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Symmetries of Spatial Graphs and Rational Twists along Spheres and Tori

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Abstract: A symmetry group of a spatial graph Γ in S^3 is a finite group consisting of orientation-preserving self-diffeomorphisms of S^3 which leave Γ setwise invariant. In this paper, we show that in many cases symmetry groups of Γ which agree on a regular neighborhood of Γ are equivalent up to conjugate by rational twists along incompressible spheres and tori in the exterior of Γ .

Keywords: 3-manifold; geometric topology; symmetry; finite group action; spatial graph; rational twist

1. Introduction

There are several approaches to the theory of graphs embedded in the 3-sphere, which are often motivated by molecular chemistry, since the chemical properties of a molecule depend on the symmetries of its molecular bond graph (see, for example, [1]). The symmetries of an abstract graph Γ are described by automorphisms. If Γ is embedded in S^3 , some of these automorphisms are induced from self-diffeomorphisms of S^3 . For example, [2–6] studied the extendabilities of the automorphisms of Γ , mainly in the case of Möbius ladders, complete graphs, and 3-connected graphs.

Even if the automorphisms of Γ extend to self-diffeomorphisms of S^3 , we face the problem of the uniqueness of the extensions. In this situation, it is enough to consider Γ to be a topological space, since we need to study self-diffeomorphisms of S^3 which agree on Γ . In the case of a non-torus knot in S^3 , there are only finitely many conjugacy classes of symmetries (see [7,8]). For a cyclic period or a free period of a knot in S^3 , it is shown in [9,10] that the cyclic group generated by the periodic self-diffeomorphism of S^3 defining the symmetry is unique up to conjugate in some cases. Moreover,

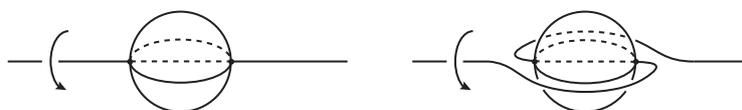
the author [11] generalized this result to the case of links in S^3 . In this paper, we generalize these results to the case of symmetries of spatial graphs in S^3 .

Suppose that any component of Γ is a non-trivial graph with no leaf. We see Γ as a geometric simplicial complex, and denote by $|\Gamma|$ the underlying topological space of Γ . A tame embedding of $|\Gamma|$ into S^3 is called a *spatial embedding* of Γ into S^3 , or simply a *spatial graph* Γ in S^3 . We say that Γ is *splittable* if there exists a sphere in S^3 disjoint from Γ that separates the components of Γ . We say that Γ is *non-splittable* if it is not splittable. Suppose that an incompressible torus in $S^3 - \Gamma$ bounds a solid torus V in S^3 containing Γ . The core of V is called a *companion knot* of Γ if it is not ambient isotopic to Γ in V . If there is no companion knot of Γ , every incompressible torus in $S^3 - \Gamma$ separates the components of Γ .

Let M be a 3-manifold, and X a submanifold of M . Denote by $N(X)$ a regular neighborhood of X , and by $E(X) = M - \text{int}N(X)$ the exterior of X . We refer to a finite subgroup G of the diffeomorphism group $\text{Diff}(M)$ as a *finite group action* on M . Finite group actions G_1 and G_2 on M are equivalent (relative to X) if some $h \in \text{Diff}(M)$ conjugates G_1 to G_2 (and restricts to the identity map on X). A *symmetry group* G of a spatial graph Γ in S^3 is a finite group action on the pair (S^3, Γ) which preserves the orientation of S^3 .

Let S^2 be the unit sphere in \mathbb{R}^3 , and S^1 the unit circle in the xy -plane in \mathbb{R}^3 . Denote by $\text{Rot}_\theta \in \text{Diff}(\mathbb{R}^3)$ the rotation about the z -axis through angle θ . Suppose that $\sigma_n \in \text{Diff}(S^2 \times I)$ and $\tau_n \in \text{Diff}(S^1 \times S^1 \times I)$, where $n \in \mathbb{R}$, is given by $\sigma_n(x, t) = (\text{Rot}_{2\pi nt}(x), t)$ and $\tau_n(x, y, t) = (\text{Rot}_{2\pi nt}(x), y, t)$. Let F be a 2-sided sphere or torus embedded in a 3-manifold M . Split M open along F into a (possibly disconnected) 3-manifold M_F . Denote by F_- and F_+ the boundary components of M_F originated from F . An n -twist along F is a discontinuous map on M induced from a diffeomorphism on $M_F - F_-$ which restricts to the identity map on $E(F_+)$ and the map on $N(F_+)$ conjugate to σ_n or τ_n according as F is a sphere or not. We say that the n -twist is *rational* if $n \in \mathbb{Q}$. Figure 1 illustrates a rotational symmetry of S^3 with a setwise invariant sphere S , and its conjugate by a 1/2-twist along S .

Figure 1. Conjugation by a 1/2-twist along a sphere S .



Our main theorem is the following:

Theorem 1.1. *Let Γ be a spatial graph in S^3 with no companion knot. Suppose that G_1 and G_2 are symmetry groups of Γ such that*

- (1) $G_1(\gamma) = G_2(\gamma) = \gamma$ for at least one component γ of Γ ,
- (2) either Γ is non-splittable, or G_1 and G_2 are cyclic groups acting on Γ freely, and
- (3) G_1 and G_2 agree on $N(\Gamma)$.

Then there is a finite sequence of rational twists along incompressible spheres and tori in $E(\Gamma)$ whose composition conjugates G_2 to a symmetry group of Γ equivalent to G_1 relative to $N(\Gamma)$.

This paper is arranged as follows. In Section 2, we study symmetry groups of non-splittable spatial graph in terms of the equivariant JSJ decomposition of the exteriors. In Section 3, we establish a canonical version of the equivariant sphere theorem for the exteriors of spatial graphs with cyclic symmetry groups, and prove Theorem 1.1.

2. Non-splittable Case

For a non-splittable spatial graph Γ in S^3 with a non-trivial symmetry group, there is a canonical method for splitting $E(\Gamma)$ equivariantly into geometric pieces by the loop theorem, the Dehn's lemma, and the JSJ decomposition theorem (see [12–14]).

