Article

Monochrome Symmetric Subsets in Colorings of Finite Abelian Groups

Yuliya Zelenyuk

School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa; E-Mail: yuliya.zelenyuk@wits.ac.za

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Abstract: A subset $S$ of a group $G$ is symmetric if there is an element $g \in G$ such that $gS^{-1}g = S$. We study some Ramsey type functions for symmetric subsets in finite Abelian groups.

Keywords: finite Abelian group; symmetric subset; Ramsey functions

Classification: MSC 2010. Primary 05A20, 05C55, 20K01; Secondary 05D10.

1. Introduction

Let $G$ be a finite group. Given an element $g \in G$, the symmetry on $G$ with the centre $g$ is the mapping

$$\eta_g : G \ni x \mapsto gx^{-1}g \in G$$

This is an old notion, which can be found in the book [1]. And it is a very natural one, since

$$\eta_g = \lambda_g \circ \iota \circ \lambda_g^{-1} = \rho_g \circ \iota \circ \rho_g^{-1}$$

where

$$\lambda_g : G \ni x \mapsto gx \in G, \rho_g : G \ni x \mapsto xg \in G,$$

are the left translation, the right translation, and the inversion, respectively. Indeed, it follows from $\lambda_g(x) = gx$ that $\lambda_g^{-1}(gx) = x$, so $\lambda_g^{-1}(x) = g^{-1}x$. Consequently, $\lambda_g^{-1} = \lambda_g^{-1}$. Similarly, $\rho_g^{-1} = \rho_g^{-1}$. Then

$$\lambda_g \circ \iota \circ \lambda_g^{-1}(x) = \lambda_g \circ \iota \circ \lambda_g^{-1}(x) = g(g^{-1}x)^{-1} = gx^{-1}g$$

and

$$\rho_g \circ \iota \circ \rho_g^{-1}(x) = \rho_g \circ \iota \circ \rho_g^{-1}(x) = (xg^{-1})^{-1}g = gx^{-1}g$$
A subset $S \subseteq G$ is symmetric if it is invariant with respect to some symmetry on $G$. Equivalently, $S$ is symmetric if there exists an element $g \in G$ (centre of symmetry) such that $gS^{-1}g = S$.

Given $r \in \mathbb{N}$, an $r$-coloring of $G$ is any mapping $\chi : G \to \{1, \ldots, r\}$.

**Definition 1.1** For every finite group and $r \in \mathbb{N}$, define the numbers $s_r(G)$ and $\sigma_r(G)$ as follows.

$s_r(G)$ is the greatest number of the form $\frac{k}{|G|}$, where $k \in \mathbb{N}$ such that for every $r$-coloring of $G$ there exists a monochrome symmetric subset of cardinality $k$.

$\sigma_r(G)$ is the greatest number of the form $\frac{k}{|G|}$, where $k \in \mathbb{N}$ such that for every $r$-coloring $\chi$ of $G$ there exists a subset $X \subseteq G$ of cardinality $k$ and element $g$ such that $\chi(x) = \chi(gx^{-1}g)$ for all $x \in X$.

It is easy to see that

$$s_r(G) \leq \frac{1}{r} + \frac{1}{|G|}, \quad \sigma_r(G) \leq 1, \quad s_r(G) \geq \frac{\sigma_r(G)}{r}$$

For every finite Abelian group $G$, $\sigma_r(G) \geq \frac{1}{r}$, and consequently, $s_r(G) \geq \frac{1}{r^2}$ [2]. In the non-Abelian case this inequality fails [3]. In this note we describe groups with $\sigma_r(G) = \frac{1}{r}$, $\sigma_r(G) = 1$, and $s_2(G) = \frac{1}{4}$. Since the journal [2] is not easy to access and it is in Ukrainian, we give here also a short proof of the inequality from [2].

2. The Inequality

In this section we prove the following theorem.

**Theorem 2.1** Let $G$ be a finite group of odd order or any finite Abelian group, and let $r \in \mathbb{N}$. Then

$$\sigma_r(G) \geq \frac{1}{r}$$

and consequently

$$s_r(G) \geq \frac{1}{r^2}$$

Let $G$ be a finite group. For every $r$-coloring $\chi : G \to \{1, \ldots, r\}$ and $g \in G$, let

$$S(\chi, g) = |\{x \in G : \chi(x) = \chi(gx^{-1}g)\}|$$

and let

$$\sigma(\chi) = \frac{1}{|G|} \max_{g \in G} S(\chi, g)$$

Then

$$\sigma_r(G) = \min_{\chi : G \to \{1, \ldots, r\}} \sigma(\chi)$$

For every $a \in G$, let

$$\nu(a) = |\{x \in G : x^2 = a\}|$$

**Lemma 2.2** For every $\chi : G \to \{1, \ldots, r\}$,

$$\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} \sum_{(x,y) \in A_i} \nu(yx^{-1})$$

where $A_i = \chi^{-1}(i)$. 
Proof Computing in two ways the number of all triples \((g, x, y) \in G \times G \times G\) such that \(gx^{-1}g = y\), we obtain

\[
\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} \sum_{(x, y) \in A_{i}^{2}} \left|\{g \in G : gx^{-1}g = y\}\right|
\]

