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Article

Long Time Behaviour on a Path Group of the Heat Semi-group Associated to a Bilaplacian

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Abstract: We show that in long-time the heat semi-group on a path group associated to a Bilaplacian on the group tends to the Haar distribution on a path group.

Keywords: heat semigroup; Haar distribution; path group

1. Introduction

Let us consider a compact connected Lie group G of dimension d endowed with its normalized biinvariant Haar measure dg. Let us consider the Laplacian Δ on it. It is equal to $\sum(\partial_{e_i})^2$ where e_i is an orthonormal basis of the Lie algebra of G. It generates a Markov semi-group P_t:

$$\frac{\partial}{\partial t}P_t f = \Delta P_t f \tag{1}$$

if f is smooth. Moreover there is a **strictly positive** heat kernel

$$P_t f(g) = \int_G p_t(g, g')f(g')dg' = \int_G p_t(e, g^{-1}g')f(g')dg' \tag{2}$$

when $t \to \infty$

$$P_t f(g) \to \int_G f(g)dg \tag{3}$$

Let us consider a Bilaplacian on G, this means a power Δ^k $k > 1$. It generates still a semi-group P_t^k. P_t^k is not a Markovian semi-group. This means that the heat kernel $p_t^k(g, g')$ associated to P_t^k can **change sign**. We have still when $t \to \infty$

$$P_t^k f(g) \to \int_G f(g)dg \tag{4}$$

In the first case, the heat semi-group is represented by the Brownian motion on G. In the second case, there is until now no stochastic process associated to it. In the case of $\mathbb{R}^d$, the path integral involved with the semi-group P_t^k is defined as a distribution in [1].

We are motivated in this work by an extension in infinite dimension of these results, by considering the case of the path-group $C([0,1],G)$ from continuous path from $[0,1]$ into G starting from e.

Let us recall that the Haar measure $d\tilde{g}$ on a topological group $\tilde{G}$ exists as a full measure **if and only if** the group is locally compact. Haar measure means that for all bounded measurable function $\tilde{F}$

$$\int_{\tilde{G}} \tilde{F}(\tilde{g}_1\tilde{g})d\tilde{g} = \int_{\tilde{G}} \tilde{F}(\tilde{g})d\tilde{g} \tag{5}$$

The difficult requirement to satisfy is the Lebesgue dominated convergence: Let $\tilde{F}_n$ be a bounded increasing sequence of measurable functions tending almost surely to $\tilde{F}$. Then

$$\int_{\tilde{G}} \tilde{F}_n d\tilde{g} \to \int_{\tilde{G}} \tilde{F} d\tilde{g} \tag{6}$$

Haar measures in infinite dimension were studied by Pickrell [2] and Asada [3]. We have defined the Haar distribution on a path group by using the Hida-Streit approach of path integrals as distribution [4–7]. We refer to the review of Albeverio for various rigorous approaches to path integrals [8] and our review on geometrical path integrals [4,9].

In the case of a path group, we can consider the Wiener process on a path-group $t \to \{s \to g_{s,t}\}$ starting from the unit path (See the work of Airault-Malliavin ([10]), the work of Baxendale [11] and the review paper of Léandre on that topic [12]). We have shown that

$$E[F(g_{.,t})] \to \int_{C([0,1],G)} F(g(.))dD \tag{7}$$

when $t \to \infty$ where dD is the Haar distribution on the path group and F is a test functional of Hida type [7].

Recently we are motivated by extending stochastic analysis tools in the non-Markovian context ([13–17]). Especially in [1], we are interested in constructing the sheet and martingales problem in distributional sense for a big-order differential operator on $\mathbb{R}^d$. We consider for that the Connes test algebra.

Let us recall what is the main difference between the Hida test algebra and the Connes test algebra.

(1) Hida considers Fock spaces and tensor product of Hilbert spaces.

(2) Connes, motivated by his work on entire cyclic cohomology, considers Banach spaces. Tensor product of Banach spaces whose theory (mainly due to Grothendieck) is much more complicated than the theory of tensor product of Hilbert spaces.