Let M be a Haken 3-manifold with incompressible boundary. The JSJ decomposition theorem and Thurston's uniformization theorem [15] assert that there is a canonical way of splitting the pair $(M, \partial M)$ along a disjoint, non-parallel, essential annuli and tori into pieces (M_i, F_i) each of which is one of the following four types:

- (1) M_i is an I -bundle over a compact surface and F_i is the ∂I -subbundle,
- (2) M_i admits a Seifert fibration in which F_i is fibered,
- (3) $\text{int}M_i$ admits a complete hyperbolic structure of finite volume, and
- (4) the double of $(M_i, F_i - \text{int}\Phi_i)$ along a non-empty compact submanifold Φ_i of F_i is of type (3).

For a finite group action G on M , the *fixed point set* $\text{Fix}(G)$ of G is the set of points in M each of which has the stabilizer G . The *singular set* $\text{Sing}(G)$ of G is the set of points in M each of which has a non-trivial stabilizer.

Lemma 2.1. *Let T be a torus embedded in S^3 . Suppose that G_1 and G_2 are orientation-preserving finite group actions on S^3 such that*

- (1) $G_1(N(T)) = G_2(N(T)) = N(T)$,
- (2) G_1 and G_2 do not interchange the components of $\partial N(T)$, and
- (3) G_1 and G_2 agree on $\partial N(T)$.

Then a rational twist along a component of $\partial N(T)$ conjugates G_2 to a finite group action \widehat{G}_2 on S^3 such that the actions of G_1 and \widehat{G}_2 on $N(T)$ are equivalent relative to $\partial N(T)$.

Proof. It is enough by Lemma 2.4 of [11] to consider the case where the actions of G_1 and G_2 on $N(T)$ are not free. For each G_i , Theorem 2.1 of [16] implies that $N(T) \cong T \times I$ admits a G_i -invariant product structure \mathcal{P}_i , in which $\text{Sing}(G_i) \cap N(T)$ consists of I -fibers. Since each element of G_i takes a meridian of T to a meridian of T , the setwise stabilizer of each I -fiber is a trivial group or a 2-fold cyclic group. Therefore, the quotient space $N(T)/G_i$ admits the induced I -bundle structure over a 2-orbifold B with underlying surface F and n cone points of index two. Since T is a torus, the orbifold Euler characteristic $\chi_{\text{orb}}(B)$ of B is calculated as follows (see [17]):

$$\chi_{\text{orb}}(B) = \chi(F) - n/2 = 0.$$

Since $n > 0$, F is a sphere and $n = 4$ holds.

Denote by $p_i: N(T) \rightarrow N(T)/G_i$ the projection map onto the quotient space for each i , and by T_t the T -fiber $T \times \{t\}$ in \mathcal{P}_1 . Connect the four cone points on $p_1(T_0)$ cyclically by a collection of arcs $\bar{a}_1, \bar{a}_2, \bar{a}_3$, and \bar{a}_4 with disjoint interiors. Each \bar{a}_i lifts to an essential loop a_i on T_0 such that a_i and a_j with $i \neq j$ are disjoint if $|j - i| = 2$, and otherwise a_i meets a_j transversally in a point. Suppose that each a_i is isotopic to a loop b_i on T_1 along an annulus B_i saturated by I -fibers in \mathcal{P}_1 , and to a loop c_i on T_1 along an annulus C_i saturated by I -fibers in \mathcal{P}_2 . Then the endpoints of each $p_2(c_i)$ is connected by $p_2(b_j)$ with $|i - j| = 0$ or 2 . Since the underlying surface of $p_2(T_1)$ is a sphere, $\bigcup_{i=1}^4 p_2(c_i)$ is isotopic to $\bigcup_{i=1}^4 p_2(b_i)$ relative to the cone points. Therefore, $G_2(\bigcup_{i=1}^4 C_i)$ is moved by an G_2 -equivariant isotopy relative to T_0 so as to agree with $G_1(\bigcup_{i=1}^4 B_i)$ on T_1 .

The I -bundle structures in \mathcal{P}_2 and \mathcal{P}_1 respectively induce orbifold isomorphisms $\varphi_1: p_2(T_1) \rightarrow p_2(T_0)$ and $\varphi_2: p_2(T_0) \rightarrow p_2(T_1)$ such that $\bar{h} = \varphi_2 \circ \varphi_1$ setwise preserves the loop $\bigcup_{i=1}^4 p_2(b_i)$. The restriction of \bar{h} on $\bigcup_{i=1}^4 p_2(b_i)$ is isotopic relative to the cone points to the identity map or an involution. Since $\bigcup_{i=1}^4 p_2(b_i)$ splits $p_2(T_1)$ into two disks with no cone point, \mathcal{P}_2 is deformed by a G_2 -equivariant isotopy so that afterwards \bar{h} is the identity map or an involution.

Take an \bar{h} -invariant S^1 -bundle structure \mathcal{S}_1 on $p_2(T_1) - p_2(b_1 \cup b_3)$ with respect to which $p_2(b_2)$ and $p_2(b_4)$ are cross sectional, and an \bar{h} -invariant S^1 -bundle structure \mathcal{S}_2 on $p_2(T_1) - p_2(b_2 \cup b_4)$ with respect to which every fiber in \mathcal{S}_1 splits into two cross sections. Then \mathcal{S}_1 and \mathcal{S}_2 induce a G_2 -invariant product structure $S^1 \times S^1$ on T_1 . Let $h: T_1 \rightarrow T_1$ be the lift of \bar{h} which takes each c_i to b_i . Then we have $h = \text{Rot}_{2\pi m} \times \text{Rot}_{2\pi n}$ for some rational numbers m and n .

Assume $(m, n) \neq (0, 0)$. Take a rational number γ so that γm and γn are coprime integers. Then $\alpha \gamma m + \beta \gamma n = 1$ holds for some integers α and β . Let $\rho: \mathbb{R}^2 \rightarrow S^1 \times S^1$ be the covering map given by $\rho(x, y) = (\text{Rot}_{2\pi x}(1, 0), \text{Rot}_{2\pi y}(1, 0))$. Denote by φ the linear transformation on \mathbb{R}^2 represented by $\begin{pmatrix} \alpha & \beta \\ -\gamma m & \gamma n \end{pmatrix}$. Then the map $\rho \circ \varphi \circ \rho^{-1} \in \text{Diff}(S^1 \times S^1)$ conjugates h to $\text{Rot}_{2\pi/\gamma} \times \text{id}_{S^1}$. Thus, h extends to $1/\gamma$ -twist τ along T_1 . Since h conjugates the action of G_2 on T_1 to itself, τ conjugates G_2 to a finite subgroup of $\text{Diff}(S^3)$. Therefore, it is enough to consider the case $(m, n) = (0, 0)$.

