Article

Positive Cosmological Constant and Quantum Theory

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Abstract: We argue that quantum theory should proceed not from a spacetime background but from a Lie algebra, which is treated as a symmetry algebra. Then the fact that the cosmological constant is positive means not that the spacetime background is curved but that the de Sitter (dS) algebra as the symmetry algebra is more relevant than the Poincare or anti de Sitter ones. The physical interpretation of irreducible representations (IRs) of the dS algebra is considerably different from that for the other two algebras. One IR of the dS algebra splits into independent IRs for a particle and its antiparticle only when Poincare approximation works with a high accuracy. Only in this case additive quantum numbers such as electric, baryon and lepton charges are conserved, while at early stages of the Universe they could not be conserved. Another property of IRs of the dS algebra is that only fermions can be elementary and there can be no neutral elementary particles. The cosmological repulsion is a simple kinematical consequence of dS symmetry on quantum level when quasiclassical approximation is valid. Therefore the cosmological constant problem does not exist and there is no need to involve dark energy or other fields for explaining this phenomenon (in agreement with a similar conclusion by Bianchi and Rovelli).

Keywords: quantum theory; de Sitter invariance; cosmological constant problem

In memory of Leonid Avksent'evich Kondratyuk

Our collaboration with Leonid Avksent'evich has considerably enhanced my understanding of quantum theory. In particular, he explained that the theory should not necessarily be based on a local Lagrangian, and symmetry on quantum level means that proper commutation relations are satisfied. I believe that the present paper is in the spirit of these ideas.

1. Introduction

The discovery of the cosmological repulsion (see e.g., [1,2]) has ignited a vast discussion on how this phenomenon should be interpreted. The majority of authors treat this phenomenon as an indication that the cosmological constant Λ in General Relativity (GR) is positive and therefore the spacetime background has a positive curvature. According to References [3,4], the observational data on the value of Λ define it with the accuracy better than 5%. Therefore the possibilities that $\Lambda = 0$ or $\Lambda < 0$ are practically excluded. We argue in Sections 2–4 that the notion of spacetime background is not physical and, from the point of view of quantum theory, one should proceed not from this notion but from a symmetry algebra. Then the fact that in classical approximation $\Lambda > 0$ is an indication that on quantum level the de Sitter (dS) algebra is a more relevant symmetry algebra than the Poincare or anti de Sitter (AdS) algebras. In particular, elementary objects in quantum theory should be described by irreducible representations (IRs) of the dS algebra by Hermitian operators. In Sections 5–7 we discuss mathematical properties of such IRs. Although there exists a vast literature on IRs of the dS group and algebra, their physical interpretation has not been widely discussed. One of the main problems is that IRs of the dS algebra are implemented on two Lorentz hyperboloids, not one as in the case of the Poincare or AdS algebras. Therefore a problem arises how IRs can be interpreted in terms of elementary particles and antiparticles. In Reference [5] we have proposed an interpretation that one IR of the dS algebra describes a particle and its antiparticle simultaneously. In Section 8 this analysis is considerably extended. In particular, we show that additive quantum numbers such as electric, baryon and lepton charges can be only approximately conserved quantities. It is also shown that only fermions can be elementary and there can be no neutral elementary particles. In Section 9 it is shown that cosmological repulsion is a simple kinematical consequence of the dS symmetry on quantum level when quasiclassical approximation is valid. For deriving this result there is no need to involve spacetime background, Riemannian geometry, de Sitter quantum field theory (QFT), Lagrangians or other sophisticated methods. In other words, the phenomenon of the cosmological repulsion can be naturally explained on the basis of existing knowledge without involving dark energy or other new fields (in agreement with a conclusion by Bianchi and Rovelli in Reference [6]). We tried to make the presentation self-contained and make it possible for readers to reproduce calculations without looking at other papers.

2. Remarks on the Cosmological Constant Problem

We would like to begin our presentation with a discussion of the following well-known problem: How many independent dimensionful constants are needed for a complete description of nature? A paper [7] represents a trialogue between three well known scientists: M.J. Duff, L.B. Okun and G. Veneziano. The results of their discussions are summarized as follows: *LBO develops the traditional approach with three constants, GV argues in favor of at most two (within superstring theory), while MJD advocates zero.* According to Reference [8], a possible definition of a fundamental constant might be such that it cannot be calculated in the existing theory. We would like to give arguments in favor of the opinion of the first author in Reference [7]. One of our goals is to argue that the cosmological constant cannot be a fundamental physical quantity.

Consider a measurement of a component of angular momentum. The result depends on the system of units. As shown in quantum theory, in units $\hbar/2 = 1$ the result is given by an integer $0, \pm 1, \pm 2, \ldots$. But we can reverse the order of units and say that in units where the momentum is an integer l, its value in kg·m²/s is $(1.05457162 \cdot 10^{-34} \cdot l/2)$ kg·m²/s. Which of those two values has more physical significance? In units where the angular momentum components are integers, the commutation relations between the components are

$$[M_x, M_y] = 2iM_z \quad [M_z, M_x] = 2iM_y \quad [M_y, M_z] = 2iM_x$$

and they do not depend on any parameters. Then the meaning of l is clear: It shows how big the angular momentum is in comparison with the minimum nonzero value 1. At the same time, the measurement of the angular momentum in units kg·m²/s reflects only a historic fact that at macroscopic conditions on the Earth in the period between the 18th and 21st centuries people measured the angular momentum in such units.

The fact that quantum theory can be written without the quantity $\hbar$ at all is usually treated as a choice of units where $\hbar = 1/2$ (or $\hbar = 1$). We believe that a better interpretation of this fact is simply that quantum theory tells us that physical results for measurements of the components of angular momentum should be given in integers. Then the question why $\hbar$ is as it is, is not a matter of fundamental physics since the answer is: Because we want to measure components of angular momentum in kg·m²/s.

Our next example is the measurement of velocity v. The fact that any relativistic theory can be written without involving c is usually described as a choice of units where $c = 1$. Then the quantity v can take only values in the range [0,1]. However, we can again reverse the order of units and say that relativistic theory tells us that results for measurements of velocity should be given by values in [0,1]. Then the question why c is as it is, is again not a matter of physics since the answer is: Because we want to measure velocity in m/s.

One might pose a question whether or not the values of $\hbar$ and c may change with time. As far as $\hbar$ is concerned, this is a question that if the angular momentum equals one then its value in kg·m²/s will always be $1.05457162 \times 10^{-34}/2$ or not. It is obvious that this is not a problem of fundamental physics but a problem how the units (kg,m,s) are defined. In other words, this is a problem of metrology and cosmology. At the same time, the value of c will always be the same since the modern *definition* of meter is the length which light passes during $(1/(3 \cdot 10^8))$ s.

It is often believed that the most fundamental constants of nature are $\hbar$, c and the gravitational constant G. The units where $\hbar = c = G = 1$ are called Planck units. Another well known notion is the $c\hbar G$ cube of physical theories. The meaning is that any relativistic theory should contain c, any quantum theory should contain $\hbar$ and any gravitational theory should contain G. However, the above remarks indicates that the meaning should be the opposite. In particular, relativistic theory *should not* contain c and quantum theory *should not contain* $\hbar$. The problem of treating G will be discussed below.

A standard phrase that relativistic theory becomes non-relativistic one when $c \to \infty$ should be understood such that if relativistic theory is rewritten in conventional (but not physical!) units then c will appear and one can take the limit $c \to \infty$. A more physical description of the transition is that all the velocities in question are much less than unity. We will see in Section 9 that those definitions are not equivalent. Analogously, a more physical description of the transition from quantum to classical theory should be that all angular momenta in question are very large rather than $\hbar \to 0$.

Consider now what happens if we assume that de Sitter symmetry is fundamental. For definiteness, we will discuss the dS SO(1,4) symmetry and the same considerations can be applied to the AdS symmetry SO(2,3). The dS space is a four-dimensional manifold in the five-dimensional space defined by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2 = R^2 \tag{1}$$

In the formal limit $R \to \infty$ the action of the dS group in a vicinity of the point $(0, 0, 0, 0, x_4 = R)$ becomes the action of the Poincare group on Minkowski space. In the literature, instead of R, the cosmological constant $\Lambda = 3/R^2$ is often used. Then $\Lambda > 0$ in the dS case and $\Lambda < 0$ in the AdS one. The dS space can be parameterized without using the quantity R at all if instead of x_a ($a = 0, 1, 2, 3, 4$) we define dimensionless variables $\xi_a = x_a/R$. It is also clear that the elements of the SO(1,4) group do not depend on R since they are products of conventional and hyperbolic rotations. So the dimensionful value of R appears only if one wishes to measure coordinates on the dS space in terms of coordinates of the flat five-dimensional space where the dS space is embedded in. This requirement does not have a fundamental physical meaning. Therefore the value of R defines only a scale factor for measuring coordinates in the dS space. By analogy with c and $\hbar$, the question why R is as it is, is not a matter of fundamental physics since the answer is: Because we want to measure distances in meters. In particular, there is no guarantee that the cosmological constant is really a constant, *i.e.*, does not change with time. It is also obvious that if the dS symmetry is assumed from the beginning then the value of Λ has no relation to the value of G.

If one assumes that spacetime background is fundamental then in the spirit of GR it is natural to think that the empty spacetime is flat, *i.e.*, that $\Lambda = 0$ and this was the subject of the well-known dispute between Einstein and de Sitter. However, as mentioned in Section 1, it is now accepted that $\Lambda \neq 0$ and, although it is very small, it is positive rather than negative. If we accept parameterization of the dS space as in Equation (1) then the metric tensor on the dS space is

$$g_{\mu\nu} = \eta_{\mu\nu} - x_\mu x_\nu / (R^2 + x_\rho x^\rho) \tag{2}$$

where $\mu, \nu, \rho = 0, 1, 2, 3$, $\eta_{\mu\nu}$ is the diagonal tensor with the components $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1$ and a summation over repeated indices is assumed. It is easy to calculate the Christoffel symbols in the approximation where all the components of the vector x are much less than R: $\Gamma_{\mu,\nu\rho} = -x_\mu \eta_{\nu\rho}/R^2$. Then a direct calculation shows that in the non-relativistic approximation the equation of motion for a single particle is

$$\mathbf{a} = \mathbf{r}c^2/R^2 \tag{3}$$

where $\mathbf{a}$ and $\mathbf{r}$ are the acceleration and the radius vector of the particle, respectively.

The fact that even a single particle in the Universe has a nonzero acceleration might be treated as contradicting the law of inertia but this law has been postulated only for Galilean or Poincare symmetries and we have $\mathbf{a} = 0$ in the limit $R \to \infty$. A more serious problem is that, according to standard experience, any particle moving with acceleration necessarily emits gravitational waves, any charged particle emits electromagnetic waves *etc.* Does this experience work in the dS world? This problem is intensively discussed in the literature (see e.g., References [9–11] and references therein). Suppose we accept that, according to GR, the loss of energy in gravitational emission is proportional to the

gravitational constant. Then one might say that in the given case it is not legitimate to apply GR since the constant G characterizes interaction between different particles and cannot be used if only one particle exists in the world. However, the majority of authors proceed from the assumption that the empty dS space cannot be literally empty. If the Einstein equations are written in the form $G_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ where $T_{\mu\nu}$ is the stress-energy tensor of matter then the case of empty space is often treated as a vacuum state of a field with the stress-energy tensor $T_{\mu\nu}^{vac}$ such that $(8\pi G/c^4)T_{\mu\nu}^{vac} = -\Lambda g_{\mu\nu}$. This field is often called dark energy. With such an approach one implicitly returns to Einstein's point of view that a curved space cannot be empty. Then the fact that $\Lambda \neq 0$ is treated as a dark energy on the flat background. In other words, this is an assumption that Poincare symmetry is fundamental while dS one is emergent.

