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Article

Monochrome Symmetric Subsets in Colorings of Finite Abelian Groups

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Abstract: A subset S of a group G is symmetric if there is an element $g \in G$ such that $gS^{-1}g = S$. We study some Ramsey type functions for symmetric subsets in finite Abelian groups.

Keywords: finite Abelian group; symmetric subset; Ramsey functions

Classification: MSC 2010. Primary 05A20, 05C55, 20K01; Secondary 05D10.

1. Introduction

Let G be a finite group. Given an element $g \in G$, the *symmetry* on G with the *centre* g is the mapping

$$\eta_g : G \ni x \mapsto gx^{-1}g \in G$$

This is an old notion, which can be found in the book [1]. And it is a very natural one, since

$$\eta_g = \lambda_g \circ \iota \circ \lambda_g^{-1} = \rho_g \circ \iota \circ \rho_g^{-1}$$

where

$$\lambda_g : G \ni x \mapsto gx \in G, \rho_g : G \ni x \mapsto xg \in G, \text{ and } \iota : G \ni x \mapsto x^{-1} \in G$$

are the left translation, the right translation, and the inversion, respectively. Indeed, it follows from $\lambda_g(x) = gx$ that $\lambda_g^{-1}(gx) = x$, so $\lambda_g^{-1}(x) = g^{-1}x$. Consequently, $\lambda_g^{-1} = \lambda_{g^{-1}}$. Similarly, $\rho_g^{-1} = \rho_{g^{-1}}$. Then

$$\lambda_g \circ \iota \circ \lambda_g^{-1}(x) = \lambda_g \circ \iota \circ \lambda_{g^{-1}}(x) = g(g^{-1}x)^{-1} = gx^{-1}g \text{ and}$$
$$\rho_g \circ \iota \circ \rho_g^{-1}(x) = \rho_g \circ \iota \circ \rho_{g^{-1}}(x) = (xg^{-1})^{-1}g = gx^{-1}g$$

A subset $S \subseteq G$ is *symmetric* if it is invariant with respect to some symmetry on G. Equivalently, S is *symmetric* if there exists an element $g \in G$ (centre of symmetry) such that $gS^{-1}g = S$.

Given $r \in \mathbb{N}$, an r-*coloring* of G is any mapping $\chi : G \to \{1, \ldots, r\}$.

Definition 1.1 For every finite group and $r \in \mathbb{N}$, define the numbers $s_r(G)$ and $\sigma_r(G)$ as follows.

$s_r(G)$ is the greatest number of the form $\frac{k}{|G|}$, where $k \in \mathbb{N}$ such that for every r-coloring of G there exists a monochrome symmetric subset of cardinality k.

$\sigma_r(G)$ is the greatest number of the form $\frac{k}{|G|}$, where $k \in \mathbb{N}$ such that for every r-coloring χ of G there exists a subset $X \subseteq G$ of cardinality k and element g such that $\chi(x) = \chi(gx^{-1}g)$ for all $x \in X$.

It is easy to see that

$$s_r(G) \leq \frac{1}{r} + \frac{1}{|G|}, \ \sigma_r(G) \leq 1, \ s_r(G) \geq \frac{\sigma_r(G)}{r}$$

For every finite Abelian group G, $\sigma_r(G) \geq \frac{1}{r}$, and consequently, $s_r(G) \geq \frac{1}{r^2}$ [2]. In the non-Abelian case this inequality fails [3]. In this note we describe groups with $\sigma_r(G) = \frac{1}{r}$, $\sigma_r(G) = 1$, and $s_2(G) = \frac{1}{4}$. Since the journal [2] is not easy to access and it is in Ukrainian, we give here also a short proof of the inequality from [2].

2. The Inequality

In this section we prove the following theorem.

Theorem 2.1 Let G be a finite group of odd order or any finite Abelian group, and let $r \in \mathbb{N}$. Then

$$\sigma_r(G) \geq \frac{1}{r}$$

and consequently

$$s_r(G) \geq \frac{1}{r^2}$$

Let G be a finite group. For every r-coloring $\chi : G \to \{1, \ldots, r\}$ and $g \in G$, let

$$S(\chi, g) = |\{x \in G : \chi(x) = \chi(gx^{-1}g)\}|$$

and let

$$\sigma(\chi) = \frac{1}{|G|} \max_{g \in G} S(\chi, g)$$

Then

$$\sigma_r(G) = \min_{\chi : G \to \{1,\ldots,r\}} \sigma(\chi)$$

For every $a \in G$, let

$$\nu(a) = |\{x \in G : x^2 = a\}|$$

Lemma 2.2 For every $\chi : G \to \{1, \ldots, r\}$,

$$\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} \sum_{(x,y) \in A_i^2} \nu(yx^{-1})$$

where $A_i = \chi^{-1}(i)$.