It is obvious that $h = \text{Rot}_{2\pi k} \times \text{Rot}_{2\pi l}$ holds for any integers k and l . By verifying that, for some choice of k and l , the above argument applied to $\text{Rot}_{2\pi k} \times \text{Rot}_{2\pi l}$ makes $G_2(\bigcup_{i=1}^4 C_i)$ isotopic to $G_1(\bigcup_{i=1}^4 B_i)$ relative to $\partial N(T)$, we may assume that they agree.

By considering an isotopy of $N(T)$ relative to $\partial N(T)$ which takes \mathcal{P}_2 to \mathcal{P}_1 on $\text{Sing}(G_1) \cap N(T)$, we may assume that G_1 and G_2 agree on $\text{Sing}(G_1) \cap N(T)$. Note that $\text{Sing}(G_1) \cap N(T)$ splits $G_1(\bigcup_{i=1}^4 B_i)$ into disks, and that $G_1(\bigcup_{i=1}^4 B_i)$ splits $N(T)$ into balls. Then the identity map on $p_2(\text{Sing}(G_2) \cap N(T))$ extends to an orbifold isomorphism $\psi: p_2(\bigcup_{i=1}^4 C_i) \rightarrow p_1(\bigcup_{i=1}^4 B_i)$. Since the quotient space of any finite group action on D^3 is isomorphic to one of the orbifolds listed on page 191 of [15], ψ and the identity map on $p_2(\partial N(T))$ extend to an orbifold isomorphism $p_2(N(T)) \rightarrow p_1(N(T))$. Thus, G_1 and G_2 are equivalent relative to $\partial N(T)$. Hence, the conclusion follows. \square

Lemma 2.2. *Let M be a Seifert manifold in S^3 with non-empty boundary, and F a non-empty closed submanifold of ∂M . Suppose that G_1 and G_2 are finite group actions on S^3 such that*

- (1) $G_1(M) = G_2(M) = M$ and $G_1(F) = G_2(F) = F$,

- (2) $G_1(T) = G_2(T) = T$ for at least one component T of F ,
- (3) G_1 and G_2 induce the same permutation on the set of the components of ∂M , and
- (4) G_1 and G_2 agree on F .

Then there is a finite sequence of rational twists along incompressible tori in M whose composition conjugates G_2 to a finite group action \widehat{G}_2 on S^3 such that the actions of G_1 and \widehat{G}_2 on M are equivalent relative to F .

Proof. The case $M = D^2 \times S^1$ and $F = \partial M$, the case $M = S^1 \times S^1 \times I$ and $F = \partial M$, and the case $M = S^1 \times S^1 \times I$ and $F \neq \partial M$ respectively follow from Lemma 2.1 of [11], Lemma 2.1 of this paper, and Theorem 8.1 of [16]. We therefore exclude these cases.

Denote by $\bigcup_k \xi_k$ the system of the exceptional fibers ξ_k in M . Let $N(\xi_k)$ be a fibered regular neighborhood of each ξ_k . It follows from Theorem 2.2 of [16] that each G_i preserves some Seifert fibration \mathcal{S}_i of M . Then the uniqueness of a Seifert fibration of M (see VI.18.Theorem of [12]) implies that $\bigcup_k N(\xi_k)$ is isotopic to a setwise G_i -invariant fibered regular neighborhood of the system of exceptional fibers in \mathcal{S}_i . Since Lemma 3.1 of [11] implies that the orders of the exceptional fibers are pairwise coprime, we may assume that $G_1(N(\xi_k)) = G_2(N(\xi_k)) = N(\xi_k)$ for each k . Therefore, it is enough by Lemma 2.1 of [11] to consider the case where M is a product S^1 -bundle.

It follows from Theorem 2.1 of [16] that M admits a G_1 -invariant product structure \mathcal{P}_1 . If $F = \partial M$, M admits a G_2 -invariant product structure \mathcal{P}_2 which agrees with \mathcal{P}_1 on F (see Theorem 2.3 of [16]). If $F \neq \partial M$, we see M as the quotient of the double \overline{M} of M along $\partial M - F$ by \mathbb{Z}_2 generated by an orientation-reversing involution, and apply the same argument to the finite group action on \overline{M} , which is the extension of \mathbb{Z}_2 by G_2 . Then we obtain a G_2 -invariant product structure \mathcal{P}_2 of M which agrees with \mathcal{P}_1 on F .

By the uniqueness of the S^1 -bundle structure of M (see VI.18.Theorem of [12]), there is a map $\varphi \in \text{Diff}(M)$ isotopic to the identity which takes the S^1 -bundle structure induced by \mathcal{P}_1 to the S^1 -bundle structure induced by \mathcal{P}_2 . Modify φ in \mathcal{P}_2 by a fiber preserving isotopy in a fibered regular neighborhood of F so as to restrict to the identity map on F . By conjugating G_2 by φ , we may therefore assume that \mathcal{P}_1 and \mathcal{P}_2 induce the same S^1 -bundle structure of M .

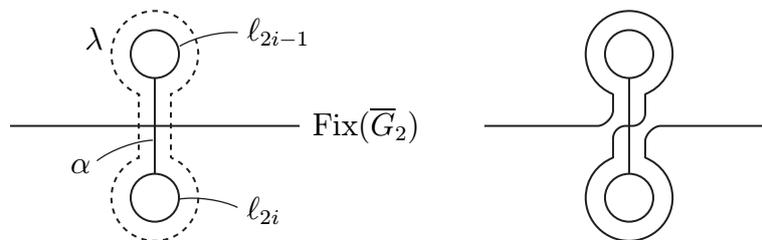
Let $p: M \rightarrow B$ be the projection map onto the base surface B . Each G_i induces a finite group action \overline{G}_i on B . We consider B to be lying on S^2 . Then each \overline{G}_i extends to an action on S^2 . Since \overline{G}_1 and \overline{G}_2 agree on $p(F)$, the quotient spaces B/\overline{G}_1 and B/\overline{G}_2 are orbifold isomorphic to suborbifolds of the same spherical orbifold listed on page 188 of [15]. We may assume that \overline{G}_1 and \overline{G}_2 are not orientation-preserving, otherwise the conclusion follows from Lemma 3.2 and Remark 3.3 of [11]. Then the assumption $G_1(T) = G_2(T) = T$ implies that each \overline{G}_i is generated by the reflection of S^2 in a loop. Since G_1 and G_2 permute the components of ∂M similarly, ∂B consists of loops $\ell_1, \dots, \ell_{2k}, \ell'_1, \dots, \ell'_n$ such that

- (1) \overline{G}_1 and \overline{G}_2 interchange ℓ_{2i-1} and ℓ_{2i} for $1 \leq i \leq k$, and
- (2) \overline{G}_1 and \overline{G}_2 setwise preserve ℓ'_i for $1 \leq i \leq n$.