In [1], we are motivated by the generalization of martingale problems in the non-Markovian context. We consider Connes spaces in [1]. In the present context, we are not motivated by that and we return in the original framework of [7].

We consider the heat semi-group on a path group associated to a bilaplacian on the group in the manner of [1]. In [1], we look at the case of $\mathbb{R}^d$. Here we consider the case of the compact Lie group G. The analysis is similar because we have analog estimates of the heat-kernel [18–20].

In order to resume, we consider an element σ of an Hida Fock space, we associate a functional $\Psi(\sigma)$ on the path group. The heat-semi group (in the **distributional** sense) Q_t^k satisfies the three next properties:

(1) $Q_t^k\Psi(\sigma)$ is still in the considered space

(2) $Q_t^k \circ Q_{t'}^k = Q_{t+t'}^k$

(3) When $t \to \infty$

$$Q_t^k\Psi(\sigma)(g_.) \to \int_{C([0,1],G)} \Psi(\sigma)dD \tag{8}$$

Q_t^k is not a Markovian semi-group on $C([0,1],G)$. Especially, Q_t^k is not represented by a stochastic process. However we expect to extend in this context (7).

2. A Brief Review on the Haar Distribution on a Path Group

Let us recall what is the Brownian motion $t \to B_t$ on $\mathbb{R}$. We consider the set of continuous path $t \to B_t$ issued from 0 from $\mathbb{R}^+$ into $\mathbb{R}$. We consider the sigma-algebra $\mathbb{F}_t$ spanned by $B_s, s \leq t$. The Brownian motion probability measure dP is characterized as the solution of the following martingale problem: if f is any bounded smooth function on $\mathbb{R}$,

$$t \to f(B_t) - \int_0^t \Delta f(B_s)ds \tag{9}$$

is a martingale associated to the filtration $\mathbb{F}_t$. This means that

$$E[(f(B_t) - \int_0^t \Delta f(B_s)ds)G] = E[(f(B_{t'}) - \int_0^{t'} \Delta f(B_{s'})ds')G] \tag{10}$$

where G is a bounded functional $\mathbb{F}_{t'}$ measurable ($t' < t$).

The Brownian motion is only continuous. However we can define stochastic integrals (as it was done by Itô). Let $s \to h_s$ be a bounded continuous process. We suppose that h_s is $\mathbb{F}_s$ measurable. Then the Itô integral is defined as follows:

$$\int_0^1 h_s\delta B_s = \lim_{k\to\infty} \sum_{l\leq k} h(\frac{l}{k})(B_{\frac{(l+1)}{k}} - B_{\frac{l}{k}}) \tag{11}$$

Moreover we have the Itô isometry

$$E[(\int_0^1 h_s\delta B_s)^2] = E[\int_0^1 h_s^2 ds] \tag{12}$$

Associated to the Brownian motion is classically associated the Bosonic Fock space.

Let H_2 be the Hilbert space of L^2 functions $h(.)$ from $\mathbb{R}^+$ into $\mathbb{R}$. We consider the symmetric tensor product $H_2^{\hat{\otimes}^n}$ of H_2. It can be realized as the set of symmetric maps h^n from $(\mathbb{R}^+)^n$ into $\mathbb{R}$ such that

$$\int_{(\mathbb{R}^+)^n} |h^n(s_1,..,s_n)|^2 ds_1..ds_n = \|h^n\|_2^2 < \infty \tag{13}$$

The symmetric Fock space WN_0 coincides with the set of formal series $\sigma = \sum h^n$ such that $\sum n!\|h^n\|^2 < \infty$. To each h^n we associate the n^{th} Wiener chaos

$$\Psi(h^n) = \int_{(\mathbb{R}^+)^n} h^n(s_1,..,s_n)\delta B_{s_1}...\delta B_{s_n} \tag{14}$$

if B_s is the standard $\mathbb{R}$-valued Brownian motion. The definition of the Wiener chaos $\Psi(h^n)$ is a small improvement of the stochastic integral $\int_0^1 h(s)\delta B_s$. By using the symmetry of h^n, we have:

$$\Psi(h^n) = n! \int_{0<s_1<s_2<...<s_n} h^n(s_1,..,s_n)\delta B_{s_1}...\delta B_{s_n} \tag{15}$$

