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Article

A New Route to the Majorana Equation

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Abstract: In this paper, we suggest an alternative strategy to derive the complex two-component Majorana equation with a mass term and elucidate the related Lorentz transformation. The Majorana equation is established completely on its own, rather than derived from the chiral Dirac equation. Thereby, use is made of the complex conjugation operator and Pauli spin matrices only. The eigenfunctions of the two-component complex Majorana equation are also calculated. The associated quantum fields are found to describe particles and antiparticles, which have opposite mean helicities and are not their own antiparticles, but correspond to two independent degrees of freedom. The four-component real Dirac equation in its Majorana representation is shown to be the natural outcome of the two-component complex Majorana equation. Both types of equations come in two forms, which correspond to the irreducible left- and right-chiral representations of the Lorentz group.

Keywords: Majorana field; chiral symmetry; Lorentz transformation

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1. Introduction

In this paper, we describe a novel route to the complex two-component Majorana equation, including a mass term. This equation is obtained here by a direct linearization of the Klein-Gordon equation (see, e.g., [1]) for a relativistic massive particle and requires the introduction of what we call rho matrices to be defined below. Our approach is similar to that used originally by Dirac [2] to obtain his famous gamma matrices and field equation. One main intention of this work is to obtain the Majorana equation "completely on its own rather than as an afterthought when treating the Dirac equation", as Case [3] phrased it half a century ago when he reformulated the Majorana theory of the neutrino on the basis of the Dirac equation in its chiral representation. We further address the question of whether Majorana particles are their own antiparticles. Following the pedagogical review on Majorana masses by Mannheim [4], we define antiparticles as being associated with the negative root of the dispersion relation and particles with the positive one, which is admittedly a non-standard definition. We consider the Majorana equation [5] from a new theoretical perspective and, thus, extend the established theoretical framework as described in the two monographs by [6,7] on this subject and on neutrino physics.

The present paper corroborates and expands the earlier work [8,9] on this subject and the largely tutorial papers by Aste [10] and Pal [11], who also discussed Weyl neutrinos [12] and defined the operation of Lorentz-invariant complex conjugation, which is important for the Majorana spinor field. At the outset, we here make use of only the complex conjugation operator and the Pauli spin matrices, corresponding to the irreducible representation of the Lorentz group. The connection to chiral symmetry is discussed, and the Lorentz transformation based on the rho matrices is established. Moreover, we show that Dirac's equation in the real Majorana [5] representation results directly from the new two-component complex Majorona equation, the eigenfunctions of which we also derive. With the results of this paper, we want to contribute to the ongoing discussion of whether massive neutrinos are Dirac or Majorana fermions and help in better understanding the latter theoretically in terms of the two-component spinor theory, which has extensively been reviewed in the recent report by Dreiner *et al.* [13]. Our approach is not yet contained in their comprehensive review paper.

Below, we will derive from scratch without recourse to the Dirac equation a complex two-component spinor field equation, from which the real four-component familiar Majorana representation of the Dirac equation naturally follows. The physical content of this new equation is essentially the same as that of the standard Majorana equation, insofar as it describes uncharged massive fermions of opposite mean kinetic helicity.

2. A New Route to the Majorana Equation

In this section, we present a direct route to the two-component-spinor complex Majorana equation, which is derived here in a novel covariant form and without recourse to the Dirac equation. We make use of standard symbols, notations and definitions and conventionally use units of $\hbar = c = 1$, with the covariant four-momentum operator denoted as $\mathcal{P}_\mu = \mathrm{i}\partial_\mu = \mathrm{i}(\partial/\partial t, \partial/\partial \mathbf{x})$, which acts on a two-component spinor field denoted as $\phi(\mathbf{x}, t)$ or a four-component spinor field denoted as $\psi(\mathbf{x}, t)$. The relativistic energy and momentum relation, $E = \sqrt{m^2 + \mathbf{p}^2}$, is, in fact, constitutive for the relativistic quantum field theory of a massive particle with mass m. This relation is usually written in manifestly covariant form as mass-shell condition, $p^\mu p_\mu = m^2$, where the contravariant four-momentum is denoted as $p^\mu = (E, \mathbf{p})$. Insertion of the differential operators into the mass shell condition yields the Klein-Gordon equation:

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \mathbf{x}^2} + m^2)\chi(\mathbf{x}, t) = 0 \tag{1}$$

for the scalar field, $\chi(\mathbf{x}, t)$. This equation only involves the second-order differential d'Alembert operator, which is known to be invariant under a Lorentz transformation.

Following Dirac [2], historically, one can derive his equation in a straightforward way from the linearization of the Klein-Gordon Equation (1), a result which is achieved by introducing four new operators, namely, the four-vector γ^μ. As shown in any text book (see, e.g., [1]), in terms of matrix algebra, one requires at least four dimensions to represent the gamma operators as 4×4 matrices. In order to satisfy the mass-shell requirement, they must algebraically obey the anti-commutator (indicated by the symbol, $\{,\}$) relation (with the Euclidean metric tensor, $g^{\mu\nu}$), reading:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2}$$

The gamma matrices have the property that $(\gamma^0)^2 = 1$ and $(\gamma^j)^2 = -1$, where j runs from one to three. In the Dirac representation, $(\gamma^0)^\dagger = \gamma^0$ is Hermitian, whereas γ^j is anti-Hermitian and, thus, obeys $(\gamma^j)^\dagger = -\gamma^j$. The resulting Dirac equation takes its standard form:

$$\mathrm{i}\gamma^\mu \partial_\mu \psi = m\psi \tag{3}$$

Majorana [5] was the first to show that there exist a purely imaginary representation of the gamma matrices, so that the corresponding four-component Dirac equation becomes real.

The new way of obtaining the two-component complex Majorana equation is here opened up by observing that there is a mathematical procedure that goes beyond pure matrix algebra, but relies on the operator of complex conjugation, which is denoted as C. It changes any complex number, z, into its complex conjugate, z^*, where the asterisk denotes complex conjugation. C will act on any complex number or spinor field, ϕ, right to it and transmutes it into its complex-conjugated version, ϕ^*. By definition, $\mathrm{C}^2 = 1$, and thus, $\mathrm{C}^\dagger = \mathrm{C}^{-1} = \mathrm{C}$. Therefore, the operator, C, has the real eigenvalues, ± 1, and the "eigenvectors", one and i, with $\mathrm{C}1 = 1$ and $\mathrm{Ci} = -\mathrm{i}$, and thus, C commutes with unity and anti-commutes with the imaginary unit. Consequently, the operator, C, is Hermitian and unitary, but anti-linear. Therefore, in any mathematical equation or operation, C can be shifted through to the outmost right side until it hits unity, from where on it can be omitted.