However, in this case a new problem arises. The corresponding quantum theory is not renormalizable and with reasonable cutoffs, the quantity Λ in units $\hbar = c = 1$ appears to be of order $1/l_P^2 = 1/G$ where l_P is the Planck length. It is obvious that since in the above theory the only dimensionful quantities in units $\hbar = c = 1$ are G and Λ, and the theory does not have other parameters, the result that $G\Lambda$ is of order unity seems to be natural. However, this value of Λ is at least by 120 orders of magnitude greater than the experimental one. Numerous efforts to solve this cosmological constant problem have not been successful so far although many explanations have been proposed.

Many physicists argue that in the spirit of GR, the theory should not depend on the choice of the spacetime background (so called a principle of background independence) and there should not be a situation when the flat background is preferable (see e.g., a discussion in Reference [12] and references therein). Moreover, although GR has been confirmed in several experiments in Solar system, it is not clear whether it can be extrapolated to cosmological distances. In other words, our intuition based on GR with $\Lambda = 0$ cannot be always correct if $\Lambda \neq 0$. In Reference [6] this point of view is discussed in details. The authors argue that a general case of Einstein's equation is when Λ is present and there is no reason to believe that a special case $\Lambda = 0$ is preferable.

In summary, numerous attempts to resolve the cosmological constant problem have not converged to any universally accepted theory. All those attempts are based on the notion of spacetime background and in the next section we discuss whether this notion is physical.

3. Should Physical Theories Involve Spacetime Background?

From the point of view of quantum theory, any physical quantity can be discussed only in conjunction with the operator defining this quantity. For example, in standard quantum mechanics the quantity t is a parameter, which has the meaning of time only in classical limit since there is no operator corresponding to this quantity. The problem of how time should be defined on quantum level is very difficult and is discussed in a vast literature (see e.g., References [13–15] and references therein). It has been also well known since the 1930s [16] that, when quantum mechanics is combined with relativity, there is no operator satisfying all the properties of the spatial position operator. In other words, the coordinates cannot be exactly measured even in situations when exact measurements are allowed by the non-relativistic uncertainty principle. In the introductory section of the well-known textbook [17] simple arguments are given that for a particle with mass m, the coordinates cannot be measured with the accuracy better than the Compton wave length $\hbar/mc$. This fact is mentioned in practically every

textbook on quantum field theory (see e.g., Reference [18]). Hence, the exact measurement is possible only either in the non-relativistic limit (when $c \to \infty$) or classical limit (when $\hbar \to 0$). There also exists a wide literature where the meaning of space is discussed (see e.g., References [12–15]).

We accept a principle that any definition of a physical quantity is a description how this quantity should be measured. In quantum theory this principle has been already implemented but we believe that it should be valid in classical theory as well. From this point of view, one can discuss if *coordinates of particles* can be measured with a sufficient accuracy, while the notion of spacetime background, regardless of whether it is flat or curved, does not have a physical meaning. Indeed, this notion implies that spacetime coordinates are meaningful even if they refer not to real particles but to points of a manifold which exists only in our imagination. However, such coordinates are not measurable. To avoid this problem one might try to treat spacetime background as a reference frame. Note that even in GR, which is a pure classical (*i.e.*, non-quantum) theory, the meaning of reference frame is not clear. In standard textbooks (see e.g., Reference [19]) the reference frame in GR is defined as a collection of weightless bodies, each of which is characterized by three numbers (coordinates) and is supplied by a clock. Such a notion (which resembles ether) is not physical even on classical level and for sure it is meaningless on quantum level. There is no doubt that GR is a great achievement of theoretical physics and has achieved great successes in describing experimental data. At the same time, it is based on the notions of spacetime background or reference frame, which do not have a clear physical meaning. Therefore it is unrealistic to expect that successful quantum theory of gravity will be based on quantization of GR. The results of GR should follow from quantum theory of gravity only in situations when spacetime coordinates of *real bodies* is a good approximation while in general the formulation of quantum theory might not involve spacetime background at all.

One might take objection that coordinates of spacetime background in GR can be treated only as parameters defining possible gauge transformations while final physical results do not depend on these coordinates. Analogously, although the quantity x in the Lagrangian density $L(x)$ is not measurable, it is only an auxiliary tool for deriving equations of motion in classical theory and constructing Hilbert spaces and operators in quantum theory. After this construction has been done, one can safely forget about background coordinates and Lagrangian. In other words, a problem is whether nonphysical quantities can be present at intermediate stages of physical theories. This problem has a long history discussed in a vast literature (see e.g., a discussion in References [12–15]). Probably Newton was the first who introduced the notion of spacetime background but, as noted in a paper in Wikipedia, "Leibniz thought instead that space was a collection of relations between objects, given by their distance and direction from one another". We believe that at the fundamental level unphysical notions should not be present even at intermediate stages. So Lagrangian can be at best treated as a hint for constructing a fundamental theory. As stated in Reference [17], local quantum fields and Lagrangians are rudimentary notion, which will disappear in the ultimate quantum theory. Those ideas have much in common with the Heisenberg S-matrix program and were rather popular till the beginning of the 1970's. Although no one took objections against those ideas, they are now almost forgotten in view of successes of gauge theories.

In summary, although the most famous successes of theoretical physics have been obtained in theories involving spacetime background, this notion does not have a physical meaning. Therefore a problem arises how to explain the fact that physics seems to be local with a good approximation. In Section 9 it

is shown that the result given by Equation (3) is simply a consequence of dS symmetry on quantum level when quasiclassical approximation works with a good accuracy. For deriving this result there is no need to involve dS space, metric, connection, dS QFT and other sophisticated methods. The first step in our approach is discussed in the next section.

4. Symmetry on Quantum Level

If we accept that quantum theory should not proceed from spacetime background, a problem arises how symmetry should be defined on quantum level. In the spirit of Dirac's paper [20], we postulate that on quantum level a symmetry means that a system is described by a set of operators, which satisfy certain commutation relations. We believe that for understanding this Dirac's idea the following example might be useful. If we define how the energy should be measured (e.g., the energy of bound states, kinetic energy *etc.*), we have a full knowledge about the Hamiltonian of our system. In particular, we know how the Hamiltonian should commute with other operators. In standard theory the Hamiltonian is also interpreted as an operator responsible for evolution in time, which is considered as a classical macroscopic parameter. In situations when this parameter is a good approximate parameter, macroscopic transformations from the symmetry group corresponding to the evolution in time have a meaning of evolution transformations. However, there is no guarantee that such an interpretation is always valid (e.g., at the very early stage of the Universe). In general, according to principles of quantum theory, self-adjoint operators in Hilbert spaces represent observables but there is no requirement that parameters defining a family of unitary transformations generated by a self-adjoint operator are eigenvalues of another self-adjoint operator. A well known example from standard quantum mechanics is that if P_x is the x component of the momentum operator then the family of unitary transformations generated by P_x is $exp(iP_x x/\hbar)$ where $x \in (-\infty, \infty)$ and such parameters can be identified with the spectrum of the position operator. At the same time, the family of unitary transformations generated by the Hamiltonian H is $exp(-iHt/\hbar)$ where $t \in (-\infty, \infty)$ and those parameters cannot be identified with a spectrum of a self-adjoint operator on the Hilbert space of our system. In the relativistic case the parameters x can be formally identified with the spectrum of the Newton-Wigner position operator [16] but it is well known that this operator does not have all the required properties for the position operator. So, although the operators $exp(iP_x x/\hbar)$ and $exp(-iHt/\hbar)$ are well defined, their physical interpretation as translations in space and time is not always valid (see also a discussion in Section 8).

The *definition* of the dS symmetry on quantum level is that the operators M^{ab} ($a, b = 0, 1, 2, 3, 4$, $M^{ab} = -M^{ba}$) describing the system under consideration satisfy the commutation relations *of the dS Lie algebra* so(1,4), *i.e.*,

$$[M^{ab}, M^{cd}] = -i(\eta^{ac}M^{bd} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}) \tag{4}$$

where η^{ab} is the diagonal metric tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$. These relations do not depend on any free parameters. One might say that this is a consequence of the choice of units where $\hbar = c = 1$. However, as noted above, any fundamental theory should not involve the quantities $\hbar$ and c.

With such a definition of symmetry on quantum level, the dS symmetry looks more natural than the Poincare symmetry. In the dS case all the ten representation operators of the symmetry algebra are

angular momenta while in the Poincare case only six of them are angular momenta and the remaining four operators represent standard energy and momentum. If we define the operators P^μ as $P^\mu = M^{4\mu}/R$ then in the formal limit when $R \to \infty$, $M^{4\mu} \to \infty$ but the quantities P^μ are finite, the relations (4) will become the commutation relations for representation operators of the Poincare algebra such that the dimensionful operators P^μ are the four-momentum operators.

A theory based on the above definition of the dS symmetry on quantum level cannot involve quantities which are dimensionful in units $\hbar = c = 1$. In particular, we inevitably come to conclusion that the dS space, the gravitational constant and the cosmological constant cannot be fundamental. The latter appears only as a parameter replacing the dimensionless operators $M^{4\mu}$ by the dimensionful operators P^μ which have the meaning of momentum operators only if R is rather large. Therefore the cosmological constant problem does not arise at all but instead we have a problem why nowadays Poincare symmetry is so good approximate symmetry. This is rather a problem of cosmology but not quantum physics.

5. IRs of the dS Algebra

If we accept dS symmetry on quantum level as described in the preceding section, a question arises how elementary particles in quantum theory should be defined. A discussion of numerous controversial approaches can be found, for example in the recent paper [21]. Although particles are observables and fields are not, in the spirit of QFT, fields are more fundamental than particles, and a possible definition is as follows [22]: *It is simply a particle whose field appears in the Lagrangian. It does not matter if it's stable, unstable, heavy, light—if its field appears in the Lagrangian then it's elementary, otherwise it's composite.* Another approach has been developed by Wigner in his investigations of unitary irreducible representations (UIRs) of the Poincare group [23]. In view of this approach, one might postulate that a particle is elementary if the set of its wave functions is the space of an IR of the symmetry group or Lie algebra in the given theory. Since we do not accept approaches based on spacetime background then by analogy with the Wigner approach we accept that, *by definition*, elementary particles in the dS invariant theory are described by IRs of the dS algebra by Hermitian operators. For different reasons, there exists a vast literature not on such IRs but on UIRs of the dS group. References to this literature can be found e.g., in our papers [5,24–26] where we used the results on UIRs of the dS group for constructing IRs of the dS algebra by Hermitian operators. In this section we will describe the construction proceeding from an excellent description of UIRs of the dS group in a book by Mensky [27]. The final result is given by explicit expressions for the operators M^{ab} in Equations (20) and (21). The readers who are not interested in technical details can skip the derivation.

The elements of the SO(1,4) group will be described in the block form

$$ g = \left\| \begin{array}{ccc} g_0^0 & \mathbf{a}^T & g_4^0 \\ \mathbf{b} & r & \mathbf{c} \\ g_0^4 & \mathbf{d}^T & g_4^4 \end{array} \right\| \tag{5} $$

where

$$ \mathbf{a} = \left\| \begin{array}{c} a^1 \\ a^2 \\ a^3 \end{array} \right\| \quad \mathbf{b}^T = \left\| \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \right\| \quad r \in SO(3) \tag{6} $$

and the subscript ^{T} means a transposed vector.