Moreover $E_P[|\Psi(h^n)|^2] = n!\|h^n\|_2^2$ and $\Psi(h^n)$ and $\Psi(h^m)$ are orthogonal in $L^2(dP)$. The L^2 of the Brownian motion can be realized as the symmetric Fock space through the isometry Ψ.

We introduce the Laplacian Δ^+ on $(\mathbb{R})^+$ and we consider the Sobolev space $H_{2,k}$ associated to $(\Delta^+ + I)^k$. On the set of formal series $\sigma = \sum h^n$, we choose a slightly different Hilbert structure:

$$\|\sigma\|_{k,C}^2 = \sum_{n=0}^{\infty} n!C^n\|h^n\|_{2,k}^2 < \infty \tag{16}$$

We get another symmetric Fock space denoted $WN_{k,C}$. We remark that if $k' \geq k, C' \geq C$

$$\|\sigma\|_{k',C'} \geq \|\sigma\|_{k,C} \tag{17}$$

The Hida test function space $W.N._{\infty-}$ is the intersection of $W.N_{k,C}$ $k \geq 1$, $C \geq 1$ endowed with the projective topology. A sequence σ_n of the Hida Fock space converges to σ for the topology of the Hida Fock space if σ_n converges to σ in all $W.N_{k,C}$. The map Wiener chaos Ψ realized a map from $W.N_{\infty-}$ into the set of continuous Brownian functional dense in $L^2(dP)$. We refer to the books [21] and [22] for an extensive study between the Fock space and the L^2 of the Wiener measure.

In infinite dimensional analysis, there are basically 3 objects:

(i) An algebraic model.

(ii) A mapping space and a map Ψ from the algebraic model into the space of functionals on this mapping space.

(iii) A path integral μ which is an element of the topological dual of the algebraic model.

In the standard case of the Brownian motion, μ is the vacuum expectation:

$$\mu[\Psi(\sigma)] = h^0 \tag{18}$$

A distribution on the Hida Fock space is a linear map μ from $W.N_{\infty-}$ into $\mathbb{R}$ which satisfies the following requirement: there exists k, C, K such that for all $\sigma \in W.N_{\infty-}$

$$|\mu(\sigma)| \leq C\|\sigma\|_{k,C} \tag{19}$$

Getzler in his seminal paper [23] is the first author who considered another map than the map Wiener chaos. Getzler is motivated by the heuristic considerations of Atiyah-Bismut-Witten relating the structure of the free loop space of a manifold and the Index theorem on a compact spin manifold. Getzler used as algebraic space a Connes space and as map Ψ the map Chen iterated integrals.

Getzler's idea was developed by Léandre ([9]) to study various path integrals in the Hida-Streit approach with a geometrical meaning. Especially Léandre ([5–6]) succeeded to define the Haar measure dD as a distribution on a current group. Let us recall quickly the definition on it. We consider a compact Riemannian manifold M ($S \in M$) and a compact Lie group G ($g \in G$). We consider the current

group $C(M,G)$ of continuous maps $S \to g(S)$ from M into G. We consider the cylindrical functional $h(g(S_1),..,g(S_r))$ on the current group. We have

$$\int_{C(M,G)} h(g(S_1),..,g(S_r))dD = \int_{G^r} h(g_1,..,g_r)dg_1..dg_r \tag{20}$$

We would like to close this operation consistently. It is the object of [5-6].

