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Article

Defining the Symmetry of the Universal Semi-Regular Autonomous Asynchronous Systems

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Abstract: The regular autonomous asynchronous systems are the non-deterministic Boolean dynamical systems and universality means the greatest in the sense of the inclusion. The paper gives four definitions of symmetry of these systems in a slightly more general framework, called semi-regularity, and also many examples.

Keywords: asynchronous system; symmetry; semi-regularity

MSC Classification: 94C10

1. Introduction

Switching theory has developed in the 1950s and the 1960s as a common effort of the mathematicians and the engineers of studying the switching circuits (a.k.a. asynchronous circuits) from digital electrical engineering. We are unaware of any existent mathematical work published after 1970 on what we call switching theory. The published works are written by engineers and their approach is always descriptive and unacceptable for the mathematicians. The label of *switching theory* has changed to *asynchronous systems* (or *circuits*) *theory*. One of the possible motivations of the situation consists in the fact that the important producers of digital equipments have stopped the dissemination of such researches.

Our interest in asynchronous systems had bibliography coming from the 1950s and the 1960s, as well as engineering works giving intuition, as well as mathematical works giving analogies. An interesting *rendez-vous* has happened when the asynchronous systems theory has met the dynamical systems theory, resulting in the so-called *regular* autonomous systems (a.k.a Boolean dynamical systems) where the vector field is $\Phi : \{0,1\}^n \to \{0,1\}^n$ and time is discrete or real, and we obtain the *unbounded delay* *model* of computation of Φ suggested by the engineers. The *synchronous* iterations of $\Phi : \Phi \circ \Phi, \Phi \circ \Phi \circ \Phi, \ldots$ of the dynamical systems are replaced by *asynchronous* iterations in which each coordinate $\Phi_1, \ldots, \Phi_n$ is iterated independently on the others, in arbitrary finite time.

We denote with $\mathbf{B} = \{0, 1\}$ the binary Boolean algebra, together with the discrete topology and with the usual algebraic laws:

$$\begin{array}{cc} & - \\ 0 & 1 \\ 1 & 0 \end{array}, \quad \begin{array}{ccc} \cdot & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}, \quad \begin{array}{ccc} \cup & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}, \quad \begin{array}{ccc} \oplus & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \tag{1}$$

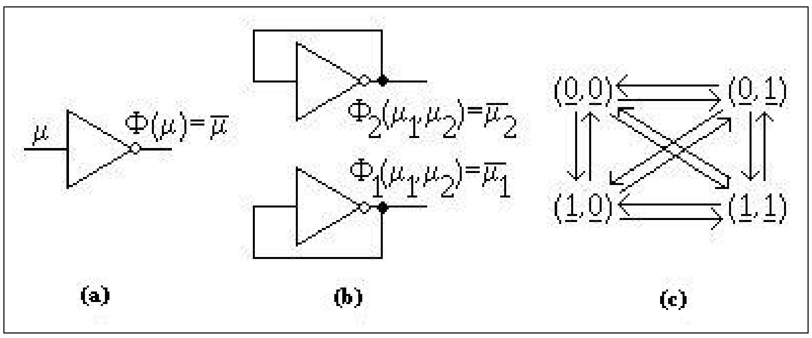
We use the same notations for the laws that are induced from $\mathbf{B}$ on other sets, for example $\forall x \in \mathbf{B}^n$, $\forall y \in \mathbf{B}^n$,

$$\overline{x} = (\overline{x_1}, \ldots, \overline{x_n})$$
$$x \cup y = (x_1 \cup y_1, \ldots, x_n \cup y_n)$$

etc. In Figure 1, we have drawn at (a) the logical gate NOT, *i.e.*, the circuit that computes the logical complement and at (b) a circuit that makes use of logical gates NOT. The asynchronous system that models the circuit from (b) has the state portrait drawn at (c). In the state portraits, the arrows show the increase of (the discrete or continuous) time. The underlined coordinates $\underline{\mu_i}$ are these coordinates for which $\Phi_i(\mu_i) \neq \mu_i$ and they are called *excited*, or *enabled*, or *unstable*. The coordinates μ_i that are not underlined fulfill by definition $\Phi_i(\mu_i) = \mu_i$ and they are called *not excited*, or *not enabled*, or *stable*. The existence of two underlined coordinates in $(0, 0)$ shows that $\Phi_1(0, 0) = 1$ may be computed first, $\Phi_2(0, 0) = 1$ may be computed first, or $\Phi_1(0, 0)$, $\Phi_2(0, 0)$ may be computed simultaneously, thus when the system is in $(0, 0)$, it may run in three different directions, which results in non-determinism.

Our present purpose is to define the symmetry of these systems.

Figure 1. (**a**) the logical gate NOT; (**b**) circuit with logical gates NOT; (**c**) state portrait.



2. Semi-Regular Systems

Notation 1 *We denote* $\mathbf{N}_{\_} = \{-1, 0, 1, 2, \ldots\}$.

Notation 2 $\chi_A : \mathbf{R} \to \mathbf{B}$ *is the notation of the characteristic function of the set* $A \subset \mathbf{R}$*:* $\forall t \in \mathbf{R}$, $\chi_A(t) = \begin{cases} 0, if\ t \notin A \\ 1, if\ t \in A \end{cases}$.

Notation 3 *We denote with* $\overline{\Pi}_n$ *the set of the sequences* $\alpha = \alpha^0, \alpha^1, \ldots, \alpha^k, \ldots \in \mathbf{B}^n$.

Notation 4 *The set of the real sequences* $t_0 < t_1 < \ldots < t_k < \ldots$ *that are unbounded from above is denoted with* Seq.