How can one construct an operator algebra similar to the Clifford algebra (2) of the Dirac gamma matrices? It can, in fact, be done by using only the two-dimensional Pauli matrices, but complemented by the operation of C. Of course, also in a two-component complex Majorana equation, we have to make use of the three Pauli matrices [14], since they are associated with the irreducible spinor representation of the rotation group. They are defined as:

$$\sigma_\mathrm{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\mathrm{y} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_\mathrm{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

Obviously, all three matrices mutually anti-commute with each other and, when being squared, are equal to the 2×2 unit matrix.

The important fourth operator is not of pure matrix nature, but involves the complex-conjugation operator, C. This new operator, which we may call τ, is defined as $\tau = \sigma_\mathrm{y}\mathrm{C}$ and obeys $\tau^2 = -1$, and thus, $\tau^\dagger = \tau^{-1} = -\tau$, which means that τ is anti-Hermitian. The Pauli matrix algebra together with the action of C have the consequence that the operation of τ on the spin vector, $\boldsymbol{\sigma}$, leads to its inversion, *i.e.*, the operation $\tau\boldsymbol{\sigma}\tau^{-1} = -\boldsymbol{\sigma}$ produces a spin flip. Equivalently, one has the anticommutator $\{\tau, \boldsymbol{\sigma}\} = \mathbf{0}$. Trivially, $\tau\mathrm{i} = -\mathrm{i}\tau$, because of the complex conjugation, C.

Similarly to the gamma matrices, we can now define the "rho" matrices, which are the components of the new contravariant four vector operator:

$$\rho^\mu = (\tau, \boldsymbol{\sigma}) \tag{5}$$

It obeys, by definition, the subsequent anti-commutator relation (earlier related versions were obtained already by Jehle [15] and Case [3]), which reads:

$$\{\rho^\mu, \rho^\nu\} = -2g^{\mu\nu} \tag{6}$$

Conversely to the gammas, the rhos have the property that $(\rho^0)^2 = -1$ and $(\rho^j)^2 = 1$, where j runs from one to three. Therefore, there appears a minus sign on the right-hand side of Equation (6), in contrast to the positive sign in Equation (2). In this representation, $(\rho^0)^\dagger = -\rho^0$ is anti-Hermitian, whereas ρ^j is Hermitian and, thus, obeys $(\rho^j)^\dagger = \rho^j$. It may appear odd that we did not choose a completely Hermitian representation, but (like for the gammas, as well) this is not possible, because a fully unitary representation of the Lorentz group does not exist [1]. In what follows, we can exploit the algebraic properties of Equation (6), yet must keep in mind that τ acts as the matrix, σ_y, but through C also transmutes any complex number to its right into its conjugated version.

When using the ρ^μ vector, the linearization of the Klein-Gordon Equation (1) is straightforward and results in a novel and non-standard form of the complex two-component Majorana equation for a massive fermion:

$$\rho^\mu \partial_\mu \phi = m\phi \tag{7}$$

More explicitly, this equation may be written as:

$$\left(\tau\frac{\partial}{\partial t} + \boldsymbol{\sigma}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\phi(\mathbf{x},t) = m\phi(\mathbf{x},t) \tag{8}$$

Apparently, this basic equation was established here without invoking the Dirac equation in the first place. Similar, but not quite the same, versions of the Majorana Equation (8) have been derived before by Marsch [8,9] and Aste [10]. As was shown many years ago by Case [3], all these previous forms of the Majorana equation can be obtained directly from Dirac's equation in its chiral representation, but not the above genuine version. By simply squaring Equation (7) and using Equation (6), we can retrieve the Klein-Gordon equation. We want to stress that the complex Majorana equation has two degrees of freedom less than the complex Dirac equation.

As is well known [1], the chiral matrix for the Dirac equation can be obtained as the four-fold product of all gamma matrices. Similarly, we can now define for the rhos the product:

$$\rho^5 = \mathrm{i}\rho^0\rho^1\rho^2\rho^3 = \mathrm{i}\tau\sigma_\mathrm{x}\sigma_\mathrm{y}\sigma_\mathrm{z} = \mathrm{i}\tau\mathrm{i} = \tau \tag{9}$$

Unlike in the case of the Dirac gamma matrices, ρ^5 gives no independent new matrix, but reproduces τ, which for the two-component complex Majorana equation apparently represents the operation of chirality conjugation.

It should be emphasized that, as a result of property (9), there exists a second version of the complex Majorana equation, which is obtained by considering that the Pauli matrix vector term in Equation (8) may have a plus or minus sign in front, as there are two irreducible representations of the Lorentz group, for which the sign at the spin operator has to be opposite. This inversion is accomplished by $\rho^5 = \tau$ and transforms (8) into another independent equation:

$$\left(\tau\frac{\partial}{\partial t} - \boldsymbol{\sigma}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\phi_-(\mathbf{x},t) = m\phi_-(\mathbf{x},t) \tag{10}$$

Consequently, the two solutions of Equations (8) and (10) are connected mathematically through the relations, $\phi_+ = \phi$, $\phi_- = \tau\phi_+$, respectively, $\phi_+ = -\tau\phi_-$; however, physically, they are to be considered as independent. The Equations (8) and (10) are the right- and left-chiral versions of the complex two-component Majorana equation, which can concisely be written as:

$$\rho^\mu_\pm \partial_\mu \phi_\pm(\mathbf{x}, t) = m\phi_\pm(\mathbf{x}, t) \tag{11}$$

This version of the rho matrix includes as a superscript the chiral index (plus for the right-chiral and minus for the left-chiral representation) and is defined as $\rho^\mu_\pm = (\tau, \pm\boldsymbol{\sigma})$. In this context, we refer again to the review of two-component spinor techniques by Dreiner *et al*. [13], which does not yet contain the present new approach.

It is worth noting that the single mass, m, in Equation (11) could in principle be different for the left- and right-chiral spinor field and, thus, may generally be replaced by $m_\pm$. However, we shall not consider this possibility here. For such an option, see, for example, the recent work by Aste [10] or the detailed books [6,7].