UIRs of the SO(1,4) group belonging to the principle series of UIRs are induced from UIRs of the subgroup H (sometimes called "little group") defined as follows [27]. Each element of H can be uniquely represented as a product of elements of the subgroups SO(3), A and **T**: $h = r\tau_A \mathbf{a_T}$ where

$$\tau_A = \left\| \begin{array}{ccc} cosh(\tau) & 0 & sinh(\tau) \\ 0 & 1 & 0 \\ sinh(\tau) & 0 & cosh(\tau) \end{array} \right\| \quad \mathbf{a_T} = \left\| \begin{array}{ccc} 1 + \mathbf{a}^2/2 & -\mathbf{a}^T & \mathbf{a}^2/2 \\ -\mathbf{a} & 1 & -\mathbf{a} \\ -\mathbf{a}^2/2 & \mathbf{a}^T & 1 - \mathbf{a}^2/2 \end{array} \right\| \tag{7}$$

The subgroup A is one-dimensional and the three-dimensional group **T** is the dS analog of the conventional translation group (see e.g., Reference [27,28]). We believe it should not cause misunderstandings when 1 is used in its usual meaning and when to denote the unit element of the SO(3) group. It should also be clear when r is a true element of the SO(3) group or belongs to the SO(3) subgroup of the SO(1,4) group. Note that standard UIRs of the Poincare group are induced from the little group, which is a semidirect product of SO(3) and four-dimensional translations and so the analogy between UIRs of the Poincare and dS groups is clear.

Let $r \rightarrow \Delta(r; \mathbf{s})$ be an UIR of the group SO(3) with the spin **s** and $\tau_A \rightarrow exp(im_{dS}\tau)$ be a one-dimensional UIR of the group A, where m_{dS} is a real parameter. Then UIRs of the group H used for inducing to the SO(1,4) group, have the form

$$\Delta(r\tau_A \mathbf{a_T}; m_{dS}, \mathbf{s}) = exp(im_{dS}\tau)\Delta(r; \mathbf{s}) \tag{8}$$

We will see below that m_{dS} has the meaning of the dS mass and therefore UIRs of the SO(1,4) group are defined by the mass and spin, by analogy with UIRs in Poincare invariant theory.

Let G=SO(1,4) and $X = G/H$ be the factor space (or coset space) of G over H. The notion of the factor space is well known (see e.g., References [27,29–34]). Each element $x \in X$ is a class containing the elements $x_G h$ where $h \in H$, and $x_G \in G$ is a representative of the class x. The choice of representatives is not unique since if x_G is a representative of the class $x \in G/H$ then $x_G h_0$, where h_0 is an arbitrary element from H, also is a representative of the same class. It is well known that X can be treated as a left G space. This means that if $x \in X$ then the action of the group G on X can be defined as follows: if $g \in G$ then gx is a class containing gx_G (it is easy to verify that such an action is correctly defined). Suppose that the choice of representatives is somehow fixed. Then $gx_G = (gx)_G(g, x)_H$ where $(g, x)_H$ is an element of H. This element is called a factor.

The explicit form of the operators M^{ab} depends on the choice of representatives in the space G/H. As explained in papers on UIRs of the SO(1,4) group (see e.g., Reference [27]), to obtain the possible closest analogy between UIRs of the SO(1,4) and Poincare groups, one should proceed as follows. Let $\mathbf{v}_L$ be a representative of the Lorentz group in the factor space SO(1,3)/SO(3) (strictly speaking, we should consider $SL(2, C)/SU(2)$). This space can be represented as the well known velocity hyperboloid with the Lorentz invariant measure

$$d\rho(\mathbf{v}) = d^3\mathbf{v}/v_0 \tag{9}$$

where $v_0 = (1 + \mathbf{v}^2)^{1/2}$. Let $I \in SO(1, 4)$ be a matrix which formally has the same form as the metric tensor η. One can show (see e.g., Reference [27] for details) that $X = G/H$ can be represented as a

union of three spaces, X_+, X_- and X_0 such that X_+ contains classes $\mathbf{v}_L h$, X_- contains classes $\mathbf{v}_L I h$ and X_0 has measure zero relative to the spaces X_+ and X_- (see also Section 7).

As a consequence, the space of UIR of the SO(1,4) group can be implemented as follows. If s is the spin of the particle under consideration, then we use $||...||$ to denote the norm in the space of UIR of the group SU(2) with the spin s. Then the space of UIR is the space of functions $\{f_1(\mathbf{v}), f_2(\mathbf{v})\}$ on two Lorentz hyperboloids with the range in the space of UIR of the group SU(2) with the spin s and such that

$$\int [||f_1(\mathbf{v})||^2 + ||f_2(\mathbf{v})||^2] d\rho(\mathbf{v}) < \infty \tag{10}$$

It is well-known that positive energy UIRs of the Poincare and AdS groups (associated with elementary particles) are implemented on an analog of X_+ while negative energy UIRs (associated with antiparticles) are implemented on an analog of X_-. Since the Poincare and AdS groups do not contain elements transforming these spaces to one another, the positive and negative energy UIRs are fully independent. At the same time, the dS group contains such elements (e.g., I [27,28]) and for this reason its UIRs can be implemented only on the union of X_+ and X_-. Even this fact is a strong indication that UIRs of the dS group cannot be interpreted in the same way as UIRs of the Poincare and AdS groups.

A general construction of the operators M^{ab} is as follows. We first define right invariant measures on $G = SO(1,4)$ and H. It is well known (see e.g., References [29–32]) that for semisimple Lie groups (which is the case for the dS group), the right invariant measure is simultaneously the left invariant one. At the same time, the right invariant measure $d_R(h)$ on H is not the left invariant one, but has the property $d_R(h_0 h) = \Delta(h_0) d_R(h)$, where the number function $h \to \Delta(h)$ on H is called the module of the group H. It is easy to show [27] that

$$\Delta(r\tau_A \mathbf{a_T}) = exp(-3\tau) \tag{11}$$

Let $d\rho(x)$ be a measure on $X = G/H$ compatible with the measures on G and H. This implies that the measure on G can be represented as $d\rho(x) d_R(h)$. Then one can show [27] that if X is a union of X_+ and X_- then the measure $d\rho(x)$ on each Lorentz hyperboloid coincides with that given by Equation (9). Let the representation space be implemented as the space of functions $\varphi(x)$ on X with the range in the space of UIR of the SU(2) group such that

$$\int ||\varphi(x)||^2 d\rho(x) < \infty \tag{12}$$

Then the action of the representation operator $U(g)$ corresponding to $g \in G$ is defined as

$$U(g)\varphi(x) = [\Delta((g^{-1}, x)_H)]^{-1/2} \Delta((g^{-1}, x)_H; m_{dS}, \mathbf{s})^{-1} \varphi(g^{-1} x) \tag{13}$$

One can directly verify that this expression defines a unitary representation. Its irreducibility can be proved in several ways (see e.g., Reference [27]).

As noted above, if X is the union of X_+ and X_-, then the representation space can be implemented as in Equation (8). Since we are interested in calculating only the explicit form of the operators M^{ab}, it suffices to consider only elements of $g \in G$ in an infinitely small vicinity of the unit element of the dS group. In that case one can calculate the action of representation operators on functions having the

carrier in X_+ and X_- separately. Namely, as follows from Equation (11), for such $g \in G$, one has to find the decompositions

$$g^{-1}\mathbf{v}_L = \mathbf{v}'_L r'(\tau')_A(\mathbf{a}')_{\mathbf{T}} \tag{14}$$

and

$$g^{-1}\mathbf{v}_L I = \mathbf{v}"{}_L I r"(\tau")_A(\mathbf{a}")_{\mathbf{T}} \tag{15}$$

where $r', r" \in SO(3)$. In this expressions it suffices to consider only elements of H belonging to an infinitely small vicinity of the unit element.

The problem of choosing representatives in the spaces SO(1,3)/SO(3) or SL(2.C)/SU(2) is well known in standard theory. The most usual choice is such that $\mathbf{v}_L$ as an element of SL(2,C) is given by

$$\mathbf{v}_L = \frac{v_0 + 1 + \mathbf{v}\sigma}{\sqrt{2(1+v_0)}} \tag{16}$$

Then by using a well known relation between elements of SL(2,C) and SO(1,3) we obtain that $\mathbf{v}_L \in SO(1,4)$ is represented by the matrix

$$\mathbf{v}_L = \left\| \begin{array}{ccc} v_0 & \mathbf{v}^T & 0 \\ \mathbf{v} & 1 + \mathbf{v}\mathbf{v}^T/(v_0+1) & 0 \\ 0 & 0 & 1 \end{array} \right\| \tag{17}$$

As follows from Equations (8) and (13), there is no need to know the expressions for $(\mathbf{a}')_{\mathbf{T}}$ and $(\mathbf{a}")_{\mathbf{T}}$ in Equations (14) and (15). We can use the fact [27] that if e is the five-dimensional vector with the components $(e^0 = 1, 0, 0, 0, e^4 = -1)$ and $h = r\tau_A\mathbf{a}_{\mathbf{T}}$, then $he = exp(-\tau)e$ regardless of the elements $r \in SO(3)$ and $\mathbf{a}_{\mathbf{T}}$. This makes it possible to easily calculate $(\mathbf{v}'_L, \mathbf{v}"{}_L, (\tau')_A, (\tau")_A)$ in Equations (14) and (15). Then one can calculate $(r', r")$ in these expressions by using the fact that the SO(3) parts of the matrices $(\mathbf{v}'_L)^{-1}g^{-1}\mathbf{v}_L$ and $(\mathbf{v}"{}_L)^{-1}g^{-1}\mathbf{v}_L$ are equal to r' and $r"$, respectively.

The relation between the operators $U(g)$ and M^{ab} is as follows. Let L_{ab} be the basis elements of the Lie algebra of the dS group. These are the matrices with the elements

$$(L_{ab})^c_d = \delta^c_d\eta_{bd} - \delta^c_b\eta_{ad} \tag{18}$$

They satisfy the commutation relations

$$[L_{ab}, L_{cd}] = \eta_{ac}L_{bd} - \eta_{bc}L_{ad} - \eta_{ad}L_{bc} + \eta_{bd}L_{ac} \tag{19}$$

Comparing Equations (4) and (19) it is easy to conclude that the M^{ab} should be the representation operators of $-iL^{ab}$. Therefore if $g = 1 + \omega_{ab}L^{ab}$, where a sum over repeated indices is assumed and the ω_{ab} are such infinitely small parameters that $\omega_{ab} = -\omega_{ba}$ then $U(g) = 1 + i\omega_{ab}M^{ab}$.