It remains to notice that

\[
\left|\{g \in G : gx^{-1}g = y\}\right| = \left|\{g \in G : gx^{-1}gx^{-1} = yx^{-1}\}\right| = \nu(yx^{-1})
\]

Proof of Theorem 2.1 Let \(\chi : G \to \{1, \ldots, r\}\) and let \(A_{i} = \chi^{-1}(i)\). By Lemma 2.2

\[
\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} \sum_{(x, y) \in A_{i}^{2}} \nu(yx^{-1})
\]

If \(G\) has odd order, then \(\nu(yx^{-1}) = 1\) for any \(x, y \in G\). Since the function \(x_{1}^{2} + \cdots + x_{r}^{2}\), where \(x_{1} + \cdots + x_{r} = C\), attains minimum when \(x_{1} = \cdots = x_{r} = C/r\),

\[
\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} |A_{i}|^{2} \geq \left(\frac{|G|}{r}\right)^{2} + \cdots + \left(\frac{|G|}{r}\right)^{2} = \frac{|G|^{2}}{r}
\]

If \(G\) is Abelian, then \(\nu(yx^{-1}) > 0\) if and only if \(yx^{-1} \in G^{2} = \{g^{2} : g \in G\}\) and in this case \(\nu(yx^{-1}) = |G : G^{2}|\). Let \(C_{j}(1 \leq j \leq k)\) be cosets of \(G\) modulo \(G^{2}\), \(C_{j,i} = C_{j} \cap A_{i}\). Then

\[
\sum_{g \in G} S(\chi, g) = \sum_{i=1}^{r} \sum_{j=1}^{k} |C_{j,i}|^{2} \cdot k \geq rk \left(\frac{|G|}{rk}\right)^{2} \cdot k = \frac{|G|^{2}}{r}
\]

Therefore, in each case, there exists an element \(g \in G\) such that \(S(\chi, g) \geq \frac{|G|}{r}\) and so \(\sigma(\chi) \geq \frac{1}{r}\).

3. Finite Abelian Groups with \(\sigma_r(G) = \frac{1}{r}\) and \(\sigma_r(G) = 1\)

In this section we describe finite Abelian groups with \(\sigma_r(G) = \frac{1}{r}\) and \(\sigma_r(G) = 1\).

Theorem 3.1 \(\sigma_r(G) = \frac{1}{r}\) if and only if \(r\) divides \(|2G|\).

Proof Define the subgroups \(2G = \{2x : x \in G\}\) and \(B(G) = \{x \in G : 2x = 0\}\). Denote \(|2G| = m\) and \(|B(G)| = k\). Obviously, \(|G| = mk\).

Consider first the case when \(r\) does not divide \(m\). Fix any \(r\)-coloring \(\chi\) of a group \(G\). Let \(C_{j}(1 \leq j \leq k)\) be cosets of \(G\) modulo \(2G\), \(C_{j,i} = C_{j} \cap \chi^{-1}(i)\). Then

\[
\sum_{i=1}^{r} |C_{j,i}|^{2} > r \left(\frac{m}{r}\right)^{2} = \frac{m^{2}}{r}
\]

Hence,

\[
\sum_{g \in G} S(\chi, g) = k \sum_{j=1}^{k} \sum_{i=1}^{r} |C_{j,i}|^{2} > k^{2} \frac{m^{2}}{r} = \frac{|G|^{2}}{r}
\]
Therefore, there exists an element \( g \in G \) such that \( S(\chi, g) > \frac{|G|}{r} \) and so \( \sigma(\chi) > \frac{1}{r} \).

Now consider the case where \( r \) divides \( m \). By Theorem 2.1, \( \sigma_r(G) \geq \frac{1}{r} \), so it suffices to construct a coloring \( \chi \) with \( \sigma(\chi) = \frac{1}{r} \). Pick subgroup \( H \) of a group \( G \) such that \( B(G) \subseteq H \) and \( |G : H| = r \). Then \( [2G : 2H] = r \). Define \( r \)-coloring \( \chi \) of \( G \) as follows:

1. every coset of \( G \) modulo \( 2H \) is monochrome;
2. every \( r \) cosets of \( G \) modulo \( 2H \) which form a coset of \( G \) modulo \( 2G \) are colored in \( r \) different colors.