(1) *Construction of the algebraic model.* We consider the positive self-adjoint Laplacian on $M \times G$ $\Delta^{M\times G}$. We consider the Sobolev space H_k of maps from h $M \times G$ into $\mathbb{R}$ such that

$$\int_{M\times G} ((\Delta^{M\times G}+2)^k h)^2 dSdg = \|h\|_k^2 \tag{21}$$

We consider the tensor product $H_k^{\otimes n}$ associated to it and we consider the natural Hilbert norm on it (dS and dg are normalized Riemannian measures on M and G respectively). $W.N_{k,C}$ is the set of formal series $\sigma = \sum h^n$ such that

$$\sum C^n \|h^n\|_k^2 = \|\sigma\|_{k,C}^2 < \infty \tag{22}$$

The Hida test functional space is the space $W.N_{\infty-} = \cap W.N_{k,C}$ endowed with the projective topology.

(2) *Construction of the map* Ψ. To h^n we associate

$$\Psi(h^n)(g(.)) = \int_{[0,1]^n} h^n(g(S_1),..,g(S_n),S_1,..,S_n)dS_1...dS_n \tag{23}$$

We put if $\sigma = \sum h^n$

$$\Psi(\sigma) = \sum_{n=0}^{\infty} \Psi(h^n) \tag{24}$$

The map Ψ realizes a continuous map from $W.N_{\infty-}$ into the set of continuous functional on $C(G,M)$.

(3) *Construction of the path integral.* We put if h^n belongs to all the Sobolev Hilbert spaces H_k

$$\int_{G,M)} \Psi(h^n)dD = \int_{M^n\times G^n} h^n(g_1,..,g_n,S_1,..,S_n)dg_1..dg_n dS_1..dS_n \tag{25}$$

This map can be extended into a linear continuous application from $W.N_{\infty-}$ (We say it is a Hida distribution) into $\mathbb{R}$. This realizes our definition ([5–7]) of the Haar distribution on the current group $C(M,G)$.

Let $I \in [0,1]^n$. We consider the normalized Lebesgue measure $d\nu^n$ on $[0,1]^n$. Let L_i be the i^{th} partial Laplacian on G^n. We consider the total operator

$$L_t^n = \prod_{i=1}^n (L_i+2) \prod_{i=1}^n (-\frac{\partial^2}{\partial s_i^2}+2) \tag{26}$$

which operates on function h^n on $G^n \times [0,1]^n$ and we consider its power $(L_t^n)^k$. Let $h^n(g^n,I)$ be a function on $G^n \times [0,1]^n$. We put

$$\|h^n\|_{C,k}^2 = C^n \int_{G^n\times[0,1]^n} |(L_t^n)^k h^n(g^n,I)|^2 dg^n d\nu^n(I) \tag{27}$$

(dg^n is the normalized Haar measure on G^n and $d\nu^n$ the normalized Lebesgue measure on $[0,1]^n$).

We put

$$\sigma = \sum h^n \tag{28}$$

and we consider the Hilbert norm

$$\|\sigma\|^2_{k,C} = \sum \|h^n\|^2_{k,C} \tag{29}$$

Definition 1: *The Hida Fock space $W.N_{\infty-}$ is the space constituted of the σ defined above such that for all $k \in \mathbb{N}$, $C > 0$ $\|\sigma\|^2_{k,C} < \infty$*

If σ belongs to $W.N_{\infty-}$, we associate

$$\Psi(\sigma)(g(.)) = \sum_{n=0}^{\infty} \int_{[0,1]^n} h^n(g(s_1), .., g(s_n), I) d\nu^n(I) \tag{30}$$

where $s \to g(s)$ belongs to $C([0,1], G)$.

Theorem 2: *If $\sigma \in W.N_{\infty-}$, $\Psi(\sigma)$ is a continuous bounded function on $C([0,1], G)$.*

We put

$$\int_{C([0,1],G)} \Psi(h^n) dD = \int_{[0,1]^n \times G^n} h^n(g_1, .., g_n, s_1, .., s_n) dg_1..dg_n ds_1..ds_n \tag{31}$$

Let us recall three of the main theorems of [7]:

Theorem 3: *dD can be extended as a distribution on the Hida Fock space. This means that there exists k, C, K such that for all $\sigma \in W.N_{\infty-}$*

$$|\int_{C([0,1],G)} \Psi(\sigma) dD| \leq K \|\sigma\|_{k,C} \tag{32}$$

Theorem 4: If $\Psi(\sigma) \geq 0$, $\int_{C([0,1];G)} \Psi(\sigma) dD \geq 0$.