We are now in position to write down the final expressions for the operators M^{ab}. Their action on functions with the carrier in X_+ has the form

$$\mathbf{M}^{(+)} = l(\mathbf{v}) + \mathbf{s}, \quad \mathbf{N}^{(+)} = -iv_0\frac{\partial}{\partial\mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},$$

$$\mathbf{B}^{(+)} = m_{dS}\mathbf{v} + i[\frac{\partial}{\partial\mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial\mathbf{v}}) + \frac{3}{2}\mathbf{v}] + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},$$

$$M^{(+)}_{04} = m_{dS}v_0 + iv_0(\mathbf{v}\frac{\partial}{\partial\mathbf{v}} + \frac{3}{2}) \tag{20}$$

where $\mathbf{M} = \{M^{23}, M^{31}, M^{12}\}$, $\mathbf{N} = \{M^{01}, M^{02}, M^{03}\}$, $\mathbf{B} = \{M^{41}, M^{42}, M^{43}\}$, $\mathbf{s}$ is the spin operator, and $\mathbf{l}(\mathbf{v}) = -i\mathbf{v} \times \partial/\partial\mathbf{v}$. At the same time, the action on functions with the carrier in X_- is given by

$$\mathbf{M}^{(-)} = l(\mathbf{v}) + \mathbf{s}, \quad \mathbf{N}^{(-)} = -iv_0 \frac{\partial}{\partial\mathbf{v}} + \frac{\mathbf{s} \times \mathbf{v}}{v_0 + 1},$$

$$\mathbf{B}^{(-)} = -m_{dS}\mathbf{v} - i[\frac{\partial}{\partial\mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial\mathbf{v}}) + \frac{3}{2}\mathbf{v}] - \frac{\mathbf{s} \times \mathbf{v}}{v_0 + 1},$$

$$M_{04}^{(-)} = -m_{dS}v_0 - iv_0(\mathbf{v}\frac{\partial}{\partial\mathbf{v}} + \frac{3}{2}) \tag{21}$$

Note that the expressions for the action of the Lorentz algebra operators on X_+ and X_- are the same and they coincide with the corresponding expressions for IRs of the Poincare algebra. At the same time, the expressions for the action of the operators $M^{4\mu}$ on X_+ and X_- differ by sign.

In deriving Equations (20) and (21) we used only the commutation relations (4), no approximations have been made and the results are exact. In particular, the dS space, the cosmological constant and the Riemannian geometry have not been involved at all. Nevertheless, the expressions for the representation operators is all we need to have the maximum possible information in quantum theory. If one defines $m = m_{dS}/R$ and the operators P^μ as in the preceding section then in the formal limit $R \to \infty$ we indeed obtain the expressions for the operators of the IRs of the Poincare algebra such that the Lorentz algebra operators are the same, $E = mv_0$ and $\mathbf{P} = m\mathbf{v}$ where E is the standard energy operator and $\mathbf{P}$ is the standard momentum operator. If we assume for definiteness that $m_{dS} > 0$ then we obtain positive energy and negative energy IRs of the Poincare algebra on X_+ and X_-, respectively. It is obvious that in that case m is the standard mass in Poincare invariant theory.

It is well known that in Poincare invariant theory the operator $W = E^2 - \mathbf{P}^2$ is the Casimir operator, *i.e.*, it commutes with all the representation operators. According to the well known Schur lemma in representation theory, all elements in the space of IR are eigenvectors of the Casimir operators with the same eigenvalue. In particular, they are the eigenvectors of the operator W with the eigenvalue m^2. As follows from Equation (4), in the dS case the Casimir operator of the second order is $I_2 = -1/2\sum_{ab} M_{ab}M^{ab}$ and this operator might be treated as a dS analog of the mass operator squared. An explicit calculation shows that for the operators given by Equations (20) and (21) the numerical value of I_2 is $m_{dS}^2 - s(s+1) + 9/4$.

6. Absence of Weyl Particles in dS Invariant Theory

According to Standard Model, only massless Weyl particles can be fundamental elementary particles in Poincare invariant theory. Therefore a problem arises whether there exist analogs of Weyl particles in dS invariant theory. In the preceding section we have shown that the dS and Poincare masses are related as $m_{dS}/R = m$. The dS mass is dimensionless while the Poincare mass has the dimension $length^{-1}$. Since the Poincare symmetry is a special case of the dS one, this fact is in agreement with the observation in Section 2 that dimensionful quantities cannot be fundamental. Let $l_C(m)$ be the Compton wave length for the particle with the mass m. Then one might think that, in view of the relation $m_{dS} = R/l_C(m)$, the dS mass shows how many Compton wave lengths are contained in the interval $(0, R)$. However, such an interpretation of the dS mass means that we wish to interpret a fundamental quantity m_{dS} in terms of our experience based on Poincare invariant theory. As already noted, the value of m_{dS} does not

depend on any quantities having the dimension $length$ or $length^{-1}$ and it is the Poincare mass which implicitly depends on R. Let us assume for estimations that the value of R is 10^{28} cm. Then even the dS mass of the electron is of order 10^{39} and this might be treated as an indication that the electron is not a true elementary particle. Moreover, the present upper level for the photon mass is 10^{-18} ev which seems to be an extremely tiny quantity. However, the corresponding dS mass is of order 10^{15} and so even the mass which is treated as extremely small in Poincare invariant theory might be very large in dS invariant theory.

In Poincare invariant theory, Weyl particles are characterized not only by the condition that their mass is zero but also by the condition that they have a definite helicity. Several authors investigated dS and AdS analogs of Weyl particles proceeding from covariant equations on the dS and AdS spaces, respectively. For example, the authors of Reference [35] show that Weyl particles arise only when the dS or AdS symmetries are broken to the Lorentz symmetry. At the level of IRs, the existence of analogs of Weyl particles is known in the AdS case. In Reference [36] we investigated such analogs by using the results of References [37] for standard IRs of the AdS algebra (*i.e.*, IRs over the field of complex numbers) and the results of Reference [38] for IRs of the AdS algebra over a Galois field. In the standard case the minimum value of the AdS energy for massless IRs with positive energy is $E_{min} = 1 + s$. In contrast with the situation in Poincare invariant theory, where massless particles cannot be in the rest state, the massless particles in the AdS theory do have rest states and the value of the z projection of the spin in such states can be $-s, -s+1, ..., s$ as usual. However, for any value of energy greater than E_{min}, the spin state is characterized only by helicity, which can take the values either s or $-s$, *i.e.*, we have the same result as in Poincare invariant theory. In contrast with IRs of the Poincare and dS algebra, IRs describing particles in AdS theory belong to the discrete series of IRs and the energy spectrum in them is discrete: $E = E_{min}, E_{min} + 1, ..., \infty$. Therefore, strictly speaking, rest states do not have measure zero. Nevertheless, the probability that the energy is exactly E_{min} is extremely small and therefore there exists a correspondence between Weyl particles in Poincare and AdS theories.

In Poincare invariant theory, IRs describing Weyl particles can be constructing by analogy with massive IRs but the little group is now E(2)instead of SO(3) (see e.g., section 2.5 in the textbook [18]). The matter is that the representation operators of the SO(3) group transform rest states into themselves but for massless particles there are no rest states. However, there exists another way of getting massless IRs: one can choose the variables for massive IRs in such a way that the operators of massless IRs can be directly obtained from the operators of massive IRs in the limit $m \to 0$. This construction has been described by several authors (see e.g., References [39–42] and references therein) and the main stages are as follows. First, instead of the $(0, 1, 2, 3)$ components of vectors, we work with the so called light front components $(+, -, 1, 2)$ where $v^{\pm} = (v^0 \pm v^3)/\sqrt{2}$ and analogously for other vectors. We choose $(v^+, \mathbf{v}_{\perp})$ as three independent components of the 4-velocity vector, where $\mathbf{v}_{\perp} = (v_x, v_y)$. In these variables the measure (9) on the Lorentz hyperboloid becomes $d\rho(v^+, \mathbf{v}_{\perp}) = dv^+ d\mathbf{v}_{\perp}/v^+$. Instead of Equation (16) we now choose representatives of the SL(2,C)/SU(2) classes as

$$v_L = \frac{1}{(v_0 + v_z)^{1/2}} \left\| \begin{array}{cc} v_0 + v_z & 0 \\ v_x + iv_y & 1 \end{array} \right\| \tag{22}$$

and by using the relation between the groups SL(2,C) and SO(1,3) we obtain that the form of this representative in the Lorentz group is

$$
v_L = \left\|
\begin{array}{cccc}
\sqrt{2}v^+ & 0 & 0 & 0 \\
\frac{\mathbf{v}_\perp^2}{\sqrt{2}v^+} & \frac{1}{\sqrt{2}v^+} & \frac{v_x}{v^+} & \frac{v_y}{v^+} \\
\sqrt{2}v_x & 0 & 1 & 0 \\
\sqrt{2}v_y & 0 & 0 & 1
\end{array}
\right\|
\tag{23}
$$

where the raws and columns are in the order $(+, -, x, y)$.

By using the scheme described in the preceding section, we can now calculate the explicit form of the representation operators of the Lorentz algebra. In this scheme the form of these operators in the IRs of the Poincare and dS algebras is the same and in the case of the dS algebra the action is the same for states with the carrier in X_+ and X_-. The results of calculations are:

$$
M^{+-} = iv^+ \frac{\partial}{\partial v^+} \quad M^{+j} = iv^+ \frac{\partial}{\partial v^j} \quad M^{12} = l_z(\mathbf{v}_\perp) + s_z
$$
$$
M^{-j} = -i(v^j \frac{\partial}{\partial v^+} + v^- \frac{\partial}{\partial v^j}) - \frac{\epsilon_{jl}}{v^+}(s^l + v^l s_z)
\tag{24}
$$

where a sum over $j, l = 1, 2$ is assumed and ϵ_{jl} has the components $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. In Poincare invariant theories one can define the standard four-momentum $p = mv$ and choose $(p^+, \mathbf{p}_\perp)$ as independent variables. Then the expressions in Equation (24) can be rewritten as

$$
M^{+-} = ip^+ \frac{\partial}{\partial p^+} \quad M^{+j} = ip^+ \frac{\partial}{\partial p^j} \quad M^{12} = l_z(\mathbf{p}_\perp) + s_z
$$
$$
M^{-j} = -i(p^j \frac{\partial}{\partial p^+} + p^- \frac{\partial}{\partial p^j}) - \frac{\epsilon_{jl}}{p^+}(ms^l + p^l s_z)
\tag{25}
$$

In dS invariant theory we can work with the same variables if m is defined as m_{dS}/R.

As seen from Equations (25), only the operators M^{-j} contain a dependence on the operators s_x and s_y but this dependence disappears in the limit $m \to 0$. In this limit the operator s_z can be replaced by its eigenvalue λ which now has the meaning of helicity. In Poincare invariant theory the four-momentum operators P^μ are simply the operators of multiplication by p^μ and therefore massless particles are characterized only by one constant—helicity.

In dS invariant theory one can calculate the action of the operators $M^{4\mu}$ by analogy with the calculation in the preceding section. The actions of these operators on states with the carrier in X_+ and X_- differ only by sign and the result for the actions on states with the carrier in X_+ is

$$
M^{4-} = m_{dS}v^- + i[v^-(v^+ \frac{\partial}{\partial v^+} + v^j \frac{\partial}{\partial v^j} + \frac{3}{2}) - \frac{\partial}{\partial v^+}] + \frac{1}{v^+}\epsilon_{jl}v^j s^l
$$
$$
M^{4j} = m_{dS}v^j + i[v^j(v^+ \frac{\partial}{\partial v^+} + v^l \frac{\partial}{\partial v^l} + \frac{3}{2}) + \frac{\partial}{\partial v^j}] - \epsilon_{jl}s^l
$$
$$
M^{4+} = m_{dS}v^+ + iv^+(v^+ \frac{\partial}{\partial v^+} + v^j \frac{\partial}{\partial v^j} + \frac{3}{2})
\tag{26}
$$

If we define $m = m_{dS}/R$ and $p^\mu = mv^\mu$ then for the operators P^μ we have

$$
P^- = p^- + \frac{ip^-}{mR}(p^+ \frac{\partial}{\partial p^+} + p^j \frac{\partial}{\partial p^j} + \frac{3}{2}) - \frac{im}{R}\frac{\partial}{\partial p^+} + \frac{1}{Rp^+}\epsilon_{jl}p^j s^l
$$

$$P^j = p^j + \frac{ip^j}{mR}(p^+\frac{\partial}{\partial p^+} + p^l\frac{\partial}{\partial p^l} + \frac{3}{2}) + \frac{im}{R}\frac{\partial}{\partial p^j} - \frac{1}{R}\epsilon_{jl}s^l$$

$$P^+ = p^+ + \frac{ip^+}{mR}(p^+\frac{\partial}{\partial p^+} + p^j\frac{\partial}{\partial p^j} + \frac{3}{2}) \tag{27}$$

Then it is clear that in the formal limit $R \to \infty$ we obtain the standard Poincare result. However, when R is finite, the dependence of the operators P^μ on s_x and s_y does not disappear. Moreover, in this case we cannot take the limit $m \to 0$. Therefore we conclude that in dS theory there are no Weyl particles, at least in the case when elementary particles are described by IRs of the principle series. Mensky conjectured [27] that massless particles in the dS invariant theory might correspond to IRs of the discrete series with $-im_{dS} = 1/2$ but this possibility has not been investigated. In any case, in contrast with the situation in Poincare invariant theory, the limit of massive IRs when $m \to 0$ does not give Weyl particles and moreover, this limit does not exist.

