in \mathbb{Z}$$

The main ingredient in the proof of the theorem is that the Bohr spectrum is nonempty if $h \neq 0$, and the arguments for this depend on the equivalence of Bohr almost periodicity with the Bochner condition that the set of translates of h is precompact for the uniform norm (see [15]).

The arguments leading to Theorem 2 can be generalised to obtain an exact counterpart theorem for $AP(\mathbb{Z}^d, \mathbb{C}^r)$. The approximating trigonometric sequences, g, now have a finite sum form:

$$g = \sum_{\omega \in F \subset \mathbb{T}^d} a_\omega \otimes e_\omega$$

where $a_\omega \in \mathbb{C}^r$ and e_ω, for $\omega = (\omega_1, \ldots, \omega_d)$ in $\mathbb{T}^d$, is the pure frequency sequence $e_{\omega_1} \otimes \cdots \otimes e_{\omega_d}$ in $AP(\mathbb{Z}^d, \mathbb{C})$ with:

$$e_\omega(k) = \omega_1^{k_1} \ldots \omega_d^{k_d}$$

The Bohr spectrum, $\Lambda(h)$, is similarly defined and is a countable subset of points ω in $\mathbb{T}^d$, which we freely identify with a countable subset of points, λ, in $[0, 2\pi)^d$. For notational convenience, we state the general theorem only in the case $d = 2$.

The metric that is appropriate in our context for the approximation of velocity fields is the uniform metric or norm $\| \cdot \|_\infty$; for velocity fields h, h', we have:

$$\|h - h'\|_\infty = \sup_{k, \kappa} \{ \|h(k, \kappa) - h'(k, \kappa)\|_2 \}$$

Theorem 3. *Let h be a sequence in $AP(\mathbb{Z}^2, \mathbb{C}^r)$, let $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots$ be a maximal subset of $\Lambda(h) \subset [0, 2\pi)^2$, which is independent over $\mathbb{Q}$, and for $n = 1, 2, \ldots$ let:*

$$K_n^{(2)} = (\prod_{k=1}^n K_{(n.n!-1, \alpha_k/n!)})(\prod_{k=1}^n K_{(n.n!-1, \beta_k/n!)}) \in AP(\mathbb{Z}^2, \mathbb{C})$$

Then, h is the uniform limit of the sequence $g_1, g_2, \ldots$ of trigonometric sequences in $AP(\mathbb{Z}^2, \mathbb{C}^r)$ given by:

$$g_n(k) = [h, R_k K_n^{(2)}]_{\mathbb{Z}^2}, \quad k \in \mathbb{Z}^2$$

Note that here, $[\cdot, \cdot]_{\mathbb{Z}^2}$ is the natural well-defined sesquilinear map from $AP(\mathbb{Z}^2, \mathbb{C}^r) \times AP(\mathbb{Z}^2, \mathbb{C})$ to $\mathbb{C}^r$.

We remark that the theory of uniformly almost periodic multi-variable functions on $\mathbb{R}^d$ and on $\mathbb{Z}^d$ is part of the abstract theory of almost periodic functions on locally compact abelian groups, due to Bochner and von Neumann [18]. For further details, see also Levitan and Zhikov [19], Loomis [20] and Shubin [21].

3.2. Almost Periodic Rigidity

We now characterise when a crystal framework admits no nontrivial almost periodic infinitesimal flexes, and in this case, we say that it is almost periodically rigid. This is evidently a form of rigidity, which is independent of any choice of translation group.

For a crystal framework, $\mathcal{C}$, with full rank translation group $\mathcal{T}$, various linear transformations may be associated with the rigidity matrix, $R(\mathcal{C})$. These transformations are restrictions of the induced linear transformation:

$$R(\mathcal{C}) : \mathbb{C}^{\mathbb{Z}^d \times |F_v|} \otimes \mathbb{C}^d \to \mathbb{C}^{\mathbb{Z}^d \times |F_e|} \otimes \mathbb{C}$$

where $\mathbb{C}^{\mathbb{Z}^d \times |F_v|} \otimes \mathbb{C}^d$ is the vector space of all velocity fields on $p(V)$, that is, the space of functions $h : \mathbb{Z}^d \times |F_v| \to \mathbb{C}^d$. The codomain of $R(\mathcal{C})$ is the vector space of complex-valued functions on the set of edges.

The right shift operators on the domain and codomain of $R(\mathcal{C})$ for the integral vector, l in $\mathbb{Z}^d$, are denoted by R_l^V and R_l^E, respectively. Here, the right shift of a sequence, $h(k, \kappa)$, by l is the sequence, $h(k - l, \kappa)$. We note that:

$$R(\mathcal{C}) \circ R_l^V = R_l^E \circ R(\mathcal{C})$$

Definition 4. *Let $h : \mathbb{Z}^d \times F_v \to \mathbb{C}^d$ be a velocity field.*

1. *An integral vector, l in $\mathbb{Z}^d$, is an ϵ-translation vector for h if $\|R_l^V(h) - h\|_\infty < \epsilon$.*

2. *The velocity field, h, is Bohr almost periodic if for every $\epsilon > 0$, the set of ϵ-translation vectors, l, is relatively dense in $\mathbb{R}^d$.*

Lemma 1. *Let g be the vector-valued trigonometric multi-sequence with finite sum representation:*

$$g = \sum_{\omega \in F \subset \mathbb{T}^d} a_\omega \otimes e_\omega$$

with nonzero coefficients, a_ω in $\mathbb{C}^{d|F_v|}$. If g is a nonzero infinitesimal flex for $\mathcal{C}$, then each component sequence $a_\omega \otimes e_\omega$ is a nonzero ω-phase-periodic infinitesimal flex.