Notation 5 *We use the notation* $\overline{P}_n$ *for the set of the functions* $\rho : \mathbf{R} \to \mathbf{B}^n$ *having the property that* $\alpha \in \overline{\Pi}_n$ *and* $(t_k) \in Seq$ *exist with* $\forall t \in \mathbf{R}$,

$$\rho(t) = \alpha^0 \chi_{\{t_0\}}(t) \oplus \alpha^1 \chi_{\{t_1\}}(t) \oplus \ldots \oplus \alpha^k \chi_{\{t_k\}}(t) \oplus \ldots \tag{2}$$

Definition 6 *Let* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *be a function. For* $\nu \in \mathbf{B}^n, \nu = (\nu_1, \ldots, \nu_n)$ *we define the function* $\Phi^\nu : \mathbf{B}^n \to \mathbf{B}^n$ *by* $\forall \mu \in \mathbf{B}^n$,

$$\Phi^\nu(\mu) = (\overline{\nu_1}\mu_1 \oplus \nu_1\Phi_1(\mu), \ldots, \overline{\nu_n}\mu_n \oplus \nu_n\Phi_n(\mu))$$

Remark 7 *For any* $\mu \in \mathbf{B}^n, \nu \in \mathbf{B}^n$ *and* $i \in \{1, \ldots, n\}$, *if* $\nu_i = 0$, *then* $\Phi_i^\nu(\mu) = \mu_i$ *i.e.,* $\Phi_i(\mu)$ *is not computed and if* $\nu_i = 1$, *then* $\Phi_i^\nu(\mu) = \Phi_i(\mu)$ *i.e.,* $\Phi_i(\mu)$ *is computed. This is the meaning of asynchronicity.*

Definition 8 *Let* $\alpha \in \overline{\Pi}_n$. *The function* $\widehat{\Phi}^\alpha : \mathbf{B}^n \times \mathbf{N}_{\_} \to \mathbf{B}^n$ *defined by* $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_{\_}$,

$$\begin{cases} \widehat{\Phi}^\alpha(\mu, -1) = \mu, \\ \widehat{\Phi}^\alpha(\mu, k+1) = \Phi^{\alpha^{k+1}}(\widehat{\Phi}^\alpha(\mu, k)) \end{cases} \tag{3}$$

is called ***discrete time*** $\alpha-$***semi-orbit of*** μ. *We consider also the sequence* $(t_k) \in Seq$ *and the function* $\rho \in \overline{P}_n$ *from Equation (2), for which the function* $\Phi^\rho : \mathbf{B}^n \times \mathbf{R} \to \mathbf{B}^n$ *is defined by:* $\forall \mu \in \mathbf{B}^n, \forall t \in \mathbf{R}$,

$$\begin{aligned} \Phi^\rho(\mu, t) = & \widehat{\Phi}^\alpha(\mu, -1)\chi_{(-\infty, t_0)}(t) \oplus \widehat{\Phi}^\alpha(\mu, 0)\chi_{[t_0, t_1)}(t) \oplus \\ & \oplus \widehat{\Phi}^\alpha(\mu, 1)\chi_{[t_1, t_2)}(t) \oplus \ldots \oplus \widehat{\Phi}^\alpha(\mu, k)\chi_{[t_k, t_{k+1})}(t) \oplus \ldots \end{aligned} \tag{4}$$

Φ^ρ *is called* ***continuous time*** $\rho-$***semi-orbit of*** μ.

Definition 9 *The* ***discrete time*** *and the* ***continuous time universal semi-regular autonomous asynchronous systems*** *associated to* Φ *are defined by*

$$\widehat{\widehat{\Xi}}_\Phi = \{\widehat{\Phi}^\alpha(\mu, \cdot) | \mu \in \mathbf{B}^n, \alpha \in \overline{\Pi}_n\}$$
$$\overline{\Xi}_\Phi = \{\Phi^\rho(\mu, \cdot) | \mu \in \mathbf{B}^n, \rho \in \overline{P}_n\}$$

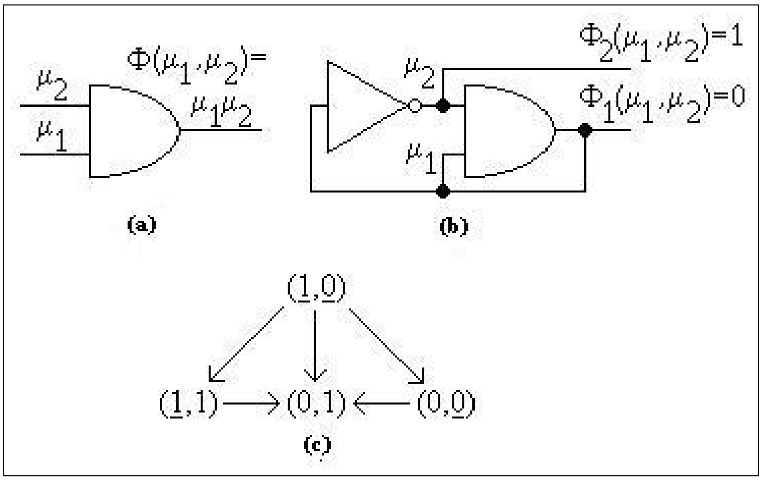
Remark 10 $\widehat{\widehat{\Xi}}_\Phi$, $\overline{\Xi}_\Phi$ *and* Φ *are usually identified.*

Example 11 *In Figure 2 we have drawn at (a) the AND gate that computes the logical intersection, at (b) a circuit with two gates and at (c) the state portrait of* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2, \forall (\mu_1, \mu_2) \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (0, 1)$. *We conclude that*

$$\begin{aligned} \overline{\Xi}_\Phi = & \{(\mu_1, \mu_2)\chi_{(-\infty, t_0)} \oplus (\mu_1\lambda_1, \mu_2 \cup \lambda_2)\chi_{[t_0, t_1)} \oplus \\ & \oplus (\mu_1\lambda_1\nu_1, \mu_2 \cup \lambda_2 \cup \nu_2)\chi_{[t_1, \infty)} | \mu, \lambda, \nu \in \mathbf{B}^2, t_0, t_1 \in \mathbf{R}, t_0 < t_1\} \end{aligned}$$

since the first coordinate might finally decrease its value and the second coordinate might finally increase its value, but the order and the time instant when these things happen are arbitrary.

Figure 2. The semi-regular system $\overline{\Xi}_\Phi$ from Example 11.