Without loss of generality, $\ell'_1 = p(T)$. Denote by $\text{Fix}(\overline{G}_i)$ the fixed point circle of the action of each \overline{G}_i on S^2 . Suppose that each $\text{Fix}(\overline{G}_i)$ is equipped with an orientation, and splits B into two pieces $B_{i,1}$ and $B_{i,2}$ so that $\ell'_1 \cap B_{1,1} = \ell'_1 \cap B_{2,1}$ and $\ell'_1 \cap B_{1,2} = \ell'_1 \cap B_{2,2}$. We may assume without loss of generality that $\ell_{2i-1} \subset B_{1,1}$ and $\ell_{2i} \subset B_{1,2}$ for $1 \leq i \leq k$, and that we meets ℓ'_1, \dots, ℓ'_n in order as we go along $\text{Fix}(\overline{G}_1)$.

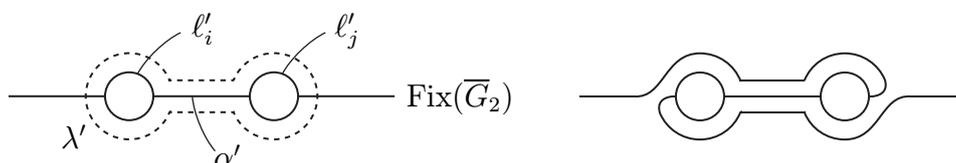
Suppose $\ell_{2i-1} \subset B_{2,2}$ and $\ell_{2i} \subset B_{2,1}$ for some i . By taking a proper arc on B/\overline{G}_2 connecting ℓ_{2i}/\overline{G}_2 and $\text{Fix}(\overline{G}_2)/\overline{G}_2$, we obtain a setwise \overline{G}_2 -invariant arc α on B which meets $\text{Fix}(\overline{G}_2)$ in a point and connects ℓ_{2i-1} and ℓ_{2i} . Then $\text{Fix}(\overline{G}_2)$ is modified by the half twist along the loop $\partial N(\ell_{2i-1} \cup \ell_{2i} \cup \alpha) \cap \text{int}B$, denoted by λ , so that afterwards $\ell_{2i-1} \subset B_{2,1}$ and $\ell_{2i} \subset B_{2,2}$, as illustrated in Figure 2. The argument presented for the proof of Lemma 2.1 implies that this modification is realized by a $1/2$ -twist along the torus $p^{-1}(\lambda)$ which conjugates G_2 to a subgroup of $\text{Diff}(S^3)$. We may therefore assume $\ell_{2i-1} \subset B_{2,1}$ and $\ell_{2i} \subset B_{2,2}$ for $1 \leq i \leq k$.

Figure 2. Half twist along λ .



Suppose that ℓ'_i and ℓ'_j are connected by an arc α' in $\text{Fix}(\overline{G}_2) \cap B$. Then $\text{Fix}(\overline{G}_2)$ is modified by the half twists along the loop $\lambda' = \partial N(\ell'_i \cup \ell'_j \cup \alpha') \cap \text{int}B$ so as to meet ℓ'_i and ℓ'_j in the reverse order, as illustrated in Figure 3, which is realized by the conjugation of G_2 by a $1/2$ -twist along the torus $p^{-1}(\lambda')$, as before. Since every permutation on the set $\{\ell'_2, \dots, \ell'_n\}$ is a product of transpositions, we may assume that $\text{Fix}(\overline{G}_2)$ meets ℓ'_1, \dots, ℓ'_n in order. Moreover, we can change the order in which $\text{Fix}(\overline{G}_2)$ meets the two points in $\ell'_i \cap \text{Fix}(\overline{G}_2)$ by the half twists along ℓ'_i , which is also realized by a $1/2$ -twist along the torus $p^{-1}(\ell'_i)$. We may therefore assume that \overline{G}_2 is equivalent to \overline{G}_1 relative to ∂B .

Figure 3. Half twist along λ' .



Now we may assume $\overline{G}_1 = \overline{G}_2$. Take a map $h \in \text{Diff}(M)$ which restricts to the identity map on F and takes \mathcal{P}_2 to \mathcal{P}_1 setwise preserving every S^1 -fiber. It is easy to verify that h is extendable to a map in $\text{Diff}(S^3)$. Hence, the conclusion follows by conjugating G_2 by h . \square

Lemma 2.3. *Let M be a compact connected 3-manifold in S^3 with non-empty boundary whose interior admits a complete hyperbolic structure of finite volume, and F a non-empty closed submanifold of ∂M . Suppose that G_1 and G_2 are finite group actions on S^3 such that*

- (1) $G_1(M) = G_2(M) = M$ and $G_1(F) = G_2(F) = F$,
- (2) G_1 and G_2 induce the same permutation on the set of the components of ∂M , and
- (3) G_1 and G_2 agree on F .

Then there is a sequence of rational twists along tori in F whose composition conjugates G_2 to a finite group action \widehat{G}_2 such that the actions of G_1 and \widehat{G}_2 on M is equivalent relative to F .

Proof. It follows from Theorem 5.5 of [18] that $\text{int}M$ admits two complete hyperbolic structures of finite volume, one is G_1 -invariant and the other is G_2 -invariant. Mostow's rigidity theorem [15] implies that complete hyperbolic structures of finite volume on $\text{int}M$ are unique up to isometry representing the identity map on $\text{Out}(\pi_1(M))$. We may therefore assume that $\text{int}M$ is endowed with the G_1 -invariant hyperbolic structure, and that G_2 is conjugate to an isometric action G'_2 by $h \in \text{Diff}(M)$ which is isotopic to the identity map.