Proof Computing in two ways the number of all triples $(g, x, y) \in G \times G \times G$ such that $gx^{-1}g = y$, we obtain

$$\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} \sum_{(x,y) \in A_i^2} |\{g \in G : gx^{-1}g = y\}|$$

It remains to notice that

$$|\{g \in G : gx^{-1}g = y\}| = |\{g \in G : gx^{-1}gx^{-1} = yx^{-1}\}| = \nu(yx^{-1})$$

Proof of Theorem 2.1 Let $\chi : G \to \{1, \dots, r\}$ and let $A_i = \chi^{-1}(i)$. By Lemma 2.2

$$\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} \sum_{(x,y) \in A_i^2} \nu(yx^{-1})$$

If G has odd order, then $\nu(yx^{-1}) = 1$ for any $x, y \in G$. Since the function $x_1^2 + \dots + x_r^2$, where $x_1 + \dots + x_r = C$, attains minimum when $x_1 = \dots = x_r = \frac{C}{r}$,

$$\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} |A_i|^2 \geq \underbrace{\left(\frac{|G|}{r}\right)^2 + \dots + \left(\frac{|G|}{r}\right)^2}_{r} = \frac{|G|^2}{r}$$

If G is Abelian, then $\nu(yx^{-1}) > 0$ if and only if $yx^{-1} \in G^2 = \{g^2 : g \in G\}$ and in this case $\nu(yx^{-1}) = [G : G^2]$. Let C_j $(1 \leq j \leq k)$ be cosets of G modulo G^2, $C_{j,i} = C_j \bigcap A_i$. Then

$$\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} \sum_{j=1}^{k} |C_{j,i}|^2 \cdot k \geq rk \left(\frac{|G|}{rk}\right)^2 \cdot k = \frac{|G|^2}{r}$$

Therefore, in each case, there exists an element $g \in G$ such that $S(\chi, g) \geq \frac{|G|}{r}$ and so $\sigma(\chi) \geq \frac{1}{r}$.

3. Finite Abelian Groups with $\sigma_r(G) = \frac{1}{r}$ and $\sigma_r(G) = 1$

In this section we describe finite Abelian groups with $\sigma_r(G) = \frac{1}{r}$ and $\sigma_r(G) = 1$.

Theorem 3.1 $\sigma_r(G) = \frac{1}{r}$ if and only if r divides $|2G|$.

Proof Define the subgroups $2G = \{2x : x \in G\}$ and $B(G) = \{x \in G : 2x = 0\}$. Denote $|2G| = m$ and $|B(G)| = k$. Obviously, $|G| = mk$.

Consider first the case when r does not divide m. Fix any r-coloring χ of a group G. Let C_j $(1 \leq j \leq k)$ be cosets of G modulo $2G$, $C_{j,i} = C_j \bigcap \chi^{-1}(i)$. Then

$$\sum_{i=1}^{r} |C_{j,i}|^2 > r \left(\frac{m}{r}\right)^2 = \frac{m^2}{r}$$

Hence,

$$\sum_{g \in G} S(\chi, g) = k \sum_{j=1}^{k} \sum_{i=1}^{r} |C_{j,i}|^2 > k^2 \frac{m^2}{r} = \frac{|G|^2}{r}$$

Therefore, there exists an element $g \in G$ such that $S(\chi, g) > \frac{|G|}{r}$ and so $\sigma(\chi) > \frac{1}{r}$.

Now consider the case where r divides m. By Theorem 2.1, $\sigma_r(G) \geq \frac{1}{r}$, so it suffices to construct a coloring χ with $\sigma(\chi) = \frac{1}{r}$. Pick subgroup H of a group G such that $B(G) \subseteq H$ and $[G : H] = r$. Then $[2G : 2H] = r$. Define r-coloring χ of G as follows:

(1) every coset of G modulo $2H$ is monochrome;

(2) every r cosets of G modulo $2H$ which form a coset of G modulo $2G$ are colored in r different colors.

Then

$$\begin{aligned}
\chi(x) = \chi(2g - x) &\Leftrightarrow x - (2g - x) \in 2H \\
&\Leftrightarrow 2(x - g) \in 2H \\
&\Leftrightarrow \exists h \in H :\ 2(x - g - h) = 0 \\
&\Leftrightarrow \exists h \in H :\ x - g - h \in B(G) \\
&\Leftrightarrow x - g \in H + B(G) = H \\
&\Leftrightarrow x \in g + H.
\end{aligned}$$

So $S(\chi, g) = |H|$ for every $g \in G$. Therefore $\sigma(\chi) = \frac{|H|}{|G|} = \frac{1}{[G:H]} = \frac{1}{r}$.

