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Article

Supersymmetric Version of the Euler System and Its Invariant Solutions

A. Michel Grundland [1,2,*] and Alexander J. Hariton [1]

[1] Centre de Recherches Mathématiques, Université de Montréal, C. P. 6128, Succ. Centre–ville, Montréal, QC H3C 3J7, Canada; E-Mail: hariton@crm.umontreal.ca

[2] Department of Mathematics and Computer Science, Université du Québec, C.P. 500, Trois–Rivières, QC G9A 5H7, Canada

* Author to whom correspondence should be addressed; E-Mail: grundlan@crm.umontreal.ca; Fax: +1-514-343-2254.

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Abstract: In this paper, we formulate a supersymmetric extension of the Euler system of equations. We compute a superalgebra of Lie symmetries of the supersymmetric system. Next, we classify the one-dimensional subalgebras of this superalgebra into 49 equivalence conjugation classes. For some of the subalgebras, the invariants have a non-standard structure. For nine selected subalgebras, we use the symmetry reduction method to find invariants, orbits and reduced systems. Through the solutions of these reduced systems, we obtain solutions of the supersymmetric Euler system. The obtained solutions include bumps, kinks, multiple wave solutions and solutions expressed in terms of arbitrary functions.

Keywords: supersymmetric models; lie superalgebras; symmetry reduction

Classification: MSC 35Q31; 76M60; 81T60

1. Introduction

Supersymmetric generalizations of fluid dynamical equations have recently generated much interest among mathematicians and physicists. Such extensions have been constructed recently for certain hydrodynamic-type models (see, e.g., [1–4]). A study of polytropic supersymmetric models has also been performed by Das and Popowicz [5] in the specific case when $p = k\rho^k$. Euler-type systems have also been analyzed from the Grassmann variable point of view by Fatyga, Kostelecky and Truax [6].

Certain solutions were found in terms of infinite series. In addition, supersymmetric versions of the Chaplygin gas in $(1 + 1)$ and $(2 + 1)$ dimensions, derived from the action for a superstring and a Nambu-Goto supermembrane, respectively, were proposed by Jackiw *et al.* (see [7] and references therein). It was suggested that non-Abelian fluid mechanics may describe a quark-gluon plasma [7]. In this context, we propose to investigate the Euler equations for a one-dimensional compressible fluid flow.

The purpose of this paper is to formulate a supersymmetric version of the Euler system of equations describing the nonstationary one-dimensional ideal (non-viscous) compressible fluid flow and construct their invariant solutions. To our knowledge, a supersymmetric extension of the Euler system has not been formulated before. The symmetry group transformations of the classical version of the Euler system of equations:

$$\begin{aligned} &\rho u_t + \rho u u_x + p_x = 0 \\ &\rho_t + u\rho_x + \rho u_x = 0 \\ &p_t + u p_x + k p u_x = 0 \end{aligned} \tag{1}$$

has been investigated by Ovsiannikov [8], where u is the flow velocity, p and ρ are the pressure and density of the fluid flow, respectively, and k is the polytropic exponent. Under the above assumptions, the examined system of Equations (1) forms a quasilinear hyperbolic system. It was proven [8,9] that the basic system (1) is invariant under the Galilean similitude group, $\mathrm{sim}(2)$ (when $k \neq 3$), and under the Galilean-projective group, $\mathrm{simPr}(2)$ (*i.e.*, the Galilean-similitude group extended by the projective transformation when $k = 3$). The Lie algebra of infinitesimal symmetries of fluid dynamics Equations (1) is spanned by:

$$\begin{aligned} &P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad B = t\partial_x + \partial_u, \qquad D = t\partial_t + x\partial_x, \\ &F = -t\partial_t + u\partial_u - 2\rho\partial_\rho, \qquad G = \rho\partial_\rho + p\partial_p, \\ &C = t^2\partial_t + tx\partial_x + (x - tu)\partial_u - t\rho\partial_\rho - 3tp\partial_p, \qquad \text{when } k = 3 \end{aligned} \tag{2}$$

The infinitesimal operators, P_0 and P_1, are translation operators in the directions of t and x, respectively. The operator, B, corresponds to the Galilean transformation, and operators D, F and G are dilation operators. Operator C corresponds to the projective conformal transformation, when the specific value of the polytropic exponent is $k = 3$. The Lie algebra, $\mathfrak{L}$, generated by (2) can be decomposed in the form:

$$\mathfrak{L} = \{C, P_0, F, D, G\} \Supset \{P_1, B\} \tag{3}$$

In the work of one of the authors of this paper [10], all subalgebras of the Lie algebra, $\mathfrak{L}$, were classified, and the symmetry reduction method (SRM) was used to generate invariant solutions of the Euler system (1), including, among other solutions, travelling waves, bumps, kinks, double-periodic solutions, *etc*. Many of these solutions had previously been found by other methods, such as the method of characteristics (see, for example, [8,9,11,12] and references therein).

The SRM, on the other hand, certainly has a broader range of application, since it leads to different sets of solutions obtained from the Lie algebra of symmetries (2) and its subalgebras classified into different conjugacy classes. This approach is of great utility in the investigation of the classical Euler Equations (1). The results obtained for the classical system (1) were so promising that it seemed worthwhile to try to apply the method to the case of a supersymmetric extension of a hydrodynamic-type system and construct invariant solutions through the use of a superspace and superfield formalism.

This paper is organized as follows. In Section 2, we construct a supersymmetric extension of the Euler-Equations (1) through a superspace and superfield formalism involving two independent fermionic variables. In Section 3, we discuss in detail a number of symmetries of the proposed supersymmetric system. For this purpose, we use a generalization of the prolongation formalism of vector fields extended in such a way as to encompass both even and odd Grassmann variables. In this case, we adapt the symmetry criterion in order to determine a superalgebra associated with the supersymmetric system. Next, we classify the one-dimensional subalgebras of the superalgebra into conjugation classes under the action of the associated supergroup. The subalgebras are classified in such a way that each representative subalgebra corresponding to a different conjugation class generates a different type of solution [13–16]. This analysis allows us to perform the symmetry reductions systematically. Section 4 contains the reduced systems corresponding to those subalgebras with its standard invariant structures. A new method is proposed for solving the reduced systems of the equations. Several new invariant solutions of the supersymmetric Euler Equation are constructed, some of which contain the freedom of arbitrary functions of one or two arguments involving expressions in terms of bosonic or fermionic variables. In the case when the arbitrary functions are expressed only in the independent fermionic variables, we are able to obtain the solution based on a truncated Taylor expansion. We give some physical interpretation of the obtained results in the context of non-linear field models. Finally, Section 5 contains some final remarks and future perspectives.

2. Supersymmetric Extension

In order to supersymmetrize the Euler Equation (1), we extend the space of independent variables, $\{x, t\}$, to the superspace, $\{x, t, \theta_1, \theta_2\}$, where θ_1 and θ_2 are independent Grassmannian-odd variables. The variables, x and t, represent the bosonic (even Grassmannian) coordinates on two-dimensional Euclidean space. The quantities, θ_1 and θ_2, are anticommuting fermionic (odd Grassmann) variables. We replace the real bosonic-valued fields of velocity, $u(x, t)$, density, $\rho(x, t)$, and pressure, $p(x, t)$, by the Grassmann-even superfields defined by:

$$\begin{aligned}
\mathcal{U}(x, t, \theta_1, \theta_2) &= F(x, t) + \theta_1\phi_1(x, t) + \theta_2\phi_2(x, t) + \theta_1\theta_2 u(x, t) \\
\mathcal{R}(x, t, \theta_1, \theta_2) &= G(x, t) + \theta_1\psi_1(x, t) + \theta_2\psi_2(x, t) + \theta_1\theta_2 \rho(x, t) \\
\mathcal{P}(x, t, \theta_1, \theta_2) &= H(x, t) + \theta_1\omega_1(x, t) + \theta_2\omega_2(x, t) + \theta_1\theta_2 p(x, t)
\end{aligned} \tag{4}$$

respectively. Here, the quantities, ϕ_i, ψ_i and ω_i $i = 1, 2$, are fermionic fields, while F, G and H are bosonic fields. The supersymmetric extension of the Euler Equations (1) is constructed in such a way that it remains invariant under the supersymmetry transformations:

$$x \to x - \underline{\eta_1}\theta_1, \qquad \theta_1 \to \theta_1 + \underline{\eta_1} \qquad \text{and} \qquad t \to t - \underline{\eta_2}\theta_2, \qquad \theta_2 \to \theta_2 + \underline{\eta_2} \tag{5}$$

where $\underline{\eta_1}$ and $\underline{\eta_2}$ are arbitrary constant odd parameters. In what follows, we use the convention that underlined constants represent Grassmann-odd parameters. These transformations are generated by the infinitesimal supersymmetry operators:

$$Q_1 = -\theta_1\partial_x + \partial_{\theta_1} \qquad \text{and} \qquad Q_2 = -\theta_2\partial_t + \partial_{\theta_2} \tag{6}$$

satisfying the anticommutation relations:

$$\{Q_i, Q_i\} = -2\partial_{x^i}, \quad i = 1, 2 \tag{7}$$

where x^1 and x^2 stand for x and t, respectively. In order to make the extended model invariant under the supersymmetry generators, Q_1 and Q_2, we introduce the covariant derivative operators:

$$D_1 = \theta_1\partial_x + \partial_{\theta_1} \qquad \text{and} \qquad D_2 = \theta_2\partial_t + \partial_{\theta_2} \tag{8}$$

which possess the property that each of the derivatives, D_i, $i = 1, 2$, anticommutes with both supersymmetry operators, Q_i, $i = 1, 2$:

$$[D_i, D_i] = 2\partial_{x^i}, \qquad \{D_1, D_2\} = \{D_i, Q_j\} = 0 \quad \text{for} \quad i, j = 1, 2 \tag{9}$$