Now, let us derive the continuity equation and the Lagrangian density of the complex two-component Majorana equation. From the anticommutator in Equation (6), one finds that $\tau\boldsymbol{\sigma}\tau = \boldsymbol{\sigma}$, and tau is anti-Hermitian. Therefore, it is appropriate to define the conjugated spinor field:

$$\bar{\phi} = (\tau\phi)^\dagger = -\phi^\dagger\tau \tag{12}$$

which is the analogue to the spinor field $\bar{\psi} = (\gamma^0\psi)^\dagger$ for the Dirac Equation (3), but, here, still involves the operator, C. Note that $\phi^\dagger = \bar{\phi}\tau$. We obtain from Equation (8) the complex conjugated and transposed Majorana equation as follows:

$$\frac{\partial}{\partial t}(\tau\phi)^\dagger + \frac{\partial}{\partial \mathbf{x}}\phi^\dagger \cdot \boldsymbol{\sigma}^\dagger = m\phi^\dagger \tag{13}$$

Consequently, by then acting on Equation (13) from the right with τ, we obtain:

$$\partial_\mu \bar{\phi}\rho^\mu = -m\bar{\phi} \tag{14}$$

Multiplying this equation from the right by ϕ and Equation (7) from the left by $\bar{\phi}$ and adding them both up yields a continuity equation in the standard form:

$$\partial_\mu J^\mu = 0 \tag{15}$$

with the associated flux density defined as:

$$J^\mu = \bar{\phi}\rho^\mu\phi = \phi^\dagger(-\tau\rho^\mu)\phi = \phi^\dagger(1, \boldsymbol{\sigma}\tau)\phi \tag{16}$$

What is the nature of this conserved current? As shown later, the kinetic helicity operator is a conserved quantity, and therefore, the continuity equation found here is interpreted as being related to the helicity current density.

Finally, at this point, we want to quote the Lagrangian density of the Majorana field, which is given by:

$$\mathcal{L} = \bar{\phi}(\rho^\mu\partial_\mu - m)\phi \tag{17}$$

Variation of the Lagrangian resulting from Equation (17) with respect to $\bar{\phi}$ yields the original Majorana Equation (7) and, with respect to ϕ, the conjugated version Equation (14). The reader should note that there is no global U(1) symmetry here. The key point is that such a standard phase factor as $\exp(\mathrm{i}\alpha)$ does not commute with the operator τ.

3. Symmetries and Lorentz Transformation of the Majorana Equation

To calculate the transformation properties of the Majorana Equation (8) under a Lorentz transformation, let us first define how the two-component spinor field ϕ is transformed. The matrix representing the Lorentz group is denoted by $S(\Lambda)$, and primed quantities indicate the new coordinates according to the standard transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$. The transformed spinor field reads:

$$\phi'(x') = S(\Lambda)\phi(x) \tag{18}$$

The requirement that the basic Equation (7) is invariant under a Lorentz transformation (see, e.g., the textbook by Kaku [1]) then leads to the equation:

$$S^{-1}\rho^{\mu}S = \Lambda^{\mu}{}_{\nu}\rho^{\nu} \tag{19}$$

This statement ensures that the expectation value of ρ^{μ} transforms as a vector. Note that the conjugated spinor field $\bar{\phi}$ by its definition (12) transforms after (18) as:

$$\bar{\phi}'(x') = (\tau S\phi(x))^{\dagger} = -\bar{\phi}(x)\tau S^{\dagger}\tau = \bar{\phi}(x)S^{-1} \tag{20}$$

Consequently, $\bar{\phi}'(x')\phi'(x') = \bar{\phi}(x)\phi(x)$ is a scalar Lorentz invariant. The property $-\tau S^{\dagger}\tau = S^{-1}$ is proven below. To obtain an explicit representation of the matrix, $S(\Lambda)$, we introduce the following matrix tensor $\sigma^{\mu\nu}$, which reads:

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\rho^{\mu}, \rho^{\nu}] = \mathrm{i}(g^{\mu\nu} + \rho^{\mu}\rho^{\nu}) \tag{21}$$

By definition, $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$. Its elements are the generators of the Lorentz group in the two-component Majorana representation. To derive the second part of this equation, the anticommutator (6) was exploited. We recall that $\tau(\rho^{\mu})^{\dagger}\tau = \rho^{\mu}$ and, therefore, obtain the helpful relation $\tau(\sigma^{\mu\nu})^{\dagger}\tau = \sigma^{\mu\nu}$. Furthermore, the important connection holds:

$$[\mathrm{i}\sigma^{\mu\nu}, \rho^{\alpha}] = 2(-g^{\alpha\mu}\rho^{\nu} + g^{\alpha\nu}\rho^{\mu}) \tag{22}$$

A similar relation holds for the Dirac gamma matrices, however with the metric tensor being replaced by its negative, because of the different anticommutator as given in Equation (2). Therefore, the Lorentz transformation matrix for the two-component Majorana field is obtained as:

$$S(\Lambda) = \exp(\frac{\mathrm{i}}{4}\sigma^{\mu\nu}\omega_{\mu\nu}) \tag{23}$$

Note that the omega parameters determining a Lorentz transformation obey $\omega_{\mu\nu} = -\omega_{\nu\mu}$. The above transformation differs from that for the gamma matrices by the appearance of a positive sign in the exponential, which is due to the negative sign at the metric tensor in Equation (6) for the rhos. Consider only an infinitesimal transformation, then $\Lambda^{\mu}_{\nu} = g^{\mu}_{\nu} + \omega^{\mu}_{\nu}$, and the exponential function in Equation (23) can be expanded to the first order in the omegas. Then, Equation (19) reduces to the form:

$$\omega^{\alpha}_{\beta}\rho^{\beta} = -\frac{1}{4}[\mathrm{i}\sigma^{\mu\nu}, \rho^{\alpha}]\,\omega_{\mu\nu} \tag{24}$$

Making use of Equation (22) completes the proof that S is the correct transformation matrix.

By exploiting the above properties for the rhos, one can now also confirm readily the previously introduced relation $-\tau S^{\dagger}\tau = S^{-1}$, where S^{-1} just has the minus sign in the exponent of Equation (23). To validate this relation, we must expand Equation (23) in a power series and exploit, again, term by term, that $\tau(\sigma^{\mu\nu})^{\dagger}\tau = \sigma^{\mu\nu}$ holds. As a further result, the Lagrangian density Equation (17) is found to be Lorentz invariant and the above defined flux density J^{μ} to transform as a four-vector.