Next, we are going to modify h in a regular neighborhood of F so as to restrict to the identity map on F . It follows from Proposition D.3.18 of [19] that F consists of tori. Let h_t be an isotopy from h to the identity map. Denote by \overline{G}_2 the finite group action on $F \times I$ whose restriction on $F \times \{t\}$ is induced from the finite group action on F given by the conjugate of G_2 by h_t . In particular, the actions of \overline{G}_2 on $F \times \{0\}$ and $F \times \{1\}$ are respectively given by G'_2 and G_2 . Note that \overline{G}_2 preserves the product structure $F \times \partial I$, and that we can embed $F \times I$ in S^3 so that \overline{G}_2 extends to a finite group action on S^3 .

We consider the partition of the set of the components of F into the orbits under the permutation induced by G_2 . Suppose that the orbits are represented by T_1, \dots, T_n . Lemma 2.1 implies that a rational twist along $T_i \times \{1\}$ conjugates the setwise stabilizer of $T_i \times I$ in \overline{G}_2 so that the action on $T_i \times I$ is equivalent relative to $T_i \times \partial I$ to the action which preserves the product structure. Suppose that the rational twists along the tori in $F \times \{1\}$ are equivariantly induced from those along $T_1 \times \{1\}, \dots, T_n \times \{1\}$. By conjugating \overline{G}_2 by their composition, it is equivalent relative to $F \times \partial I$ to the action which preserves the product structure. This implies that h is modified equivariantly so as to restrict to the identity map on F .

Suppose that $g_1 \in G_1$ and $g_2 \in G_2$ agree on F . Then $g_1 \circ g_2^{-1}$ restricts to the identity map on F . Since the isometry group of $\text{int}M$ is finite (see [15]), Newman's theorem [20] implies $g_1 = g_2$. Hence, G_1 and G'_2 agree on M . This completes the proof. \square

Lemma 2.4. *Let M be a compact connected 3-manifold in S^3 with non-empty boundary such that the double \overline{M} of M along a non-empty compact submanifold Φ of ∂M admits a complete hyperbolic structure of finite volume in its interior. Let F be a closed submanifold of ∂M containing Φ . Suppose that G_1 and G_2 are finite group actions on S^3 such that*

- (1) $G_1(M) = G_2(M) = M$ and $G_1(F) = G_2(F) = F$,
- (2) G_1 and G_2 induce the same permutation on the set of the components of ∂M , and
- (3) G_1 and G_2 agree on F .

Then there is a finite sequence of rational twists along tori in F whose composition conjugates G_2 to a finite group action \widehat{G}_2 such that the actions of G_1 and \widehat{G}_2 on M are equivalent relative to F .

Proof. We see M as the quotient of \overline{M} by \mathbb{Z}_2 generated by an orientation-reversing involution. Each G_i induces a finite group action \overline{G}_i on \overline{M} which is an extension of \mathbb{Z}_2 by G_i . As in the proof of Lemma 2.3, we consider $\text{int}\overline{M}$ endowed with a \overline{G}_1 -invariant hyperbolic structure. Then some $\overline{h} \in \text{Diff}(\overline{M})$, which is isotopic to the identity map, conjugates \overline{G}_2 to an isometric action \overline{G}'_2 . Clearly, Φ meets $\text{int}\overline{M}$ in a totally geodesic surface, and therefore $\overline{h}(\Phi) = \Phi$ holds.

Suppose that $\overline{g}_1 \in \overline{G}_1$ and $\overline{g}_2 \in \overline{G}'_2$ respectively induce $g_1 \in G_1$ and $g_2 \in G_2$ which agree on F . Then $\overline{g}_1^{-1} \circ \overline{g}_2$ restricts to an isometry on each component Φ_i of Φ , which is a compact surface of negative Euler characteristic (see Proposition D.3.18 of [19]). Since $\overline{g}_1^{-1} \circ \overline{g}_2$ is trivial in $\text{Out}(\pi_1(\Phi_i))$, \overline{g}_1 and \overline{g}_2 agree on Φ_i . Therefore, [20] implies $g_1 = g_2$. Hence, some $h \in \text{Diff}(M)$, which setwise preserves Φ and is isotopic to the identity map, conjugates the action of G_1 on M to G_2 .

It follows from Proposition D.3.18 of [19] that $F - \Phi$ consists of tori. As in the proof of Lemma 2.3, modify h in $N(F - \Phi)$ by rational twists along tori in $F - \Phi$ so that afterwards h restricts to the identity map on $F - \Phi$ and conjugates the action of G_1 on M to G_2 . Moreover, we may assume by Lemma 2.3 of [11] that h restricts to the identity map on Φ . Since h extends to an automorphism of S^3 which is diffeomorphic outside M , the conclusion follows. \square

Proposition 2.5. *Theorem 1.1 is true, if Γ is non-splittable.*

Proof. The equivariant loop theorem (see Chapter VII of [15] and [21]) implies that there is a G_1 -invariant system \mathcal{D}_1 of disjoint disks properly embedded in $E(\Gamma)$ which splits $E(\Gamma)$ into pieces with incompressible boundary. The equivariant Dehn's lemma [21,22] implies that the boundary loops of \mathcal{D}_1 bound a G_2 -invariant system \mathcal{D}_2 of disjoint disks properly embedded in $E(\Gamma)$. Since Γ is non-splittable, $E(\Gamma)$ is irreducible. Therefore, there is an isotopy of $E(\Gamma)$ relative to $\partial E(\Gamma)$ which takes \mathcal{D}_2 to \mathcal{D}_1 . Since any finite group action on D^2 is orthogonal [15], we may assume that G_1 and G_2 agree on \mathcal{D}_1 . Moreover, the induced actions on the balls obtained by splitting $E(\Gamma)$ along \mathcal{D}_1 are equivalent relative to the boundary (see [15]). Therefore, it is enough to consider the case where $E(\Gamma)$ is a Haken manifold with incompressible boundary.

We may assume by the equivariant JSJ decomposition theorem (see Theorem 8.6 of [16]) and by the uniqueness of the JSJ decomposition [13,14] that there is a G_1 -invariant and G_2 -invariant system \mathcal{T} of essential annuli and tori in $E(\Gamma)$ realizing the canonical JSJ decomposition of the pair $(E(\Gamma), \partial E(\Gamma))$.