Theorem 5: If $\Psi(\sigma) = 0$, $\int_{C([0,1];G)} \Psi(\sigma) dD = 0$.

3. A Non-Markovian Semi-group on a Path Group

In the sequel, we will suppose that $4k \geq d$. In such a case ([20]), we have

$$|p^k_t(g, g')| \leq \frac{C}{t^{d/4k}} G_{2k,a}(\frac{d(g, g')}{t^{1/4k}}) \tag{33}$$

where $G_{m,a}(u) = \exp[-au^{2m/2m-1}]$. $p^k_t(g, g')$ is the heat-kernel associated to the heat semi-group P^k_t and d is the biinvariant Riemannian distance on G.

$$P^k_t f(g) = \int_G p^k_t(g, g') f(g') dg' \tag{34}$$

Moreover, since Δ^k is biinvariant

$$p^k_t(gg^1, g'g^1) = p^k_t(g^1 g, g^1 g') = p^k_t(g, g') \tag{35}$$

Since it is an heat kernel associated to a semi-group, it satisfies the Kolmogorov equation:

$$p^k_{t+s}(g, g') = \int_G p^k_t(g, g^1) p^k_s(g^1, g') dg^1 \tag{36}$$

This shows that if $t \in [0,1]$ that

$$\|P_t^k f\|_\infty \leq C\|f\|_\infty \tag{37}$$

and that

$$|P_t^k|[|d(e,.)|^p](e) \leq Ct^{\alpha(k,p)} \tag{38}$$

Remark: We could get in the sequel more general convolution semi-groups [20] with generators of degree $2k$ whose associated heat-kernels satisfied still (33).

Let us divide the interval time $[0,1]$ into in time intervals $[t_l, t_{l+1}]$ of length $1/m$. Let F be a cylindrical functional $h(g_{t_1}, g_{t_2}, g_{t_m})$. Let us introduce

$$P_t^{k,m} h(g_{t_1}, .., g_{t_m}) = \int_{G^m} h(g_{t_1} g_1, .., g_{t_m} g_m) \prod_{i=1}^{m} p_{t/m}^k(g_{i-1}, g_i) dg_i \tag{39}$$

($g_0 = e$). This defines a semi-group on G^m. Let us show this statement. We remark

$$P_s^{k,m} P_t^{k,m} F^m(g_{t_1}, .., t_m) = \int_{G^m \times G^m} h(g_{t_1}\overline{g}_1 g_1, .., g_{t_m}\overline{g}_m g_m) \prod_{i=0}^{m-1} p_{t/m}^k(g_i, g_{i+1}) \prod_{i=0}^{m-1} p_{s/m}^k(\overline{g}_i, \overline{g}_{i+1}) dg_i d\overline{g}_i \tag{40}$$

We do the change of variable $\tilde{g}_i = \overline{g}_i g_i$; $g_i = g_i$. We recognize in the last expression

$$\int_{G^m \times G^m} h(g_{t_1}\tilde{g}_1, .., g_{t_m}\tilde{g}_m) \prod_{i=0}^{m-1} p_{t/m}^k(g_i, g_{i+1}) \prod_{i=0}^{m-1} p_{s/m}^k(\tilde{g}_i g_i^{-1}, \tilde{g}_{i+1} g_{i+1}^{-1}) dg_i d\tilde{g}_i \tag{41}$$