Theorem 3.2 $\sigma_r(G) = 1$ if and only if one of the following cases holds:

(1) $r = 1$;

(2) $r = 2$ and G is a cyclic group of order either 3 or 5;

(3) G is a Boolean group.

Proof Sufficiency is obvious. We need to prove Necessity. Assume on the contrary that neither of cases (1)–(3) holds.

Suppose first that $|G|$ is even. Then both subgroups $2G$ and the elementary Abelian 2-group $B(G)$ are different from G. Pick $a, b \in G$ such that $a + b \notin 2G$ and $a - b \notin B(G)$. Define $\chi : G \to \{1, 2\}$ by

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \{a, b\} \\ 2 & \text{otherwise} \end{cases}$$

Let $g \in G$. Since $a + b \notin 2G$, $2g - a \neq b$. If $2g - a = a$, then $2g - b \neq b$, because $a - b \notin B(G)$. It follows that either $\chi(a) \neq \chi(2g - a)$ or $\chi(b) \neq \chi(2g - b)$, a contradiction.

Now suppose that $|G|$ is odd. Then $2G = G$. Since $|G| \geq 7$, we can choose distinct $a, b, c \in G$ such that for any distinct $g, x \in \{a, b, c\}$, $2g - x \notin \{a, b, c\}$.

To see this, pick any distinct $a, b \in G$. There is a unique $g \in G$ such that $b = 2g - a$. We then pick $c \in G \setminus \{a, b, g, 2a - b, 2b - a\}$.

Define $\chi : G \to \{1, 2\}$ by

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \{a, b, c\} \\ 2 & \text{otherwise} \end{cases}$$

Let $g \in G$. If $g \notin \{a, b, c\}$ and $2g - a = b$, then $2g - c \notin \{a, b, c\}$. If $g \in \{a, b, c\}$, say $g = a$, then $2g - b \notin \{a, b, c\}$. It follows that there is $x \in \{a, b, c\}$ such that $\chi(x) \neq \chi(2g - x)$, again a contradiction.

Remark Theorem 3.2 describes finite Abelian groups where each r-coloring is symmetric. A coloring χ of G is *symmetric* if there exists $g \in G$ such that

$$\chi(gx^{-1}g) = \chi(x) \text{ for all } x \in G$$

Obviously, the number of all r-colorings of a group G of order n equals r^n. To find the number of all symmetric r-colorings of a group G is a quite complicated exercise involving Möbious inversion on the lattice of subgroups. Precise formula for the number of all symmetric r-colorings of finite Abelian group was established in [4], and the corresponding formula for every finite group has been found only recently [5] (for the quaternion group, see [6]).

4. Finite Abelian Groups with $s_2(G) = \frac{1}{4}$

In this section we prove the following.

Theorem 4.1 Let G be a finite Abelian group and $n \in \mathbb{N}$.

(1) If G contains subgroup $\bigoplus_n \mathbb{Z}_4$, then $s_{2^n}(G) = \frac{1}{4^n}$;

(2) If G does not contain subgroup $\mathbb{Z}_4$, then $s_2(G) > \frac{1}{4}$.

We first prove some auxiliary statements.

Lemma 4.2 $s_{2^n}(\bigoplus_n \mathbb{Z}_4) = \frac{1}{4^n}$.

Proof Define the coloring $\chi : \bigoplus_n \mathbb{Z}_4 \to \bigoplus_n \mathbb{Z}_2$ by

$$(\chi(x))_i = \begin{cases} 0 & \text{if } (x)_i \in \{0, 1\} \\ 1 & \text{if } (x)_i \in \{2, 3\} \end{cases}$$

Fix $g \in \bigoplus_n \mathbb{Z}_4$. If $\chi(x) = \chi(2g - x)$, then $(\chi(x))_i = (\chi(2g - x))_i$. It remains to notice that $s_2(\mathbb{Z}_4) = \frac{1}{4}$.

For every group G and a coloring χ, let $s(\chi)$ denote the cardinality of the largest monochrome symmetric subset of G divided by $|G|$.

Lemma 4.3 Let G be a finite group, let $f : G \to H$ be a surjective homomorphism and let χ be a coloring of H. Define coloring φ of G by $\varphi = \chi \circ f$. Then $s(\varphi) = s(\chi)$.

Proof Let S be a monochrome subset of G symmetric with respect to $g \in G$. By definition of φ it follows that $\varphi(x) = \varphi(gx^{-1}g)$ if and only if $\chi(f(x)) = \chi(f(g)f(x)^{-1}f(g))$. So, $f(S)$ is a monochrome subset of H symmetric with respect to $f(g)$. Since $|S| \leq |\ker f| \cdot |f(S)|$,

$$\frac{|S|}{|G|} \leq \frac{|\ker f| \cdot |f(S)|}{|G|} = \frac{f(S)}{|H|}$$

Thus $s(\varphi) \leq s(\chi)$.