The argument presented for the proof of Proposition 3.10 of [11] implies that some $h \in \text{Diff}(S^3)$, which is isotopic to the identity map relative to $N(\Gamma)$, conjugates G_2 to a finite group action which agree with G_1 on the annuli in \mathcal{T} . We may therefore assume that \mathcal{T} contains no annuli.

The rest of the proof proceeds by induction on the number of tori in \mathcal{T} . Take a piece M_k attaching $\partial E(\Gamma)$. By Lemmas 2.2, 2.3 and 2.4, it is enough to consider the case where G_2 agrees with G_1 on $G_1(M_k)$. Moreover, we may assume by Lemma 2.1 that G_1 and G_2 agree on the components of $\text{cl}(E(\Gamma) - G_1(M_k))$ each of which is a product I -bundle over a torus. Hence, the conclusion follows by the induction hypothesis. \square

3. Possibly Splittable Case

For a symmetry group G of a splittable spatial graph Γ in S^3 , there is a setwise G -invariant system \mathcal{S} of spheres realizing the prime factorization of $E(\Gamma)$ (see [23]). However, \mathcal{S} is not unique in contrast

to the JSJ decomposition of a Haken 3-manifold. If some component of Γ is setwise invariant and every essential sphere in $E(\Gamma)$ has a trivial stabilizer, there is a canonical choice of \mathcal{S} (see [11]). We first prove that this is possible also in the setting of Theorem 1.1.

Lemma 3.1. *Let Γ be a splittable spatial graph in S^3 . Suppose that G_1 and G_2 are symmetry groups of Γ such that*

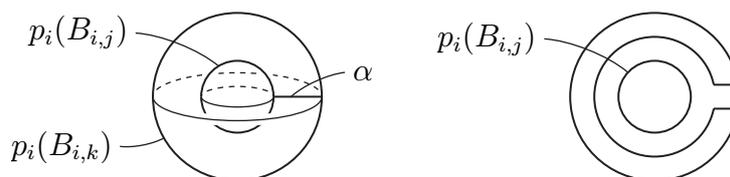
- (1) $G_1(\gamma) = G_2(\gamma) = \gamma$ for at least one component γ of Γ ,
- (2) G_1 and G_2 are cyclic groups acting on Γ freely, and
- (3) G_1 and G_2 agree on $N(\Gamma)$.

Then each G_i admits a setwise G_i -invariant system \mathcal{B}_i of disjoint balls in S^3 not containing γ such that each $\partial\mathcal{B}_i$ realizes the prime factorization of $E(\Gamma)$. Moreover, for some choice of \mathcal{B}_1 and \mathcal{B}_2 , there is a finite sequence of rational twists along incompressible tori in $E(\Gamma) - \text{int}\mathcal{B}_2$ and a map in $\text{Diff}(S^3)$ which restricts to the identity map on $N(\Gamma)$ whose composition conjugates the action of G_2 on $S^3 - \text{int}\mathcal{B}_2$ to the action of G_1 on $S^3 - \text{int}\mathcal{B}_1$.

Proof. Denote by Γ_γ the non-splittable spatial subgraph of Γ containing γ which is obtained by the prime factorization of $E(\Gamma)$. It follows from the equivariant sphere theorem [23] that each G_i admits a setwise G_i -invariant system $\mathcal{S}_i = S_{i,1} \cup \dots \cup S_{i,n}$ of disjoint, non-parallel, essential spheres in $E(\Gamma)$ realizing the prime factorization. Suppose that each $S_{i,j}$ bounds a ball $B_{i,j}$ disjoint from γ . Note that $\text{Sing}(G_i)$ avoids $B_{i,j}$ or meets $B_{i,j}$ in a trivial 1-string tangle (see [15]).

Suppose $B_{i,j} \subset B_{i,k}$ for some distinct j and k . Denote by $p_i: S^3 \rightarrow S^3/G_i = S^3$ the projection map onto the quotient space. Take an arc α properly embedded in $p_i(B_{i,k} - \text{int}B_{i,j})$ which connects $p_i(\partial B_{i,k})$ and $p_i(\partial B_{i,j})$. Suppose that α lies on $p_i(\text{Sing}(G_i))$ if $\text{Sing}(G_i)$ connects $\partial B_{i,j}$ and $\partial B_{i,k}$. By replacing $B_{i,j}$ with another ball in $\text{int}B_{i,k}$ if necessary, α meets \mathcal{S}_i in its endpoints. By drilling into $p_i(B_{i,k})$ along $\alpha \cup p_i(B_{i,j})$, $B_{i,k}$ is deformed to a ball disjoint from $B_{i,j}$, as illustrated in Figure 4 in which the result of the deformation is presented in a cross-sectional view. By a finite repetition of this operation, we obtain a system $\mathcal{B}_i = B_{i,1} \cup \dots \cup B_{i,n}$ of disjoint balls. This proves the first half of the lemma. Without loss of generality, $\Gamma \cap B_{2,j} = \Gamma \cap B_{1,j}$ for $1 \leq j \leq n$.

Figure 4. Modification of $p_i(B_{i,k})$ which makes $p_i(B_{i,k})$ disjoint from $p_i(B_{i,j})$.

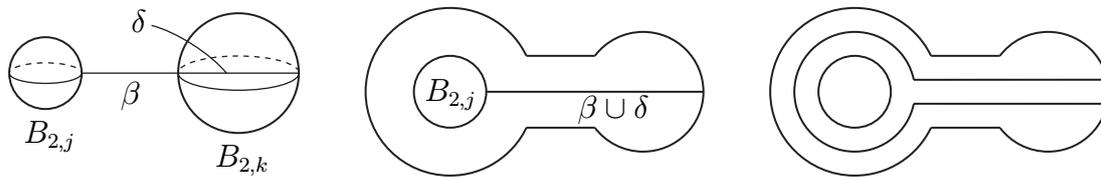


Proposition 2.5 implies that there is a finite sequence of rational twists along incompressible tori in $E(\Gamma_\gamma)$ whose composition h conjugates G_2 to a symmetry group \widehat{G}_2 of Γ_γ equivalent to G_1 relative to $N(\Gamma_\gamma)$. By a G_2 -equivariant isotopy, we may assume that these incompressible tori are disjoint from \mathcal{B}_2 . Then h restricts to the identity map on \mathcal{B}_2 . Suppose that $H \in \text{Diff}(S^3, \Gamma_\gamma)$ realizes the above equivalence of \widehat{G}_2 and G_1 . Then H takes $\text{Sing}(\widehat{G}_2)$ to $\text{Sing}(G_1)$. As a consequence of the affirmative

answer to the Smith conjecture [15], $\text{Sing}(G_1)$ is either an empty set, a trivial knot, or a Hopf link whose components have different indices. Suppose that the orientation of $\text{Sing}(G_1)$ is induced from the orientation of $\text{Sing}(\widehat{G}_2)$ by H .