Proof. For $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{T}^d$ and $N \in \mathbb{N}$, let $R^V(\omega, N)$ be the linear map on the normed space $\ell^\infty(\mathbb{Z}^d \times F_v, \mathbb{C}^d)$ given in terms of the right shift operators, R_k^V, $k \in \mathbb{Z}^d$, by:

$$R^V(\omega, N) = \frac{1}{(N+1)^d} \sum_{k:0 \leq k_i \leq N} \bar{\omega}^k R_{-k}^V$$

Similarly, let $R^E(\omega, N)$ be the linear map on $\ell^\infty(\mathbb{Z}^d \times F_e, \mathbb{C})$ given by,

$$R^E(\omega, N) = \frac{1}{(N+1)^d} \sum_{k:0 \leq k_i \leq N} \overline{\omega}^k R_{-k}^E$$

The sequence $R(\omega', N)(a_\omega \otimes e_\omega)$ converges uniformly to $a_\omega \otimes e_\omega$, when $\omega' = \omega$, and to the zero sequence otherwise, since for each $(l, v_\kappa) \in \mathbb{Z}^d \times F_v$,

$$
\begin{aligned}
\lim_{N \to \infty} R^V(\omega', N)(a_\omega \otimes e_\omega)(l, v_\kappa) &= \lim_{N \to \infty} \frac{1}{(N+1)^d} \sum_{k:0 \leq k_i \leq N} \overline{\omega'}^k R_{-k}^V(a_\omega \otimes e_w)(l, v_\kappa) \\
&= \lim_{N \to \infty} \frac{1}{(N+1)^d} \sum_{k:0 \leq k_i \leq N} \overline{\omega'}^k \omega^{l+k} a_\omega \\
&= \left(\lim_{N \to \infty} \frac{1}{(N+1)^d} \sum_{k:0 \leq k_i \leq N} \overline{\omega'}^k \omega^k \right) \omega^l a_\omega
\end{aligned}
$$

and

$$\lim_{N \to \infty} \frac{1}{(N+1)^d} \sum_{k:0 \leq k_i \leq N} \overline{\omega'}^k \omega^k = \begin{cases} 1 & \text{if } \omega = \omega' \\ 0 & \text{if } \omega \neq \omega' \end{cases}$$

Note also that $R(\mathcal{C})$ is a bounded linear transformation from $\ell^\infty(\mathbb{Z}^d \times F_v, \mathbb{C}^d)$ to $\ell^\infty(\mathbb{Z}^d \times F_e, \mathbb{C})$, which commutes with the right shift operators. Thus, if ω is a multi-frequency for g, then:

$$R(\mathcal{C})(a_\omega \otimes e_\omega) = \lim_{N \to \infty} R(\mathcal{C})(R^V(\omega, N)g) = \lim_{N \to \infty} R^E(\omega, N)R(\mathcal{C})g = 0 \quad \square$$

Lemma 2. *Let K be a trigonometric polynomial in $AP(\mathbb{Z}^d \times F_v, \mathbb{C})$, and let h be an infinitesimal flex for $\mathcal{C}$ in $AP(\mathbb{Z}^d \times F_v, \mathbb{C}^d)$. Then, the mean convolution multi-sequence $g : \mathbb{Z}^d \times F_v \to \mathbb{C}^d$ given by $g(k, v_\kappa) = [h, R_k(K)]_{\mathbb{Z}^d}$ is an infinitesimal flex for $\mathcal{C}$.*

Proof. By linearity, it suffices to assume that K is the elementary multi-sequence, e_ω, so that $K(k) = \omega^k$ for $k \in \mathbb{Z}^d$. Then, g is the uniform limit of the sequence (g_N), where:

$$g_N(k, v_\kappa) = \frac{1}{(N+1)^d} \sum_{1 \le s_i \le N} \bar{\omega}^s (R_{-s}h)(k, v_\kappa)$$

To see this, note that the convergence is uniform if h is a trigonometric sequence. Since the linear maps, $h \to g_N$, are contractive for the uniform norm, uniform convergence holds for a general almost periodic velocity sequence. Thus, since the vector space of infinitesimal flexes is invariant under translation, it follows that $R(\mathcal{C})g_N = 0$ for each N and, hence, that $R(\mathcal{C})g = 0$. $\quad \square$

The following theorem shows that a crystal framework is almost periodically rigid if and only if it is periodically rigid and the RUM spectrum is trivial.

Theorem 4. *Let $\mathcal{C}$ be a crystallographic bar-joint framework in $\mathbb{R}^d$. The following statements are equivalent:*

(i) Every almost periodic infinitesimal flex of $\mathcal{C}$ is trivial;

(ii) Every strictly periodic infinitesimal flex of $\mathcal{C}$ is trivial, and $\Omega(\mathcal{C}) = \{\hat{1}\}$.

Proof. $(i) \Rightarrow (ii)$ This follows, since every phase-periodic infinitesimal flex is also an almost periodic infinitesimal flex.