Let us now briefly discuss the symmetries of the Majorana equation as given in Equation (8). Generally speaking, if the Majorana equation is invariant under the symmetry operation, $\mathcal{S}$, then the spinor field $\phi^{\mathcal{S}} = \mathcal{S}\phi$ also fulfills that equation. We recall an important symmetry operator, namely the chirality operator $\rho^5 = \tau$ derived before, and we also consider, in particular, the parity, $\mathcal{P}$, and time reversal, $\mathcal{T}$, operations. We define conventionally the time and space coordinate inversion operations as $\mathbb{T}$ and $\mathbb{P}$, acting on a spinor field ϕ as follows, $\mathbb{T}\phi(\mathbf{x}, t) = \phi(\mathbf{x}, -t)$, and $\mathbb{P}\phi(\mathbf{x}, t) = \phi(-\mathbf{x}, t)$. We note that the coordinate reversal operators, $\mathbb{T}$ and $\mathbb{P}$, commute with τ and $\boldsymbol{\sigma}$. With these preparations in mind, it is easy to see which operators provide the requested symmetry operations. The operator of time reversal is $\mathcal{T} = \mathrm{i}\mathbb{T}$, of parity $\mathcal{P} = \tau\mathbb{P}$, and of chiral conjugation $\mathcal{C} = \tau$.

Let us first consider in Equation (8) the time reversal, $\mathcal{T}$. Apparently, it neither affects the mass term nor the momentum term, but it changes the sign of the first term twice and, thus, has no net effect. Therefore, also $\phi^{\mathcal{T}} = \mathrm{i}\phi(\mathbf{x}, -t)$ solves the Majorana equation. The parity operation, $\mathcal{P}$, commutes with the mass term and the first term in Equation (8) and it also leaves the momentum term invariant, since $\boldsymbol{\sigma}$ and $\mathbf{x}$ both change signs together, as τ anticommutes with the sigmas. Therefore, also $\phi^{\mathcal{P}} = \sigma_{\mathrm{y}}\phi^{*}(-\mathbf{x}, t)$ solves the Majorana equation. Finally, we consider the chiral conjugation defined as $\mathcal{C} = \rho^5 = \tau$. It changes the signs of the spatial derivative term in Equation (8), but has no effect on the time-derivate term and mass term. Therefore, $\phi^{\mathcal{C}} = \sigma_{\mathrm{y}}\phi^{*}(\mathbf{x}, t)$ does not solve the Majorana equation, but, actually, its chirality-conjugated version Equation (10). Apparently, the Majorana equation obeys parity and time reversal, yet maximally violates chiral symmetry.

4. Derivation of the Real Four-Component Majorana Equation

Here, we derive the well-known real four-component Majorana representation [5] of the Dirac equation from the complex two-component Majorana Equation (8). We can reformulate this equation and transfer it into a real form by decomposing the spinor field ϕ into its real and imaginary part: $\phi = \phi_R + \mathrm{i}\phi_I$. Making use of the three real 2×2 matrices defined as $\alpha = \sigma_{\mathrm{z}}, \beta = \sigma_{\mathrm{x}}, \gamma = -\mathrm{i}\sigma_{\mathrm{y}}$, we can decompose the complex Majorana Equation (8) into two real equations corresponding to the real and imaginary part of Equation (8) as follows:

$$\gamma\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial y}\right)\phi_I + \left(\beta\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial z}\right)\phi_R = m\phi_R \tag{25}$$

and similarly, we obtain:

$$\gamma\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}\right)\phi_R+\left(\beta\frac{\partial}{\partial x}+\alpha\frac{\partial}{\partial z}\right)\phi_I=m\phi_I \tag{26}$$

both of which can be combined, by introducing $\phi_{\pm}=\phi_R\pm\phi_I$, in a single equation:

$$\gamma\frac{\partial}{\partial t}\phi_{\pm}\pm\left(\beta\frac{\partial}{\partial x}+\alpha\frac{\partial}{\partial z}\right)\phi_{\pm}+\gamma\frac{\partial}{\partial y}\phi_{\mp}=\pm m\phi_{\pm} \tag{27}$$

Note that $\gamma^2=-1$, $\alpha^2=1$ and $\beta^2=1$. Furthermore, these matrices all anticommute, and thus, by squaring (27), one immediately retains for both ϕ_+ and ϕ_- the Klein-Gordon Equation (1), which, in fact, was our starting point when deriving a relativistic wave equation.

From these two real two-component equations (27), when both are combined into a single four-component one, the standard Dirac equation in the real Majorana representation follows immediately. Namely, we may arrange the two spinor fields $\phi_{\pm}$ into a single four-component real Dirac spinor field, $\psi^{\mathrm{T}}=(\phi_+^{\mathrm{T}},\phi_-^{\mathrm{T}})$, where the superscript, T, indicates the transposed spinor. Then, the coupled system of Equation (27) transforms into a 4×4 real matrix differential equation, which reads:

$$\begin{aligned}\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix}\frac{\partial\psi}{\partial t}+\begin{pmatrix}\beta & 0\\ 0 & -\beta\end{pmatrix}\frac{\partial\psi}{\partial x}+\begin{pmatrix}0 & \gamma\\ \gamma & 0\end{pmatrix}\frac{\partial\psi}{\partial y}&\\ +\begin{pmatrix}\alpha & 0\\ 0 & -\alpha\end{pmatrix}\frac{\partial\psi}{\partial z}=m\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\psi&\end{aligned} \tag{28}$$

Consequently, we can now introduce the subsequent real 4×4 Majorana matrices in their natural (as deduced from the Pauli matrices) representation:

$$\bar{\gamma}^{\mu}=\left(\begin{pmatrix}-\gamma & 0\\ 0 & \gamma\end{pmatrix},\begin{pmatrix}-\beta & 0\\ 0 & -\beta\end{pmatrix},\begin{pmatrix}0 & -\gamma\\ \gamma & 0\end{pmatrix},\begin{pmatrix}-\alpha & 0\\ 0 & -\alpha\end{pmatrix}\right) \tag{29}$$

They all mutually anticommute and obey:

$$\bar{\gamma}^{\mu}\bar{\gamma}^{\nu}+\bar{\gamma}^{\nu}\bar{\gamma}^{\mu}=-2g^{\mu\nu} \tag{30}$$

We use the barred symbols for the real part of the Dirac matrices in their purely imaginary Majorana representation. These real gamma matrices have the property that $(\bar{\gamma}^0)^2=-1$ and $(\bar{\gamma}^j)^2=1$, where j runs from one to three. Thus, in the Majorana representation, $(\bar{\gamma}^0)^{\mathrm{T}}=-\bar{\gamma}^0$ is antisymmetric, whereas $(\bar{\gamma}^j)^{\mathrm{T}}=\bar{\gamma}^j$ is symmetric, where the superscript, T, indicates the transposed matrix.