3. Anti-Semi-Regular Systems

Definition 12 *Let $\Phi : \mathbf{B}^n \to \mathbf{B}^n, \alpha \in \overline{\Pi}_n, (t_k) \in Seq$ and $\rho \in \overline{P}_n$ from Equation (2). The function ${}^*\widehat{\Phi}^\alpha : \mathbf{B}^n \times \mathbf{N}_- \to \mathbf{B}^n$ that satisfies $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_-$,*

$$\begin{cases} {}^*\widehat{\Phi}^\alpha(\mu, -1) = \mu \\ \Phi^{\alpha^{k+1}}({}^*\widehat{\Phi}^\alpha(\mu, k+1)) = {}^*\widehat{\Phi}^\alpha(\mu, k) \end{cases} \tag{5}$$

*is called **discrete time $\alpha-$anti-semi-orbit of** μ and the function ${}^*\Phi^\rho : \mathbf{B}^n \times \mathbf{R} \to \mathbf{B}^n$ that satisfies $\forall \mu \in \mathbf{B}^n, \forall t \in \mathbf{R}$,*

$$\begin{aligned} {}^*\Phi^\rho(\mu, t) = {}^*\widehat{\Phi}^\alpha(\mu, -1)\chi_{(-\infty,t_0)}(t) \oplus {}^*\widehat{\Phi}^\alpha(\mu, 0)\chi_{[t_0,t_1)}(t) \oplus \\ \oplus {}^*\widehat{\Phi}^\alpha(\mu, 1)\chi_{[t_1,t_2)}(t) \oplus \ldots \oplus {}^*\widehat{\Phi}^\alpha(\mu, k)\chi_{[t_k,t_{k+1})}(t) \oplus \ldots \end{aligned} \tag{6}$$

*is called **continuous time $\rho-$anti-semi-orbit of** μ.*

Remark 13 *We compare the semi-orbits and the anti-semi-orbits now and see that they run both from the past to the future, but the cause-effect relation is different: in $\widehat{\Phi}^\alpha, \Phi^\rho$ the cause is in the past and the effect is in the future, while in ${}^*\widehat{\Phi}^\alpha, {}^*\Phi^\rho$ the cause is in the future and the effect is in the past.*

Definition 14 *The **discrete time** and the **continuous time universal anti-semi-regular autonomous asynchronous systems** associated to Φ are defined by*

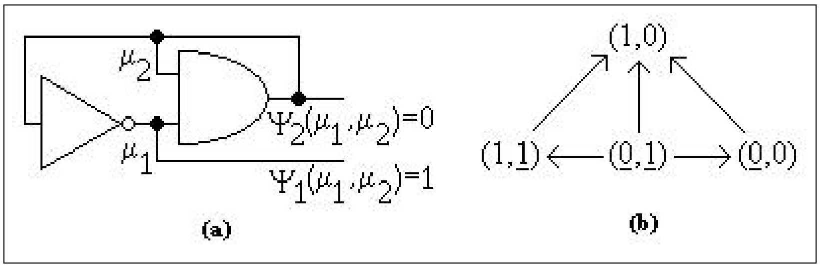
$$\begin{aligned} {}^*\widehat{\overline{\Xi}}_\Phi &= \{{}^*\widehat{\Phi}^\alpha(\mu, \cdot) | \mu \in \mathbf{B}^n, \alpha \in \overline{\Pi}_n\} \\ {}^*\overline{\Xi}_\Phi &= \{{}^*\Phi^\rho(\mu, \cdot) | \mu \in \mathbf{B}^n, \rho \in \overline{P}_n\} \end{aligned}$$

Example 15 *In Figure 3 we have drawn at (a) the circuit and at (b) the state portrait of $\Psi : \mathbf{B}^2 \to \mathbf{B}^2$, $\forall(\mu_1, \mu_2) \in \mathbf{B}^2$, $\Psi(\mu_1, \mu_2) = (1, 0)$ for which*

$$\begin{aligned} \overline{\Xi}_\Psi = \{&(\mu_1, \mu_2)\chi_{(-\infty,t_0)} \oplus (\mu_1 \cup \lambda_1, \mu_2\lambda_2)\chi_{[t_0,t_1)} \oplus \\ &\oplus (\mu_1 \cup \lambda_1 \cup \nu_1, \mu_2\lambda_2\nu_2)\chi_{[t_1,\infty)} | \mu, \lambda, \nu \in \mathbf{B}^2, t_0, t_1 \in \mathbf{R}, t_0 < t_1\} \end{aligned}$$

The arrows in Figures 2(c) and 3(b) are the same, but with a different sense and we note that $\overline{\Xi}_\Psi = {}^*\overline{\Xi}_\Phi$, where Φ is the one from Example 11.

Figure 3. The semi-regular system $\overline{\Xi}_{\Psi}$ from Example 15.



4. Isomorphisms and Anti-Isomorphisms

Definition 16 *Let* $g : \mathbf{B}^n \to \mathbf{B}^n$. *It defines the functions* $\widehat{g} : \overline{\Pi}_n \to \overline{\Pi}_n, \forall \alpha \in \overline{\Pi}_n, \forall k \in \mathbf{N}$,

$$\widehat{g}(\alpha)(k) = g(\alpha^k)$$

$\widetilde{g} : \overline{P}_n \to \overline{P}_n, \forall \rho \in \overline{P}_n, \forall t \in \mathbf{R}$,

$$\widetilde{g}(\rho)(t) = \begin{cases} (0,\ldots,0), & \textit{if } \rho(t) = (0,\ldots,0) \\ g(\rho(t)), & \textit{otherwise} \end{cases}$$

and $g : (\mathbf{B}^n)^{\mathbf{R}} \to (\mathbf{B}^n)^{\mathbf{R}}, \forall x \in (\mathbf{B}^n)^{\mathbf{R}}, \forall t \in \mathbf{R}$,

$$g(x)(t) = g(x(t))$$

Theorem 17 *Let* $\Phi, \Psi, g, g' : \mathbf{B}^n \to \mathbf{B}^n$. *The following statements are equivalent:*
(a) $\forall \nu \in \mathbf{B}^n$, *the diagram*

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi^{\nu}}{\to} & \mathbf{B}^n \\ g \downarrow & & \downarrow g \\ \mathbf{B}^n & \overset{\Psi^{g'(\nu)}}{\to} & \mathbf{B}^n \end{array}$$

is commutative;
(b) $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \overline{\Pi}_n, \forall k \in \mathbf{N}_{\_}$,

$$g(\widehat{\Phi}^{\alpha}(\mu, k)) = \widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\mu), k)$$

(c) $\forall \mu \in \mathbf{B}^n$,

$$g(\mu) = \Psi^{g'(0,\ldots,0)}(g(\mu))$$

and $\forall \mu \in \mathbf{B}^n, \forall \rho \in \overline{P}_n, \forall t \in \mathbf{R}$,

$$g(\Phi^{\rho}(\mu, t)) = \Psi^{\widetilde{g'}(\rho)}(g(\mu), t)$$

Proof. (a)$\Longrightarrow$(b): We fix arbitrarily $\mu \in \mathbf{B}^n$, $\alpha \in \overline{\Pi}_n$ and we use the induction on $k \geq -1$. For $k = -1$, (b) becomes $g(\mu) = g(\mu)$, thus we suppose that it is true for k and we prove it for $k + 1$:

$$g(\widehat{\Phi}^{\alpha}(\mu, k+1)) = g(\Phi^{\alpha^{k+1}}(\widehat{\Phi}^{\alpha}(\mu, k))) = \Psi^{g'(\alpha^{k+1})}(g(\widehat{\Phi}^{\alpha}(\mu, k)))$$
$$= \Psi^{g'(\alpha^{k+1})}(\widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\mu), k)) = \widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\mu), k+1)$$