$(ii) \Rightarrow (i)$ Let u be an almost periodic infinitesimal flex. Then, by Theorem 3, for d dimensions, u is a uniform limit of the sequence, (g_n), of trigonometric sequences in $AP(\mathbb{Z}^d \times F_v, \mathbb{C}^d)$ given by:

$$g_n(k, v_\kappa) = [h, R_k(K_n^{(d)})]_{\mathbb{Z}_d}, \quad k \in \mathbb{Z}^d$$

where $K_n^{(d)}$, $n = 1, 2, \ldots$, is the sequence of Bochner–Fejér kernels for u. By Lemma 2, each trigonometric sequence, g_n, is an infinitesimal flex of $\mathcal{C}$, and so, by Lemma 1, each g_n is a finite linear combination of phase-periodic infinitesimal flexes of $\mathcal{C}$. By hypothesis, the RUM spectrum, $\Omega(\mathcal{C})$, contains the single multi-phase, $\hat{1}$, and so, every phase-periodic infinitesimal flex is strictly periodic. In particular, g_n is strictly periodic. It follows that u is strictly periodic and, hence, trivial, as desired. $\quad \square$

We note that it follows from the proof of the theorem that any almost periodic infinitesimal flex is approximable by a sequence of finite linear combinations of phase-periodic flexes, with approximation in the sense of uniform convergence for the uniform norm given before in Theorem 3. For supercell periodic flexes, one can be more specific, as in the following proposition, which essentially follows from Theorem 1 and Lemma 1 (see also [8]).

Proposition 5. *Let $\mathcal{C}$ be a crystallographic bar-joint framework in $\mathbb{R}^d$. Then, the space of supercell periodic infinitesimal flexes for m-fold periodicity with $m = (m_1, \ldots, m_d)$ is equal to the linear span of:*

$$\{b \otimes e_\omega : \Phi(\overline{\omega})b = 0, \ \omega \in \Omega_m(\mathcal{C})\}$$

where $\Omega_m(\mathcal{C})$ is the finite subset of the RUM spectrum given by the multi-phases ω, whose k-th component is an m_{k-th} root of unity.

In particular, every supercell periodic infinitesimal flex for $\mathcal{C}$ is an almost periodic infinitesimal flex for $\mathcal{C}$.

The Bohr spectrum of an almost periodic infinitesimal flex, u, of the crystal framework, $\mathcal{C}$, is the finite or countable set given by:

$$\Lambda(u, C) = \{\lambda \in [0, 1)^d : [u, e_\omega]_{\mathbb{Z}^d} \neq 0, \text{ for } \omega = e^{2\pi i \lambda} \in \mathbb{T}^d\}$$

It follows from Theorem 3, Lemmas 1 and 2, as in the proof above, that $\Lambda(u, C)$ is contained in the RUM spectrum of $\mathcal{C}$. Furthermore, since phase-periodic flexes are almost periodic, it follows that the RUM spectrum as a subset of $[0, 1)^d$ is the union of the Bohr spectra of all almost periodic infinitesimal flexes. Note that the spectra here depend on the translation group in the following manner. If $\mathcal{C}'$ has the same underlying bar-joint framework as $\mathcal{C}$, but full rank translation group $\mathcal{T}' \subseteq \mathcal{T}$, then the infinitesimal flex, u, is represented anew as a sequence in $AP(\mathbb{Z}^d \times F_v', \mathbb{C}^d)$, where F_v' is a vertex motif for $\mathcal{T}'$. The Bohr spectrum, $\Lambda(u, \mathcal{C}')$, is then the image of $\Lambda(u, \mathcal{C})$ under the surjective map, $\mathbb{T}^d \to \mathbb{T}^d$, induced by the inclusion, $\mathcal{T}' \subseteq \mathcal{T}$. This follows the same relationship as that for the RUM spectrum noted in [8]. It also follows from this that the dimension of $\Omega(\mathcal{C})$, as a topological space or as an algebraic variety, is independent of the translation group, and we refer to this integer, which takes values between zero and d, as the RUM dimension of $\mathcal{C}$.

We see in the next section that $\Omega(\mathcal{C})$, as a subset of $[0, 1)^d$, often decomposes as a union of linear components. This is the case, for example, in two dimensions if $\Phi_{\mathcal{C}}(z)$ is a square matrix function, whose determinant polynomial, $\det \Phi_{\mathcal{C}}(z)$, either vanishes identically or factorises into simple factors of the form $(z^n - \lambda w^m)$ with $|\lambda| = 1$. It follows that each almost periodic flex, u, of $\mathcal{C}$ admits a finite sum decomposition $u_1 + \cdots + u_r$ in which each component, u_i, is an almost periodic flex, whose Bohr spectrum lies in the i-th linear component. Such component flexes are partially periodic, being periodic in certain directions of translational symmetry.

4. Gallery of Crystal Frameworks

We now exhibit a number of illustrative examples. The first two of these show two extreme cases: firstly, where the RUM spectrum is a singleton, and secondly, where the RUM spectrum is $\mathbb{T}^d$.

Example 1. *Let $\mathcal{C} = (G, p)$ be the crystallographic bar-joint framework with motif (F_v, F_e) and translation group $\mathcal{T}$ indicated in Table 1. Simplifying earlier notation, the motif vertex is labelled v and the motif edges are labelled $e_0 = v(0, 0)v(1, 0)$, $e_1 = v(1, 0)v(0, 1)$ and $e_2 = v(0, 0)v(0, 1)$. The*

translation group is $\mathcal{T} = \{k_1 a_1 + k_2 a_2 : k_1, k_2 \in \mathbb{Z}\}$, where $a_1 = (1, 0)$ and $a_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. The symbol function for $\mathcal{C}$ is,

$$\Phi_\mathcal{C}(z, w) = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \end{array} \begin{bmatrix} \overset{v_x}{\bar{z} - 1} & \overset{v_y}{0} \\ \frac{1}{2}(\bar{z} - \bar{w}) & \frac{\sqrt{3}}{2}(\bar{w} - \bar{z}) \\ \frac{1}{2}(\bar{w} - 1) & \frac{\sqrt{3}}{2}(\bar{w} - 1) \end{bmatrix}$$

Note that $\Phi_\mathcal{C}(z, w)$ has rank two unless $z = w = 1$, and so, the RUM spectrum of $\mathcal{C}$ is the singleton, $(1, 1) \in \mathbb{T}^2$. Furthermore, there are no non-trivial strictly periodic infinitesimal flexes of $\mathcal{C}$, and so, by Theorem 4, $\mathcal{C}$ is almost periodically infinitesimally rigid.