Suppose that $B_{2,j}$ and $B_{2,k}$ are connected by an arc β in $\text{Sing}(\widehat{G}_2) - \text{int}\mathcal{B}_2$, and that $\text{Sing}(\widehat{G}_2)$ meets $B_{2,k}$ in an arc δ . Then $B_{2,k}$ can be modified by a \widehat{G}_2 -equivariant deformation along $\beta \cup B_{2,j}$ similar to the inverse of that mentioned above so as to contain $B_{2,j}$. Moreover, it can be deformed along $\beta \cup \delta \cup B_{2,j}$ so as to avoid $B_{2,j}$ again, as illustrated in Figure 5. Note that this operation changes the order in which the circle in $\text{Sing}(\widehat{G}_2)$ containing β meets the balls in $\{B_{2,1}, \dots, B_{2,n}\}$.

Figure 5. Modification of $B_{2,k}$ realizing the transposition of $B_{2,j}$ and $B_{2,k}$.



Let C be a component of $\text{Sing}(G_1)$. Without loss of generality, C meets $B_{1,1}, \dots, B_{1,r}$ in order, and avoids $B_{1,r+1}, \dots, B_{1,n}$. Since G_1 and G_2 agree on $N(\Gamma)$, the component $H^{-1}(C)$ of $\text{Sing}(\widehat{G}_2)$ meets $B_{2,1}, \dots, B_{2,r}$ possibly not in order. Since every permutation on the set $\{B_{2,1}, \dots, B_{2,r}\}$ is a product of transpositions realized by the above operation, we may assume that $H^{-1}(C)$ meets $B_{2,1}, \dots, B_{2,r}$ in order. Apply this argument to each component of $\text{Sing}(G_1)$. Since each $\partial\mathcal{B}_i$ realizes the prime factorization of $E(\Gamma)$, we can modify H by a G_1 -equivariant isotopy relative to $N(\Gamma_\gamma)$ so that we have $H(\text{Sing}(\widehat{G}_2)) = \text{Sing}(G_1)$ and $H(B_{2,j}) = B_{1,j}$ for each j . Thus, H is modified so as to conjugate the action of \widehat{G}_2 on $S^3 - \text{int}\mathcal{B}_2$ to the action of G_1 on $S^3 - \text{int}\mathcal{B}_1$.

After this modification, H restricts to an orientation-preserving homeomorphism on \mathcal{B}_1 . Therefore, $H(\Gamma \cap \mathcal{B}_2)$ is ambient isotopic to $\Gamma \cap \mathcal{B}_1$ in \mathcal{B}_1 . Hence, H can be modified in \mathcal{B}_1 so as to restrict to the identity map on $N(\Gamma)$. This completes the proof. \square

Lemma 3.2. *Suppose that G_1 and G_2 are orientation-preserving finite cyclic group actions on $S^2 \times I$ such that*

- (1) G_1 and G_2 do not interchange the components of $S^2 \times \partial I$, and
- (2) G_1 and G_2 agree on $S^2 \times \partial I$.

Then a rational twist along $S^2 \times \{1\}$ conjugates G_2 to a finite group action equivalent to G_1 relative to $S^2 \times \partial I$.

Proof. It is enough to consider the case where G_1 is not trivial. It follows from the remark after Theorem 8.1 of [16] that $S^2 \times I$ admits a G_1 -invariant product structure \mathcal{P}_1 and a G_2 -invariant product structure \mathcal{P}_2 . Since the actions of G_1 and G_2 on $S^2 \times \{0\}$ are conjugate to a rotation of S^2 (see [15]), each $\text{Fix}(G_i)$ consists of two I -fibers in \mathcal{P}_i . Since G_1 and G_2 agree on $S^2 \times \partial I$, we have $\partial\text{Fix}(G_1) = \partial\text{Fix}(G_2)$.

Denote by $p_i: S^2 \times I \rightarrow S^2 \times I/G_i$ the projection map onto the quotient space for each i , and by S_t the S^2 -fiber $S^2 \times \{t\}$ in \mathcal{P}_1 . Connect the two cone points of $p_1(S_0)$ by an arc \bar{a} embedded in $p_1(S_0)$.

Then $p_1^{-1}(\bar{a})$ is a spatial θ_n -curve consisting of two vertices on the fixed points and $n > 1$ edges each connecting them. Denote by A_i the branched surface consisting of I -fibers in \mathcal{P}_i attaching $p_1^{-1}(\bar{a})$ for each i . Then each $p_1(A_i \cap S_1)$ is an arc connecting the two cone points on $p_1(S_1)$. Since the underlying space of $p_1(S_1)$ is a sphere, $p_1(A_2 \cap S_1)$ is isotopic to $p_1(A_1 \cap S_1)$ relative to the cone points. Therefore, A_2 is deformed by a G_2 -equivariant isotopy relative to S_0 so that $A_1 \cap S_1 = A_2 \cap S_1$. There are two cases depending on whether $\text{Fix}(G_1)$ and $\text{Fix}(G_2)$ are isotopic relative to the endpoints or not.