But

$$\int_{G^m} \prod_{i=0}^{m-1} p_{t/m}^k(g_i, g_{i+1}) \prod_{i=0}^{m-1} p_{s/m}^k(\tilde{g}_i g_i^{-1}, \tilde{g}_{i+1} g_{i+1}^{-1}) dg_i = \int_{G^m} \prod_{i=0}^{m-1} p_{t/m}^k(g_i, g_{i+1}) p_{s/m}^k(\tilde{g}_i, \tilde{g}_{i+1} g_{i+1}^{-1} g_i) dg_i = \int_{G^m} \prod_{i=0}^{m-1} p_{s/m}^k(\tilde{g}_i, \overline{g}_i) p_{t/m}^k(\overline{g}_i, \tilde{g}_{i+1}) d\overline{g}_i = \prod_{i=0}^{m-1} p_{\frac{s+t}{m}}^k(\tilde{g}_i, \tilde{g}_{i+1}) \tag{42}$$

We have used the semi-group property (36) of P_t^k and the fact that P_t^k is biinvariant (35).

We would like to extend by continuity this formula for functionals which depend on an infinite number of variables $\Psi(\sigma)$ of the previous type. We put for h^n:

$$\mu[\Psi(h^n)] = \int_{G^n \times [0,1]^n} h^n(g_1, .., g_n, s_1, .., s_n) \prod_{i=0}^{n-1} p_{s_{i+1}-s_i}^k(g_i, g_{i+1}) dg_i d\nu^n(s_1, .., s_n) \tag{43}$$

($s_0 = 0$). We order $s_1 < s_2 < .. < s_n$ without to loose generality.

We extend μ by linearity.

Theorem 6: *μ is a Hida distribution . Moreover if $\Psi(\sigma) = 0$, $\mu[\Psi(\sigma] = 0$.*

Proof: By the property of the cylindrical semi-group listed in the beginning of this part, we have

$$\int_{G^n} |h^n(g_1,..,g_n,s_1,..,s_n)| \prod_{i=0}^{n-1} |p^k_{s_{i+1}-s_i}(g_i,g_{i+1})| dg_i \leq C^n \|h^n\|_\infty \tag{44}$$

where $\|\|_\infty$ is the uniform norm of h^n. This uniform norm can be estimated by Sobolev imbedding theorem by $\|h^n\|_{k,C}$ for some big k and C independent of n. It follows clearly from that μ is an Hida distribution.

Let us give some details in order to estimate $\|h^n\|_\infty$. We introduce the ordered set of eigenvalues λ_i of Δ. Let $(\alpha) = (i_1,..,i_n)$. Let ϕ_i be the normalized eigenvectors associated to λ_i. We consider $\mathbb{C}$-valued functions to do that. We introduce $\phi_{(\alpha)}(g_1,..,g_n) = \prod \phi_{i_j}(g_j)$. We get

$$h^n = \sum_{(\alpha)} \lambda_{(\alpha)} \phi_{(\alpha)} \tag{45}$$

Therefore

$$\|h^n\|_\infty \leq \sum_{(\alpha)} \|\lambda_{(\alpha)}\|_\infty \|\phi_{(\alpha)}\|_\infty \tag{46}$$

By Garding and Sobolev inequality, the right-hand side of the previous inequality is smaller than

$$C^n \sum_{(\alpha)} K^{-l}_{(\alpha)} \|\lambda_{(\alpha)}\|_{k,C} \|\phi_{(\alpha)}\|_{k,C} \tag{47}$$

for some big k, some big C and some big l.

$$K_{(\alpha)} = \prod_{i \in (\alpha)} (2 + \lambda_i) \tag{48}$$

Let us recall that $\lambda_i \geq 0$ and that $\lambda_i \geq Ci^m$for some m ([24]). We apply Cauchy-Schwartz inequality in (47). We deduce that

$$\|h^n\|_\infty \leq C^n \{\sum_{(\alpha)} K^{-l}_{(\alpha)}\}^{1/2} \{\sum_{(\alpha)} \|\lambda_{(\alpha)}|^2_{k,C} \|\phi_{(\alpha)}\|^2_{k,C}\}^{1/2} \tag{49}$$

But

$$\sum_{(\alpha)} \|\lambda_{(\alpha)}\|^2_{k,C} \|\phi_{(\alpha)}\|^2_{k,C} = \|h^n\|^2_{k,C} \tag{50}$$

Moreover, by [8], $\lambda_i \geq Ci^m$ for some i. Therefore if l is big enough, $\sum_{(\alpha)} K^{-l}_{(\alpha)}$ is finite bounded independently of n.