In fact, $\mathcal{C}$ is sequentially infinitesimally rigid in the sense that there exists an increasing chain of finite subgraphs $G_1 \subset G_2 \subset \cdots$ of G, such that every vertex of G is contained in some G_n and each sub-framework (G_n, p) is infinitesimally rigid. For example, for each n, take G_n to be the vertex-induced subgraph on $\{v(k_1, k_2) : (k_1, k_2) \in \mathbb{Z}^2, |k_i| \leq n\}$. It follows that $\mathcal{C}$ admits no nontrivial infinitesimal flexes and, so, is (absolutely) infinitesimally rigid as a bar-joint framework. In [22], we obtain a general characterisation of countable simple graphs, G, whose locally generic placements are infinitesimally rigid in this sense. The condition is that G should contain a vertex-complete chain of $(2, 3)$-tight subgraphs. The crystal framework, $\mathcal{C}$, may be viewed as a non-generic placement of such a graph, which remains infinitesimally rigid despite the crystallographic symmetry.

Table 1. An infinitesimally rigid crystal framework.

Motif	Translation group	Crystal framework
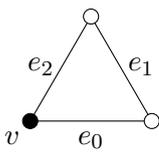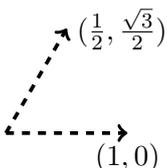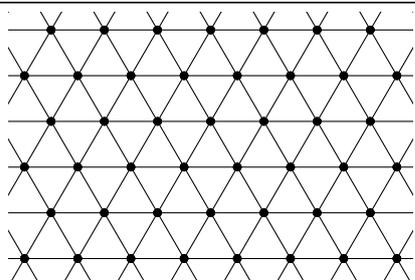
If the symbol function, $\Phi_\mathcal{C}(z)$, is a square matrix, then the determinant of $\Phi_\mathcal{C}(z)$ gives rise to the crystal polynomial, $p_\mathcal{C}(z)$ (see [8]). It is shown in [6] that such a crystal framework has a local infinitesimal flex if and only if the crystal polynomial, $p_\mathcal{C}(z)$, is identically zero.

Example 2. *Consider the crystal framework, $\mathcal{C}$, with the motif and translation group shown in Table 2. The motif vertices are $v_0 = (0, 0)$ and $v_1 = (\frac{1}{2}, \frac{1}{2})$. The motif edges are $e_0 = v_0(0, 0)v_0(1, 0)$, $e_1 = v_0(0, 0)v_0(0, 1)$, $e_2 = v_0(0, 1)v_1(0, 0)$ and $e_3 = v_0(1, 0)v_1(0, 0)$. Note that the symbol function, $\Phi_\mathcal{C}(z, w)$, is a square matrix,*

$$\Phi_\mathcal{C}(z, w) = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{array} \left[\begin{array}{cc|cc} \overset{v_{0,x}}{\bar{z} - 1} & \overset{v_{0,y}}{0} & \overset{v_{1,x}}{0} & \overset{v_{1,y}}{0} \\ 0 & \bar{w} - 1 & 0 & 0 \\ -\frac{1}{2}\bar{w} & \frac{1}{2}\bar{w} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2}\bar{z} & -\frac{1}{2}\bar{z} & -\frac{1}{2} & \frac{1}{2} \end{array} \right]$$

The determinant of $\Phi_{\mathcal{C}}(z, w)$ vanishes identically, and so, the RUM spectrum of $\mathcal{C}$ is $\mathbb{T}^2$. A local infinitesimal flex of $\mathcal{C}$ is evident by defining $u(v_1) = (1, 1)$ and $u_v = 0$ for all $v \neq v_1$. A phase-periodic infinitesimal flex of $\mathcal{C}$ for $\omega = (\omega_1, \omega_2)$ is obtained by taking $u(v_0, k) = 0$ and $u(v_1, k) = \omega_1^{k_1}\omega_2^{k_2}(1, 1)$ for each $k = (k_1, k_2) \in \mathbb{Z}^2$. In particular, any finite linear combination of such phase-periodic flexes will be an almost periodic infinitesimal flex for $\mathcal{C}$.

Table 2. A crystal framework with the full rigid unit mode (RUM) spectrum.

Motif	**Translation group**	**Crystal framework**
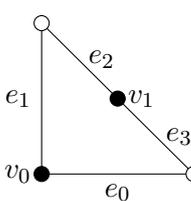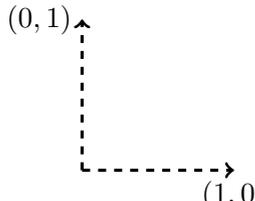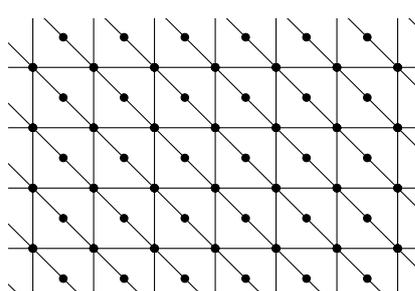
Recall that a velocity vector u is supercell periodic for a crystal framework, $\mathcal{C}$, if $u(v_\kappa, 0) = u(v_\kappa, k)$ for each motif vertex, v_κ, and all k in a full rank subgroup of $\mathbb{Z}^d$.