7. Other Implementations of IRs

In this section we will briefly describe two more implementations of IRs of the dS algebra. The first one is based on the fact that since SO(1,4)=SO(4)A**T** and H=SO(3)A**T** [27], there also exists a choice of representatives which is probably even more natural than those described above. Namely, we can choose as representatives the elements from the coset space SO(4)/SO(3). Since the universal covering group for SO(4) is SU(2) × SU(2) and for SO(3) — SU(2), we can choose as representatives the elements of the first multiplier in the product SU(2) × SU(2). Elements of SU(2) can be represented by the points $u = (\mathbf{u}, u_4)$ of the three-dimensional sphere S^3 in the four-dimensional space as $u_4 + i\sigma\mathbf{u}$ where σ are the Pauli matrices and $u_4 = \pm(1 - \mathbf{u}^2)^{1/2}$ for the upper and lower hemispheres, respectively. Then the calculation of the operators is similar to that described above and the results are as follows. The Hilbert space is now the space of functions $\varphi(u)$ on S^3 with the range in the space of the IR of the su(2) algebra with the spin s and such that

$$\int ||\varphi(u)||^2 du < \infty \tag{28}$$

where du is the SO(4) invariant volume element on S^3. The explicit calculation shows that in this case the operators have the form

$$\mathbf{M} = l(\mathbf{u}) + \mathbf{s} \quad \mathbf{B} = iu_4\frac{\partial}{\partial\mathbf{u}} - \mathbf{s} \quad M_{04} = (m_{dS} + 3i/2)u_4 + iu_4\mathbf{u}\frac{\partial}{\partial\mathbf{u}}$$

$$\mathbf{N} = -i[\frac{\partial}{\partial\mathbf{u}} - \mathbf{u}(\mathbf{u}\frac{\partial}{\partial\mathbf{u}})] + (m_{dS} + 3i/2)\mathbf{u} - \mathbf{u} \times \mathbf{s} + u_4\mathbf{s} \tag{29}$$

Since Equations (10), (20) and (21) on one hand and Equations (28) and (29) on the other are the different implementations of one and the same representation, there exists a unitary operator transforming functions $f(v)$ into $\varphi(u)$ and operators (20,21) into operators (29). For example in the spinless case the operators (20) and (29) are related to each other by a unitary transformation

$$\varphi(u) = exp(-im_{dS}lnv_0)v_0^{3/2}f(v) \tag{30}$$

where the relation between the points of the upper hemisphere and X_+ is $\mathbf{u} = \mathbf{v}/v_0$ and $u_4 = (1-\mathbf{u}^2)^{1/2}$. The relation between the points of the lower hemisphere and X_- is $\mathbf{u} = -\mathbf{v}/v_0$ and $u_4 = -(1-\mathbf{u}^2)^{1/2}$.

The equator of S^3 where $u_4 = 0$ corresponds to X_0 and has measure zero with respect to the upper and lower hemispheres. For this reason one might think that it is of no interest for describing particles in dS theory. Nevertheless, an interesting observation is that while none of the components of u has the magnitude greater than unity, the set X_0 in terms of velocities is characterized by the condition that $|\mathbf{v}|$ is infinitely large and therefore the standard Poincare momentum $\mathbf{p} = m\mathbf{v}$ is infinitely large too. This poses a question whether $\mathbf{p}$ always has a physical meaning. From mathematical point of view Equation (29) might seem more convenient than Equations (20) and (21) since S^3 is compact and there is no need to break it into the upper and lower hemispheres. In addition, Equation (29) is an explicit implementation of the idea that since in dS invariant theory all the variables (x^1, x^2, x^3, x^4) are on equal footing and so(4) is the maximal compact kinematical algebra, the operators $\mathbf{M}$ and $\mathbf{B}$ do not depend on m_{dS}. However, those expressions are not convenient for investigating Poincare approximation since the Lorentz boost operators $\mathbf{N}$ depend on m_{dS}.

Finally, we describe an implementation of IRs based on the explicit construction of the basis in the representation space. This construction is based on the method of su(2)×su(2) shift operators, developed by Hughes [43] for constructing UIRs of the group SO(5). It will be convenient for us to deal with the set of operators $(\mathbf{J}', \mathbf{J}'', R_{ij})$ $(i, j = 1, 2)$ instead of M^{ab}. Here $\mathbf{J}'$ and $\mathbf{J}''$ are two independent su(2) algebras (*i.e.*, $[\mathbf{J}', \mathbf{J}''] = 0$). In each of them one chooses as the basis the operators (J_+, J_-, J_3) such that $J_1 = J_+ + J_-$, $J_2 = -i(J_+ - J_-)$ and the commutation relations have the form

$$[J_3, J_+] = 2J_+, \quad [J_3, J_-] = -2J_-, \quad [J_+, J_-] = J_3 \tag{31}$$

The commutation relations of the operators $\mathbf{J}'$ and $\mathbf{J}''$ with R_{ij} have the form

$$\begin{aligned}
&[J_3', R_{1j}] = R_{1j}, \quad [J_3', R_{2j}] = -R_{2j}, \quad [J_3'', R_{i1}] = R_{i1}, \\
&[J_3'', R_{i2}] = -R_{i2}, \quad [J_+', R_{2j}] = R_{1j}, \quad [J_+'', R_{i2}] = R_{i1}, \\
&[J_-', R_{1j}] = R_{2j}, \quad [J_-'', R_{i1}] = R_{i2}, \\
&[J_+', R_{1j}] = [J_+'', R_{i1}] = [J_-', R_{2j}] = [J_-'', R_{i2}] = 0
\end{aligned} \tag{32}$$

and the commutation relations of the operators R_{ij} with each other have the form

$$\begin{aligned}
&[R_{11}, R_{12}] = 2J_+', \quad [R_{11}, R_{21}] = 2J_+'', \\
&[R_{11}, R_{22}] = -(J_3' + J_3''), \quad [R_{12}, R_{21}] = J_3' - J_3'' \\
&[R_{11}, R_{22}] = -2J_-'', \quad [R_{21}, R_{22}] = -2J_-'
\end{aligned} \tag{33}$$

The relation between the sets $(\mathbf{J}', \mathbf{J}'', R_{ij})$ and M^{ab} is given by

$$\begin{aligned}
&\mathbf{M} = (\mathbf{J}' + \mathbf{J}'')/2 \quad \mathbf{B} = (\mathbf{J}' - \mathbf{J}'')/2 \quad M_{01} = i(R_{11} - R_{22})/2, \\
&M_{02} = (R_{11} + R_{22})/2 \quad M_{03} = -i(R_{12} + R_{21})/2 \quad M_{04} = (R_{12} - R_{21})/2
\end{aligned} \tag{34}$$

Then it is easy to see that Equation (4) follows from Equations (32–34) and *vice versa*.

Consider the space of maximal $su(2) \times su(2)$ vectors, *i.e.*, such vectors x that $J_+'x = J_+''x = 0$. Then from Equations (32) and (33) it follows that the operators

$$\begin{aligned}
&A^{++} = R_{11} \quad A^{+-} = R_{12}(J_3'' + 1) - J_-''R_{11} \quad A^{-+} = R_{21}(J_3' + 1) - J_-'R_{11} \\
&A^{--} = -R_{22}(J_3' + 1)(J_3'' + 1) + J_-''R_{21}(J_3' + 1) + \\
&J_-'R_{12}(J_3'' + 1) - J_-'J_-''R_{11}
\end{aligned} \tag{35}$$

act invariantly on this space. The notations are related to the property that if x^{kl} ($k, l > 0$) is the maximal su(2)×su(2) vector and simultaneously the eigenvector of operators J_3' and J_3'' with the eigenvalues k and l, respectively, then $A^{++}x^{kl}$ is the eigenvector of the same operators with the values $k+1$ and $l+1$, $A^{+-}x^{kl}$ - the eigenvector with the values $k+1$ and $l-1$, $A^{-+}x^{kl}$ - the eigenvector with the values $k-1$ and $l+1$ and $A^{--}x^{kl}$ - the eigenvector with the values $k-1$ and $l-1$.

The basis in the representation space can be explicitly constructed assuming that there exists a vector e^0 which is the maximal su(2)×su(2) vector such that

$$J_3'e_0 = 0 \quad J_3''e_0 = se_0 \quad A^{--}e_0 = A^{-+}e_0 = 0 \quad I_2e^0 = [m_{dS}^2 - s(s+1) + 9/4]e^0 \tag{36}$$

Then, as shown in Reference [26], the full basis of the representation space consists of vectors

$$e_{ij}^{nr} = (J_-')^i(J_-'')^j(A^{++})^n(A^{+-})^r e^0 \tag{37}$$

where $n = 0, 1, 2, ..., r$ can take only the values $0, 1, ..., 2s$ and for the given n and s, i can take the values $0, 1, ..., n + r$ and j can take the values $0, 1, ..., n + 2s - r$.

These results show that IRs of the dS algebra can be constructed purely algebraically without involving analytical methods of the theory of UIRs of the dS group. As shown in Reference [26], this implementation is convenient for generalizing standard quantum theory to a quantum theory over a Galois field.

8. Physical Interpretation of IRs of the dS Algebra

In Section 5–7 we discussed mathematical properties of IRs of the dS algebra. In particular it has been noted that they are implemented on two Lorentz hyperboloids, not one as IRs of the Poincare algebra. Therefore the number of states in IRs of the dS algebra is twice as big as in IRs of the Poincare algebra. A problem arises whether this is compatible with a requirement that any dS invariant theory should become a Poincare invariant one in the formal limit $R \to \infty$. Although there exists a wide literature on IRs of the dS group and algebra, their physical interpretation has not been widely discussed. Probably one of the reasons is that physicists working on dS QFT treat fields as more fundamental objects than particles (although the latter are observables while the former are not).

In his book [27] Mensky notes that, in contrast with IRs of the Poincare and AdS groups, IRs of the dS group characterized by m_{dS} and $-m_{dS}$ are unitarily equivalent and therefore the energy sign cannot be used for distinguishing particles and antiparticles. He proposes an interpretation where a particle and its antiparticle are described by the same IRs but have different space-time descriptions (defined by operators intertwining IRs with representations induced from the Lorentz group). Mensky shows that in the general case his two solutions still cannot be interpreted as a particle and its antiparticle, respectively, since they are nontrivial linear combinations of functions with different energy signs. However, such an interpretation is recovered in Poincare approximation.