Then

\[
\chi(x) = \chi(2g - x) \iff x - (2g - x) \in 2H \\
\iff 2(x - g) \in 2H \\
\iff \exists h \in H : 2(x - g - h) = 0 \\
\iff \exists h \in H : x - g - h \in B(G) \\
\iff x - g \in H + B(G) = H \\
\iff x \in g + H.
\]

So \( S(\chi, g) = |H| \) for every \( g \in G \). Therefore \( \sigma(\chi) = \frac{|H|}{|G|} = \frac{1}{|G : H|} = \frac{1}{r} \).

**Theorem 3.2** \( \sigma_r(G) = 1 \) if and only if one of the following cases holds:

1. \( r = 1 \);
2. \( r = 2 \) and \( G \) is a cyclic group of order either 3 or 5;
3. \( G \) is a Boolean group.

**Proof** Sufficiency is obvious. We need to prove Necessity. Assume on the contrary that neither of cases (1)–(3) holds.

Suppose first that \( |G| \) is even. Then both subgroups \( 2G \) and the elementary Abelian 2-group \( B(G) \) are different from \( G \). Pick \( a, b \in G \) such that \( a + b \notin 2G \) and \( a - b \notin B(G) \). Define \( \chi : G \to \{1, 2\} \) by

\[
\chi(x) = \begin{cases} 
1 & \text{if } x \in \{a, b\} \\
2 & \text{otherwise}
\end{cases}
\]

Let \( g \in G \). Since \( a + b \notin 2G \), \( 2g - a \neq b \). If \( 2g - a = a \), then \( 2g - b \neq b \), because \( a - b \notin B(G) \). It follows that either \( \chi(a) \neq \chi(2g - a) \) or \( \chi(b) \neq \chi(2g - b) \), a contradiction.

Now suppose that \( |G| \) is odd. Then \( 2G = G \). Since \( |G| \geq 7 \), we can choose distinct \( a, b, c \in G \) such that for any distinct \( g, x \in \{a, b, c\} \), \( 2g - x \notin \{a, b, c\} \).

To see this, pick any distinct \( a, b \in G \). There is a unique \( g \in G \) such that \( b = 2g - a \). We then pick \( c \in G \setminus \{a, b, 2a - b, 2b - a\} \).

Define \( \chi : G \to \{1, 2\} \) by

\[
\chi(x) = \begin{cases} 
1 & \text{if } x \in \{a, b, c\} \\
2 & \text{otherwise}
\end{cases}
\]
Let $g \in G$. If $g \not\in \{a, b, c\}$ and $2g - a = b$, then $2g - c \notin \{a, b, c\}$. If $g \in \{a, b, c\}$, say $g = a$, then $2g - b \notin \{a, b, c\}$. It follows that there is $x \in \{a, b, c\}$ such that $\chi(x) \neq \chi(2g - x)$, again a contradiction.

**Remark** Theorem 3.2 describes finite Abelian groups where each $r$-coloring is symmetric. A coloring $\chi$ of $G$ is symmetric if there exists $g \in G$ such that $\chi(g^{-1}x) = \chi(x)$ for all $x \in G$.

Obviously, the number of all $r$-colorings of a group $G$ of order $n$ equals $r^n$. To find the number of all symmetric $r$-colorings of a group $G$ is a quite complicated exercise involving Mōbious inversion on the lattice of subgroups. Precise formula for the number of all symmetric $r$-colorings of finite Abelian group was established in [4], and the corresponding formula for every finite group has been found only recently [5] (for the quaternion group, see [6]).

### 4. Finite Abelian Groups with $s_2(G) = \frac{1}{4}$

In this section we prove the following.

**Theorem 4.1** Let $G$ be a finite Abelian group and $n \in \mathbb{N}$.

1. If $G$ contains subgroup $\bigoplus_n \mathbb{Z}_4$, then $s_2^n(G) = \frac{1}{4^n}$;
2. If $G$ does not contain subgroup $\mathbb{Z}_4$, then $s_2(G) > \frac{1}{4}$.

We first prove some auxiliary statements.