Example 3. *Let $\mathcal{C}$ be the crystallographic bar-joint framework with motif (F_v, F_e) and translation group $\mathcal{T}$ indicated in Table 3. Note that $\mathcal{C}$ has symbol function,*

$$\Phi_{\mathcal{C}}(z, w) = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{cc} v_{0,x} \quad\;\; v_{0,y} & v_{1,x} \quad v_{1,y} \\ \left[\begin{array}{cc|cc} -1 & 0 & \bar{z}w & 0 \\ 0 & -1 & 0 & 1 \\ \bar{z} - 1 & \bar{z} - 1 & 0 & 0 \\ \bar{z} & 0 & -1 & 0 \\ 0 & \bar{w} & 0 & -1 \end{array}\right] \end{array}$$

The RUM spectrum of $\mathcal{C}$ is $\Omega(\mathcal{C}) = \{(1, 1), (-1, 1)\}$. Note that $\mathcal{C}$ does not admit any non-trivial infinitesimal flexes, which are strictly periodic with respect to $\mathcal{T}$. However, $\mathcal{C}$ does admit non-trivial supercell periodic infinitesimal flexes, which may be constructed from the motif and RUM spectrum. Assign velocity vectors $u(v_0, 0)$ and $u(v_1, 0)$ to the motif vertices and consider the multi-phase $\omega = (\omega_1, \omega_2) = (-1, 1) \in \Omega(\mathcal{C})$. Define for each $k = (k_1, k_2) \in \mathbb{Z}^2$,

$$u(v_0, k) = \omega^k u(v_0, 0) = (-1)^{k_1} u(v_0, 0)$$

$$u(v_1, k) = \omega^k u(v_1, 0) = (-1)^{k_1} u(v_1, 0)$$

Then, u is supercell periodic with respect to the full rank subgroup $2\mathbb{Z} \times \mathbb{Z}$. If, for example, we set $u(v_0, 0) = (1, -1)$ and $u(v_1, 0) = (-1, -1)$, then u is also an infinitesimal flex for $\mathcal{C}$. Note that u consists of alternating rotational motions. In the notation of Proposition 5, u has m-fold periodicity, where we have taken $m = (2, 1)$; the multi-phase $\omega = (-1, 1)$ is contained in,

$$\Omega_m(\mathcal{C}) = \{\omega \in \Omega(\mathcal{C}) : \omega_1^2 = 1, \ \omega_2 = 1\}$$

and u is the ω-phase-periodic velocity vector, $b \otimes e_\omega$, where $b = (u(v_0, 0), u(v_1, 0)) \in \ker \Phi_{\mathcal{C}}(\bar{\omega})$.

Table 3. A crystal framework with the RUM spectrum $\Omega(\mathcal{C}) = \{(1, 1), (-1, 1)\}$.

Motif	Translation group	Crystal framework
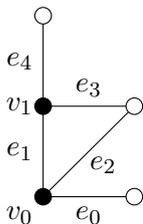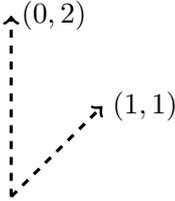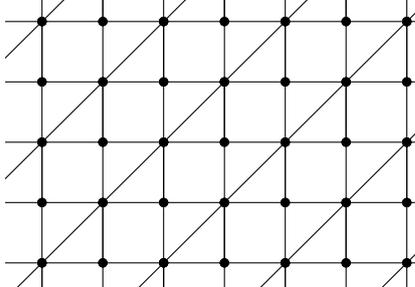
Example 4. *Let $\mathcal{C}$ be the crystallographic bar-joint framework in $\mathbb{R}^2$, which is indicated in Table 4. The motif vertices are $p(v_0) = (0, 0)$ and $p(v_1) = (\frac{1}{3}, -\frac{2}{3})$. The symbol function, $\Phi_{\mathcal{C}}(z, w)$ is,*

$$\Phi_{\mathcal{C}}(z, w) = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{bmatrix} \overset{v_{0,x}}{-\frac{1}{3}} & \overset{v_{0,y}}{\frac{2}{3}} & \overset{v_{1,x}}{\frac{1}{3}} & \overset{v_{1,y}}{-\frac{2}{3}} \\ \bar{z} - 1 & 0 & 0 & 0 \\ 0 & \bar{w} - 1 & 0 & 0 \\ \bar{z} - \bar{w} & \bar{w} - \bar{z} & 0 & 0 \\ 0 & 0 & \bar{z} - 1 & 0 \\ 0 & 0 & 0 & \bar{w} - 1 \\ 0 & 0 & \bar{z} - \bar{w} & \bar{w} - \bar{z} \end{bmatrix}$$

The RUM spectrum, $\Omega(\mathcal{C})$, is the singleton, $(1, 1) \in \mathbb{T}^2$. However, every strictly periodic velocity field, u, with $(p(v_0) - p(v_1)) \cdot (u(v_0) - u(v_1)) = 0$ and $u(v_0) \neq u(v_1)$ is a non-trivial strictly periodic infinitesimal flex of $\mathcal{C}$.

As we have noted, the RUM dimension of a crystal framework, $\mathcal{C}$, is the dimension of the RUM spectrum, $\Omega(\mathcal{C})$, as a real algebraic variety (see [8]).