In view of the above discussion, it is desirable to give an interpretation of IRs which does not involve spacetime. In Reference [5] we have proposed an interpretation such that one IR describes a particle and its antiparticle simultaneously. In this section this analysis is extended.

8.1. Problems with Physical Interpretation of IRs

Consider first the case when the quantity m_{dS} is very large. Then, as follows from Equations (20) and (21), the action of the operators $M^{4\mu}$ on states localized on X_+ or X_- can be approximately written as $\pm m_{dS} v^\mu$, respectively. Therefore a question arises whether the standard Poincare energy E can be defined as $E = M_{04}/R$. Indeed, with such a definition, states localized on X_+ will have a positive energy while states localized on X_- will have a negative energy. Then a question arises whether this is compatible with the standard interpretation of IRs, according to which the following requirements should be satisfied:

Standard-Interpretation Requirements: Each element of the full representation space represents a possible physical state for the given elementary particle. The representation describing a system of N free elementary particles is the tensor product of the corresponding single-particle representations.

Recall that the operators of the tensor product are given by sums of the corresponding single-particle operators. For example, if $M_{04}^{(1)}$ is the operator M_{04} for particle 1 and $M_{04}^{(2)}$ is the operator M_{04} for particle 2 then the operator M_{04} for the free system $\{12\}$ is given by $M_{04}^{(12)} = M_{04}^{(1)} + M_{04}^{(2)}$. Here it is assumed that the action of the operator $M_{04}^{(j)}$ ($j = 1, 2$) in the two-particle space is defined as follows. It acts according to Equation (20) or (21) over its respective variables while over the variables of the other particle it acts as the identity operator.

One could try to satisfy the standard interpretation as follows.

A) Assume that in Poincare approximation the standard energy should be defined as $E = \pm M_{04}/R$ where the plus sign should be taken for the states with the carrier in X_+ and as the minus sign—for the states with the carrier in X_-. Then the energy will always be positive definite.

B) One might say that the choice of the energy sign is only a matter of convention. Indeed, to measure the energy of a particle with the mass m one has to measure its momentum $\mathbf{p}$ and then the energy can be defined not only as $(m^2 + \mathbf{p}^2)^{1/2}$ but also as $-(m^2 + \mathbf{p}^2)^{1/2}$. In that case the standard energy in the Poincare approximation could be defined as $E = M_{04}/R$ regardless of whether the carrier of the given state is in X_+ or X_-.

It is easy to see that either of the above possibilities is incompatible with Standard-Interpretation Requirements. Consider, for example, a system of two free particles in the case when m_{dS} is very large. Then with a high accuracy the operators M_{04}/R and $\mathbf{B}/R$ can be chosen diagonal simultaneously.

Let us first assume that the energy should be treated according to B). Then a system of two free particles with the equal masses can have the same quantum numbers as the vacuum (for example, if the first particle has the energy $E_0 = (m^2 + \mathbf{p}^2)^{1/2}$ and momentum $\mathbf{p}$ while the second one has the energy $-E_0$ and the momentum $-\mathbf{p}$) what obviously contradicts experiment. For this and other reasons it is well known that in Poincare invariant theory the particles should have the same energy sign. Analogously, if the single-particle energy is treated according to A) then the result for the two-body energy of a particle-antiparticle system will contradict experiment.

We conclude that IRs of the dS algebra cannot be interpreted in the standard way since such an interpretation is physically meaningless even in Poincare approximation. The above discussion indicates that the problem we have is similar to that with the interpretation of the fact that the Dirac equation has solutions with both, positive and negative energies.

As already noted, in Poincare and AdS theories there exist positive energy IRs implemented on the upper hyperboloid and negative energy IRs implemented on the lower hyperboloid. In the latter case Standard-Interpretation Requirements are not satisfied for the reasons discussed above. However, we cannot declare such IRs unphysical and throw them away. In QFT quantum fields necessarily contain both types of IRs such that positive energy IRs are associated with particles while negative energy IRs are associated with antiparticles. Then the energy of antiparticles can be made positive after proper second quantization. In view of this observation, we will investigate whether IRs of the dS algebra can be interpreted in such a way that one IR describes a particle and its antiparticle simultaneously such that states localized on X_+ are associated with a particle while states localized on X_- are associated with its antiparticle.

By using Equation (10), one can directly verify that the operators (20) and (21) are Hermitian if the scalar product in the space of IR is defined as follows. Since the functions $f_1(\mathbf{v})$ and $f_2(\mathbf{v})$ in Equation (10) have the range in the space of IR of the su(2) algebra with the spin s, we can replace them by the sets of functions $f_1(\mathbf{v}, j)$ and $f_2(\mathbf{v}, j)$, respectively, where $j = -s, -s + 1, ..., s$. Moreover, we can combine these functions into one function $f(\mathbf{v}, j, \epsilon)$ where the variable ϵ can take only two values, say +1 or -1, for the components having the carrier in X_+ or X_-, respectively. If now $\varphi(\mathbf{v}, j, \epsilon)$ and $\psi(\mathbf{v}, j, \epsilon)$ are two elements of our Hilbert space, their scalar product is defined as

$$(\varphi, \psi) = \sum_{j, \epsilon} \int \varphi(\mathbf{v}, j, \epsilon)^* \psi(\mathbf{v}, j, \epsilon) d\rho(\mathbf{v}) \tag{38}$$

where the subscript * applied to scalar functions means the usual complex conjugation.

At the same time, we use * to denote the operator adjoint to a given one. Namely, if A is the operator in our Hilbert space then A^* means the operator such that

$$(\varphi, A\psi) = (A^*\varphi, \psi) \tag{39}$$

for all such elements φ and ψ that the left hand side of this expression is defined.

Even in the case of the operators (20) and (21) we can formally treat them as integral operators with some kernels. Namely, if $A\varphi = \psi$, we can treat this relation as

$$\sum_{j', \epsilon'} \int A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')\varphi(\mathbf{v}', j', \epsilon')d\rho(\mathbf{v}') = \psi(\mathbf{v}, j, \epsilon) \tag{40}$$

where in the general case the kernel $A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')$ of the operator A is a distribution.

As follows from Equations (38–40), if $B = A^*$ then the relation between the kernels of these operators is

$$B(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon') = A(\mathbf{v}', j', \epsilon'; \mathbf{v}, j, \epsilon)^* \tag{41}$$

In particular, if the operator A is Hermitian then

$$A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')^* = A(\mathbf{v}', j', \epsilon'; \mathbf{v}, j, \epsilon) \tag{42}$$

and if, in addition, its kernel is real then the kernel is symmetric, *i.e.*,

$$A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon') = A(\mathbf{v}', j', \epsilon'; \mathbf{v}, j, \epsilon) \tag{43}$$

In particular, this property is satisfied for the operators $m_{dS}v_0$ and $m_{dS}\mathbf{v}$ in Equations (20) and (21). At the same time, the operators

$$l(\mathbf{v}) \quad -iv_0\frac{\partial}{\partial\mathbf{v}} \quad -i[\frac{\partial}{\partial\mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial\mathbf{v}}) + \frac{3}{2}\mathbf{v}] \quad -iv_0(\mathbf{v}\frac{\partial}{\partial\mathbf{v}} + \frac{3}{2}) \tag{44}$$

which are present in Equations (20) and (21), are Hermitian but have imaginary kernels. Therefore, as follows from Equation (42), their kernels are antisymmetric:

$$A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon') = -A(\mathbf{v}', j', \epsilon'; \mathbf{v}, j, \epsilon) \tag{45}$$

In standard approach to quantum theory, the operators of physical quantities act in the Fock space of the given system. Suppose that the system consists of free particles and their antiparticles. Strictly speaking, in our approach it is not clear yet what should be treated as a particle or antiparticle. The considered IRs of the dS algebra describe objects such that $(\mathbf{v}, j, \epsilon)$ is the full set of their quantum numbers. Therefore we can define the annihilation and creation operators $(a(\mathbf{v}, j, \epsilon), a(\mathbf{v}, j, \epsilon)^*)$ for these objects. If the operators satisfy the anticommutation relations then we require that

$$\{a(\mathbf{v}, j, \epsilon), a(\mathbf{v}', j', \epsilon')^*\} = \delta_{jj'}\delta_{\epsilon\epsilon'}v_0\delta^{(3)}(\mathbf{v} - \mathbf{v}') \tag{46}$$

while in the case of commutation relations

$$[a(\mathbf{v}, j, \epsilon), a(\mathbf{v}', j', \epsilon')^*] = \delta_{jj'}\delta_{\epsilon\epsilon'}v_0\delta^{(3)}(\mathbf{v} - \mathbf{v}') \tag{47}$$

In the first case, any two a-operators or any two a^* operators anticommute with each other while in the second case they commute with each other.

The problem of second quantization can now be formulated such that IRs should be implemented as Fock spaces, *i.e.*, states and operators should be expressed in terms of the (a, a^*) operators. A possible implementation is as follows. We define the vacuum state Φ_0 such that is has a unit norm and satisfies the requirement

$$a(\mathbf{v}, j, \epsilon)\Phi_0 = 0 \quad \forall\, \mathbf{v}, j, \epsilon \tag{48}$$

The image of the state $\varphi(\mathbf{v}, j, \epsilon)$ in the Fock space is defined as

$$\varphi_F = \sum_{j,\epsilon}\int \varphi(\mathbf{v}, j, \epsilon)a(\mathbf{v}, j, \epsilon)^*d\rho(\mathbf{v})\Phi_0 \tag{49}$$

and the image of the operator with the kernel $A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')$ in the Fock space is defined as

$$A_F = \sum_{j,\epsilon,j',\epsilon'}\int\int A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')a(\mathbf{v}, j, \epsilon)^*a(\mathbf{v}', j', \epsilon')d\rho(\mathbf{v})d\rho(\mathbf{v}') \tag{50}$$

One can directly verify that this is an implementation of IR in the Fock space. In particular, the commutation relations in the Fock space will be preserved regardless of whether the (a, a^*) operators satisfy commutation or anticommutation relations and, if any two operators are adjoint in the implementation of IR described above, they will be adjoint in the Fock space as well. In other words, we have a * homomorphism of Lie algebras of operators acting in the space of IR and in the Fock space.

We now require that in Poincare approximation the energy should be positive definite. Recall that the operators (20) and (21) act on their respective subspaces or in other words, they are diagonal in the quantum number ϵ.

Suppose that $m_{dS} > 0$ and consider the quantized operator corresponding to the dS energy M_{04} in Equation (20). In Poincare approximation, $M_{04}^{(+)} = m_{dS}v_0$ is fully analogous to the standard free energy and therefore, as follows from Equation (50), its quantized form is

$$(M_{04}^{(+)})_F = m_{dS} \sum_j \int v_0 a(\mathbf{v}, j, 1)^* a(\mathbf{v}, j, 1) d\rho(\mathbf{v}) \tag{51}$$

This expression is fully analogous to the quantized Hamiltonian in standard theory and it is well known that the operator defined in such a way is positive definite.

Consider now the operator $M_{04}^{(-)}$. In Poincare approximation its quantized form is

$$(M_{04}^{(-)})_F == -m_{dS} \sum_j \int v_0 a(\mathbf{v}, j, -1)^* a(\mathbf{v}, j, -1) d\rho(\mathbf{v}) \tag{52}$$

and this operator is negative definite, what is unacceptable.

One might say that the operators $a(\mathbf{v}, j, -1)$ and $a(\mathbf{v}, j, -1)^*$ are "nonphysical": $a(\mathbf{v}, j, -1)$ is the operator of object's annihilation with the negative energy, and $a(\mathbf{v}, j, -1)^*$ is the operator of object's creation with the negative energy.