**Lemma 4.2** $s_2^n(\bigoplus_n \mathbb{Z}_4) = \frac{1}{4^n}$.

**Proof** Define the coloring $\chi : \bigoplus_n \mathbb{Z}_4 \rightarrow \bigoplus_n \mathbb{Z}_2$ by

$$(\chi(x))_i = \begin{cases} 0 & \text{if } (x)_i \in \{0, 1\} \\ 1 & \text{if } (x)_i \in \{2, 3\} \end{cases}$$

Fix $g \in \bigoplus_n \mathbb{Z}_4$. If $\chi(x) = \chi(2g - x)$, then $(\chi(x))_i = (\chi(2g - x))_i$. It remains to notice that $s_2(\mathbb{Z}_4) = \frac{1}{4}$.

For every group $G$ and a coloring $\chi$, let $s(\chi)$ denote the cardinality of the largest monochrome symmetric subset of $G$ divided by $|G|$.

**Lemma 4.3** Let $G$ be a finite group, let $f : G \rightarrow H$ be a surjective homomorphism and let $\chi$ be a coloring of $H$. Define coloring $\varphi$ of $G$ by $\varphi = \chi \circ f$. Then $s(\varphi) = s(\chi)$.

**Proof** Let $S$ be a monochrome subset of $G$ symmetric with respect to $g \in G$. By definition of $\varphi$ it follows that $\varphi(x) = \varphi(gx^{-1}g)$ if and only if $\chi(f(x)) = \chi(f(g)f(x^{-1})f(g))$. So, $f(S)$ is a monochrome subset of $H$ symmetric with respect to $f(g)$. Since $|S| \leq |\ker f| \cdot |f(S)|$,

$$\frac{|S|}{|G|} \leq \frac{|\ker f| \cdot |f(S)|}{|G|} = \frac{f(S)}{|H|}$$

Thus $s(\varphi) \leq s(\chi)$. 

Conversely, let $S$ be a monochrome subset of $H$ symmetric with respect to $h \in H$. Then $f^{-1}(S)$ is a monochrome subset of $G$ symmetric with respect to any $g \in f^{-1}(h)$. Since $|f^{-1}(S)| = |\ker f| \cdot |f(S)|$, 

$$\frac{|f^{-1}(S)|}{|G|} = \frac{|S|}{|H|}$$

Thus $s(\varphi) \geq s(\chi)$.

**Corollary 4.4** Let $G$ be a finite group and let $H$ be a homomorphic image of $G$. Then $s_\tau(G) \leq s_\tau(H)$.

**Proof of Theorem 4.1** (1) By Theorem 2.1, we have that $s_2 \tau(G) \geq \frac{1}{4^n}$. If $G$ contains subgroup $\bigoplus_n \mathbb{Z}_4$, then there exists a homomorphism from $G$ onto $\bigoplus_n \mathbb{Z}_4$. Thus, by Corollary 4.4, $s_2 \tau(G) \leq s_2 \tau(\bigoplus_n \mathbb{Z}_4)$.

By Lemma 4.2, $s_2 \tau(\bigoplus_n \mathbb{Z}_4) = \frac{1}{4^n}$. Thus $s_2 \tau(G) = \frac{1}{4^n}$.

(2) Suppose that $G$ does not contain $\mathbb{Z}_4$. Then $G = H \times B$ for some subgroup $H$ of odd order and Boolean group $B$ (which can be trivial). Let $\chi$ be an arbitrary 2-coloring of $G$. For every $b \in B$ define the coloring $\chi_b$ on $H$ by $\chi_b(x) = \chi(x, b)$. Then

$$\sum_{h \in H} S(\chi, h) = \sum_{h \in H} \sum_{b \in B} S(\chi_b, h) = \sum_{b \in B} \sum_{h \in H} S(\chi_b, h)$$

Since $H$ has odd order, we obtain that

$$\sum_{h \in H} S(\chi_b, h) > \frac{|H|^2}{2}$$

Then

$$\sum_{h \in H} S(\chi, h) = \sum_{h \in H} \sum_{b \in B} S(\chi_b, h) > |B| \cdot \frac{|H|^2}{2}$$

It follows that there exists $h \in H$ such that

$$S(\chi, h) > \frac{|B| \cdot |H|}{2} = \frac{|G|}{2}$$

Consequently,

$$\sigma(\chi) > \frac{1}{2}$$

and hence

$$s_2(G) > \frac{1}{4}$$

Theorem 4.1 implies the following two criteria.

**Corollary 4.5** For every finite Abelian group $G$, $s_2(G) = \frac{1}{4}$ if and only if $G$ contains subgroup $\mathbb{Z}_4$.

**Corollary 4.6** $s_2(\mathbb{Z}_n) = \frac{1}{4}$ if and only if $4|n$.

Below we give the corresponding coloring.
We conclude the paper with the following table (Table 1).

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References


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