Finally, the real four-component-spinor Majorana equation can be written as:

$$\bar{\gamma}^{\mu}\partial_{\mu}\psi+m\psi=0 \tag{31}$$

which can, with the help of the purely imaginary Dirac matrices, $\gamma^{\mu}=\mathrm{i}\bar{\gamma}^{\mu}=(\gamma^0,\boldsymbol{\gamma})$, easily be brought into the standard form of the Dirac equation as quoted in Equation (3). For completeness, we also give the Dirac matrices in their Majorana representation as derived here:

$$\boldsymbol{\gamma}=\left(\begin{pmatrix}-\mathrm{i}\sigma_{\mathrm{x}} & 0\\ 0 & -\mathrm{i}\sigma_{\mathrm{x}}\end{pmatrix},\begin{pmatrix}0 & -\sigma_{\mathrm{y}}\\ \sigma_{\mathrm{y}} & 0\end{pmatrix},\begin{pmatrix}-\mathrm{i}\sigma_{\mathrm{z}} & 0\\ 0 & -\mathrm{i}\sigma_{\mathrm{z}}\end{pmatrix}\right) \tag{32}$$

where the spatial components are written as a three-vector $\boldsymbol{\gamma}$ and the temporal component is obtained as:

$$\gamma^0 = \begin{pmatrix} -\sigma_{\mathrm{y}} & 0 \\ 0 & \sigma_{\mathrm{y}} \end{pmatrix} \tag{33}$$

Therefore, the four-component Dirac equation is, in its above real Majorana representation (31), a direct consequence of the basic two-component complex Majorana Equation (8), which was derived here without invoking the Dirac equation in the first place. In analogy to the rho matrices used in Equation (11), we can also define the right- and left-chiral Majorana gamma matrices as follows:

$$\bar{\gamma}^{\mu}_{\pm} = (\bar{\gamma}^0, \pm\bar{\boldsymbol{\gamma}}) \tag{34}$$

with the three vector $\bar{\boldsymbol{\gamma}} = (\bar{\gamma}^1, \bar{\gamma}^2, \bar{\gamma}^3)$. The left- and right-chiral real Majorana equations, which are the analogues to complex Equation (11), then read:

$$\bar{\gamma}^{\mu}_{\pm}\partial_{\mu}\psi_{\pm}(\mathbf{x}, t) + m\psi_{\pm}(\mathbf{x}, t) = 0 \tag{35}$$

Consequently, the two solutions of Equation (35) are connected mathematically through the relations, $\psi_- = \bar{\gamma}^0\psi_+$, respectively $\psi_+ = -\bar{\gamma}^0\psi_-$; however, they must be considered as physically independent. Generally, we may again replace m by $m_{\pm}$, thus permitting different masses for the left- and right-chiral equation. We emphasize that the above Majorana equation is real. We can therefore define in analogy to Equation (12) the conjugated real spinor field:

$$\bar{\psi} = (\bar{\gamma}^0\psi)^{\mathrm{T}} = -\psi^{\mathrm{T}}\bar{\gamma}^0 \tag{36}$$

which appears in the transposed equation:

$$\frac{\partial}{\partial t}(\bar{\gamma}^0\phi)^{\mathrm{T}} + \frac{\partial}{\partial \mathbf{x}}\psi^{\mathrm{T}} \cdot \bar{\boldsymbol{\gamma}}^{\mathrm{T}} + m\psi^{\mathrm{T}} = 0 \tag{37}$$

Consequently, by then acting on Equation (37) from the right with $\bar{\gamma}^0$ and observing that $\bar{\gamma}^0\bar{\boldsymbol{\gamma}}^{\mathrm{T}}\bar{\gamma}^0 = \bar{\boldsymbol{\gamma}}$, we finally obtain:

$$\partial_{\mu}\bar{\psi}\bar{\gamma}^{\mu} - m\bar{\psi} = 0 \tag{38}$$

Multiplying this equation now from the right by ψ and Equation (31) from the left by $\bar{\psi}$ and adding them both up yields a continuity equation in the standard form $\partial_{\mu}\bar{J}^{\mu} = 0$, with the particle flux density defined as:

$$\bar{J}^{\mu} = \bar{\psi}\bar{\gamma}^{\mu}\psi = \psi^{\mathrm{T}}(-\bar{\gamma}^0\bar{\gamma}^{\mu})\psi = \psi^{\mathrm{T}}(1, \bar{\boldsymbol{\gamma}}\bar{\gamma}^0)\psi \tag{39}$$

At this point, we can again quote the Lagrangian density of the real Majorana field, which is given by:

$$\bar{\mathcal{L}} = \bar{\psi}(\bar{\gamma}^{\mu}\partial_{\mu} + m)\psi \tag{40}$$

Variation of the Lagrangian resulting from Equation (40) with respect to $\bar{\psi}$ yields the original real four-component Majorana Equation (31) and, with respect to ψ, conjugated version Equation (38).

5. Eigenfunctions and Quantum Fields of the Complex Two-Component Majorana Equation

We return again to the basic Majorana Equation (8) and now derive its eigenfunctions. For that purpose, we make the usual plane-wave ansatz:

$$\phi(\mathbf{x},t) = u\exp(-\mathrm{i}Et + \mathrm{i}\mathbf{p}\cdot\mathbf{x}) + v\exp(\mathrm{i}Et - \mathrm{i}\mathbf{p}\cdot\mathbf{x}) \tag{41}$$

We need both the plane wave function, as well as its complex conjugate, because of the complex conjugation operator C in τ. The two-component spinors, u and v, cannot simply be complex conjugated, since the wave function ϕ is not expected to be real. The resulting linked spinor equations are:

$$(m - \mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{p})u(\mathbf{p},E) = -\mathrm{i}E\tau v(\mathbf{p},E) \tag{42}$$

$$(m + \mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{p})v(\mathbf{p},E) = +\mathrm{i}E\tau u(\mathbf{p},E) \tag{43}$$

By insertion of the first into the second equation, or *vice versa*, the relativistic dispersion relation is obtained from:

$$\left((m - \mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{p})(m + \mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{p}) + (\mathrm{i}E\tau)^2\right)u(\mathbf{p},E) = 0 \tag{44}$$

which yields the two eigenvalues:

$$E_{1,2}(\mathbf{p}) = \pm E_0(p) = \pm\sqrt{m^2 + \mathbf{p}^2} \tag{45}$$

The negative root in Equation (45) is related to antiparticles, the positive one to particles. For the subsequent discussion, we introduce here the Majorana operator, $\mathcal{M}$, which is defined as follows:

$$\mathcal{M} = \tau\frac{\partial}{\partial t} + \boldsymbol{\sigma}\cdot\frac{\partial}{\partial\mathbf{x}} - m \tag{46}$$

The solution ϕ then must obey $\mathcal{M}\phi = 0$. Note that the four-momentum operator, $\mathcal{P}^\mu$, does not commute with $\mathcal{M}$, because of τ. Therefore, the desired eigenfunction ϕ of $\mathcal{M}$ cannot be an eigenfunction of the energy or momentum operator. However, the helicity operator, $\mathcal{H} = \boldsymbol{\sigma}\cdot\mathbf{P}$, with $\mathbf{P} = -\mathrm{i}\frac{\partial}{\partial\mathbf{x}}$, does commute with $\mathcal{M}$, as $\mathrm{i}\boldsymbol{\sigma}$ commutes with τ.