Thus, if we write our supersymmetry equations in terms of the superfields, $\mathcal{U}$, $\mathcal{R}$ and $\mathcal{P}$, and their covariant derivatives of various orders, then they will be invariant under the transformations generated by Q_1 and Q_2. The most general supersymmetric extension of the Euler Equation (1) invariant under the transformations generated by Q_1 and Q_2 is constructed by considering linear combinations of the products of the various covariant derivatives of the superfields, $\mathcal{U}$, $\mathcal{R}$ and $\mathcal{P}$. These multiply together to produce the given terms as coefficients, whose components reproduce each term of the classical Equation (1). The result of this analysis provides the following form of the supersymmetric Euler Equations:

$$\begin{aligned}
&- a_1\mathcal{R}(D_1{D_2}^3\mathcal{U}) + a_2(D_1\mathcal{R})({D_2}^3\mathcal{U}) - a_3(D_2\mathcal{R})(D_1{D_2}^2\mathcal{U}) + \\
&(a_1 + a_2 + a_3 - 1)(D_1D_2\mathcal{R})({D_2}^2\mathcal{U}) + a_4\mathcal{R}(D_1D_2\mathcal{U})({D_1}^3D_2\mathcal{U}) \\
&- a_5(D_1\mathcal{R})(D_2\mathcal{U})({D_1}^3D_2\mathcal{U}) - a_6(D_1\mathcal{R})(D_1D_2\mathcal{U})({D_1}^2D_2\mathcal{U}) \\
&+ a_7(D_2\mathcal{R})(D_1\mathcal{U})({D_1}^3D_2\mathcal{U}) + a_8(D_2\mathcal{R})(D_1D_2\mathcal{U})({D_1}^3\mathcal{U}) + a_9(D_1D_2\mathcal{R})\mathcal{U}({D_1}^3D_2\mathcal{U}) \\
&- a_{10}(D_1D_2\mathcal{R})(D_1\mathcal{U})({D_1}^2D_2\mathcal{U}) + a_{11}(D_1D_2\mathcal{R})(D_2\mathcal{U})({D_1}^3\mathcal{U}) \\
&+ \left(1 - \sum_{i=4}^{11} a_i\right)(D_1D_2\mathcal{R})(D_1D_2\mathcal{U})({D_1}^2\mathcal{U}) + {D_1}^2\mathcal{P} = 0
\end{aligned}$$

$$\begin{aligned}
&{D_2}^2\mathcal{R} - a_{12}\mathcal{U}({D_1}^3D_2\mathcal{R}) + a_{13}(D_1\mathcal{U})({D_1}^2D_2\mathcal{R}) - a_{14}(D_2\mathcal{U})({D_1}^3\mathcal{R}) \\
&+ (a_{12} + a_{13} + a_{14} - 1)(D_1D_2\mathcal{U})({D_1}^2\mathcal{R}) - a_{15}\mathcal{R}({D_1}^3D_2\mathcal{U}) + a_{16}(D_1\mathcal{R})({D_1}^2D_2\mathcal{U}) \\
&- a_{17}(D_2\mathcal{R})({D_1}^3\mathcal{U}) + (a_{15} + a_{16} + a_{17} - 1)(D_1D_2\mathcal{R})({D_1}^2\mathcal{U}) = 0
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&{D_2}^2\mathcal{P} - a_{18}\mathcal{U}({D_1}^3D_2\mathcal{P}) + a_{19}(D_1\mathcal{U})({D_1}^2D_2\mathcal{P}) - a_{20}(D_2\mathcal{U})({D_1}^3\mathcal{P}) \\
&+ (a_{18} + a_{19} + a_{20} - 1)(D_1D_2\mathcal{U})({D_1}^2\mathcal{P}) - a_{21}k\mathcal{P}({D_1}^3D_2\mathcal{U}) + a_{22}k(D_1\mathcal{P})({D_1}^2D_2\mathcal{U}) \\
&- a_{23}k(D_2\mathcal{P})({D_1}^3\mathcal{U}) + (a_{21} + a_{22} + a_{23} - 1)k(D_1D_2\mathcal{P})({D_1}^2\mathcal{U}) = 0
\end{aligned}$$

where $a_1, a_2, \ldots, a_{23}$ are twenty-three arbitrary bosonic parameters. In this paper, we focus our analysis on the simplest case, where all of the parameters, a_i, $i = 1, 2, \ldots, 23$, vanish:

$$\begin{aligned}(D_1D_2\mathcal{R})(D_2{}^2\mathcal{U}) - (D_1D_2\mathcal{R})(D_1D_2\mathcal{U})(D_1{}^2\mathcal{U}) - D_1{}^2\mathcal{P} = 0\\ D_2{}^2\mathcal{R} - (D_1D_2\mathcal{U})(D_1{}^2\mathcal{R}) - (D_1D_2\mathcal{R})(D_1{}^2\mathcal{U}) = 0\\ D_2{}^2\mathcal{P} - (D_1D_2\mathcal{U})(D_1{}^2\mathcal{P}) - k(D_1D_2\mathcal{P})(D_1{}^2\mathcal{U}) = 0\end{aligned} \tag{11}$$

In what follows, we will refer to system (11) as the supersymmetric extension of the Euler Equations, since it reduces to the classical Euler Equation (1) when θ_1 and θ_2 tend to zero.

In this paper, we use the convention that partial derivatives involving odd variables satisfy the Leibniz rule:

$$\partial_{\theta_i}(fg) = (\partial_{\theta_i}f)g + (-1)^{\deg(f)}f(\partial_{\theta^i}g) \tag{12}$$

where:

$$\deg(f) = \begin{cases} 0 & \text{if} \quad f \text{ is even,} \\ 1 & \text{if} \quad f \text{ is odd,} \end{cases} \tag{13}$$

and the notation:

$$f_{\theta_1\theta_2} = \frac{\partial}{\partial\theta_2}\left(\frac{\partial f}{\partial\theta_1}\right) \tag{14}$$

The partial derivatives with respect to odd coordinates satisfy $\partial_{\theta_i}\theta_j = \delta^i_j$. The operators, ∂_{θ_i}, Q_1, Q_2, D_1 and D_2, change the parity of the function acted on in the sense that it converts a bosonic function to a fermionic function and *vice versa*. For example, $\partial_{\theta_1}\mathcal{U}$ is an odd superfield, while $\partial_{\theta_1\theta_2}\mathcal{U}$ is an even superfield, and so on. For further details, see the book by Cornwell [17] and the reference by DeWitt [18]. The chain rule for a Grassmann-valued composite function, $f(g(x))$, is [18,19]

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}\cdot\frac{\partial f}{\partial g} \tag{15}$$

The interchangeability of mixed derivatives (with proper respect to the ordering of odd variables) is assumed throughout.

The even supernumbers, variables, fields, *etc.*, are assumed to be elements of the even part, Λ_{even}, of the underlying abstract real Grassmann ring, Λ. The odd supernumbers, variables, fields, *etc.*, lie in its odd part, Λ_{odd}. We shall assume throughout the paper that the functions, $u(x,t)$, $\rho(x,t)$ and $p(x,t)$, have values in the invertible subset of Λ_{even} plus $\{0\}$, *i.e.*, nonvanishing nilpotent values are ruled out. This technical assumption allows us to perform the necessary simplifications in our calculations without splitting off of singular subcases.

Re-writing Equation (11) in terms of the independent variables, x, t, θ_1 and θ_2, we obtain the form:

$$\begin{aligned}&\theta_1\theta_2\mathcal{R}_{xt}\mathcal{U}_t + \theta_1\mathcal{R}_{x\theta_2}\mathcal{U}_t - \theta_2\mathcal{R}_{t\theta_1}\mathcal{U}_t - \mathcal{R}_{\theta_1\theta_2}\mathcal{U}_t + \theta_1\theta_2\mathcal{R}_{xt}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_x - \theta_1\theta_2\mathcal{R}_{x\theta_2}\mathcal{U}_{t\theta_1}\mathcal{U}_x\\ &+ \theta_1\mathcal{R}_{x\theta_2}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_x + \theta_1\theta_2\mathcal{R}_{t\theta_1}\mathcal{U}_{x\theta_2}\mathcal{U}_x - \theta_2\mathcal{R}_{t\theta_1}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_x + \theta_1\theta_2\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{xt}\mathcal{U}_x\\ &+ \theta_1\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{x\theta_2}\mathcal{U}_x - \theta_2\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{t\theta_1}\mathcal{U}_x - \mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_x - \mathcal{P}_x = 0\end{aligned} \tag{16}$$

$$\begin{aligned}&\mathcal{R}_t - \theta_1\theta_2\mathcal{U}_{xt}\mathcal{R}_x - \theta_1\mathcal{U}_{x\theta_2}\mathcal{R}_x + \theta_2\mathcal{U}_{t\theta_1}\mathcal{R}_x + \mathcal{U}_{\theta_1\theta_2}\mathcal{R}_x - \theta_1\theta_2\mathcal{R}_{xt}\mathcal{U}_x - \theta_1\mathcal{R}_{x\theta_2}\mathcal{U}_x\\ &+ \theta_2\mathcal{R}_{t\theta_1}\mathcal{U}_x + \mathcal{R}_{\theta_1\theta_2}\mathcal{U}_x = 0\end{aligned} \tag{17}$$

$$\mathcal{P}_t - \theta_1\theta_2\mathcal{U}_{xt}\mathcal{P}_x - \theta_1\mathcal{U}_{x\theta_2}\mathcal{P}_x + \theta_2\mathcal{U}_{t\theta_1}\mathcal{P}_x + \mathcal{U}_{\theta_1\theta_2}\mathcal{P}_x - k\theta_1\theta_2\mathcal{P}_{xt}\mathcal{U}_x - k\theta_1\mathcal{P}_{x\theta_2}\mathcal{U}_x + k\theta_2\mathcal{P}_{t\theta_1}\mathcal{U}_x + k\mathcal{P}_{\theta_1\theta_2}\mathcal{U}_x = 0 \quad (18)$$

The system (16)–(18) is a second-order nonlinear system of partial differential Equations, whose coefficients depend on the fermionic variables.

3. Symmetries of the Supersymmetric Euler Equations

A symmetry supergroup, G, of a supersymmetric system is a (local) supergroup of transformations acting on a Cartesian product of supermanifolds, $X \times U$, where X is the space of independent variables, $(x, t, \theta_1, \theta_2)$, and U is the space of dependent superfields, $(\mathcal{U}, \mathcal{R}, \mathcal{P})$. The action of the group, G, on the function, $\mathcal{U}(x, t, \theta_1, \theta_2)$, $\mathcal{R}(x, t, \theta_1, \theta_2)$ and $\mathcal{P}(x, t, \theta_1, \theta_2)$, maps solutions of (11) to solutions of (11). If we assume that G is a Lie supergroup, as described in [20,21], one can associate with it its Lie superalgebra, $\mathfrak{G}$, whose elements are infinitesimal symmetries of (11).