(b)$\Longrightarrow$(c): The first statement results from (b) if we take $\alpha^0 = (0, \ldots, 0)$ and $k = 0$. In order to prove the second statement, let $\mu \in \mathbf{B}^n$ and $\rho \in \overline{P}_n$ be arbitrary, thus Equation (2) holds with $(t_k) \in Seq, \rho(t_0), \ldots, \rho(t_k), \ldots \in \overline{\Pi}_n$. If $\forall t \in \mathbf{R}, \rho(t) = (0, \ldots, 0)$ the statement to prove takes the form $g(\mu) = g(\mu)$ so that we can suppose now that a finite or an infinite number of $\rho(t_k)$ are $\neq (0, \ldots, 0)$. In the case $\forall k \in \mathbf{N}, \rho(t_k) \neq (0, \ldots, 0)$ that does not restrict the generality of the proof, we have that

$$\widetilde{g'}(\rho)(t) = g'(\rho(t_0))\chi_{\{t_0\}}(t) \oplus \ldots \oplus g'(\rho(t_k))\chi_{\{t_k\}}(t) \oplus \ldots \tag{7}$$

is an element of $\overline{P}_n$ and

$$\begin{aligned} g(\Phi^\rho(\mu, t)) &= g(\mu\chi_{(-\infty,t_0)}(t) \oplus \widehat{\Phi}^\alpha(\mu, 0)\chi_{[t_0,t_1)}(t) \oplus \ldots \oplus \widehat{\Phi}^\alpha(\mu, k)\chi_{[t_k,t_{k+1})}(t) \oplus \ldots) \\ &= g(\mu)\chi_{(-\infty,t_0)}(t) \oplus g(\widehat{\Phi}^\alpha(\mu, 0))\chi_{[t_0,t_1)}(t) \oplus \ldots \oplus g(\widehat{\Phi}^\alpha(\mu, k))\chi_{[t_k,t_{k+1})}(t) \oplus \ldots \\ &= g(\mu)\chi_{(-\infty,t_0)}(t) \oplus \widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\mu), 0)\chi_{[t_0,t_1)}(t) \oplus \ldots \oplus \widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\mu), k)\chi_{[t_k,t_{k+1})}(t) \oplus \ldots \\ &= \Psi^{\widetilde{g'}(\rho)}(g(\mu), t) \end{aligned}$$

(c)$\Longrightarrow$(a): Let $\nu, \mu \in \mathbf{B}^n$ be arbitrary and fixed and we consider $\rho \in \overline{P}_n$ given by Equation (2), with $(t_k) \in Seq$ fixed, $\rho(t_0) = \nu$ and $\forall k \geq 1, \rho(t_k) \neq (0, \ldots, 0)$. We have

$$\begin{aligned} g(\Phi^\rho(\mu, t)) &= g(\mu\chi_{(-\infty,t_0)}(t) \oplus \Phi^\nu(\mu)\chi_{[t_0,t_1)}(t) \oplus \widehat{\Phi}^\alpha(\mu, 1)\chi_{[t_1,t_2)}(t) \oplus \ldots) \\ &= g(\mu)\chi_{(-\infty,t_0)}(t) \oplus g(\Phi^\nu(\mu))\chi_{[t_0,t_1)}(t) \oplus g(\widehat{\Phi}^\alpha(\mu, 1))\chi_{[t_1,t_2)}(t) \oplus \ldots \end{aligned} \tag{8}$$

Case (i) $\nu = (0, \ldots, 0)$, the commutativity of the diagram is equivalent with the first statement of (c).
Case(ii) $\nu \neq (0, \ldots, 0)$,

$$\begin{aligned} \widetilde{g'}(\rho)(t) &= g'(\rho(t)) \\ &= g'(\nu)\chi_{\{t_0\}}(t) \oplus g'(\rho(t_1))\chi_{\{t_1\}}(t) \oplus \ldots \\ \Psi^{\widetilde{g'}(\rho)}(g(\mu), t) &= g(\mu)\chi_{(-\infty,t_0)}(t) \oplus \Psi^{g'(\nu)}(g(\mu))\chi_{[t_0,t_1)}(t) \oplus \widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\mu), 1)\chi_{[t_1,t_2)}(t) \oplus \ldots \end{aligned}$$

and from Equation (8), for $t \in [t_0, t_1)$, we obtain

$$g(\Phi^\nu(\mu)) = \Psi^{g'(\nu)}(g(\mu))$$

Definition 18 *We consider the functions $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$. If $g, g' : \mathbf{B}^n \to \mathbf{B}^n$ bijective exist such that one of the equivalent properties (a), (b) or (c) from Theorem 17 is satisfied, then we say that the couple (g, g') defines an* ***isomorphism*** *from $\widehat{\widehat{\Xi}}_\Phi$ to $\widehat{\widehat{\Xi}}_\Psi$, or from Ξ_Φ to Ξ_Ψ, or from Φ to Ψ. We use the notation $\overline{Iso}(\Phi, \Psi)$ for the set of these couples and we also denote with $\overline{Aut}(\Phi) = \overline{Iso}(\Phi, \Phi)$ the set of the* ***automorphisms*** *of $\widehat{\widehat{\Xi}}_\Phi$, Ξ_Φ, or Φ.*

Theorem 19 *For $\Phi, \Psi, g, g' : \mathbf{B}^n \to \mathbf{B}^n$, the following statements are equivalent:*
(a) $\forall \nu \in \mathbf{B}^n$, the diagram is commutative;

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi^\nu}{\rightarrow} & \mathbf{B}^n \\ g \downarrow & & \downarrow g \\ \mathbf{B}^n & \overset{\Psi^{g'(\nu)}}{\longleftarrow} & \mathbf{B}^n \end{array}$$

(b) $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \overline{\Pi}_n, \forall k \in \mathbf{N}_{\_}$,

$$g(\mu) = {}^*\widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\widehat{\Phi}^\alpha(\mu, k)), k)$$