Table 4. A crystal framework with $\Omega(\mathcal{C}) = \{(1, 1)\}$ and nontrivial strictly periodic flexes.

Motif	Translation group	Crystal framework
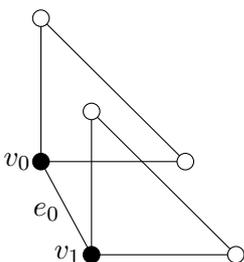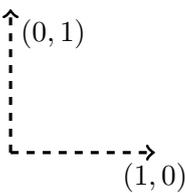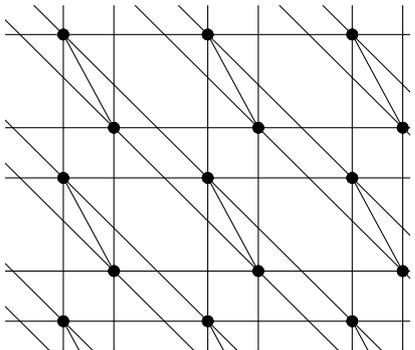
Example 5. *The crystal framework, $\mathcal{C}$, illustrated in Table 5 has motif vertices $v_0 = (0,0)$, $v_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $v_2 = (0,1)$. The symbol function is a square matrix,*

$$\Phi(z,w) = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{cc} \overset{v_{0,x}}{} & \overset{v_{0,y}}{} \\ \left[\begin{array}{cc} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -1 \\ \frac{1}{2}\bar{w} & \frac{\sqrt{3}}{2}\bar{w} \\ 0 & \bar{w} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2}\bar{z} & -\frac{\sqrt{3}}{2}\bar{z} \end{array}\right. \end{array} \begin{array}{cc} \overset{v_{1,x}}{} & \overset{v_{1,y}}{} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{array} \begin{array}{cc} \overset{v_{2,x}}{} & \overset{v_{2,y}}{} \\ \left.\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 \\ \frac{1}{2}w\bar{z} & -\frac{\sqrt{3}}{2}w\bar{z} \\ 0 & 0 \end{array}\right] \end{array}$$

The crystal polynomial factors in to linear parts,

$$p_{\mathcal{C}}(z,w) = (z-1)(w-1)(z-w)$$

and so, the RUM spectrum is a proper subset of $\mathbb{T}^2$, whose representation in $[0,1)^2$ consists of the points (s,t) in the line segments given by:

$$s = 0, \quad t = 0, \quad s = t$$

In particular, $\mathcal{C}$ is almost periodically infinitesimally flexible, but has no local infinitesimal flexes. Furthermore, the RUM dimension of $\mathcal{C}$ is one. It follows that every almost periodic infinitesimal flex decomposes as a sum $u_1 + u_2 + u_3$ of three almost periodic flexes corresponding to this ordered decomposition. Furthermore, u_1, with the Bohr spectrum in the line $s = 0$, is periodic in the direction of the period vector $a_1 = (1,0)$, while u_2, with the Bohr spectrum in the line $t = 0$, is periodic in the direction of the period vector $a_2 = (1/2, (2+\sqrt{3})/2)$, and u_3, with the Bohr spectrum, in the line $s = t$ is periodic in the direction $a_1 - a_2$.

Table 5. A crystal framework with RUM dimension one.

Motif	Translation group	Crystal framework
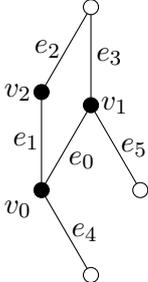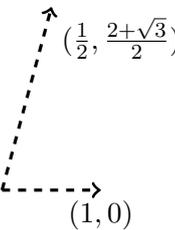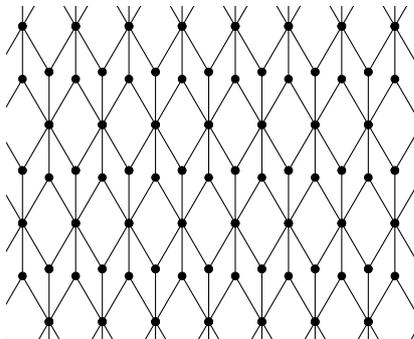
Example 6. *Let $\mathcal{C}$ be the crystal framework illustrated in Table 6. The symbol function, $\Phi(z,w)$, is the square matrix,*

	$v_{1,x}$	$v_{1,y}$	$v_{2,x}$	$v_{2,y}$	$v_{3,x}$	$v_{3,y}$	$v_{4,x}$	$v_{4,y}$	$v_{5,x}$	$v_{5,y}$	$v_{6,x}$	$v_{6,y}$
e_1	0	0	0	0	0	0	$\bar{z}$	$\frac{1}{2}\bar{z}$	0	0	-1	$-\frac{1}{2}$
e_2	$-\frac{1}{2}$	-1	0	0	$\frac{1}{2}$	1	0	0	0	0	0	0
e_3	0	0	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	0	0	0	0	0
e_4	0	0	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}\bar{z}$	0	0	0	0	0
e_5	0	0	0	0	0	0	-1	$\frac{1}{2}$	0	0	1	$-\frac{1}{2}$
e_6	0	$\frac{1}{2}\bar{w}$	0	0	0	0	0	0	0	$-\frac{1}{2}$	0	0
e_7	0	0	0	0	0	0	-1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	0
e_8	0	0	0	0	0	0	$\bar{z}$	$-\frac{1}{2}\bar{z}$	-1	$\frac{1}{2}$	0	0
e_9	$\frac{1}{2}\bar{w}$	$\bar{w}$	$-\frac{1}{2}$	-1	0	0	0	0	0	0	0	0
e_{10}	$-\frac{1}{2}\bar{w}$	$\bar{w}$	0	0	$\frac{1}{2}$	-1	0	0	0	0	0	0
e_{11}	0	$-\frac{1}{2}$	0	0	0	0	0	0	0	0	0	$\frac{1}{2}$
e_{12}	$\frac{1}{2}$	-1	$-\frac{1}{2}$	1	0	0	0	0	0	0	0	0