The supersymmetric Euler Equation (11) are invariant under the Lie superalgebra, $\mathfrak{S}$, generated by the following seven infinitesimal vector fields:

$$\begin{aligned} &L_1 = 2x\partial_x + \theta_1\partial_{\theta_1} + 3\mathcal{U}\partial_{\mathcal{U}} + 4\mathcal{P}\partial_{\mathcal{P}}, \qquad L_2 = 2t\partial_t + \theta_2\partial_{\theta_2} - \mathcal{U}\partial_{\mathcal{U}} - 4\mathcal{P}\partial_{\mathcal{P}}, \\ &L_3 = \mathcal{R}\partial_{\mathcal{R}} + \mathcal{P}\partial_{\mathcal{P}}, \qquad P_1 = \partial_x, \qquad P_2 = \partial_t, \qquad Q_1 = -\theta_1\partial_x + \partial_{\theta_1} \\ &Q_2 = \theta_2\partial_t + \partial_{\theta_2} \end{aligned} \quad (19)$$

The generators, P_1 and P_2, represent translations in space and time, respectively, while L_1, L_2 and L_3 generate dilations in both even and odd variables. In addition, we recover the supersymmetry transformations, Q_1 and Q_2, which we identified previously in (6). In order to determine this superalgebra of infinitesimal symmetries, we have made use of the theory described in the book by Olver [22]. The commutation (anticommutation in the case of two odd operators) relations of the superalgebra, $\mathfrak{S}$, of the supersymmetric Euler Equations are given in Table 1.

Table 1. Supercommutation table for the Lie superalgebra, $\mathfrak{S}$, spanned by the vector fields (19).

	$\mathbf{L_1}$	$\mathbf{P_1}$	$\mathbf{Q_1}$	$\mathbf{L_2}$	$\mathbf{P_2}$	$\mathbf{Q_2}$	$\mathbf{L_3}$
$\mathbf{L_1}$	0	$-2P_1$	$-Q_1$	0	0	0	0
$\mathbf{P_1}$	$2P_1$	0	0	0	0	0	0
$\mathbf{Q_1}$	Q_1	0	$-2P_1$	0	0	0	0
$\mathbf{L_2}$	0	0	0	0	$-2P_2$	$-Q_2$	0
$\mathbf{P_2}$	0	0	0	$2P_2$	0	0	0
$\mathbf{Q_2}$	0	0	0	Q_2	0	$-2P_2$	0
$\mathbf{L_3}$	0	0	0	0	0	0	0

It should be noted that the generators of the superalgebra, $\mathfrak{S}$, listed in (19), contain three dilations, L_1, L_2 and L_3. This is analogous to the Lie algebra, $\mathfrak{L}$, of the classical version, whose generators (in Equation (2)) also contain three dilations, D, F and G. The decomposition of the superalgebra, $\mathfrak{S}$, of the supersymmetric system is very different from the decomposition of the Lie algebra of the classical system, $\mathfrak{L}$.

The Lie superalgebra, $\mathfrak{S}$, can be decomposed into the following combination of direct and semi-direct sums:

$$\mathfrak{S} = (\{L_1\} \supset\!\!\!+ \{P_1, Q_1\}) \oplus (\{L_2\} \supset\!\!\!+ \{P_2, Q_2\}) \oplus \{L_3\} \tag{20}$$

The one-dimensional subalgebras of $\mathfrak{S}$ can be classified into conjugacy classes under the action of the group generated by $\mathfrak{S}$. We obtain the following list of subalgebras:

$$\begin{aligned}
&\mathcal{L}_1 = \{P_2\}, \quad \mathcal{L}_2 = \{\underline{\nu} Q_2\}, \quad \mathcal{L}_3 = \{P_2 + \underline{\nu} Q_2\}, \quad \mathcal{L}_4 = \{L_2\},\\
&\mathcal{L}_5 = \{P_1\}, \quad \mathcal{L}_6 = \{\underline{\mu} Q_1\}, \quad \mathcal{L}_7 = \{P_1 + \underline{\mu} Q_1\}, \quad \mathcal{L}_8 = \{L_1\},\\
&\mathcal{L}_9 = \{P_1 + \varepsilon P_2\}, \quad \mathcal{L}_{10} = \{P_1 + \underline{\nu} Q_2\}, \quad \mathcal{L}_{11} = \{P_1 + \varepsilon P_2 + \underline{\nu} Q_2\},\\
&\mathcal{L}_{12} = \{L_2 + \varepsilon P_1\}, \quad \mathcal{L}_{13} = \{P_2 + \underline{\mu} Q_1\}, \quad \mathcal{L}_{14} = \{\underline{\mu} Q_1 + \underline{\nu} Q_2\},\\
&\mathcal{L}_{15} = \{P_2 + \underline{\mu} Q_1 + \underline{\nu} Q_2\}, \quad \mathcal{L}_{16} = \{L_2 + \underline{\mu} Q_1\}, \quad \mathcal{L}_{17} = \{P_1 + \varepsilon P_2 + \underline{\mu} Q_1\},\\
&\mathcal{L}_{18} = \{P_1 + \underline{\mu} Q_1 + \underline{\nu} Q_2\}, \quad \mathcal{L}_{19} = \{P_1 + \varepsilon P_2 + \underline{\mu} Q_1 + \underline{\nu} Q_2\},\\
&\mathcal{L}_{20} = \{L_2 + \varepsilon P_1 + \underline{\mu} Q_1\} \quad \mathcal{L}_{21} = \{L_1 + \varepsilon P_2\}, \quad \mathcal{L}_{22} = \{L_1 + \underline{\nu} Q_2\},\\
&\mathcal{L}_{23} = \{L_1 + \varepsilon P_2 + \underline{\nu} Q_2\}, \quad \mathcal{L}_{24} = \{L_1 + k L_2\}, \quad \mathcal{L}_{25} = \{L_3\},\\
&\mathcal{L}_{26} = \{L_3 + \varepsilon P_2\}, \quad \mathcal{L}_{27} = \{L_3 + \underline{\nu} Q_2\}, \quad \mathcal{L}_{28} = \{L_3 + \varepsilon P_2 + \underline{\nu} Q_2\},\\
&\mathcal{L}_{29} = \{L_3 + k L_2\} \quad \mathcal{L}_{30} = \{L_3 + \varepsilon P_1\}, \quad \mathcal{L}_{31} = \{L_3 + \underline{\mu} Q_1\},\\
&\mathcal{L}_{32} = \{L_3 + \varepsilon P_1 + \underline{\mu} Q_1\}, \quad \mathcal{L}_{33} = \{L_3 + k L_1\} \quad \mathcal{L}_{34} = \{L_3 + \varepsilon_1 P_1 + \varepsilon_2 P_2\},\\
&\mathcal{L}_{35} = \{L_3 + \varepsilon P_1 + \underline{\nu} Q_2\}, \quad \mathcal{L}_{36} = \{L_3 + \varepsilon_1 P_1 + \varepsilon_2 P_2 + \underline{\nu} Q_2\},\\
&\mathcal{L}_{37} = \{L_3 + k L_2 + \varepsilon P_1\}, \quad \mathcal{L}_{38} = \{L_3 + \varepsilon P_2 + \underline{\mu} Q_1\},\\
&\mathcal{L}_{39} = \{L_3 + \underline{\mu} Q_1 + \underline{\nu} Q_2\}, \quad \mathcal{L}_{40} = \{L_3 + \varepsilon P_2 + \underline{\mu} Q_1 + \underline{\nu} Q_2\},\\
&\mathcal{L}_{41} = \{L_3 + k L_2 + \underline{\mu} Q_1\}, \quad \mathcal{L}_{42} = \{L_3 + \varepsilon_1 P_1 + \varepsilon_2 P_2 + \underline{\mu} Q_1\},\\
&\mathcal{L}_{43} = \{L_3 + \varepsilon P_1 + \underline{\mu} Q_1 + \underline{\nu} Q_2\}, \quad \mathcal{L}_{44} = \{L_3 + \varepsilon_1 P_1 + \varepsilon_2 P_2 + \underline{\mu} Q_1 + \underline{\nu} Q_2\},\\
&\mathcal{L}_{45} = \{L_3 + k L_2 + \varepsilon P_1 + \underline{\mu} Q_1\} \quad \mathcal{L}_{46} = \{L_3 + k L_1 + \varepsilon P_2\},\\
&\mathcal{L}_{47} = \{L_3 + k L_1 + \underline{\nu} Q_2\}, \quad \mathcal{L}_{48} = \{L_3 + k L_1 + \varepsilon P_2 + \underline{\nu} Q_2\},\\
&\mathcal{L}_{49} = \{L_3 + k L_1 + \ell L_2\}
\end{aligned} \tag{21}$$

where ε, ε_1 and ε_2 are either one or -1, the parameters, k and ℓ, are non-zero, and $\underline{\mu}$ and $\underline{\nu}$ are fermionic constants. These representative subalgebras allow us to determine invariant solutions of the supersymmetric Euler Equations (11) using the symmetry reduction method. It should be noted that some of the subalgebras of the Lie symmetry superalgebra have non-standard invariant structures in the sense that they do not lead to symmetry reductions in the usual sense. For instance, the subalgebra, $\mathcal{L}_2 = \{\underline{\nu} Q_2\}$, has the nonstandard invariant, $\underline{\nu} f(x, t, \theta_1, \theta_2, \mathcal{U}, \mathcal{R}, \mathcal{P})$, where f is an arbitrary function of its arguments. Such non-standard invariants were found by the authors for several other supersymmetric hydrodynamic-type systems, including the supersymmetric sinh-Gordon Equation [23], the supersymmetric Klein-Gordon polynomial Equation [23] and supersymmetric polytropic gas dynamics [4].

4. Invariant Solutions of the Supersymmetric Euler Equations

We now proceed to use the symmetry reduction method to determine the invariants and orbits of nine different subalgebras. Consequently, we obtain the reduced system specific to the given subalgebra.

(1) For subalgebra $\mathcal{L}_1 = \{P_2\}$, the invariants are

$$\sigma = x, \qquad \theta_1, \qquad \theta_2, \qquad \mathcal{U}, \qquad \mathcal{R}, \qquad \mathcal{P}.$$

Therefore, the superfields are of the form:

$$\mathcal{U}(x,\theta_1,\theta_2), \qquad \mathcal{R}(x,\theta_1,\theta_2), \qquad \mathcal{P}(x,\theta_1,\theta_2),$$

and the reduced system is:

$$\begin{aligned}
&\theta_1\mathcal{R}_{x\theta_2}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_x + \theta_1\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{x\theta_2}\mathcal{U}_x - \mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_x - \mathcal{P}_x = 0,\\
&-\theta_1\mathcal{U}_{x\theta_2}\mathcal{R}_x + \mathcal{U}_{\theta_1\theta_2}\mathcal{R}_x - \theta_1\mathcal{R}_{x\theta_2}\mathcal{U}_x + \mathcal{R}_{\theta_1\theta_2}\mathcal{U}_x = 0\\
&-\theta_1\mathcal{U}_{x\theta_2}\mathcal{P}_x + \mathcal{U}_{\theta_1\theta_2}\mathcal{P}_x - k\theta_1\mathcal{P}_{x\theta_2}\mathcal{U}_x + k\mathcal{P}_{\theta_1\theta_2}\mathcal{U}_x = 0
\end{aligned} \tag{22}$$

One solution of this system is:

$$\begin{aligned}
\mathcal{U} &= k_1 - \underline{k_3\mu}x + \theta_1\underline{k_3} + \theta_2\underline{k_2}\\
\mathcal{R} &= \gamma(x) + \theta_1\psi_1(x) + \theta_2\underline{k_4}\\
\mathcal{P} &= k_5 + \theta_1\underline{k_6} + \theta_2\underline{k_7}
\end{aligned} \tag{23}$$

where k_1 and k_5 are arbitrary bosonic constants, $\underline{k_2}$, $\underline{k_3}$, $\underline{k_4}$, $\underline{k_6}$ and $\underline{k_7}$ are arbitrary fermionic constants, γ is an arbitrary bosonic function of x and ψ_1 is an arbitrary fermionic function of x. This solution is stationary. The choice of arbitrary functions in the density superfield, $\mathcal{R}$, does not influence the velocity and pressure superfields ($\mathcal{U}$ and $\mathcal{P}$, respectively), which are linear in x, θ_1 and θ_2.