Consequently, it is convenient to express the spinors, u or v, in ϕ in terms of the eigenfunctions of the helicity operator. The eigenvalue equation of the helicity operator in Fourier space reads:

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})u_\pm(\hat{\mathbf{p}}) = \pm u_\pm(\hat{\mathbf{p}}) \tag{47}$$

These two eigenvectors depend only on the momentum unit vector $\hat{\mathbf{p}} = \mathbf{p}/p$ and can be written as:

$$u_+(\hat{\mathbf{p}}) = \begin{pmatrix} \cos\frac{\theta}{2}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\phi} \\ \sin\frac{\theta}{2}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\phi} \end{pmatrix}, \quad u_-(\hat{\mathbf{p}}) = \begin{pmatrix} -\sin\frac{\theta}{2}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\phi} \\ \cos\frac{\theta}{2}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\phi} \end{pmatrix} \tag{48}$$

in which the half angles of θ and ϕ appear. The eigenvectors for the same $\hat{\mathbf{p}}$ are orthogonal to each other and normalized to unity. They further obey the relation $u_\pm(-\hat{\mathbf{p}}) = \mathrm{i}u_\mp(\hat{\mathbf{p}})$. This is a consequence of Equation (48) and eigenvalue Equation (47), which implies that $u_\pm(\hat{\mathbf{p}})$ is an eigenvector of the helicity operator, corresponding to a right-handed, respectively, left-handed, screw with respect to the momentum direction. According to Equation (48), we have $u_\pm^\dagger(\hat{\mathbf{p}})u_\pm(\hat{\mathbf{p}}) = 1$ and $u_\mp^\dagger(\hat{\mathbf{p}})u_\pm(\hat{\mathbf{p}}) = 0$. The dagger denotes, as usual, the transposed and complex conjugated vector, respectively, matrix. The trace matrix elements of the spin operator between these two eigenvectors read:

$$u_{\pm}^{\dagger}(\hat{\mathbf{p}})\boldsymbol{\sigma}u_{\pm}(\hat{\mathbf{p}}) = \pm\hat{\mathbf{p}} \tag{49}$$

Straightforward calculation shows that the spin-flip operator $\tau = \sigma_y \mathrm{C}$, when operating on the above eigenspinors, leads to:

$$\tau u_{\pm}(\hat{\mathbf{p}}) = \pm \mathrm{i} u_{\mp}(\hat{\mathbf{p}}) \tag{50}$$

i.e., it connects the eigenfunctions of opposite helicity. Both are necessary to represent the polarization spinor of a massive relativistic fermion. To solve the coupled Equations (42) and (43), we can, because of Equation (50), make the ansatz:

$$u = u_+ \qquad \text{or} \qquad u = u_- \tag{51}$$

and, correspondingly:

$$v = c_+ u_- \qquad \text{or} \qquad v = c_- u_+ \tag{52}$$

with some coefficients, $c_{\pm}$, still to be determined. Insertion of these relations into Equations (42) and (43) yields the solutions:

$$c_+ = -\frac{m + \mathrm{i}p}{E} \tag{53}$$

$$c_- = +\frac{m - \mathrm{i}p}{E} \tag{54}$$

By putting these u and v back into the original ansatz Equation (41), we obtain two solutions of the Majorana equation $\mathcal{M}\phi = 0$, with $\phi = \phi(\mathbf{x}, t; \mathbf{p}, E)$. As an abbreviation, we write for the argument in the exponential plane wave function: $px = p^{\mu}x_{\mu} = Et - \mathbf{p}\mathbf{x}$. The two basic wave functions, $\phi_{1,2}$, are then given by:

$$\phi_1 = \frac{1}{\sqrt{2}}\left(u_+ \exp(-\mathrm{i}px) - \frac{m + \mathrm{i}p}{E} u_- \exp(\mathrm{i}px)\right) \tag{55}$$

$$\phi_2 = \frac{1}{\sqrt{2}}\left(u_- \exp(-\mathrm{i}px) + \frac{m - \mathrm{i}p}{E} u_+ \exp(\mathrm{i}px)\right) \tag{56}$$

Conveniently, we normalized these functions, such that their squared modules are equal to unity. When adding them together, we find a more general solution:

$$\phi = \frac{u_+ + u_-}{2}\exp(-\mathrm{i}px) + \frac{c_- u_+ + c_+ u_-}{2}\exp(\mathrm{i}px) \tag{57}$$

which also solves the Majorana equation. For the squared module of ϕ, we obtain the formula:

$$\phi^{\dagger}\phi = 1 + \frac{p}{E}\sin\left[2(Et - \mathbf{p}\cdot\mathbf{x})\right] \tag{58}$$

Addressing the continuity equation now, we find, for the time derivative of Equation (58), the result:

$$\frac{\partial}{\partial t}(\phi^{\dagger}\phi) = 2p\cos\left[2(Et - \mathbf{p}\cdot\mathbf{x})\right] \tag{59}$$

A somewhat more lengthy calculation for the divergence of the particle flux density gives the final result:

$$\frac{\partial}{\partial \mathbf{x}}\cdot(\phi^{\dagger}\boldsymbol{\sigma}\tau\phi) = -2p\cos\left[2(Et - \mathbf{p}\cdot\mathbf{x})\right] \tag{60}$$

and, therefore, the particle current density (16) obeys continuity Equation (15), as shown upon insertion of the general solution (57) into that equation. In the derivation of Equation (60), use was made of the relation, $\hat{\mathbf{p}} \cdot (u_\pm^\dagger \boldsymbol{\sigma} u_\mp) = 0$, which follows from Equation (47).