The crystal polynomial is:

$$p_{\mathcal{C}}(z,w) = (z-1)(z+1)(w-1)(w+1)$$

As in the last example, the RUM spectrum decomposes as a union of linear subsets. In the $[0,1)^2$ representation, it yields two horizontal and two vertical lines. From this, it follows that any almost periodic infinitesimal flex decomposes as a sum of two flexes, each of which is supercell periodic in one of the axial directions.

The following two examples, shown in Tables 7 and 8, have the same underlying graph and the same crystallographic point group, the dihedral group, $\mathcal{C}_{2v}$. However, the former has RUM dimension one and is linearly indecomposable while the latter has RUM dimension zero. Note that four-regular crystal frameworks such as these have a square symbol function.

Table 6. A crystal framework with supercell periodic flexes.

| **Motif**	**Translation group**	**Crystal framework**
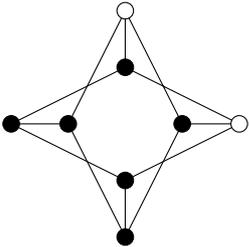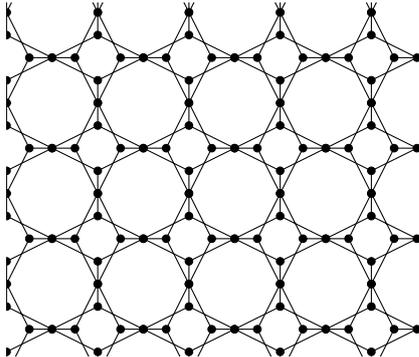
Example 7. *Let $\mathcal{C}$ be the crystal framework with $|F_e| = d|F_v|$ illustrated in Table 7. The symbol function, $\Phi_{\mathcal{C}}(z, w)$, is,*

$$
\begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & \tfrac{1}{2} & \tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & -\tfrac{1}{2} & \tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{\sqrt{3}}{2} & -\tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\tfrac{1}{2} & \tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\tfrac{1}{2} & \tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \tfrac{1}{2} & \tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\tfrac{1}{2} & \tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{\bar{z}}{2} & -\tfrac{\sqrt{3}\bar{z}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{\bar{z}}{2} & \tfrac{\sqrt{3}\bar{z}}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{z} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & -\tfrac{\bar{w}}{2} & \tfrac{\sqrt{3}\bar{w}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & 0 & 0 & \tfrac{\bar{w}}{2} & \tfrac{\sqrt{3}\bar{w}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & -\tfrac{1}{2} & \tfrac{\sqrt{3}}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{\sqrt{3}}{2} & 0 & 0 & -\tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}
$$

The crystal polynomial factorizes in to linear factors,

$$
p_{\mathcal{C}}(z, w) = (z + 1)(z - 1)^3
$$

Table 7. A crystal framework with the linearly decomposable RUM spectrum.

Motif	Translation group	Crystal framework
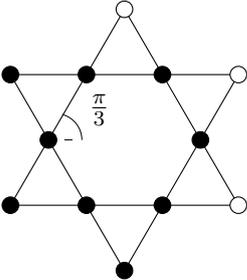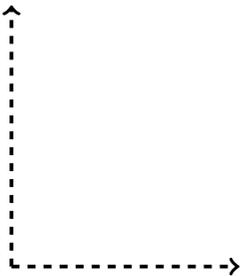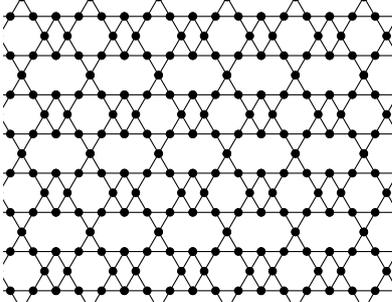
Example 8. *Let $\mathcal{C}$ be the crystal framework illustrated in Table 8. The framework motif satisfies $|F_e| = d|F_v|$, and so, the symbol function, $\Phi_{\mathcal{C}}(z, w)$, is a square matrix,*

$$
\begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -b & -a & b & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b & -a & -b & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & -b & -a & 0 & 0 & 0 & 0 & 0 & 0 \\
b & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & a & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a\bar{z} & -b\bar{z} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\bar{z}}{\sqrt{2}} & -\frac{\bar{z}}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\bar{z}}{\sqrt{2}} & \frac{\bar{z}}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & -a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a\bar{z} & -b\bar{z} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & -\frac{\bar{w}}{2} & \frac{\sqrt{3}\bar{w}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & \frac{\bar{w}}{2} & \frac{\sqrt{3}\bar{w}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 & 0 & 0 & 0 & 0 & 0 & -a & -b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
a & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & b & 0 & 0
\end{bmatrix}
$$

where $a = \frac{\sqrt{3}+1}{2\sqrt{2}}$ and $b = \frac{\sqrt{3}-1}{2\sqrt{2}}$. The crystal polynomial takes the form,

$$
p_{\mathcal{C}}(z, w) = z^4 w - \frac{1}{\sqrt{3}} z^3 w^2 + \left(\frac{\sqrt{3}}{2} - 2 \right) z^3 w - \frac{1}{2\sqrt{3}} z^3 + \left(\frac{1}{2} + \frac{1}{2\sqrt{3}} \right) z^2 w^2
$$

$$
+ \frac{1}{2\sqrt{3}} z^2 w - \frac{1}{\sqrt{3}} z^2 - \frac{1}{2} z w^2 + \left(\frac{3}{2} - \frac{1}{\sqrt{3}} \right) z w + \frac{1}{\sqrt{3}} z - \frac{1}{2} w
$$

which leads to a finite RUM spectrum.