(2) For subalgebra, $\mathcal{L}_3 = \{P_2 + \underline{\nu}Q_2\}$, the invariants are:

$$\sigma = x, \qquad \theta_1, \qquad \tau_2 = \theta - \underline{\nu}t, \qquad \mathcal{U}, \qquad \mathcal{R}, \qquad \mathcal{P},$$

Therefore, the superfields are of the form:

$$\mathcal{U}(x,\theta_1,\tau_2), \qquad \mathcal{R}(x,\theta_1,\tau_2), \qquad \mathcal{P}(x,\theta_1,\tau_2),$$

and the reduced system is:

$$
\begin{aligned}
&\theta_1\underline{\nu}\mathcal{R}_{x\tau_2}\mathcal{U}_{\tau_2} + \underline{\nu}\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{\tau_2} - \theta_1\tau_2\underline{\nu}\mathcal{R}_{x\tau_2}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_x + \theta_1\tau_2\mathcal{R}_{x\tau_2}\underline{\nu}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_x + \theta_1\mathcal{R}_{x\tau_2}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_x\\
&- \theta_1\tau_2\underline{\nu}\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{x\tau_2}\mathcal{U}_x + \tau_2\underline{\nu}\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_x - \theta_1\tau_2\underline{\nu}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_{x\tau_2}\mathcal{U}_x + \theta_1\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{x\tau_2}\mathcal{U}_x\\
&+ \tau_2\underline{\nu}\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_x - \mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_x - \mathcal{P}_x = 0\\[2ex]
&- \underline{\nu}\mathcal{R}_{\tau_2} + \theta_1\tau_2\underline{\nu}\mathcal{U}_{x\tau_2}\mathcal{R}_x - \theta_1\mathcal{U}_{x\tau_2}\mathcal{R}_x - \tau_2\underline{\nu}\mathcal{U}_{\theta_1\tau_2}\mathcal{R}_x + \mathcal{U}_{\theta_1\tau_2}\mathcal{R}_x + \theta_1\tau_2\underline{\nu}\mathcal{R}_{x\tau_2}\mathcal{U}_x\\
&- \theta_1\mathcal{R}_{x\tau_2}\mathcal{U}_x - \tau_2\underline{\nu}\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_x + \mathcal{R}_{\theta_1\tau_2}\mathcal{U}_x = 0\\[2ex]
&- \underline{\nu}\mathcal{P}_{\tau_2} + \theta_1\tau_2\underline{\nu}\mathcal{U}_{x\tau_2}\mathcal{P}_x - \theta_1\mathcal{U}_{x\tau_2}\mathcal{P}_x - \tau_2\underline{\nu}\mathcal{U}_{\theta_1\tau_2}\mathcal{P}_x + \mathcal{U}_{\theta_1\tau_2}\mathcal{P}_x + k\theta_1\tau_2\underline{\nu}\mathcal{P}_{x\tau_2}\mathcal{U}_x\\
&- k\theta_1\mathcal{P}_{x\tau_2}\mathcal{U}_x - k\tau_2\underline{\nu}\mathcal{P}_{\theta_1\tau_2}\mathcal{U}_x + k\mathcal{P}_{\theta_1\tau_2}\mathcal{U}_x = 0
\end{aligned}
\tag{24}
$$

We have two solutions. The first solution is:

$$
\begin{aligned}
\mathcal{U} &= \underline{\nu}h(x) + \theta_1\phi_1(x) + \theta_2\underline{\nu}\ell(x) + \theta_1\theta_2\underline{\nu}f(x)\\
\mathcal{R} &= \underline{\nu}j(x) + \theta_2\underline{\nu}g(x)\\
\mathcal{P} &= C_1 + \theta_2\underline{\nu}C_2
\end{aligned}
\tag{25}
$$

where C_1 and C_2 are arbitrary bosonic constants, ℓ, and g are arbitrary bosonic functions of x and f and h, j and ϕ_1 are arbitrary fermionic functions of x. This solution represents a stationary flow. The choice of arbitrary functions in $\mathcal{U}$ and $\mathcal{R}$ does not influence the pressure superfield, $\mathcal{P}$, which is linear in θ_2. The arbitrary functions can be chosen completely arbitrarily. In particular, we can choose bumps, kinks and multiple wave solutions. The postulated form of the arbitrary functions will change as the solution evolves in time. For example, as an illustration, if we choose the arbitrary function:

$$
g(x) = \mathrm{cn}\left[1 + \cosh\left(\arctan\left(ax\right)\right)^{-1}, \kappa\right], \quad a \in \mathbb{R} \tag{26}
$$

and the modulus, κ, of the elliptic function is such that $0 \leq \kappa^2 \leq 1$, then the solution has one purely real and one purely imaginary period. For a real argument of the elliptic function, we have the relation:

$$
-1 \leq \mathrm{cn}\left[1 + \cosh\left(\arctan\left(ax\right)\right)^{-1}, \kappa\right] \leq 1 \tag{27}
$$

and so, the solution represents a cnoidal wave bump. Asymptotically, this solution approaches a constant. The second solution is:

$$
\begin{aligned}
\mathcal{U} &= \underline{h}(\mu x + C_3\theta_1 + \underline{\mu}\underline{\nu}\theta_2)\\
\mathcal{R} &= \gamma(x) + \theta_1\psi_1(x) + \theta_2(\underline{\nu}x + \underline{\nu}) + \theta_1\theta_2(\underline{h}\underline{\mu}) - \theta_1\underline{\nu}t(\underline{h}\underline{\mu})\\
\mathcal{P} &= C_1 + \theta_1\underline{C_4} + \theta_2\underline{C_2} - \underline{\nu}t\underline{C_2} + \theta_1\theta_2\underline{\nu}\underline{C_5}
\end{aligned}
\tag{28}
$$

where C_1 and C_3 are arbitrary bosonic constants, $\underline{h}$, $\underline{C_2}$, $\underline{C_4}$ and $\underline{C_5}$ are arbitrary fermionic constants, γ is an arbitrary bosonic function of x and ψ_1 is an arbitrary fermionic function of x. This solution is non-stationary, and the choice of $\gamma(x)$ and $\psi_1(x)$ in the density superfield, $\mathcal{R}$, does not influence $\mathcal{U}$ and $\mathcal{P}$. The time dependence in the pressure superfield, $\mathcal{P}$, does not affect $\mathcal{U}$ and $\mathcal{R}$.

(3) For subalgebra, $\mathcal{L}_4 = \{L_2\}$, the invariants are:

$$\sigma = x, \qquad \theta_1, \qquad \tau_2 = t^{-1/2}\theta_2, \qquad \mathcal{A} = t^{1/2}\mathcal{U}, \qquad \mathcal{R}, \qquad \mathcal{C} = t^2\mathcal{P}.$$

Therefore, we have the superfields:

$$\mathcal{U} = t^{-1/2}\mathcal{A}(x,\theta_1,\tau_2), \qquad \mathcal{R}(x,\theta_1,\tau_2), \qquad \mathcal{P} = t^{-2}\mathcal{C}(x,\theta_1,\tau_2).$$

The corresponding system is given by:

$$\begin{aligned}&-\frac{1}{2}\theta_1\mathcal{R}_{x\tau_2}\mathcal{A} + \frac{1}{2}\theta_1\tau_2\mathcal{R}_{x\tau_2}\mathcal{A}_{\tau_2} + \frac{1}{2}\mathcal{R}_{\theta_1\tau_2}\mathcal{A} + \frac{1}{2}\tau_2\mathcal{R}_{\theta_1\tau_2}\mathcal{A}_{\tau_2} + \frac{1}{2}\theta_1\tau_2\mathcal{R}_{x\tau_2}\mathcal{A}_{\theta_1}\mathcal{A}_x\\&+\theta_1\mathcal{R}_{x\tau_2}\mathcal{A}_{\theta_1\tau_2}\mathcal{A}_x - \frac{1}{2}\theta_1\tau_2\mathcal{R}_{\theta_1\tau_2}(\mathcal{A}_x)^2 + \frac{1}{2}\tau_2\mathcal{R}_{\theta_1\tau_2}\mathcal{A}_{\theta_1}\mathcal{A}_x + \theta_1\mathcal{R}_{\theta_1\tau_2}\mathcal{A}_{x\tau_2}\mathcal{A}_x\\&-\mathcal{R}_{\theta_1\tau_2}\mathcal{A}_{\theta_1\tau_2}\mathcal{A}_x - \mathcal{C}_x = 0\end{aligned} \tag{29}$$

$$\frac{1}{2}\theta_1\tau_2\mathcal{A}_x\mathcal{R}_x - \frac{1}{2}\tau_2\mathcal{A}_{\theta_1}\mathcal{R}_x - \theta_1\mathcal{A}_{x\tau_2}\mathcal{R}_x + \mathcal{A}_{\theta_1\tau_2}\mathcal{R}_x - \theta_1\mathcal{R}_{x\tau_2}\mathcal{A}_x + \mathcal{R}_{\theta_1\tau_2}\mathcal{A}_x - \frac{1}{2}\tau_2\mathcal{R}_{\tau_2} = 0 \tag{30}$$

$$\begin{aligned}&-2\mathcal{C} - \frac{1}{2}\tau_2\mathcal{C}_{\tau_2} + \frac{1}{2}\theta_1\tau_2\mathcal{A}_x\mathcal{C}_x - \frac{1}{2}\tau_2\mathcal{A}_{\theta_1}\mathcal{C}_x - \theta_1\mathcal{A}_{x\tau_2}\mathcal{C}_x + \mathcal{A}_{\theta_1\tau_2}\mathcal{C}_x + 2k\theta_1\tau_2\mathcal{C}_x\mathcal{A}_x\\&-2k\tau_2\mathcal{C}_{\theta_1}\mathcal{A}_x - k\theta_1\mathcal{C}_{x\tau_2}\mathcal{A}_x + k\mathcal{C}_{\theta_1\tau_2}\mathcal{A}_x = 0\end{aligned} \tag{31}$$

We obtain the solution:

$$\begin{aligned}&\mathcal{U} = t^{-1/2}\theta_1\underline{\mu}f(x)\\&\mathcal{R} = \theta_1\underline{\mu}g(x)\\&\mathcal{P} = 0\end{aligned} \tag{32}$$

where f and g are arbitrary bosonic functions of x. This solution is non-stationary. If the functions, $f(x)$ and $g(x)$, are chosen to be bounded or periodic, then they remain bounded, and the velocity superfield, $\mathcal{U}$, is dampened with time, t.