Let us consider the polygonal approximation of mesh $1/l$ $g^l_.$ of $g_.$. If $\Psi(\sigma) = 0$, we get $\Psi(\sigma)(g^l_.) = 0$. But $\Psi(\sigma)(g^l_.)$ is a cylindrical functional which depends only of $g_{t_1},..,g_{t_l} = x_1$. We use the properties listed in the beginning of this part. We get

$$P^{k,l}_1[\Psi(\sigma)(g^l_.)](e) = 0 \tag{51}$$

by the property listed of the beginning of the cylindrical semi-group $P_t^{k,l}$. It remains to show that when $l \to \infty$ that $P_1^{k,l}[\Psi(\sigma)(g_.^l)]$ is very close from $\mu[\Psi(\sigma)(g_.)]$. This follows from the next consideration. Let h^n be an elementary tensor product. We get clearly

$$|\mu[\Psi(h^n)(g_.^l)] - \mu[\Psi(h^n)(g_.)]| \leq C^n \|h^n\|_{1,\infty} \sum_{i=0}^{n} \int_{[0,1]^n \times G} |d(e,g_i)|(|p^k_{s_i-[s_i]_-}(e,g_i)| + |p^k_{[s_i]_+-s_i}(e,g_i)|)dg_i d\nu^n(s_1,..,s_n) \quad (52)$$

where $[s]-$ denotes the supremum of the time of the subdivision smaller to s and $[s]_+$ denotes the infimum of the time of the subdivision larger to s. $\|h^n\|_{1,\infty}$ is the uniform C^1 norm of h^n. This norm can be estimated by the Sobolev imbedding theorem by $\|h^n\|_{k^1,C_1}$ for k^1 and C^1 independent of n as in (50).

It remains to use the inequality (35) to conclude.$\diamondsuit$

Definition 7: *μ is called the Wiener distribution issued from the unit path associated to Δ^k.*

Let h^n be a smooth function from $G^n \times [0,1]^n$ into $\mathbb{R}$. We suppose that $0 < s_1 < s_2 .. < s_n$ in order to simplify the exposition. We put

$$P_t^{k,n} F^n(g_1,..,g_n,s_1,..,s_n) = \int_{G^m} h^n(g_1 y_1,..,g_m y_n,s_1,..,s_n) \prod_{i=0}^{n-1} p^k_{t(s_{i+1}-s_i)}(y_i,y_{i+1})dy_i \quad (53)$$

$P_t^{k,n}$ is the cylindrical semi-group on cylindrical functional associated to $g_{s_1},..,g_{s_n}$.

lemma 8: *There exist a C' bounded when t is bounded and which depend not of n, a k' which depend only of k and not on n such that*

$$\|P_t^{k,n} h^n\|_{C,k} \leq \|h^n\|_{C',k'} \quad (54)$$

Proof: If we take derivative in g_i, the result comes by taking derivative under the sign integral in (43). The result arises then from (37). Let us take first of all derivative in s_i. Either we take derivative of h^n and the result goes by the same way. Or we take derivative in s_{i+1} or s_i of the heat kernel p_s^k. We represent in the way (43) the integral, we remark that the heat kernel satisfies the heat-equation and we integrate by parts in order to conclude.$\diamondsuit$