Table 8. A crystal framework with finite RUM spectrum.

Motif	Translation group	Crystal framework

Example 9. *The crystal framework illustrated in Table 9 is based on a motif consisting of a regular octagon of equilateral triangles. The crystal polynomial is,*

$$p_{\mathcal{C}}(z, w) = p_1(z, w) p_2(z, w)$$

where

$$p_1(z, w) = (\sqrt{3} - \sqrt{2})z^2 w - zw^2 + 2(\sqrt{2} - \sqrt{3} + 1)zw + (\sqrt{3} - \sqrt{2})w - z$$

$$p_2(z, w) = (\sqrt{3} + \sqrt{2})z^2 w - zw^2 - 2(\sqrt{2} + \sqrt{3} - 1)zw + (\sqrt{3} + \sqrt{2})w - z$$

The RUM spectrum consists of points $(z, w) \in \mathbb{T}^2$ that satisfy $\Re(w) = a\Re(z) + (1 - a)$ for either $a = \sqrt{3} - \sqrt{2}$ or $a = \sqrt{3} + \sqrt{2}$. This set is illustrated in Figure 1 as a subset of the torus $[0, 1)^2$, which consists of four closed curves with the common intersection point $(0, 0)$.

**Table 9.** A crystal framework with the linearly indecomposable RUM spectrum.

Motif	Translation group	Crystal framework

Let $\tilde{\mathcal{C}}$ be the basic one-dimensional grid framework for the lattice, $\mathbb{Z}$ in $\mathbb{R}$. For any crystal framework, $\mathcal{C}$ in $\mathbb{R}^d$, one may construct a product framework $\tilde{\mathcal{C}} = \mathcal{C} \times \mathcal{C}_{\mathbb{Z}}$ in $\mathbb{R}^{d+1}$, whose intersection with the hyperplanes $\mathbb{R}^d \times \{n\}$ are copies of $\mathcal{C}$ and where these copies are connected by the edges $((p(v), n), (p(v), n + 1))$. In the case that $\mathcal{C}$ has square matrix symbol function $\Phi_{\mathcal{C}}(z_1, \ldots, z_d)$, it is straightforward to verify that:

$$p_{\tilde{\mathcal{C}}}(z_1, \ldots, z_{d+1}) = (z_{d+1} - 1)^{|F_v|} p_{\mathcal{C}}(z_1, \ldots, z_d)$$

This leads readily to the identification of the RUM spectrum in $\mathbb{T}^3$ of such frameworks. Further, three-dimensional examples not of this product form may be found in Power [8] and Wegner [12].

Example 10. *Let $\tilde{\mathcal{C}}$ be the three-dimensional framework derived from the regular octagon framework of Example 9. Then the crystal polynomial admits a three-fold factorisation and it follows that the RUM spectrum has the topological structure of four two-dimensional tori connected over the common circle of points $(1, 1, z)$ in $\mathbb{T}^3$.*

Figure 1. The RUM spectrum for Example 9.
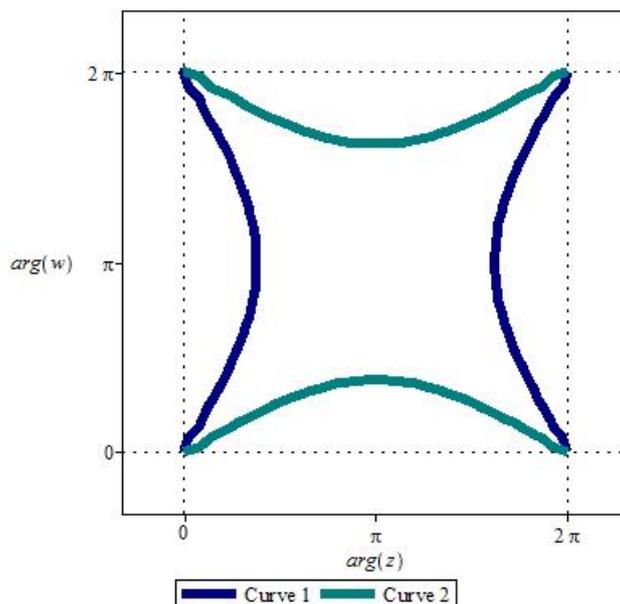
Further Work

The examples above and the foregoing analysis suggest a number of intriguing lines of investigation.

On the computational side, it would be of interest to explore the variation of the RUM spectrum with respect to parameters determining distinct placements, as in the pair given in Tables 7 and 8. Parameter curves of constant RUM dimension and their connections with crystallographic symmetry would be of particular interest.

On the theoretical side, it is natural now to seek characterisations of almost periodic infinitesimal rigidity and flexibility for frameworks that are almost periodic or quasicrystallographic. One class of such frameworks would require a vertex set that is an almost periodic perturbation (or even an incommensurate periodic perturbation) of a periodic framework vertex set, with the bar lengths adjusted accordingly.

Acknowledgments

Conflicts of Interest

The authors declare no conflicts of interest.

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