(4) For subalgebra, $\mathcal{L}_7 = \{P_1 + \underline{\mu}Q_1\}$, the invariants are:

$$\sigma = t, \qquad \tau_1 = \theta_1 - \underline{\mu}x, \qquad \theta_2, \qquad \mathcal{U}, \qquad \mathcal{R}, \qquad \mathcal{P}.$$

Therefore, we have the superfields:

$$\mathcal{U} = \mathcal{U}(t,\tau_1,\theta_2), \qquad \mathcal{R} = \mathcal{R}(t,\tau_1,\theta_2), \qquad \mathcal{P} = \mathcal{P}(t,\tau_1,\theta_2).$$

The corresponding system is given by:

$$\begin{aligned}&-\underline{\mu}\tau_1\theta_2\mathcal{R}_{t\tau_1}\mathcal{U}_t - \underline{\mu}\tau_1\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_t - \theta_2\mathcal{R}_{t\tau_1}\mathcal{U}_t - \mathcal{R}_{\tau_1\theta_2}\mathcal{U}_t + \underline{\mu}\theta_2\mathcal{R}_{t\tau_1}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_{\tau_1}\\&+\underline{\mu}\theta_2\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{t\tau_1}\mathcal{U}_{\tau_1} + \underline{\mu}\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_{\tau_1} + \underline{\mu}\mathcal{P}_{\tau_1} = 0\\[2ex]&\mathcal{R}_t - \underline{\mu}\theta_2\mathcal{U}_{t\tau_1}\mathcal{R}_{\tau_1} - \underline{\mu}\mathcal{U}_{\tau_1\theta_2}\mathcal{R}_{\tau_1} - \underline{\mu}\theta_2\mathcal{R}_{t\tau_1}\mathcal{U}_{\tau_1} - \underline{\mu}\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\tau_1} = 0\\[2ex]&\mathcal{P}_t - \underline{\mu}\theta_2\mathcal{U}_{t\tau_1}\mathcal{P}_{\tau_1} - \underline{\mu}\mathcal{U}_{\tau_1\theta_2}\mathcal{P}_{\tau_1} - k\underline{\mu}\theta_2\mathcal{P}_{t\tau_1}\mathcal{U}_{\tau_1} - k\underline{\mu}\mathcal{P}_{\tau_1\theta_2}\mathcal{U}_{\tau_1} = 0\end{aligned} \tag{33}$$

We have two solutions. The first solution is:

$$\begin{aligned}
\mathcal{U} &= \alpha(t) + (\theta_1 - \underline{\mu}x)\phi_1(t) + \theta_2\phi_2(t) + (\theta_1 - \underline{\mu}x)\theta_2\beta(t) \\
\mathcal{R} &= C_4 + \theta_1\underline{\mu}\ell \\
\mathcal{P} &= C_3 + \theta_1\underline{\mu}C_2
\end{aligned} \tag{34}$$

where C_2, C_3, C_4 and ℓ are arbitrary bosonic constants, α and β are arbitrary bosonic functions of t and ϕ_1 and ϕ_2 are arbitrary fermionic functions of t. This solution is non-stationary, and the velocity, $\mathcal{U}$, does not influence the other superfields, $\mathcal{R}$ and $\mathcal{P}$.

It should be noted that both supersymmetries, Q_1 and Q_2, are broken by the solution (34). Indeed, the supersymmetric transformation generated by Q_1 is: $x \to x - \underline{\eta_1}\theta_1, \quad \theta_1 \to \theta_1 + \underline{\eta_1}$, so that the superfield, $\mathcal{U}(x, t, \theta_1, \theta_2)$, is transformed to $\mathcal{U}(x + \underline{\eta_1}\theta_1, t, \theta_1 - \underline{\eta_1}, \theta_2)$. Therefore, if the superfield, $\mathcal{U}$, is to be preserved by the supersymmetry, Q_1, we must have:

$$(-\underline{\eta_1} - \underline{\mu\eta_1}\theta_1)\phi_1(t) = 0, \qquad \text{and} \qquad (-\underline{\eta_1} - \underline{\mu\eta_1}\theta_1)\theta_2\beta(t) = 0$$

Similarly, if the superfields, $\mathcal{R}$ and $\mathcal{P}$, are to be preserved, we must have $\underline{\eta_1\mu}\ell = 0$ and $\underline{\eta_1}\underline{\mu}C_2 = 0$, respectively. Therefore, the solution (34) is only preserved if $\underline{\eta_1} = 0$, so the supersymmetry, Q_1, is broken by the solution (34). In a similar way, it can be shown that the supersymmetry, Q_2, is also broken by the solution (34).

The second solution of system (33) is:

$$\begin{aligned}
\mathcal{U} &= (\underline{\mu}t\underline{C_1} + C_5 + \underline{C_7}\underline{\mu}t) + (\theta_1 - \underline{\mu}x)(-\underline{\mu}tC_2 - \underline{C_6} + C_8\underline{\mu}t) + \theta_2(-\underline{\mu}tC_3 - \underline{C_7}) \\
&\quad + (\theta_1 - \underline{\mu}x)\theta_2(\underline{\mu}t\underline{C_4} + C_8) \\
\mathcal{R} &= (\underline{\mu}C_8\underline{C_9}t - \underline{\mu}\underline{C_6}C_{10}t + C_{13}) + (\theta_1 - \underline{\mu}x)\underline{C_9} + \theta_2(-2\underline{\mu}C_8C_{10}t + \underline{C_{14}}) + C_{10} \\
\mathcal{P} &= (\underline{\mu}C_8\underline{C_{11}}t - k\underline{\mu}\underline{C_6}C_{12}t + C_{15}) + (\theta_1 - \underline{\mu}x)\underline{C_{11}} + \theta_2(-(k+1)\underline{\mu}C_8C_{12}t + \underline{C_{16}}) \\
&\quad + C_{12}
\end{aligned} \tag{35}$$

where C_2, C_3, C_5, C_8, C_{10}, C_{12}, C_{13} and C_{15} are arbitrary bosonic constants and $\underline{C_1}$, $\underline{C_4}$, $\underline{C_6}$, $\underline{C_7}$, $\underline{C_9}$, $\underline{C_{11}}$, $\underline{C_{14}}$ and $\underline{C_{16}}$ are arbitrary fermionic constants. The solution is non-stationary. All of the superfields are linear in x and t. In this case, they influence each other, since the superfields have parameters in common.

(5) For subalgebra, $\mathcal{L}_8 = \{L_1\}$, the invariants are:

$$\sigma = t, \qquad \tau_1 = x^{-1/2}\theta_1, \qquad \theta_2, \qquad \mathcal{A} = x^{-3/2}\mathcal{U}, \qquad \mathcal{R}, \qquad \mathcal{C} = x^{-2}\mathcal{P}.$$

Therefore, we have the superfields:

$$\mathcal{U} = x^{3/2}\mathcal{A}(t, \tau_1, \theta_2), \qquad \mathcal{R}(t, \tau_1, \theta_2), \qquad \mathcal{P} = x^2\mathcal{C}(t, \tau_1, \theta_2).$$

The corresponding system is given by:

$$\begin{aligned}
&-\theta_2\mathcal{R}_{t\tau_1}\mathcal{A}_t - \mathcal{R}_{\tau_1\theta_2}\mathcal{A}_t + \frac{9}{4}\tau_1\theta_2\mathcal{R}_{t\tau_1}\mathcal{A}_{\theta_2}\mathcal{A} - \frac{3}{2}\theta_2\mathcal{R}_{t\tau_1}\mathcal{A}_{\tau_1\theta_2}\mathcal{A} + \frac{1}{2}\theta_2\mathcal{R}_{t\tau_1}\mathcal{A}_{\tau_1\theta_2}\tau_1\mathcal{A}_{\tau_1} \\
&+ \frac{9}{4}\tau_1\theta_2\mathcal{R}_{\tau_1\theta_2}\mathcal{A}_t\mathcal{A} + \frac{9}{4}\tau_1\mathcal{R}_{\tau_1\theta_2}\mathcal{A}_{\theta_2}\mathcal{A} - \frac{3}{2}\theta_2\mathcal{R}_{\tau_1\theta_2}\mathcal{A}_{t\tau_1}\mathcal{A} + \frac{1}{2}\theta_2\mathcal{R}_{\tau_1\theta_2}\mathcal{A}_{t\tau_1}\tau_1\mathcal{A}_{\tau_1} \\
&- \frac{3}{2}\mathcal{R}_{\tau_1\theta_2}\mathcal{A}_{\tau_1\theta_2}\mathcal{A} + \frac{1}{2}\mathcal{R}_{\tau_1\theta_2}\mathcal{A}_{\tau_1\theta_2}\tau_1\mathcal{A}_{\tau_1} - 2\mathcal{C} + \frac{1}{2}\tau_1\mathcal{C}_{\tau_1} = 0
\end{aligned} \tag{36}$$

$$\theta_2\mathcal{A}_{t\tau_1}\tau_1\mathcal{R}_{\tau_1}+\mathcal{A}_{\tau_1\theta_2}\tau_1\mathcal{R}_{\tau_1}-3\theta_2\mathcal{R}_{t\tau_1}\mathcal{A}+\theta_2\mathcal{R}_{t\tau_1}\tau_1\mathcal{A}_{\tau_1}-3\mathcal{R}_{\tau_1\theta_2}\mathcal{A}+\mathcal{R}_{\tau_1\theta_2}\tau_1\mathcal{A}_{\tau_1}$$
$$-2\mathcal{R}_t=0 \tag{37}$$

$$\begin{aligned}&\mathcal{C}_t-3\tau_1\theta_2\mathcal{A}_t\mathcal{C}-3\tau_1\mathcal{A}_{\theta_2}\mathcal{C}+2\theta_2\mathcal{A}_{t\tau_1}\mathcal{C}-\frac{1}{2}\theta_2\mathcal{A}_{t\tau_1}\tau_1\mathcal{C}_{\tau_1}+2\mathcal{A}_{\tau_1\theta_2}\mathcal{C}-\frac{1}{2}\mathcal{A}_{\tau_1\theta_2}\tau_1\mathcal{C}_{\tau_1}\\&-3k\tau_1\theta_2\mathcal{C}_t\mathcal{A}-3k\tau_1\mathcal{C}_{\theta_2}\mathcal{A}+\frac{3}{2}k\mathcal{C}_{t\tau_1}\mathcal{A}-\frac{1}{2}k\mathcal{C}_{t\tau_1}\tau_1\mathcal{A}_{\tau_1}+\frac{3}{2}k\mathcal{C}_{\tau_1\theta_2}\mathcal{A}\\&-\frac{1}{2}k\mathcal{C}_{\tau_1\theta_2}\tau_1\mathcal{A}_{\tau_1}=0\end{aligned} \tag{38}$$