We will interpret the operator $(M_{04}^{(-)})_F$ as that related to antiparticles. In QFT the annihilation and creation operators for antiparticles are present in quantized fields with the coefficients describing negative energy solutions of the corresponding covariant equation. This is an implicit implementation of the idea that the creation or annihilation of an antiparticle can be treated, respectively as the annihilation or creation of the corresponding particle with the negative energy. In our case this idea can be implemented explicitly.

Instead of the operators $a(\mathbf{v}, j, -1)$ and $a(\mathbf{v}, j, -1)^*$, we define new operators $b(\mathbf{v}, j)$ and $b(\mathbf{v}, j)^*$. If $b(\mathbf{v}, j)$ is treated as the "physical" operator of antiparticle annihilation then, according to the above idea, it should be proportional to $a(\mathbf{v}, -j, -1)^*$. Analogously, if $b(\mathbf{v}, j)^*$ is the "physical" operator of antiparticle creation, it should be proportional to $a(\mathbf{v}, -j, -1)$. Therefore

$$b(\mathbf{v}, j) = \eta(j) a(\mathbf{v}, -j, -1)^* \quad b(\mathbf{v}, j)^* = \eta(j)^* a(\mathbf{v}, -j, -1) \tag{53}$$

where $\eta(j)$ is a phase factor such that

$$\eta(j)\eta(j)^* = 1 \tag{54}$$

As follows from this relations, if a particle is characterized by additive quantum numbers (e.g., electric, baryon or lepton charges) then its antiparticle is characterized by the same quantum numbers but with the minus sign. The transformation described by Equations (53) and (54) can also be treated as a special case of the Bogolubov transformation discussed in a wide literature on many-body theory (see, e.g., Chapter 10 in Reference [44] and references therein).

Since we treat $b(\mathbf{v}, j)$ as the annihilation operator and $b(\mathbf{v}, j)^*$ as the creation one, instead of Equation (48) we should define a new vacuum state $\tilde{\Phi}_0$ such that

$$a(\mathbf{v}, j, 1)\tilde{\Phi}_0 = b(\mathbf{v}, j)\tilde{\Phi}_0 = 0 \quad \forall \, \mathbf{v}, j, \tag{55}$$

and the images of states localized in X_- should be defined as

$$\varphi_F^{(-)} = \sum_{j,\epsilon} \int \varphi(\mathbf{v}, j, -1) b(\mathbf{v}, j)^* d\rho(\mathbf{v}) \tilde{\Phi}_0 \tag{56}$$

In that case the (b, b^*) operators should be such that in the case of anticommutation relations

$$\{b(\mathbf{v}, j), b(\mathbf{v}', j')^*\} = \delta_{jj'} v_0 \delta^{(3)}(\mathbf{v} - \mathbf{v}'), \tag{57}$$

and in the case of commutation relations

$$[b(\mathbf{v}, j), b(\mathbf{v}', j')^*] = \delta_{jj'} v_0 \delta^{(3)}(\mathbf{v} - \mathbf{v}') \tag{58}$$

We have to verify whether the new definition of the vacuum and one-particle states is a correct implementation of IR in the Fock space. A necessary condition is that the new operators should satisfy the commutation relations of the dS algebra. Since we replaced the (a, a^*) operators by the (b, b^*) operators only if $\epsilon = -1$, it is obvious from Equation (50) that the images of the operators (20) in the Fock space satisfy Equation (4). Therefore we have to verify that the images of the operators (21) in the Fock space also satisfy Equation (4).

Consider first the case when the operators $a(\mathbf{v}, j, \epsilon)$ satisfy the anticommutation relations. By using Equation (53) one can express the operators $a(\mathbf{v}, j, -1)$ in terms of the operators $b(\mathbf{v}, j)$. Then it follows from the condition (53) that the operators $b(\mathbf{v}, j)$ indeed satisfy Equation (57). If the operator A_F is defined by Equation (50) and is expressed only in terms of the (a, a^*) operators at $\epsilon = -1$, then in terms of the (b, b^*)-operators it acts on states localized in X_- as

$$A_F = \sum_{j,j'} \int \int A(\mathbf{v}, j, -1; \mathbf{v}', j', -1) \eta(j') \eta(j)^* b(\mathbf{v}, -j) b(\mathbf{v}', -j')^* d\rho(\mathbf{v}) d\rho(\mathbf{v}') \tag{59}$$

As follows from Equation (57), this operator can be written as

$$A_F = C - \sum_{j,j'} \int \int A(\mathbf{v}', -j', -1; \mathbf{v}, -j, -1) \eta(j) \eta(j')^* b(\mathbf{v}, j)^* b(\mathbf{v}', j') d\rho(\mathbf{v}) d\rho(\mathbf{v}') \tag{60}$$

where C is the trace of the operator A_F:

$$C = \sum_j \int A(\mathbf{v}, j, -1; \mathbf{v}, j, -1) d\rho(\mathbf{v}) \tag{61}$$

In general, C is some indefinite constant. It can be eliminated by requiring that all quantized operators should be written in the normal form or by using another prescriptions. The existence of infinities in the standard approach is the well known problem and we will not discuss it. Therefore we will always assume that if the operator A_F is defined by Equation (50) then in the case of anticommutation relations its action on states localized in X_- can be written as in Equation (60) with $C = 0$. Then, taking into account the properties of the kernels discussed above, we conclude that in terms of the (b, b^*)-operators the kernels of the operators $(m_{dS}v)_F$ change their sign while the kernels of the operators in Equation (44) remain the same. In particular, the operator $(-m_{dS}v_0)_F$ acting on states localized on X_- has the same kernel as the operator $(m_{ds}v_0)_F$ acting on states localized in X_+ has in terms of the a-operators. This implies that in Poincare approximation the energy of the states localized in X_- is positive definite, as well as the energy of the states localized in X_+.

Consider now how the spin operator changes when the a-operators are replaced by the b-operators. Since the spin operator is diagonal in the variable $\mathbf{v}$, it follows from Equation (60) that the transformed spin operator will have the same kernel if

$$s_i(j, j') = -\eta(j)\eta(j')^* s_i(-j', -j) \tag{62}$$

where $s_i(j, j')$ is the kernel of the operator s_i. For the z component of the spin operator this relation is obvious since s_z is diagonal in (j, j') and its kernel is $s_z(j, j') = j\delta_{jj'}$. If we choose $\eta(j) = (-1)^{(s-j)}$ then the validity of Equation (62) for $s = 1/2$ can be verified directly while in the general case it can be verified by using properties of $3j$ symbols.

The above results for the case of anticommutation relations can be summarized as follows. If we replace m_{dS} by $-m_{dS}$ in Equation (21) then the new set of operators

$$\mathbf{M}' = l(\mathbf{v}) + \mathbf{s}, \quad \mathbf{N}' = -iv_0\frac{\partial}{\partial\mathbf{v}} + \frac{\mathbf{s} \times \mathbf{v}}{v_0 + 1},$$

$$\mathbf{B}' = m_{dS}\mathbf{v} - i[\frac{\partial}{\partial\mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial\mathbf{v}}) + \frac{3}{2}\mathbf{v}] - \frac{\mathbf{s} \times \mathbf{v}}{v_0 + 1},$$

$$M'_{04} = m_{dS}v_0 - iv_0(\mathbf{v}\frac{\partial}{\partial\mathbf{v}} + \frac{3}{2}) \tag{63}$$

obviously satisfies the commutation relations (4). The kernels of these operators define quantized operators in terms of the (b, b^*)-operators in the same way as the kernels of the operators (20) define quantized operators in terms of the (a, a^*)-operators. In particular, in Poincare approximation the energy operator acting on states localized in X_- can be defined as $E' = M'_{04}/R$ and in this approximation it is positive definite.

At the same time, in the case of commutation relation the replacement of the (a, a^*)-operators by the (b, b^*)-operators is unacceptable for several reasons. First of all, if the operators $a(\mathbf{v}, j, \epsilon)$ satisfy the commutation relations (47), the operators defined by Equation (53) will not satisfy Equation (58). Also, the r.h.s. of Equation (60) will now have the opposite sign. As a result, the transformed operator M_{04} will remain negative definite in Poincare approximation and the operators (44) will change their sign. In particular, the angular momentum operators will no longer satisfy correct commutation relations.

We have shown that if the definitions (48) and (49) are replaced by (55) and (56), respectively, then the images of both sets of operators in Equation (20) and Equation (21) satisfy the correct commutation relations in the case of anticommutators. A question arises whether the new implementation in the Fock space is equivalent to the IR described in Section 5. For understanding the essence of the problem, the following very simple pedagogical example might be useful.

Consider a representation of the SO(2) group in the space of functions $f(\varphi)$ on the circumference $\varphi \in [0, 2\pi]$ where φ is the polar angle and the points $\varphi = 0$ and $\varphi = 2\pi$ are identified. The generator of counterclockwise rotations is $A = -id/d\varphi$ while the generator of clockwise rotations is $id/d\varphi$. The equator of the circumference contains two points, $\varphi = 0$ and $\varphi = \pi$ and has measure zero. Therefore we can represent each $f(\varphi)$ as a superposition of functions with the carriers in the upper and lower semi circumferences, S_+ and S_-. The operators A and B are defined only on differentiable functions. The Hilbert space H contains not only such functions but a set of differentiable functions is dense in H. If a function $f(\varphi)$ is differentiable and has the carrier in S_+ then $Af(\varphi)$ and $Bf(\varphi)$ also have the carrier

in S_+ and analogously for functions with the carrier in S_-. However, we cannot define a representation of the SO(2) group such that its generator is A on functions with the carrier in S_+ and B on functions with the carrier in S_- because a counterclockwise rotation on S_+ should be counterclockwise on S_- and analogously for clockwise rotations. In other words, the actions of the generator on functions with the carriers in S_+ and S_- cannot be independent.

In the case of finite dimensional representations, any IR of a Lie algebra by Hermitian operators can be always extended to an UIR of the corresponding Lie group. In that case the UIR has a property that any state is its cyclic vector *i.e.*, the whole representation space can be obtained by acting by representation operators on this vector and taking all possible linear combinations. For infinite dimensional IRs this is not always the case and there should exist conditions for IRs of Lie algebras by Hermitian operators to be extended to corresponding UIRs. This problem has been extensively discussed in the mathematical literature (see e.g., References [29–32]). By analogy with finite dimensional IRs, one might think that in the case of infinite dimensional IRs there should exist an analog of the cyclic vector. In Section 7 we have shown that for infinite dimensional IRs of the dS algebra this idea can be explicitly implemented by choosing a cyclic vector and acting on this vector by operators of the enveloping algebra of the dS algebra. Therefore if IRs are implemented as described in Section 5, one might think that the action of representation operators on states with the carrier in X_+ should define its action on states with the carrier in X_-.

8.2. *Example of Transformation Mixing Particles and Antiparticles*

We treated states localized in X_+ as particles and states localized in X_- as corresponding antiparticles. However, the space of IR contains not only such states. There is no rule prohibiting states with the carrier having a nonempty intersection with both, X_+ and X_-. Suppose that there exists a unitary transformation belonging to the UIR of the dS group such that it transform a state with the carrier in X_+ to a state with the carrier in X_-. If the Fock space is implemented according to Equations (48) and (49) then the transformed state will have the form

$$\varphi_F^{(-)} = \sum_j \int \varphi(\mathbf{v}, j) a(\mathbf{v}, j, -1)^* d\rho(\mathbf{v}) \Phi_0 \tag{64}$$

while with the implementation in terms of the (b, b^*) operators it should have the form (56). Since the both states are obtained from the same state with the carrier in X_+, they should be the same. However, they cannot be the same. This is clear even from the fact that in Poincare approximation the former has a negative energy while the latter has a positive energy.