When using the τ operator, wave functions (55) and (56) (while indicating here explicitly in the subscript that we are dealing with the right-chiral field) can be written more concisely as:

$$\phi_{\mathrm{R}\,1,2} = \frac{1}{\sqrt{2}} \left(1 \mp \frac{p \mp \mathrm{i}m}{E} \tau\right) u_\pm \exp(-\mathrm{i}px) \tag{61}$$

Operating now with the Majorana operator, $\mathcal{M}$, on the wave function in this form, it is readily shown, with some algebra and the use of the properties of τ and $\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}$ given in Equations (47) and (50), that $\phi_{\mathrm{R}\,1,2}$ indeed obeys the Majorana Equation (8).

At this stage, we remind again that there is also the left-chiral Majorana Equation (10), the solution of which can be obtained as before from the above solution as $\phi_\mathrm{L} = \tau \phi_\mathrm{R}$. We also quote the two possible wave functions for the sake of completeness:

$$\phi_{\mathrm{L}\,1,2} = \frac{1}{\sqrt{2}} \left(\tau \pm \frac{p \pm \mathrm{i}m}{E}\right) u_\pm \exp(-\mathrm{i}px) \tag{62}$$

We return to the discussion of the right-chiral field and omit again the index, R, and now, recall that we found the two eigenvalues (45), which we identified with particles and antiparticles. The spinor vector space that is spanned by u_+ and u_- is only two-dimensional. Thus, we can only get two linearly-independent solutions for ϕ. It appears natural to identify the solution ϕ_1 with particles and, then, ϕ_2 with antiparticles, where we will conventionally use the inverse momentum, $-\mathbf{p}$, for the antiparticle. Consequently, the Majorana fermions discussed here are not their own antiparticles, but independent degrees of freedom of the massive Majorana field. Thus, we we get the two linearly-independent solutions,

$$\phi_\mathrm{P} = \phi_1(\mathbf{x}, t; \mathbf{p}, E_0(p)) \tag{63}$$

$$\phi_\mathrm{A} = \phi_2(\mathbf{x}, t; -\mathbf{p}, -E_0(p)) \tag{64}$$

Inserting the respective energy and momentum variables into Equation (61), we obtain:

$$\phi_{\mathrm{P,A}} = \frac{1}{\sqrt{2}} \left(1 - \frac{p \mp im}{E_0(p)} \tau\right) u_\pm(\pm\hat{\mathbf{p}}) \exp[\mp\mathrm{i}(E_0(p)t - \mathbf{p} \cdot \mathbf{x})]$$

For the subsequent calculations, it is useful to define the scalar product of two spinor fields or wave functions as follows:

$$(\phi, \phi') = \frac{1}{V} \int d^3x \phi^\dagger(\mathbf{x}) \phi'(\mathbf{x}) = (\phi', \phi)^* \tag{65}$$

which may be used to normalize any wave function to unity, by adjusting its free amplitude. The normalization volume is V. It is by their definition clear that $(\phi_{\mathrm{P,A}}, \phi_{\mathrm{P,A}}) = 1$, and we further get that the two wave functions are orthogonal, *i.e.*, we have $(\phi_{\mathrm{P,A}}, \phi_{\mathrm{A,P}}) = 0$, as the spatially varying parts of the product of the two wave functions are, on average, zero. That the helicity spinors, $u_\pm$, form an orthonormal set has also been used.

We may now complete wave functions (63) and (64) by noting that one can still multiply their exponentials with a complex amplitude, a, for the particle, and b, for the antiparticle, which must have a module of unity, $aa^* = |a|^2 = 1$ and $bb^* = |b|^2 = 1$, to ensure normalization, but which may depend on $\mathbf{p}$. The wave function written out more explicitly for the particle then reads:

$$\phi_{\rm P}(\mathbf{x},t;\mathbf{p}) = \tfrac{1}{\sqrt{2}}u_+(\hat{\mathbf{p}})a(\mathbf{p})\exp(-\mathrm{i}E_0(p)t + \mathrm{i}\mathbf{p}\cdot\mathbf{x}) \\ -\tfrac{1}{\sqrt{2}}\varepsilon_+(p)u_-(\hat{\mathbf{p}})a^*(\mathbf{p})\exp(\mathrm{i}E_0(p)t - \mathrm{i}\mathbf{p}\cdot\mathbf{x}) \tag{66}$$

which is normalized to unity. A similar expression for the antiparticle is obtained by inserting b and b^* into the spinor (64), yielding:

$$\phi_{\rm A}(\mathbf{x},t;\mathbf{p}) = \tfrac{\mathrm{i}}{\sqrt{2}}u_+(\hat{\mathbf{p}})b^*(\mathbf{p})\exp(\mathrm{i}E_0(p)t - \mathrm{i}\mathbf{p}\cdot\mathbf{x}) \\ -\tfrac{\mathrm{i}}{\sqrt{2}}\varepsilon_-(p)u_-(\hat{\mathbf{p}})b(\mathbf{p})\exp(-\mathrm{i}E_0(p)t + \mathrm{i}\mathbf{p}\cdot\mathbf{x}) \tag{67}$$

We introduced the useful abbreviation:

$$\varepsilon_\pm(p) = \frac{m \pm \mathrm{i}p}{E_0(p)} \tag{68}$$

which, by definition, obeys $\varepsilon_+\varepsilon_- = 1$, since $E_0^2 = (m + \mathrm{i}p)(m - \mathrm{i}p)$. Furthermore, one has $\varepsilon_\pm^* = \varepsilon_\mp$.

At this stage, the transition from a classical to a quantum Majorana field is obvious. We just have to replace the amplitudes, which are functions of $\mathbf{p}$ in Equations (66) and (67), by the canonical anti-commuting fermion operators obeying for the particle:

$$\{a(\mathbf{p}), a^\dagger(\mathbf{p}')\} = \delta_{\mathbf{p},\mathbf{p}'} \tag{69}$$

respectively, for the antiparticle:

$$\{b(\mathbf{p}), b^\dagger(\mathbf{p}')\} = \delta_{\mathbf{p},\mathbf{p}'} \tag{70}$$

Of course, all possible anticommutators between either a pair of creation, respectively annihilation, operators are zero. Mutually, between the as and bs, all anticommutators must vanish, since they represent independent degrees of freedom of the Majorana quantum field. As shown below, the operator $a^\dagger(\mathbf{p})$ creates and *vice versa* $a(\mathbf{p})$ annihilates the plane-wave state of a particle of positive helicity, with momentum $\mathbf{p}$ and energy $E_0 = \sqrt{m^2 + p^2}$. The b-operators do the same, yet for the associated antiparticle of opposite negative helicity. The operation of τ on the amplitudes, a and b, is clear. As we have $\tau a = a^*\tau$ and $\tau^{-1} = -\tau$, we conclude that $\tau^{-1}a(\mathbf{p})\tau = a^\dagger(\mathbf{p})$ and $a(\mathbf{p})\tau = \tau a^\dagger(\mathbf{p})$, which means that commuting a creation operator with τ transmutes it into its Hermitian conjugate, which is the related annihilation operator and *vice versa*.