(c) $\forall \mu \in \mathbf{B}^n$,

$$g(\mu) = \Psi^{g'(0,\ldots,0)}(g(\mu))$$

and $\forall \mu \in \mathbf{B}^n, \forall \rho \in \overline{P}_n, \forall t \in \mathbf{R}$,

$$g(\mu) = {}^*\Psi^{\widetilde{g'}(\rho)}(g(\Phi^\rho(\mu, t)), t)$$

Proof. (a)$\Longrightarrow$(b): We fix arbitrarily $\mu \in \mathbf{B}^n$, $\alpha \in \overline{\Pi}_n$ and we use the induction on $k \geq -1$. In the case $k = -1$ the equality to be proved is satisfied

$$g(\mu) = g(\widehat{\Phi}^\alpha(\mu, -1)) = \widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\widehat{\Phi}^\alpha(\mu, -1)), -1)$$

thus we presume that the statement is true for k and we prove it for $k+1$. We have:

$$\begin{aligned} g(\mu) &= {}^*\widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\widehat{\Phi}^\alpha(\mu, k)), k) \\ &= {}^*\widehat{\Psi}^{\widehat{g'}(\alpha)}(\Psi^{g'(\alpha^{k+1})}(g(\Phi^{\alpha^{k+1}}(\widehat{\Phi}^\alpha(\mu, k)))), k) \\ &= {}^*\widehat{\Psi}^{\widehat{g'}(\alpha)}(g(\widehat{\Phi}^\alpha(\mu, k+1)), k+1) \end{aligned}$$

The proof is similar with the proof of Theorem 17.

Definition 20 *Let* $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$. *If* $g, g' : \mathbf{B}^n \to \mathbf{B}^n$ *bijective exist such that one of the equivalent properties (a), (b) or (c) from Theorem 19 is fulfilled, we say that the couple* (g, g') *defines an* ***anti-isomorphism*** *from* $\widehat{\widehat{\Xi}}_\Phi$ *to* ${}^*\widehat{\widehat{\Xi}}_\Psi$, *or from* Ξ_Φ *to* ${}^*\Xi_\Psi$, *or from* Φ *to* Ψ. *We use the notation* ${}^*\overline{Iso}(\Phi, \Psi)$ *for these couples and we also denote with* ${}^*\overline{Aut}(\Phi) = {}^*\overline{Iso}(\Phi, \Phi)$ *the set of the* ***anti-automorphisms*** *of* $\widehat{\widehat{\Xi}}_\Phi$, Ξ_Φ *or* Φ.

5. Symmetry and Anti-Symmetry

Remark 21 *The fact that* $(1_{\mathbf{B}^n}, 1_{\mathbf{B}^n}) \in \overline{Aut}(\Phi)$ *implies* $\overline{Aut}(\Phi) \neq \emptyset$, *but all of* $\overline{Iso}(\Phi, \Psi), {}^*\overline{Iso}(\Phi, \Psi)$ *and* ${}^*\overline{Aut}(\Phi)$ *may be empty.*

Definition 22 *Let* $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$, $\Phi \neq \Psi$. *If* $\overline{Iso}(\Phi, \Psi) \neq \emptyset$, *then* $\widehat{\widehat{\Xi}}_\Phi$, $\widehat{\widehat{\Xi}}_\Psi$; Ξ_Φ, Ξ_Ψ; Φ, Ψ *are called* ***symmetrical***, *or* ***conjugated***; *if* ${}^*\overline{Iso}(\Phi, \Psi) \neq \emptyset$, *then* $\widehat{\widehat{\Xi}}_\Phi$, ${}^*\widehat{\widehat{\Xi}}_\Psi$; $\Xi_\Phi, {}^*\Xi_\Psi$; Φ, Ψ *are called* ***anti-symmetrical***, *or* ***anti-conjugated***.

If $card(\overline{Aut}(\Phi)) > 1$, *then* $\widehat{\widehat{\Xi}}_\Phi$, Ξ_Φ *and* Φ *are called* ***symmetrical*** *and if* ${}^*\overline{Aut}(\Phi) \neq \emptyset$, *then* $\widehat{\widehat{\Xi}}_\Phi$, Ξ_Φ *and* Φ *are called* ***anti-symmetrical***.

Remark 23 *The symmetry of* Φ, Ψ *means that* $(g, g') \in \overline{Iso}(\Phi, \Psi)$ *maps the transfers* $\mu \to \Phi^\nu(\mu)$ *in transfers* $g(\mu) \to g(\Phi^\nu(\mu)) = \Psi^{g'(\nu)}(g(\mu))$; *the situation when* Φ *is symmetrical and* $(g, g') \in \overline{Aut}(\Phi)$ *is similar. Anti-symmetry may be understood as mirroring:* $(g, g') \in {}^*\overline{Iso}(\Phi, \Psi)$ *maps the transfers (or arrows)* $\mu \to \Phi^\nu(\mu)$ *in transfers* $g(\mu) \longleftarrow g(\Phi^\nu(\mu)) = \Psi^{g'(\nu)}(g(\mu))$ *and similarly for* $(g, g') \in {}^*\overline{Aut}(\Phi)$.

Theorem 24 *Let* $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$.
(a) If $(g, g') \in \overline{Iso}(\Phi, \Psi)$*, then* $(g^{-1}, g'^{-1}) \in \overline{Iso}(\Psi, \Phi)$.
(b) If $(g, g') \in {}^*\overline{Iso}(\Phi, \Psi)$*, then* $(g^{-1}, g'^{-1}) \in {}^*\overline{Iso}(\Psi, \Phi)$.

Proof. (a): The hypothesis states that $\forall \nu \in \mathbf{B}^n$, the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi^\nu}{\to} & \mathbf{B}^n \\ g \downarrow & & \downarrow g \\ \mathbf{B}^n & \overset{\Psi^{g'(\nu)}}{\to} & \mathbf{B}^n \end{array}$$

commutes, with g, g' bijective. We fix arbitrarily $\nu \in \mathbf{B}^n, \mu \in \mathbf{B}^n$. We denote $\mu' = g(\mu), \nu' = g'(\nu)$ and we note that

$$g^{-1}(\Psi^{\nu'}(\mu')) = \Phi^{g'^{-1}(\nu')}(g^{-1}(\mu')) \tag{9}$$

As ν, μ were chosen arbitrarily and on the other hand, when ν runs in $\mathbf{B}^n$, ν' runs in $\mathbf{B}^n$ and when μ runs in $\mathbf{B}^n$, μ' runs in $\mathbf{B}^n$, we infer that Equation (9) is equivalent with the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Psi^{\nu'}}{\to} & \mathbf{B}^n \\ g^{-1} \downarrow & & \downarrow g^{-1} \\ \mathbf{B}^n & \overset{\Phi^{g'^{-1}(\nu')}}{\to} & \mathbf{B}^n \end{array}$$

for any $\nu' \in \mathbf{B}^n$. We have proved that $(g^{-1}, g'^{-1}) \in \overline{Iso}(\Psi, \Phi)$.