Conversely, let S be a monochrome subset of H symmetric with respect to $h \in H$. Then $f^{-1}(S)$ is a monochrome subset of G symmetric with respect to any $g \in f^{-1}(h)$. Since $|f^{-1}(S)| = |\ker f| \cdot |f(S)|$,

$$\frac{|f^{-1}(S)|}{|G|} = \frac{|S|}{|H|}$$

Thus $s(\varphi) \geq s(\chi)$.

Corollary 4.4 Let G be a finite group and let H be a homomorphic image of G. Then $s_r(G) \leq s_r(H)$.

Proof of Theorem 4.1 (1) By Theorem 2.1, we have that $s_{2^n}(G) \geq \frac{1}{4^n}$. If G contains subgroup $\bigoplus_n \mathbb{Z}_4$, then there exists a homomorphism from G onto $\bigoplus_n \mathbb{Z}_4$. Thus, by Corollary 4.4, $s_{2^n}(G) \leq s_{2^n}(\bigoplus_n \mathbb{Z}_4)$. By Lemma 4.2, $s_{2^n}(\bigoplus_n \mathbb{Z}_4) = \frac{1}{4^n}$. Thus $s_{2^n}(G) = \frac{1}{4^n}$.

(2) Suppose that G does not contain $\mathbb{Z}_4$. Then $G = H \times B$ for some subgroup H of odd order and Boolean group B (which can be trivial). Let χ be an arbitrary 2-coloring of G. For every $b \in B$ define the coloring χ_b on H by $\chi_b(x) = \chi(x, b)$. Then

$$\sum_{h\in H} S(\chi, h) = \sum_{h\in H}\sum_{b\in B} S(\chi_b, h) = \sum_{b\in B}\sum_{h\in H} S(\chi_b, h)$$

Since H has odd order, we obtain that

$$\sum_{h\in H} S(\chi_b, h) > \frac{|H|^2}{2}$$

Then

$$\sum_{h\in H} S(\chi, h) = \sum_{h\in H}\sum_{b\in B} S(\chi_b, h) > |B| \cdot \frac{|H|^2}{2}$$

It follows that there exists $h \in H$ such that

$$S(\chi, h) > \frac{|B| \cdot |H|}{2} = \frac{|G|}{2}$$

Consequently,

$$\sigma(\chi) > \frac{1}{2}$$

and hence

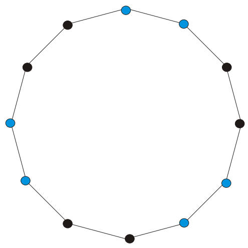
$$s_2(G) > \frac{1}{4}$$

Theorem 4.1 implies the following two criteria.

Corollary 4.5 For every finite Abelian group G, $s_2(G) = \frac{1}{4}$ if and only if G contains subgroup $\mathbb{Z}_4$.

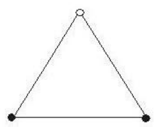
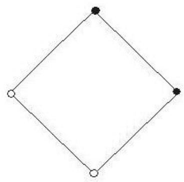
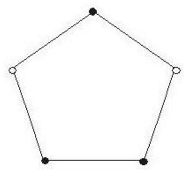
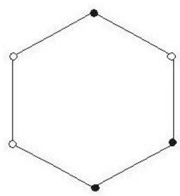
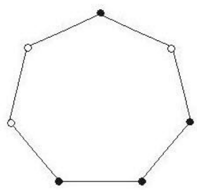
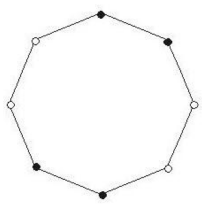
Corollary 4.6 $s_2(\mathbb{Z}_n) = \frac{1}{4}$ if and only if $4|n$.

Below we give the corresponding coloring.



We conclude the paper with the following table (Table 1).

Table 1. Ramsey functions $s_2(\mathbb{Z}_n)$ and $\sigma_2(\mathbb{Z}_n)$ for $n \leq 8$.

n	k	$s_2(\mathbb{Z}_n) = \frac{k}{n}$	m	$\sigma_2(\mathbb{Z}_n) = \frac{m}{n}$	χ
1	1	1	1	1	
2	1	0.5	2	1	
3	2	0.66666666	3	1	
4	1	0.25	2	0.5	
5	3	0.6	5	1	
6	2	0.33333333	4	0.66666666	
7	3	0.42857142	5	0.71428571	
8	2	0.25	4	0.5	

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