Assume that $\text{Fix}(G_1)$ and $\text{Fix}(G_2)$ are isotopic relative to the endpoints. Then \mathcal{P}_2 is deformed by an isotopy relative to $S^2 \times \partial I$ so as to agree with \mathcal{P}_1 on a setwise G_1 -invariant tubular neighborhood $N(\text{Fix}(G_1))$ saturated in the I -bundle structure induced from \mathcal{P}_1 . Since each A_i meets the solid torus $(S^2 \times I) - \text{int}N(\text{Fix}(G_1))$ in the system of meridian disks, A_2 is moved to A_1 by an isotopy relative to $(S^2 \times \partial I) \cup N(\text{Fix}(G_1))$. We may therefore assume $A_1 = A_2$, and that G_1 and G_2 agree on $N(\text{Fix}(G_1))$. Then the I -bundle structures in \mathcal{P}_1 and \mathcal{P}_2 respectively induce the orbifold isomorphisms $\varphi_1: p_2(S_1) \rightarrow p_2(S_0)$ and $\varphi_2: p_2(S_0) \rightarrow p_1(S_1)$ such that $\varphi_2 \circ \varphi_1$ is isotopic to the identity map by an isotopy relative to the cone points which setwise preserves $p_2(A_2 \cap S_1)$. Then we can deform \mathcal{P}_2 by an isotopy on $p_2(S^2 \times I)$ relative to $p_2(S_0)$ which setwise preserves $p_2(A_2 \cap S_1)$ so that \mathcal{P}_1 and \mathcal{P}_2 induce the same ∂I -bundle structure on $S^1 \times \partial I$. Hence, the diffeomorphism of $S^2 \times I$ which takes \mathcal{P}_2 to \mathcal{P}_1 induces the equivalence of G_1 and G_2 relative to $S^2 \times \partial I$, as required.

Assume that $\text{Fix}(G_1)$ and $\text{Fix}(G_2)$ are not isotopic relative to the endpoints. Let $h: S_1 \rightarrow S_1$ be a lift of an orientation-preserving involution on $p_1(S_1)$ which interchanges the cone points. Then h is a diffeomorphism isotopic to the identity map which conjugates the action of G_2 on S_1 to itself and is realized by a 1/2-twist along the sphere S_1 . We may therefore assume that $\text{Fix}(G_1)$ and $\text{Fix}(G_2)$ are isotopic relative to the endpoints. Hence, the conclusion follows by the argument presented for the previous case. \square

Proof of Theorem 1.1. It is enough by Proposition 2.5 to prove the theorem in the case where Γ is splittable. Then G_1 and G_2 are cyclic groups acting on Γ freely. We may assume by Lemma 3.1 that there is a setwise G_1 -invariant and setwise G_2 -invariant system \mathcal{B} of disjoint balls in S^3 not containing γ such that $\partial\mathcal{B}$ realizes the prime factorization of $E(\Gamma)$, and that G_1 and G_2 agree on $E(\mathcal{B})$.

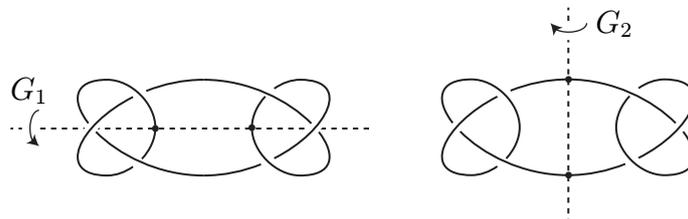
Suppose that \mathcal{B} consists of balls B_1, \dots, B_n . Each $\Gamma \cap B_i$ is a non-empty, non-splittable, spatial subgraph of Γ . By applying Proposition 2.5 to the actions of the setwise stabilisers of B_i in G_1 and G_2 on B_i , we may assume that G_1 and G_2 agree on $E(\partial\mathcal{B})$. Hence the conclusion follows by applying Lemma 3.2 to the actions of G_1 and G_2 on $N(\partial\mathcal{B})$ equivariantly. \square

Remark 3.3. Theorem 1.1 requires the spatial graph Γ to have no companion knot, and the symmetry groups G_1 and G_2 of Γ to act on Γ freely if Γ is splittable. These requirements are needed because of the following examples.

- (1) Suppose that Γ is a granny knot. Then Γ has two companion knots K_1 and K_2 , both of which are trefoil knots. We obtain $E(K_1)$, $E(K_2)$, and a 2-fold composing space by the JSJ decomposition of $E(\Gamma)$. Figure 6 illustrates \mathbb{Z}_2 -symmetries G_1 and G_2 of Γ such that G_2 interchanges $E(K_1)$ and $E(K_2)$ but G_1 does not. By conjugating G_1 by a map in $\text{Diff}(S^3)$ which moves $N(\Gamma)$ in the longitudinal direction, G_1 and G_2 are not equivalent but agree on $\partial N(\Gamma)$. Moreover, any rational

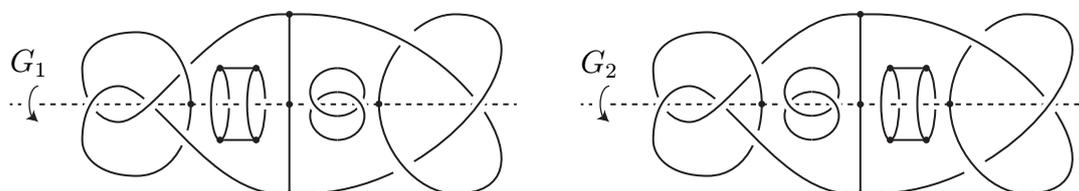
twists along incompressible tori in $E(\Gamma)$ cannot change the induced symmetries of $E(K_1)$ and $E(K_2)$, since the trefoil knot exterior is atoroidal.

Figure 6. \mathbb{Z}_2 -symmetries of a spatial graph with companion knots.



- (2) Suppose that Γ is a spatial graph which splits into non-splittable spatial graphs γ_1 , γ_2 and γ_3 , as illustrated in Figure 7, where γ_1 is a spatial θ -curve. According to the choice of two edges of γ_1 , we obtain a trefoil knot K_1 , a figure-eight knot K_2 , or their connected sum $K_1\#K_2$. Then any map in $\text{Diff}(S^3, \Gamma)$ does not permute these edges. The \mathbb{Z}_2 -symmetries G_1 and G_2 of Γ illustrated in Figure 7 are not equivalent, since there is no map in $\text{Diff}(S^3, \Gamma)$ which takes $\text{Sing}(G_1)$ to $\text{Sing}(G_2)$ and interchanges γ_2 and γ_3 . Moreover, we cannot perform rational twists along incompressible spheres and tori in $E(\Gamma)$ to make G_2 equivalent to G_1 , since any setwise G_2 -invariant incompressible sphere in $E(\Gamma)$ separates γ_2 and γ_3 .

Figure 7. \mathbb{Z}_2 -symmetries which are not free on a splittable spatial graph.



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