(b): By hypothesis $\forall \nu \in \mathbf{B}^n$, the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi^\nu}{\to} & \mathbf{B}^n \\ g \downarrow & & \downarrow g \\ \mathbf{B}^n & \overset{\Psi^{g'(\nu)}}{\longleftarrow} & \mathbf{B}^n \end{array}$$

is commutative, g, g' bijective and we prove that $\forall \nu' \in \mathbf{B}^n$, the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Psi^{\nu'}}{\to} & \mathbf{B}^n \\ g^{-1} \downarrow & & \downarrow g^{-1} \\ \mathbf{B}^n & \overset{\Phi^{g'^{-1}(\nu')}}{\longleftarrow} & \mathbf{B}^n \end{array}$$

is commutative.

Theorem 25 $\overline{Aut}(\Phi)$ *is a group relative to the law:* $\forall (g, g') \in \overline{Aut}(\Phi), \forall (h, h') \in \overline{Aut}(\Phi)$,

$$(h, h') \circ (g, g') = (h \circ g, h' \circ g')$$

Proof. The fact that $\forall (g, g') \in \overline{Aut}(\Phi), \forall (h, h') \in \overline{Aut}(\Phi), (h \circ g, h' \circ g') \in \overline{Aut}(\Phi)$ is proved like this: $\forall \nu \in \mathbf{B}^n$,

$$\begin{aligned} (h \circ g) \circ \Phi^\nu &= h \circ (g \circ \Phi^\nu) = h \circ (\Phi^{g'(\nu)} \circ g) = (h \circ \Phi^{g'(\nu)}) \circ g \\ &= (\Phi^{h'(g'(\nu))} \circ h) \circ g = \Phi^{(h' \circ g')(\nu)} \circ (h \circ g) \end{aligned}$$

the fact that $(1_{\mathbf{B}^n}, 1_{\mathbf{B}^n}) \in \overline{Aut}(\Phi)$ was mentioned before; and the fact that $\forall (g, g') \in \overline{Aut}(\Phi)$, $(g^{-1}, g'^{-1}) \in \overline{Aut}(\Phi)$ was shown at Theorem 24(a).

Definition 26 *Any subgroup* $G \subset \overline{Aut}(\Phi)$ *with* $card(G) > 1$ *is called a* ***group of symmetry*** *of* $\widehat{\widehat{\Xi}}_\Phi$*, of* $\widehat{\Xi}_\Phi$ *or of* Φ.

6. Examples

Example 27 *$\Phi, \Psi : \mathbf{B}^2 \to \mathbf{B}^2$ are given by, see Figure 4*

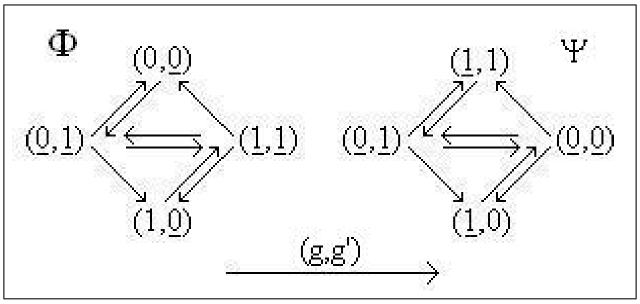
$$\forall(\mu_1,\mu_2) \in \mathbf{B}^2, \Phi(\mu_1,\mu_2) = (\mu_1 \oplus \mu_2, \overline{\mu_2})$$
$$\forall(\mu_1,\mu_2) \in \mathbf{B}^2, \Psi(\mu_1,\mu_2) = (\overline{\mu_1}, \overline{\mu_1}\,\overline{\mu_2} \cup \mu_1\mu_2)$$

and the bijections $g, g' : \mathbf{B}^2 \to \mathbf{B}^2$ are $\forall(\mu_1,\mu_2) \in \mathbf{B}^2$,

$$g(\mu_1,\mu_2) = (\overline{\mu_2}, \overline{\mu_1})$$
$$g'(\mu_1,\mu_2) = (\mu_2, \mu_1)$$

(in order to understand the choice of g', to be remarked in Figure 4 the positions of the underlined coordinates for Φ and Ψ). Φ and Ψ are conjugated.

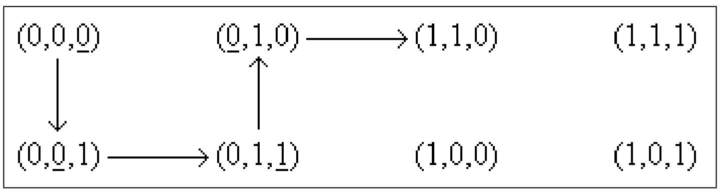
Figure 4. Symmetrical systems, Example 27.



Example 28 *The system from Figure 5 is symmetrical and a group of symmetry is generated by the couples $(g, 1_{\mathbf{B}^3}), (u, 1_{\mathbf{B}^3}), (v, 1_{\mathbf{B}^3})$, see Equation (10); g, u, v are transpositions that permute the isolated fixed points $(1,0,0), (1,0,1), (1,1,1)$.*

$$\begin{array}{ccccc}
(\mu_1,\mu_2,\mu_3) & 1_{\mathbf{B}^3} & g & u & v \\
(0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\
(0,1,0) & (0,1,0) & (0,1,0) & (0,1,0) & (0,1,0) \\
(0,1,1) & (0,1,1) & (0,1,1) & (0,1,1) & (0,1,1) \\
(1,0,0) & (1,0,0) & (1,0,0) & (1,0,1) & (1,1,1) \\
(1,0,1) & (1,0,1) & (1,1,1) & (1,0,0) & (1,0,1) \\
(1,1,0) & (1,1,0) & (1,1,0) & (1,1,0) & (1,1,0) \\
(1,1,1) & (1,1,1) & (1,0,1) & (1,1,1) & (1,0,0)
\end{array} \tag{10}$$

Figure 5. Symmetrical system, Example 28.