We obtain the solution:

$$\begin{aligned}&\mathcal{U}=x^{3/2}\left(\underline{\mu}f(t)+x^{-1/2}\theta_1\underline{\nu}q(t)+\theta_2\underline{\mu}h(t)+x^{-1/2}\theta_1\theta_2\underline{\mu}g(t)\right),\\&\mathcal{R}=C_1+\theta_2\underline{C_2},\\&\mathcal{P}=x^2\theta_2\underline{\mu}\end{aligned} \tag{39}$$

where C_1 and C_2 are arbitrary bosonic constants, h and q are arbitrary bosonic functions of t and f and g are arbitrary fermionic functions of t. This solution is non-stationary. The velocity and pressure superfields are subject to infinite growth in the variable, x. The arbitrary functions, $f(t)$, $q(t)$, $h(t)$ and $g(t)$, do not influence the superfields, $\mathcal{R}$ and $\mathcal{P}$.

(6) For subalgebra, $\mathcal{L}_9=\{P_1+\varepsilon P_2\}$, the invariants are:

$$\sigma=x-\varepsilon t,\qquad \theta_1,\qquad \theta_2,\qquad \mathcal{U},\qquad \mathcal{R},\qquad \mathcal{P}.$$

Obtaining the superfields:

$$\mathcal{U}(\sigma,\theta_1,\theta_2),\qquad \mathcal{R}(\sigma,\theta_1,\theta_2),\qquad \mathcal{P}(\sigma,\theta_1,\theta_2),$$

we are led to the reduced system:

$$\begin{aligned}&\theta_1\theta_2\mathcal{R}_{\sigma\sigma}\mathcal{U}_\sigma-\varepsilon\theta_1\mathcal{R}_{\sigma\theta_2}\mathcal{U}_\sigma-\theta_2\mathcal{R}_{\sigma\theta_1}\mathcal{U}_\sigma+\varepsilon\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_\sigma-\varepsilon\theta_1\theta_2\mathcal{R}_{\sigma\sigma}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_\sigma\\&+\varepsilon\theta_1\theta_2\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\sigma\theta_1}\mathcal{U}_\sigma+\theta_1\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_\sigma-\varepsilon\theta_1\theta_2\mathcal{R}_{\sigma\theta_1}\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma+\varepsilon\theta_2\mathcal{R}_{\sigma\theta_1}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_\sigma\\&-\varepsilon\theta_1\theta_2\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{\sigma\sigma}\mathcal{U}_\sigma+\theta_1\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma+\varepsilon\theta_2\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{\sigma\theta_1}\mathcal{U}_\sigma-\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_{\theta_1\theta_2}\mathcal{U}_\sigma-\mathcal{P}_\sigma=0\end{aligned} \tag{40}$$

$$\begin{aligned}&-\varepsilon\mathcal{R}_\sigma+\varepsilon\theta_1\theta_2\mathcal{U}_{\sigma\sigma}\mathcal{R}_\sigma-\theta_1\mathcal{U}_{\sigma\theta_2}\mathcal{R}_\sigma-\varepsilon\theta_2\mathcal{U}_{\sigma\theta_1}\mathcal{R}_\sigma+\mathcal{U}_{\theta_1\theta_2}\mathcal{R}_\sigma+\varepsilon\theta_1\theta_2\mathcal{R}_{\sigma\sigma}\mathcal{U}_\sigma\\&-\theta_1\mathcal{R}_{\sigma\theta_2}\mathcal{U}_\sigma-\varepsilon\theta_2\mathcal{R}_{\sigma\theta_1}\mathcal{U}_\sigma+\mathcal{R}_{\theta_1\theta_2}\mathcal{U}_\sigma=0\end{aligned} \tag{41}$$

$$\begin{aligned}&-\varepsilon\mathcal{P}_\sigma+\varepsilon\theta_1\theta_2\mathcal{U}_{\sigma\sigma}\mathcal{P}_\sigma-\theta_1\mathcal{U}_{\sigma\theta_2}\mathcal{P}_\sigma-\varepsilon\theta_2\mathcal{U}_{\sigma\theta_1}\mathcal{P}_\sigma+\mathcal{U}_{\theta_1\theta_2}\mathcal{P}_\sigma+\varepsilon k\theta_1\theta_2\mathcal{P}_{\sigma\sigma}\mathcal{U}_\sigma\\&-k\theta_1\mathcal{P}_{\sigma\theta_2}\mathcal{U}_\sigma-\varepsilon k\theta_2\mathcal{P}_{\sigma\theta_1}\mathcal{U}_\sigma+k\mathcal{P}_{\theta_1\theta_2}\mathcal{U}_\sigma=0\end{aligned} \tag{42}$$

We have two solutions. The first solution is:

$$\begin{aligned}&\mathcal{U}=C_0+\theta_1C_1\underline{\mu}+\theta_2C_1\underline{\nu}+\theta_1\theta_2C_2\underline{\mu\nu}\\&\mathcal{R}=C_6+\theta_1\underline{C_7}+\theta_2\underline{C_8}+\theta_1\theta_2C_9\\&\mathcal{P}=C_{10}+\theta_1\underline{C_3}+\theta_2\underline{C_4}+\theta_1\theta_2C_5\end{aligned} \tag{43}$$

where C_0, C_1, C_2, C_5, C_6, C_9 and C_{10} are arbitrary bosonic constants and $\underline{C_3}$, $\underline{C_4}$, $\underline{C_7}$ and $\underline{C_8}$ are arbitrary fermionic constants. This solution does not depend on the symmetry variable, σ. However, the superfields are expressible in terms of the Grassmann variables, θ_1 and θ_2. The second solution is:

$$\begin{aligned} \mathcal{U} &= \underline{\mu}\underline{\nu}\ell(x-\varepsilon t) + \theta_1\underline{\mu}\underline{\nu}q(x-\varepsilon t) + \theta_2\underline{\mu}\underline{\nu}q(x-\varepsilon t) + \theta_1\theta_2\underline{\mu}j(x-\varepsilon t) \\ \mathcal{R} &= C_3 + \theta_1\underline{C_4} + \theta_2\underline{C_4} \\ \mathcal{P} &= C_0 + \theta_1 C_1\underline{\mu} + \theta_2 C_1\underline{\nu} + \theta_1\theta_2 C_2\underline{\mu}\underline{\nu} \end{aligned} \tag{44}$$

where C_0, C_1, C_2 and C_3 are arbitrary bosonic constants, $\underline{C_4}$ is an arbitrary fermionic constant, ℓ is an arbitrary bosonic function of its argument and q and j are arbitrary fermionic functions of their arguments. This solution represents a travelling wave. The choice of the functions, ℓ, q and j, in $\mathcal{U}$ does not affect the superfields, $\mathcal{R}$ and $\mathcal{P}$, which depend only on θ_1 and θ_2.

(7) For subalgebra, $\mathcal{L}_{10} = \{P_1 + \underline{\nu}Q_2\}$, the invariants are:

$$\sigma = t + \underline{\nu}\theta_2 x, \qquad \theta_1, \qquad \tau_2 = \theta_2 - \underline{\nu}x, \qquad \mathcal{U}, \qquad \mathcal{R}, \qquad \mathcal{P},$$

and the superfields are:

$$\mathcal{U}(\sigma,\theta_1,\tau_2), \qquad \mathcal{R}(\sigma,\theta_1,\tau_2), \qquad \mathcal{P}(\sigma,\theta_1,\tau_2),$$

which allows us to reduce to the system:

$$\begin{aligned} &-\nu\theta_1\theta_2\mathcal{R}_{\sigma\tau_2}\mathcal{U}_\sigma + \nu\theta_1\mathcal{R}_\sigma\mathcal{U}_\sigma - \nu\theta_1\theta_2\mathcal{R}_{\sigma\tau_2}\mathcal{U}_\sigma - \theta_2\mathcal{R}_{\sigma\theta_1}\mathcal{U}_\sigma + \nu x\mathcal{R}_{\sigma\theta_1}\mathcal{U}_\sigma - \mathcal{R}_{\theta_1\tau_2}\mathcal{U}_\sigma \\ &+\nu\theta_2\mathcal{R}_{\sigma\theta_1}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_{\tau_2} + \nu\theta_2\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{\sigma\theta_1}\mathcal{U}_{\tau_2} - \nu\theta_2\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_\sigma + \nu\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{\theta_1\tau_2}\mathcal{U}_{\tau_2} - \nu\theta_2\mathcal{P}_\sigma \\ &+\nu\mathcal{P}_{\tau_2} = 0 \end{aligned} \tag{45}$$

$$\begin{aligned} &\mathcal{R}_\sigma - \nu\theta_2\mathcal{U}_{\sigma\theta_1}\mathcal{R}_{\tau_2} + \nu\theta_2\mathcal{U}_{\theta_1\tau_2}\mathcal{R}_\sigma - \nu\mathcal{U}_{\theta_1\tau_2}\mathcal{R}_{\tau_2} - \nu\theta_2\mathcal{R}_{\sigma\theta_1}\mathcal{U}_{\tau_2} + \nu\theta_2\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_\sigma \\ &-\nu\mathcal{R}_{\theta_1\tau_2}\mathcal{U}_{\tau_2} = 0 \end{aligned} \tag{46}$$

$$\begin{aligned} &\mathcal{P}_\sigma - \nu\theta_2\mathcal{U}_{\sigma\theta_1}\mathcal{P}_{\tau_2} + \nu\theta_2\mathcal{U}_{\theta_1\tau_2}\mathcal{P}_\sigma - \nu\mathcal{U}_{\theta_1\tau_2}\mathcal{P}_{\tau_2} - k\nu\theta_2\mathcal{P}_{\sigma\theta_1}\mathcal{U}_{\tau_2} + k\nu\theta_2\mathcal{P}_{\theta_1\tau_2}\mathcal{U}_\sigma \\ &-k\nu\mathcal{P}_{\theta_1\tau_2}\mathcal{U}_{\tau_2} = 0 \end{aligned} \tag{47}$$

We obtain the solution:

$$\begin{aligned} \mathcal{U} &= \alpha(t+\underline{\nu}\theta_2 x) + \theta_1\phi_1(t+\underline{\nu}\theta_2 x) + (\theta_2 - \underline{\nu}x)\phi_2(t+\underline{\nu}\theta_2 x) + \theta_1(\theta_2-\underline{\nu}x)\beta(t+\underline{\nu}\theta_2 x) \\ \mathcal{R} &= \underline{C_1}\underline{\nu} + \theta_2 C_2\underline{\nu} \\ \mathcal{P} &= k_1\underline{C_1}\underline{\nu} + \theta_2 k_1 C_2\underline{\nu} \end{aligned} \tag{48}$$

where k_1 and C_2 are arbitrary bosonic constants, $\underline{C_1}$ is an arbitrary fermionic constant, α and β are arbitrary bosonic function of their arguments and ϕ_1 and ϕ_2 are arbitrary fermionic functions of their arguments. This solution is a travelling wave involving the Grassmann variable, θ_2. The superfields, $\mathcal{R}$ and $\mathcal{P}$, depend linearly on θ_2.