The resulting quantum field operators corresponding to the above wavefunctions, $\phi_{\rm P,A}$, are denoted by a capital $\Phi_{\rm P,A}$. We use for the sake of notational simplicity again the abbreviation px for $E_0(p)t - \mathbf{px}$ in the exponentials, and, also, omit here the momentum arguments. The Majorana field operators read:

$$\Phi_{\rm P}(\mathbf{x},t;\mathbf{p}) = \frac{1}{\sqrt{2}}\left[u_+a\exp(-\mathrm{i}px) - \varepsilon_+u_-a^\dagger\exp(\mathrm{i}px)\right] \tag{71}$$

$$\Phi_{\rm A}(\mathbf{x},t;\mathbf{p}) = \frac{\mathrm{i}}{\sqrt{2}}\left[u_+b^\dagger\exp(\mathrm{i}px) - \varepsilon_-u_-b\exp(-\mathrm{i}px)\right] \tag{72}$$

whereby $\Phi_{\rm P}^\dagger\Phi_{\rm P} = 1/2$, as $a^\dagger a + aa^\dagger = 1$. The same relation holds for the antiparticle field operator. Remember that the four-momentum operator is defined as:

$$\mathcal{P}^\mu = \mathrm{i}(\frac{\partial}{\partial t}, -\frac{\partial}{\partial \mathbf{x}}) \tag{73}$$

With this operator, we get the following binary forms:

$$\Phi_{\rm P}^{\dagger}\mathcal{P}^{\mu}\Phi_{\rm P} = (E_0(p), \mathbf{p})(a^{\dagger}a - \frac{1}{2}) \tag{74}$$

$$\Phi_{\rm A}^{\dagger}\mathcal{P}^{\mu}\Phi_{\rm A} = (E_0(p), \mathbf{p})(b^{\dagger}b - \frac{1}{2}) \tag{75}$$

and for the helicity in Fourier representation:

$$\mathcal{H}_{\rm P}(\mathbf{p}) = \Phi_{\rm P}^{\dagger}(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\Phi_{\rm P} = +(a^{\dagger}a - \frac{1}{2}) \tag{76}$$

$$\mathcal{H}_{\rm A}(\mathbf{p}) = \Phi_{\rm A}^{\dagger}(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\Phi_{\rm A} = -(b^{\dagger}b - \frac{1}{2}) \tag{77}$$

Not unexpectedly, the particle has a positive mean helicity and the antiparticle, the opposite negative mean helicity. For the net total helicity, we thus obtain the sum:

$$\sum_{\mathbf{p}}(\mathcal{H}_{\rm P}(\mathbf{p}) + \mathcal{H}_{\rm A}(\mathbf{p})) = \sum_{\mathbf{p}}\left(a^{\dagger}(\mathbf{p})a(\mathbf{p}) - b^{\dagger}(\mathbf{p})b(\mathbf{p})\right) \tag{78}$$

We can now sum up over all momenta to obtain the complete Majorana quantum field operator as follows:

$$\Phi(\mathbf{x}, t) = \sum_{\mathbf{p}}(\Phi_{\rm P}(\mathbf{x}, t; \mathbf{p}) + \Phi_{\rm A}(\mathbf{x}, t; \mathbf{p})) \tag{79}$$

Note that this is not a Hermitian operator. The expectation value of any Hermitian operator, $\mathcal{O}$, for the Majorana quantum field Φ is defined through the scalar product of the related wave functions. Therefore, we have the definition:

$$< \mathcal{O} >= (\Phi, \mathcal{O}\Phi) \tag{80}$$

For the total four momentum operator, we thus obtain:

$$< \mathcal{P}^{\mu} >= \sum_{\mathbf{p}}(E_0(p), \mathbf{p})\left(a^{\dagger}(\mathbf{p})a(\mathbf{p}) + b^{\dagger}(\mathbf{p})b(\mathbf{p})\right) \tag{81}$$

where, conventionally, an infinite constant, $\sum_{\mathbf{p}} E_0(p)$, has been discarded as the irrelevant zero-point energy. The sum over all momenta vanishes naturally. In the derivation of Equation (81), using the anticommutation relations and relations (69) and (70) were essential, and further, spatially oscillating mixing terms yield on average no contribution to the expectation value.

Finally, we note that the equal-time anticommutator of the field (79) can be written as:

$$\{\Phi_{\alpha}(\mathbf{x}, t), \Phi_{\beta}^{\dagger}(\mathbf{x}', t)\} = \delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{x}') \tag{82}$$

which is a consequence of relations (69) and (70) and the completeness relation, which is obeyed by the polarization eigenvectors and which reads:

$$u_{+\alpha}^{\dagger}u_{+\beta} + u_{-\alpha}^{\dagger}u_{-\beta} = \delta_{\alpha\beta} \tag{83}$$

where the indices, α and β, run over one and two. This equation can be readily verified by insertion of the components of the helicity eigenvectors given in Equation (48). As a result, the field, $\mathrm{i}\Phi^{\dagger}(\mathbf{x}, t)$, is the conjugated generalized momentum of the Majorana field $\Phi(\mathbf{x}, t)$, as is to be expected for any fermion quantum field.

6. Summary and Conclusions

The principal goal of this paper was to present an alternative strategy to derive the complex two-component Majorana equation and to discuss the related Lorentz transformation. The Majorana equation was here established completely on its own, rather than as a derivative of the chiral Dirac equation. Thereby, we made use only of the complex conjugation operator and the Pauli spin matrices. We also calculated the eigenfunctions of the two-component complex Majorana equation and, thereby, exploited the properties of the spin-flip or chiral conjugation operator. The associated Majorana quantum fields were found to describe particles and antiparticles with opposite mean helicities. The massive Majorana fermions as discussed here are not their own antiparticles, but they correspond to the two independent possible degrees of freedom of the complex two-component Majorana spinor field. We showed that the four-component Dirac equation in its real Majorana representation is the natural outcome of the genuine, two-component complex Majorana equation. Both types of equations come in two forms, corresponding to the irreducible left- and right-chiral representations of the Lorentz group, are independent and may thus involve different particle masses. The Majorana fields cannot be further decomposed and are to be considered as fundamental.

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Conflicts of Interest

The author declares no conflict of interest.

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