Example 29 *The function* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$ *defined by* $\forall \mu \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (\overline{\mu_1}, \overline{\mu_2})$ *fulfills for* $\nu \in \mathbf{B}^2$:

$$\begin{aligned}
\Phi^{\nu}(\mu_1, \mu_2) &= (\overline{\nu_1}\mu_1 \oplus \nu_1\overline{\mu_1}, \overline{\nu_2}\mu_2 \oplus \nu_2\overline{\mu_2}) \\
(\Phi^{\nu} \circ \Phi^{\nu})(\mu_1, \mu_2) &= (\overline{\nu_1}\Phi_1^{\nu_1}(\mu_1, \mu_2) \oplus \nu_1\overline{\Phi_1^{\nu_1}(\mu_1, \mu_2)}), \overline{\nu_2}\Phi_2^{\nu_2}(\mu_1, \mu_2) \oplus \nu_2\overline{\Phi_2^{\nu_2}(\mu_1, \mu_2)}) \\
&= (\overline{\nu_1}(\overline{\nu_1}\mu_1 \oplus \nu_1\overline{\mu_1}) \oplus \nu_1(\overline{\nu_1}\mu_1 \oplus \nu_1\overline{\mu_1} \oplus 1), \overline{\nu_2}(\overline{\nu_2}\mu_2 \oplus \nu_2\overline{\mu_2}) \oplus \nu_2(\overline{\nu_2}\mu_2 \oplus \nu_2\overline{\mu_2} \oplus 1)) \\
&= ((\nu_1 \oplus 1)\mu_1 \oplus \nu_1(\mu_1 \oplus 1) \oplus \nu_1, (\nu_2 \oplus 1)\mu_2 \oplus \nu_2(\mu_2 \oplus 1) \oplus \nu_2) \\
&= (\nu_1\mu_1 \oplus \mu_1 \oplus \nu_1\mu_1 \oplus \nu_1 \oplus \nu_1, \nu_2\mu_2 \oplus \mu_2 \oplus \nu_2\mu_2 \oplus \nu_2 \oplus \nu_2) \\
&= (\mu_1, \mu_2)
\end{aligned}$$

thus $(1_{\mathbf{B}^2}, 1_{\mathbf{B}^2}) \in {}^{*}\overline{Aut}(\Phi)$ *and* Φ *is anti-symmetrical. The state portrait of* Φ *was drawn in Figure 1(c).*

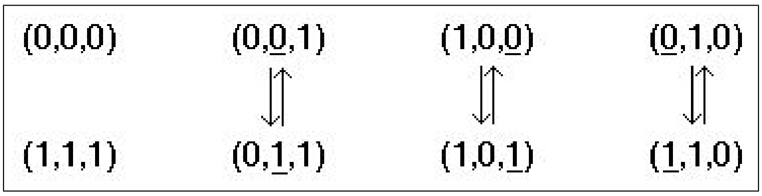
Notation 30 *Let* $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ *be a bijection. We use the notation* $\pi_\sigma : \mathbf{B}^n \to \mathbf{B}^n$ *for the bijection given by* $\forall \mu \in \mathbf{B}^n$,

$$\pi_\sigma(\mu_1, \dots, \mu_n) = (\mu_{\sigma(1)}, \dots, \mu_{\sigma(n)})$$

Definition 31 *Any of* $\widehat{\widehat{\Xi}}_\Phi$, $\overline{\Xi}_\Phi$ *and* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *is called* ***symmetrical relative to the coordinates*** *if the bijection* σ *exists,* $\sigma \neq 1_{\{1,\dots,n\}}$ *such that* $(\pi_\sigma, \pi_\sigma) \in \overline{Aut}(\Phi)$.

Example 32 *We consider the function* $\Phi : \mathbf{B}^3 \to \mathbf{B}^3$ *defined by* $\forall \mu \in \mathbf{B}^3$, $\Phi(\mu_1, \mu_2, \mu_3) = (\mu_2\mu_3 \oplus \mu_1 \oplus \mu_2, \mu_1\mu_3 \oplus \mu_2 \oplus \mu_3, \mu_1\mu_2 \oplus \mu_1 \oplus \mu_3)$ *and the permutation* $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. *A group of symmetry of* $\overline{\Xi}_\Phi$ *is represented by* $G = \{(1_{\mathbf{B}^3}, 1_{\mathbf{B}^3}), (\pi_\sigma, \pi_\sigma), (\pi_{\sigma\circ\sigma}, \pi_{\sigma\circ\sigma})\}$. *We have given in Figure 6 the state portrait of* Φ.

Figure 6. System that is symmetrical relative to the coordinates, Example 32.



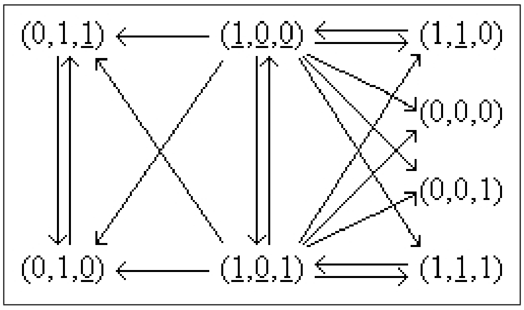
Notation 33 *For* $\lambda \in \mathbf{B}^n$, *we denote by* $\theta^\lambda : \mathbf{B}^n \to \mathbf{B}^n$ *the translation of vector* $\lambda : \forall \mu \in \mathbf{B}^n$,

$$\theta^\lambda(\mu) = \mu \oplus \lambda$$

Definition 34 *If* $(\theta^\lambda, g') \in \overline{Aut}(\Phi)$ *holds for some* $(\theta^\lambda, g') \neq (1_{\mathbf{B}^n}, 1_{\mathbf{B}^n})$, *we say that any of* $\widehat{\widehat{\Xi}}_\Phi$, $\overline{\Xi}_\Phi$ *and* Φ *is* ***symmetrical relative to translations***.

Example 35 *In Figure 7*

Figure 7. Φ has the automorphism $(\theta^{(0,0,1)}, 1_{\mathbf{B}^3})$, Example 35.



we have the system with Φ given by Equation (11)

$$\begin{array}{cc} (\mu_1,\mu_2,\mu_3) & \Phi \\ (0,0,0) & (0,0,0) \\ (0,0,1) & (0,0,1) \\ (0,1,0) & (0,1,1) \\ (0,1,1) & (0,1,0) \\ (1,0,0) & (0,1,1) \\ (1,0,1) & (0,1,0) \\ (1,1,0) & (1,0,0) \\ (1,1,1) & (1,0,1) \end{array} \tag{11}$$

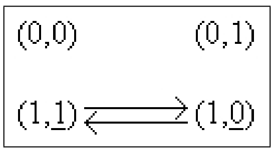
and $(\theta^{(0,0,1)}, 1_{\mathbf{B}^3}) \in \overline{Aut}(\Phi)$, as resulting from the state portrait.