Let us suppose that the time subdivision is fixed. Clearly

$$P_{t'}^{k,n} \circ P_t^{k,n} = P_{t+t'}^{k,n} \quad (55)$$

Let h^n be a function from $G^n \times [O,1]^n$ into $\mathbb{R}$. We put

$$Q_t^k[\Psi(h^n)](g_.) = \int_{[0,1]^n} P_t^{k,n} h^n(g_{s_1},..,g_{s_n},s_1,..,s_n)d\nu^n(s_1,..,s_n) \quad (56)$$

Theorem 9: *Q_t^k can be extended by linearity as a continuous linear operator on the Hida Fock space. If $\Psi(\sigma)(g_.) = 0$, $Q_t^k[\Psi(\sigma)](g_.) = 0$ and we get the semi-group property*

$$Q_t^k[Q_{t'}^k[\Psi(\sigma)]](g_.) = Q_{t+t'}^k[\Psi(\sigma)](g_.) \quad (57)$$

if σ belong to $W.N_{\infty-}$.

Proof: The fact that Q_t^k can be extended by linearity follows from the previous lemma. $Q_t^k[\Psi(\sigma)](x_.) = 0$ if $\Psi(\sigma) = 0$ holds exactly as in the proof of Theorem 6. For a simple element h^n of the Hida Fock space, we have clearly:

$$Q_t^k[Q_{t'}^k[\Psi(h^n)]](g_.) = Q_{t+t'}^k[\Psi(h^n)](g_.) \tag{58}$$

This result can be extended by continuity.♢

4. Long Time Behaviour

The main theorem of this paper is the following:

Theorem 10: *If σ belong to $W.N_{\infty-}$, then when $t \to \infty$*

$$Q_t^k[\Psi(\sigma)](e_.) \to \int_{C([0,1];G)} \Psi(\sigma) dD \tag{59}$$

where $e_.$ is the unit path.

Proof: Let us decompose $L^2(G)$ in an orthonormal basis of eigenvectors ϕ_i of Δ associated to the eigenvalues λ_i. Classically [24], $\sup_g |\phi_i(g)| \leq Ci^{m_0}$ and $\lambda_i \geq Ci^{m_1}$ for some positive m_0 and m_1. Classically the heat kernel is given by

$$p_t^k(g, g') = 1 + \sum_{i>0} \exp[-\lambda_i^k] \phi_i(g) \phi_i(g') \tag{60}$$

>From the previous bound, we deduce if $t \geq 1$

$$\sup_{g,g'} |p_t^k(g, g')| \leq C < \infty \tag{61}$$

$P_t^{k,n}$ is associated if $s_1 < s_2 < .. < s_n < 1$ to a invariant elliptic operator on G^n. It has therefore the unique invariant measure $\otimes dg_i$. This shows that if h^n is an element of the Hida Fock space that

$$P_t^{k,n} h^n(e, .., e, s_1, s_n) \to \int_{G^n} h^n(g_1, .., g_n, s_1, .., s_n) \prod_{i=1}^n dg_i \tag{62}$$

provided all s_i are different.

By the previous estimates, if $t \geq 1$

$$\sup |P_t^{k,n} h^n| \leq C^n \|h^n\|_\infty \tag{63}$$

where $\|h^n\|_\infty$ is the supremum norm of h^n which can be estimated by Sobolev imbedding theorem by $\|h^n\|_{C',k'}$ for some C', some k' independent of n. Therefore

$$Q_t^k[\Psi(\sigma)](e_.) = \sum_{n=0}^\infty \int_{[0,1]^n} P_t^k[h^n](e, .., e, s_1, ..s_n) ds_1 .. ds_n \tag{64}$$

By the dominated Lebesgue convergence, this tends when $t \to \infty$ to

$$\sum_{n=0}^\infty \int_{G^n \times [0,1]^n} h^n(g_1, .., g_n, s_1, .., s_n) \prod_{i=1}^n dg_i \prod_{i=1}^n ds_i = \int_{C([0,1];G)} \Psi(\sigma) dD \tag{65}$$

♢

5. Conclusions

We define a non-Markovian semi-group on a path group which acts on a Hida type test algebra on the path group and we study its long time behaviour related to the Haar distribution on the path group.

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