(8) For subalgebra, $\mathcal{L}_{17} = \{P_1 + \varepsilon P_2 \underline{\mu} Q_1\}$, the invariants:

$$\sigma = x - \varepsilon t + \varepsilon \underline{\mu} t \theta_1, \qquad \tau_1 = \theta_1 - \varepsilon \underline{\mu} t, \qquad \theta_2, \qquad \mathcal{U}, \qquad \mathcal{R}, \qquad \mathcal{P},$$

give us superfields of the form:

$$\mathcal{U}(\sigma, \tau_1, \theta_2), \qquad \mathcal{R}(\sigma, \tau_1, \theta_2), \qquad \mathcal{P}(\sigma, \tau_1, \theta_2),$$

which lead us to the following reduced system:

$$\begin{aligned}
&\tau_1\theta_2\mathcal{R}_{\sigma\sigma}\mathcal{U}_\sigma + \tau_1\theta_2\underline{\mu}\mathcal{R}_{\sigma\sigma}\mathcal{U}_{\tau_1} + \tau_1\theta_2\underline{\mu}\mathcal{R}_{\sigma\tau_1}\mathcal{U}_\sigma + \varepsilon\underline{\mu}t\theta_2\mathcal{R}_{\sigma\sigma}\mathcal{U}_\sigma - \varepsilon\tau_1\mathcal{R}_{\sigma\theta_2}\mathcal{U}_\sigma \\
&- \varepsilon\tau_1\mathcal{R}_{\sigma\theta_2}\underline{\mu}\mathcal{U}_{\tau_1} - \underline{\mu}t\mathcal{R}_{\sigma\theta_2}\mathcal{U}_\sigma + \varepsilon\theta_2\underline{\mu}t\mathcal{R}_{\sigma\sigma}\mathcal{U}_\sigma - \theta_2\underline{\mu}\mathcal{R}_\sigma\mathcal{U}_\sigma + \theta_2\underline{\mu}\tau_1\mathcal{R}_{\sigma\tau_1}\mathcal{U}_\sigma \\
&+ \theta_2\mathcal{R}_{\sigma\tau_1}\underline{\mu}\tau_1\mathcal{U}_\sigma - \theta_2\mathcal{R}_{\sigma\tau_1}\mathcal{U}_\sigma - \theta_2\mathcal{R}_{\sigma\tau_1}\underline{\mu}\mathcal{U}_{\tau_1} + \underline{\mu}t\mathcal{R}_{\sigma\theta_2}\mathcal{U}_\sigma - \varepsilon\underline{\mu}\tau_1\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_\sigma + \varepsilon\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_\sigma \\
&+ \varepsilon\underline{\mu}\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\tau_1} - \tau_1\theta_2\mathcal{R}_{\sigma\sigma}\underline{\mu}t\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma - \varepsilon\tau_1\theta_2\mathcal{R}_{\sigma\sigma}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma - \varepsilon\tau_1\theta_2\underline{\mu}\mathcal{R}_{\sigma\tau_1}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma \\
&- \underline{\mu}t\theta_2\mathcal{R}_{\sigma\sigma}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma - \tau_1\theta_2\mathcal{R}_{\sigma\theta_2}\underline{\mu}t\mathcal{U}_{\sigma\sigma}\mathcal{U}_\sigma + \varepsilon\tau_1\theta_2\mathcal{R}_{\sigma\theta_2}\underline{\mu}(\mathcal{U}_\sigma)^2 + \varepsilon\tau_1\theta_2\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\sigma\tau_1}\mathcal{U}_\sigma \\
&+ \underline{\mu}t\theta_2\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\sigma\tau_1}\mathcal{U}_\sigma + \varepsilon\tau_1\mathcal{R}_{\sigma\theta_2}\underline{\mu}t\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma + \tau_1\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma + \varepsilon\underline{\mu}t\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma \\
&+ \tau_1\theta_2\underline{\mu}t\mathcal{R}_{\sigma\sigma}\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma - \varepsilon\tau_1\theta_2\underline{\mu}\mathcal{R}_\sigma\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma - \varepsilon\tau_1\theta_2\mathcal{R}_{\sigma\tau_1}\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma - \underline{\mu}t\theta_2\mathcal{R}_{\sigma\tau_1}\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma \\
&- \theta_2\underline{\mu}t\mathcal{R}_{\sigma\sigma}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma + \varepsilon\theta_2\underline{\mu}\mathcal{R}_\sigma\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma - \varepsilon\theta_2\underline{\mu}\tau_1\mathcal{R}_{\sigma\tau_1}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma + \theta_2\mathcal{R}_{\sigma\tau_1}\underline{\mu}t\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma \\
&+ \varepsilon\theta_2\mathcal{R}_{\sigma\tau_1}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma - \tau_1\theta_2\underline{\mu}t\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\sigma\sigma}\mathcal{U}_\sigma - \varepsilon\tau_1\theta_2\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\sigma\sigma}\mathcal{U}_\sigma - \varepsilon\tau_1\theta_2\mathcal{R}_{\tau_1\theta_2}\underline{\mu}\mathcal{U}_{\sigma\tau_1}\mathcal{U}_\sigma \\
&- \underline{\mu}t\theta_2\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\sigma\sigma}\mathcal{U}_\sigma + \varepsilon\tau_1\underline{\mu}t\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma + \tau_1\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma + \varepsilon\underline{\mu}t\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma \\
&+ \theta_2\underline{\mu}t\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\sigma\tau_1}\mathcal{U}_\sigma - \theta_2\mathcal{R}_{\tau_1\theta_2}\underline{\mu}t\mathcal{U}_{\sigma\sigma}\mathcal{U}_\sigma + \varepsilon\theta_2\mathcal{R}_{\tau_1\theta_2}\underline{\mu}(\mathcal{U}_\sigma)^2 - \varepsilon\theta_2\mathcal{R}_{\tau_1\theta_2}\underline{\mu}\tau_1\mathcal{U}_{\sigma\tau_1}\mathcal{U}_\sigma \\
&+ \varepsilon\theta_2\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\sigma\tau_1}\mathcal{U}_\sigma - \varepsilon\underline{\mu}t\mathcal{R}_{\sigma\theta_2}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma - \varepsilon\underline{\mu}t\mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\sigma\theta_2}\mathcal{U}_\sigma - \mathcal{R}_{\tau_1\theta_2}\mathcal{U}_{\tau_1\theta_2}\mathcal{U}_\sigma - \mathcal{P}_\sigma = 0
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\varepsilon\underline{\mu}\tau_1\mathcal{R}_\sigma - \varepsilon\mathcal{R}_\sigma - \varepsilon\underline{\mu}\mathcal{R}_{\tau_1} + \varepsilon\tau_1\theta_2\mathcal{U}_{\sigma\sigma}\mathcal{R}_\sigma + \varepsilon\tau_1\theta_2\underline{\mu}\mathcal{U}_{\sigma\tau_1}\mathcal{R}_\sigma + \underline{\mu}t\theta_2\mathcal{U}_{\sigma\sigma}\mathcal{R}_\sigma - \tau_1\mathcal{U}_{\sigma\theta_2}\mathcal{R}_\sigma \\
&- \varepsilon\underline{\mu}t\mathcal{U}_{\sigma\theta_2}\mathcal{R}_\sigma + \theta_2\underline{\mu}t\mathcal{U}_{\sigma\sigma}\mathcal{R}_\sigma - \varepsilon\theta_2\underline{\mu}\mathcal{U}_\sigma\mathcal{R}_\sigma + \varepsilon\theta_2\underline{\mu}\tau_1\mathcal{U}_{\sigma\tau_1}\mathcal{R}_\sigma - \varepsilon\theta_2\mathcal{U}_{\sigma\tau_1}\mathcal{R}_\sigma \\
&+ \varepsilon\underline{\mu}t\mathcal{U}_{\sigma\theta_2}\mathcal{R}_\sigma + \mathcal{U}_{\tau_1\theta_2}\mathcal{R}_\sigma + \varepsilon\tau_1\theta_2\mathcal{R}_{\sigma\sigma}\mathcal{U}_\sigma + \varepsilon\tau_1\theta_2\underline{\mu}\mathcal{R}_{\sigma\tau_1}\mathcal{U}_\sigma + \underline{\mu}t\theta_2\mathcal{R}_{\sigma\sigma}\mathcal{U}_\sigma - \tau_1\mathcal{R}_{\sigma\theta_2}\mathcal{U}_\sigma \\
&- \varepsilon\underline{\mu}t\mathcal{R}_{\sigma\theta_2}\mathcal{U}_\sigma + \theta_2\underline{\mu}t\mathcal{R}_{\sigma\sigma}\mathcal{U}_\sigma - \varepsilon\theta_2\underline{\mu}\mathcal{R}_\sigma\mathcal{U}_\sigma + \varepsilon\theta_2\underline{\mu}\tau_1\mathcal{R}_{\sigma\tau_1}\mathcal{U}_\sigma - \varepsilon\theta_2\mathcal{R}_{\sigma\tau_1}\mathcal{U}_\sigma \\
&+ \varepsilon\underline{\mu}t\mathcal{R}_{\sigma\theta_2}\mathcal{U}_\sigma + \mathcal{R}_{\tau_1\theta_2}\mathcal{U}_\sigma = 0
\end{aligned} \tag{50}$$