Example 36 *In Equation (12) we have a function $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$ for which four functions $g_1', g_2', g_3', g_4' : \mathbf{B}^2 \to \mathbf{B}^2$ exist:*

$$\begin{array}{cccccc} (\mu_1,\mu_2) & \Phi & g_1' & g_2' & g_3' & g_4' \\ (0,0) & (0,0) & (0,0) & (1,0) & (0,0) & (1,0) \\ (0,1) & (0,1) & (0,1) & (0,1) & (1,1) & (1,1) \\ (1,0) & (1,1) & (1,0) & (0,0) & (1,0) & (0,0) \\ (1,1) & (1,0) & (1,1) & (1,1) & (0,1) & (0,1) \end{array} \tag{12}$$

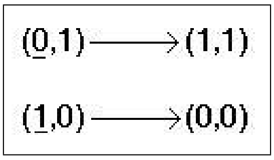
such that $(1_{\mathbf{B}^2}, g_1'), (1_{\mathbf{B}^2}, g_2'), (1_{\mathbf{B}^2}, g_3'), (1_{\mathbf{B}^2}, g_4') \in \overline{Aut}(\Phi)$. The state portrait of Φ is drawn Figure 8.

Figure 8. Φ is symmetrical relative to translations with $(0,0)$, Example 36.



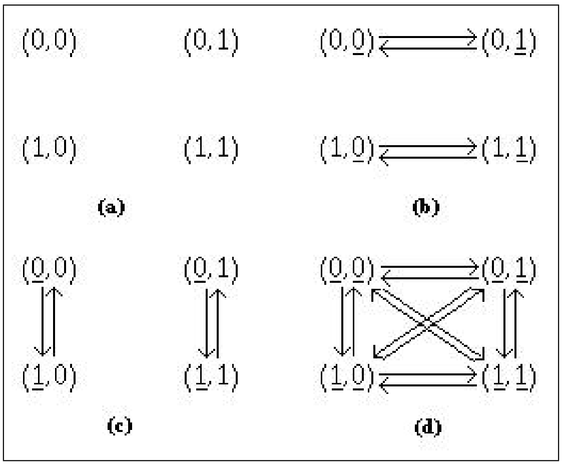
Example 37 *The system from Figure 9 is symmetrical relative to translations, since it has the group of symmetry $G = \{(1_{\mathbf{B}^2}, 1_{\mathbf{B}^2}), (\theta^{(1,1)}, 1_{\mathbf{B}^2})\}$. Φ is self-dual $\Phi = \Phi^*$, where the dual Φ^* of Φ is defined by $\Phi^*(\mu) = \overline{\Phi(\overline{\mu})}$.*

Figure 9. Function Φ that is self dual, $(\theta^{(1,1)}, 1_{\mathbf{B}^2}) \in \overline{Aut}(\Phi)$, Example 37.



Example 38 *Functions* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$ *exist, see Figure 10, that are symmetrical relative to the translations with any* $\lambda \in \mathbf{B}^2$*, thus their group of symmetry is* $G = \{(1_{\mathbf{B}^2}, 1_{\mathbf{B}^2}), (\theta^{(0,1)}, 1_{\mathbf{B}^2}), (\theta^{(1,0)}, 1_{\mathbf{B}^2}), (\theta^{(1,1)}, 1_{\mathbf{B}^2})\}$. *The fact that* $(\theta^{(1,1)}, 1_{\mathbf{B}^2}) \in G$ *shows that all these functions:* $\Phi(\mu) = (\mu_1, \mu_2)$, $\Phi(\mu) = (\mu_1, \overline{\mu_2})$, $\Phi(\mu) = (\overline{\mu_1}, \mu_2)$, $\Phi(\mu) = (\overline{\mu_1}, \overline{\mu_2})$ *are self-dual,* $\Phi = \Phi^*$.

Figure 10. Functions Φ that are self dual, $(\theta^{(1,1)}, 1_{\mathbf{B}^2}) \in \overline{Aut}(\Phi)$, Example 38.



Example 39 *The group of symmetry* G *of the system from Figure 11 has four elements given by Equation (13)*

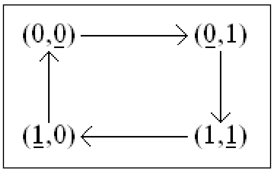
$$\begin{array}{ccccc} (\mu_1, \mu_2) & 1_{\mathbf{B}^2} & g & h & \theta^{(1,1)} \\ (0,0) & (0,0) & (0,1) & (1,0) & (1,1) \\ (0,1) & (0,1) & (1,1) & (0,0) & (1,0) \\ (1,0) & (1,0) & (0,0) & (1,1) & (0,1) \\ (1,1) & (1,1) & (1,0) & (0,1) & (0,0) \end{array} \tag{13}$$

and we remark that $h = g^{-1}, \theta^{(1,1)} = (\theta^{(1,1)})^{-1}$ *hold, see also Equation (14).*

$$\begin{array}{ccccc} (\nu_1, \nu_2) & (1_{\mathbf{B}^2})' & g' & h' & (\theta^{(1,1)})' \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,1) & (0,1) & (1,0) & (1,0) & (0,1) \\ (1,0) & (1,0) & (0,1) & (0,1) & (1,0) \\ (1,1) & (1,1) & (1,1) & (1,1) & (1,1) \end{array} \tag{14}$$

We have $\Phi = \Phi^*$ *like previously.*

Figure 11. Symmetry including symmetry relative to translations, Example 39.



7. Conclusions

The paper defines the universal semi-regular autonomous asynchronous systems and the universal anti-semi-regular autonomous asynchronous systems. It also defines and characterizes the isomorphisms (automorphisms) and the anti-isomorphisms (anti-automorphisms) of these systems. Symmetry is defined as the existence of such isomorphisms (automorphisms), while anti-symmetry is defined as the existence of such anti-isomorphisms (anti-automorphisms). Many examples are given. A by-pass product in this study is anti-symmetry, which is related with systems having the cause in the future and the effect in the past. Another by-pass product consists in semi-regularity, since important examples of isomorphisms (automorphisms) are of semi-regular systems only and do not keep progressiveness and regularity [2,3].

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