$$\begin{aligned}
&\varepsilon\underline{\mu}\tau_1\mathcal{P}_\sigma - \varepsilon\mathcal{P}_\sigma - \varepsilon\underline{\mu}\mathcal{P}_{\tau_1} + \varepsilon\tau_1\theta_2\mathcal{U}_{\sigma\sigma}\mathcal{P}_\sigma + \varepsilon\tau_1\theta_2\underline{\mu}\mathcal{U}_{\sigma\tau_1}\mathcal{P}_\sigma + \underline{\mu}t\theta_2\mathcal{U}_{\sigma\sigma}\mathcal{P}_\sigma - \tau_1\mathcal{U}_{\sigma\theta_2}\mathcal{P}_\sigma \\
&- \varepsilon\underline{\mu}t\mathcal{U}_{\sigma\theta_2}\mathcal{P}_\sigma + \theta_2\underline{\mu}t\mathcal{U}_{\sigma\sigma}\mathcal{P}_\sigma - \varepsilon\theta_2\underline{\mu}\mathcal{U}_\sigma\mathcal{P}_\sigma + \varepsilon\theta_2\underline{\mu}\tau_1\mathcal{U}_{\sigma\tau_1}\mathcal{P}_\sigma - \varepsilon\theta_2\mathcal{U}_{\sigma\tau_1}\mathcal{P}_\sigma \\
&+ \varepsilon\underline{\mu}t\mathcal{U}_{\sigma\theta_2}\mathcal{P}_\sigma + \mathcal{U}_{\tau_1\theta_2}\mathcal{P}_\sigma + \varepsilon k\tau_1\theta_2\mathcal{P}_{\sigma\sigma}\mathcal{U}_\sigma + \varepsilon k\tau_1\theta_2\underline{\mu}\mathcal{P}_{\sigma\tau_1}\mathcal{U}_\sigma + \underline{\mu}kt\theta_2\mathcal{P}_{\sigma\sigma}\mathcal{U}_\sigma \\
&- k\tau_1\mathcal{P}_{\sigma\theta_2}\mathcal{U}_\sigma - \varepsilon\underline{\mu}kt\mathcal{P}_{\sigma\theta_2}\mathcal{U}_\sigma + k\theta_2\underline{\mu}t\mathcal{P}_{\sigma\sigma}\mathcal{U}_\sigma - \varepsilon k\theta_2\underline{\mu}\mathcal{P}_\sigma\mathcal{U}_\sigma + \varepsilon k\theta_2\underline{\mu}\tau_1\mathcal{P}_{\sigma\tau_1}\mathcal{U}_\sigma \\
&- \varepsilon k\theta_2\mathcal{P}_{\sigma\tau_1}\mathcal{U}_\sigma + \varepsilon k\underline{\mu}t\mathcal{P}_{\sigma\theta_2}\mathcal{U}_\sigma + k\mathcal{P}_{\tau_1\theta_2}\mathcal{U}_\sigma = 0
\end{aligned} \tag{51}$$

We obtain the solution:

$$\begin{aligned}
&\mathcal{U} = C_0 + \theta_1\underline{\mu} + \theta_2\underline{C_1}, \\
&\mathcal{R} = \underline{\mu}\underline{C_8} - \underline{\mu}\underline{C_7}(x - \varepsilon t) + (\theta_1 - \varepsilon\underline{\mu}t)\underline{C_7} + \theta_2\underline{\mu}C_{10} + \theta_2\underline{\mu}C_9(x - \varepsilon t) + (\theta_1 - \varepsilon\underline{\mu}t)\theta_2 C_9, \\
&\mathcal{P} = C_3 + \theta_1\underline{\mu}\ell + \theta_2\underline{C_5} + \theta_1\theta_2\underline{C_2\mu}
\end{aligned} \tag{52}$$

where C_0, C_3, C_9, C_{10} and ℓ are arbitrary bosonic constants and $\underline{C_1}$, $\underline{C_2}$, $\underline{C_5}$, $\underline{C_7}$ and $\underline{C_8}$ are arbitrary fermionic constants. This solution is non-stationary. The density, $\mathcal{R}$, is a travelling wave. The superfields, $\mathcal{U}$ and $\mathcal{P}$, depend only on θ_1 and θ_2.

(9) For subalgebra, $\mathcal{L}_{30} = \{L_3 + \varepsilon P_1\}$, the invariants are:

$$t, \qquad \theta_1, \qquad \theta_2, \qquad \mathcal{U}, \qquad \mathcal{B} = e^{-\varepsilon x}\mathcal{R}, \qquad \mathcal{C}e^{-\varepsilon x}\mathcal{P}.$$

Therefore, we have the superfields:

$$\mathcal{U}(t,\theta_1,\theta_2), \qquad \mathcal{R} = e^{\varepsilon x}\mathcal{B}(t,\theta_1,\theta_2), \qquad \mathcal{P} = e^{\varepsilon x}\mathcal{C}(t,\theta_1,\theta_2),$$

and the reduced system is:

$$\begin{aligned}
&\theta_1\theta_2\varepsilon\mathcal{B}_t\mathcal{U}_t - \theta_2\mathcal{B}_{t\theta_1}\mathcal{U}_t + \theta_1\varepsilon\mathcal{B}_{\theta_2}\mathcal{U}_t - \mathcal{B}_{\theta_1\theta_2}\mathcal{U}_t - \varepsilon\mathcal{C} = 0,\\
&\mathcal{B}_t + \theta_2\mathcal{U}_{t\theta_1}\varepsilon\mathcal{B} + \mathcal{U}_{\theta_1\theta_2}\varepsilon\mathcal{B} = 0, \qquad (53)\\
&\mathcal{C}_t + \theta_2\mathcal{U}_{t\theta_1}\varepsilon\mathcal{C} + \mathcal{U}_{\theta_1\theta_2}\varepsilon\mathcal{C} = 0
\end{aligned}$$

We have two solutions. The first solution is:

$$\begin{aligned}
&\mathcal{U} = C_1 + \theta_1\underline{C_2} + \theta_2\underline{C_3} + \theta_1\theta_2 C_4,\\
&\mathcal{R} = e^{\varepsilon(x-C_4t)}\left[k_1 + \theta_1\underline{k_3} + \theta_2\underline{k_4} + \theta_1\theta_2 k_5\right], \qquad (54)\\
&\mathcal{P} = 0
\end{aligned}$$

where C_1, C_4, k_1 and k_5 are arbitrary bosonic constants and $\underline{C_2}$, $\underline{C_3}$, $\underline{k_3}$ and $\underline{k_4}$ are arbitrary fermionic constants. This is a non-stationary solution. The density superfield, $\mathcal{R}$, contains a factor of $e^{\varepsilon(x-C_4t)}$. Depending on whether ε is -1 or one, at a fixed position, x, we have either damping or growth. The velocity is not influenced, however. The second solution is:

$$\begin{aligned}
&\mathcal{U} = At + B + \theta_1\underline{C_6} + \theta_1\theta_2\underline{\mu}\underline{C_5},\\
&\mathcal{R} = e^{\varepsilon x}\left[\theta_1\underline{\mu} + \theta_2\underline{\mu} + \theta_1\theta_2\underline{\mu}\underline{C_5}\right], \qquad (55)\\
&\mathcal{P} = Ae^{\varepsilon x}\left[-\varepsilon\underline{\mu}\underline{C_5} + \theta_1\underline{\mu}\right]
\end{aligned}$$

where A and B are arbitrary bosonic constants and $\underline{C_5}$ and $\underline{C_6}$ are arbitrary fermionic constants. This is a non-stationary solution. The density and pressure contain a damping (or growth) factor. The velocity superfield is linear in time t. The velocity and pressure are linked through the constant, A. Depending on the sign of A, the velocity, $\mathcal{U}$, is either increasing or decreasing.

5. Final Remarks and Open Problems

In this paper, we discuss first the construction of a supersymmetric extension of the Euler system (1) through a superspace and superfield formalism. This analysis includes a supersymmetric extension of a one-dimensional ideal compressible non-viscous fluid flow. This allows us to determine a Lie superalgebra of infinitesimal symmetries, which generate the Lie point symmetries of the Euler system of Equations. We observe that, in analogy with the classical Euler system, the Lie symmetry superalgebra of the supersymmetric Euler system contains three independent dilations. Next, we discuss the classification of its subalgebras. Through the use of a generalized version of the symmetry reduction method, we have demonstrated how to find exact invariant solutions of the supersymmetric model. A systematic use of the structure of the invariance supergroup of the Euler Equations allows us to generate all symmetry variables. For certain subalgebras, the invariants have a nonstandard structure and, therefore, do not lead to invariant solutions. This phenomenon of nonstandard invariants has also been observed, among other places, in the analysis of the symmetries of the supersymmetric sine-Gordon Equation [24] and of supersymmetric polytropic gas dynamics [4]. The SRM enables us to reduce, after some transformations, the basic system of PDEs to many possible ODEs. In each case, the three superfields can be decomposed in terms of its independent fermionic variables, θ_1 and θ_2. This allows us to determine the invariant solutions of the supersymmetric Euler system component-wise. For nine specific subalgebras involving different types of generators, a number of invariant solutions were found. These included solutions involving several arbitrary functions of one variable. In particular, this includes bumps, kinks and multiple wave solutions. If these arbitrary functions depend on x, the postulated form will change as the solution evolves in time. Such a large number of arbitrary function degrees of freedom were not found in the previous analyses by the authors of supersymmetric hydrodynamic systems (*i.e.*, polytropic gas [4], integrable models (sine-Gordon, sinh-Gordon, polynomial Klein-Gordon [23,24]). Some of the obtained solutions involved damping and growth. However, no blow-up phenomenon (gradient catastrophe) was observed for any of our solutions.

In the future, it would be interesting to expand our analysis in several directions. One such possibility would be to apply the above supersymmetric extension methods to the Euler system in higher dimensions. Due to the complexity of the calculations involved, this would require the development of a computer algebra Lie symmetry package capable of handling odd and even Grassmann variables. To the best of our knowledge, such a package does not presently exist. Conservation laws are well-established for the classical Euler Equations, but it has been observed that, for many hydrodynamic-type systems, such conservation laws are broken in their corresponding supersymmetric extensions (see, e.g., [4] and references therein). The question of which quantities are conserved by the supersymmetric model still remains an open question for hydrodynamic-type Equations. These topics will be investigated in our future work.

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Conflict of Interest

The authors declare no conflict of interest.

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