Our construction shows that the interpretation of states as particles and antiparticles is not always consistent. We can only guarantee that this interpretation is consistent when we consider only states localized either in X_+ or in X_- and only transformations which do not mix such states. In quantum theory there is a superselection rule (SSR) prohibiting states which are superpositions of states with different electric, baryon or lepton charges. In general, if states ψ_1 and ψ_2 are such that there are no physical operators A such that $(\psi_2, A\psi_1) \neq 0$ then the SSR says that the state $\psi = \psi_1 + \psi_2$ is prohibited. The meaning of the SSR is now widely discussed (see e.g., Reference [45] and references therein). Since the SSR implies that the superposition principle, which is a key principle of quantum theory, is

not universal, several authors argue that the SSR should not be present in quantum theory. Other authors argue that the SSR is only a dynamical principle since, as a result of decoherence, the state ψ will quickly disappear and so it cannot be observable.

We now give an example of a transformation, which transform states localized in X_+ to ones localized in X_- and *vice versa*. Let $I \in SO(1,4)$ be a matrix which formally coincides with the metric tensor η. If this matrix is treated as a transformation of the dS space, it transforms the North pole $(0,0,0,0,x^4 = R)$ to the South pole $(0,0,0,0,x^4 = -R)$ and *vice versa*. As already explained, in our approach the dS space is not involved and in Sections 5–7 the results for UIRs of the dS group have been used only for constructing IRs of the dS algebra. This means that the unitary operator $U(I)$ corresponding to I is well defined and we can consider its action without relating I to a transformation of the dS space.

If $\mathbf{v}_L$ is a representative defined by Equation (17) then it is easy to verify that $I\mathbf{v}_L = (-\mathbf{v})_L I$ and, as follows from Equation (13), if ψ_1 is localized in X_+ then $\psi_2 = U(I)\psi_1$ will be localized in X_-. Therefore $U(I)$ transforms particles into antiparticles and *vice versa*. In Section 3 we argued that the notion of the spacetime background is unphysical and that unitary transformations generated by self-adjoint operators may not have a usual interpretation. The example with $U(I)$ gives a good illustration of this point. Indeed, if we work with the dS space, we might expect that all unitary transformations corresponding to the elements of the group SO(1,4) act in the space of IR only kinematically, in particular they transform particles to particles and antiparticles to antiparticles. However, in QFT in curved spacetime this is not the case. Nevertheless, this is not treated as an indication that standard notion of the dS space is not physical. Although fields are not observable, in QFT in curved spacetime they are treated as fundamental and single-particle interpretations of field equations are not tenable (moreover, some QFT theorists state that particles do not exist). For example, as shown in References [9–11,46,47], solutions of fields equations are superpositions of states which usually are interpreted as a particle and its antiparticle, and in the dS space neither coefficient in the superposition can be zero. This result is compatible with the Mensky's one [27] described in the beginning of this section. One might say that our result is in agreement with those in dS QFT since UIRs of the dS group describe not a particle or antiparticle but an object such that a particle and its antiparticle are different states of this object (at least in Poincare approximation). However, as noted above, in dS QFT this is not treated as the fact that the dS space is unphysical.

The matrix I belongs to the component of unity of the group SO(1,4). For example, the transformation I can be obtained as a product of rotations by 180 degrees in planes $(1,2)$ and $(3,4)$. Therefore, $U(I)$ can be obtained as a result of continuous transformations $exp[i(M_{12}\varphi_1 + M_{34}\varphi_2)]$ when the values of φ_1 and φ_2 change from zero to π. Any continuous transformation transforming a state with the carrier in X_+ to the state with the carrier in X_- is such that the carrier should cross X_0 at some values of the transformation parameters. As noted in the preceding section, the set X_0 is characterized by the condition that the standard Poincare momentum is infinite and therefore, from the point of view of intuition based on Poincare invariant theory, one might think that no transformation when the carrier crosses X_0 is possible. However, as we have seen in the preceding section, in variables (u_1, u_2, u_3, u_4) the condition $u_4 = 0$ defines the equator of S^3 corresponding to X_0 and this condition is not singular. So from the point of view of dS theory, nothing special happens when the carrier crosses X_0. We observe only either particles or antiparticles but not their linear combinations because Poincare approximation works

with a very high accuracy and it is very difficult to perform transformations mixing states localized in X_+ and X_-.

8.3. Summary

As follows from the above discussion, *objects belonging to IRs of the dS algebra can be treated as particles or antiparticles only if Poincare approximation works with a high accuracy*. As a consequence, *the conservation of electric, baryon and lepton charges can be only approximate*.

At the same time, our discussion shows that the approximation when one IR of the dS algebra splits into independent IRs for a particle and its antiparticle can be valid only in the case of anticommutation relations. Since it is a reasonable requirement that dS theory should become the Poincare one at certain conditions, the above results show that *in dS invariant theory only fermions can be elementary*.

Let us now consider whether there exist neutral particles in dS invariant theory. In AdS and Poincare invariant theories, neutral particles are described as follows. One first construct a covariant field containing both IRs, with positive and negative energies. Therefore the number of states is doubled in comparison with the IR. However, to satisfy the requirement that neutral particles should be described by real (not complex) fields, one has to impose a relation between the creation and annihilation operators for states with positive and negative energies. Then the number of states describing a neutral field again becomes equal to the number of states in the IR. In contrast with those theories, IRs of the dS algebra are implemented on both, upper and lower Lorentz hyperboloids and therefore the number of states in IRs is twice as big as for IRs of the Poincare and AdS algebras. Even this fact shows that in dS invariant theory there can be no neutral particles since it is not possible to reduce the number of states in IR. Another argument is that, as follows from the above construction, dS invariant theory is not C invariant. Indeed, C invariance in standard theory means that representation operators are invariant under the interchange of a-operators and b-operators. However, in our case when a-operators are replaced by b-operators, the operators (20) become the operators (63). Those sets of operators coincide only in Poincare approximation while in general the operators $M^{4\mu}$ in Equations (20) and (63) are different. Therefore a particle and its antiparticle are described by different sets of operators. We conclude that *in dS invariant theory neutral particles cannot be elementary*.

9.   dS Quantum Mechanics and Cosmological Repulsion

The results on IRs can be applied not only to elementary particles but even to macroscopic bodies when it suffices to consider their motion as a whole. This is the case when the distances between the bodies are much greater that their sizes. In this section we will consider the operators $M^{4\mu}$ not only in Poincare approximation but taking into account dS corrections. If those corrections are small, one can neglect transformations mixing states on the upper and lower Lorentz hyperboloids (see the discussion in the preceding section) and describe the representation operators for a particle and its antiparticle by Equations (20) and (63), respectively.

We define $E = M_{04}/R$, $\mathbf{P} = \mathbf{B}/R$ and $m = m_{dS}/R$. Consider the non-relativistic approximation when $|\mathbf{v}| \ll 1$. If we wish to work with units where the dimension of velocity is m/s, we should replace $\mathbf{v}$ by $\mathbf{v}/c$. If $\mathbf{p} = m\mathbf{v}$ then it is clear from the expressions for $\mathbf{B}$ in Equations (20) and (63) that $\mathbf{p}$

becomes the real momentum $\mathbf{P}$ only in the limit $R \to \infty$. Now by analogy with nonrelativistic quantum mechanics, we *define* the position operator $\mathbf{r}$ as $i\partial/\partial\mathbf{p}$. At this stage we do not have any coordinate space yet. However, if we assume that quasiclassical approximation is valid, we can treat $\mathbf{p}$ and $\mathbf{r}$ as usual vectors and neglect their commutators. Then as follows from Equation (20)

$$\mathbf{P} = \mathbf{p} + mc\mathbf{r}/R \quad H = \mathbf{p}^2/2m + c\mathbf{pr}/R \tag{65}$$

where $H = E - mc^2$ is the classical non-relativistic Hamiltonian and, as follows from Equations (63)

$$\mathbf{P} = \mathbf{p} - mc\mathbf{r}/R \quad H = \mathbf{p}^2/2m - c\mathbf{pr}/R \tag{66}$$

As follows from these expressions, in both cases

$$H(\mathbf{P}, \mathbf{r}) = \frac{\mathbf{P}^2}{2m} - \frac{mc^2\mathbf{r}^2}{2R^2} \tag{67}$$

The last term in Equation (67) is the dS correction to the non-relativistic Hamiltonian. It is interesting to note that the non-relativistic Hamiltonian depends on c although it is usually believed that c can be present only in relativistic theory. This illustrates the fact mentioned in Section 2 that the transition to non-relativistic theory understood as $|\mathbf{v}| \ll 1$ is more physical than that understood as $c \to \infty$. The presence of c in Equation (67) is a consequence of the fact that this expression is written in standard units. In non-relativistic theory c is usually treated as a very large quantity. Nevertheless, the last term in Equation (67) is not large since we assume that R is very large.

The result given by Equation (3) is now a consequence of the equations of motion for the Hamiltonian given by Equation (67). In our approach this result has been obtained without using dS space and Riemannian geometry while the fact that $\Lambda \neq 0$ should be treated not such that the spacetime background has a curvature (since the notion of the spacetime background is meaningless) but as an indication that the symmetry algebra is the dS algebra rather than the Poincare one. *Therefore for explaining the fact that $\Lambda \neq 0$ there is no need to involve dark energy or any other quantum fields.*

Another way to show that our results are compatible with GR is as follows. The well known result of GR is that if the metric is stationary and differs slightly from the Minkowskian one then in the non-relativistic approximation the curved spacetime can be effectively described by a gravitational potential $\varphi(\mathbf{r}) = (g_{00}(\mathbf{r}) - 1)/2c^2$. We now express x_0 in Equation (1) in terms of a new variable t as $x_0 = t + t^3/6R^2 - t\mathbf{x}^2/2R^2$. Then the expression for the interval becomes

$$ds^2 = dt^2(1 - \mathbf{r}^2/R^2) - d\mathbf{r}^2 - (\mathbf{r}d\mathbf{r}/R)^2 \tag{68}$$

Therefore, the metric becomes stationary and $\varphi(\mathbf{r}) = -\mathbf{r}^2/2R^2$ in agreement with Equation (67).

Consider now a system of two free particles described by the variables $\mathbf{p}_j$ and $\mathbf{r}_j$ ($j = 1, 2$). Define the standard non-relativistic variables

$$\mathbf{P}_{12} = \mathbf{p}_1 + \mathbf{p}_2 \quad \mathbf{q}_{12} = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/(m_1 + m_2)$$
$$\mathbf{R}_{12} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2) \quad \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2 \tag{69}$$

Then if the particles are described by Equation (65), the two-particle operators $\mathbf{P}$ and $\mathbf{E}$ in the non-relativistic approximation are given by

$$\mathbf{P} = \mathbf{P}_{12} + M\mathbf{R}_{12}/R, \quad E = M + \mathbf{P}_{12}^2/2M + \mathbf{P}_{12}\mathbf{R}_{12}/R \tag{70}$$

where

$$M = M(\mathbf{q}_{12}, \mathbf{r}_{12}) = m_1 + m_2 + \mathbf{q}_{12}^2/2m_{12} + \mathbf{q}_{12}\mathbf{r}_{12}/R \tag{71}$$

and m_{12} is the reduced two-particle mass. Comparing Equations (65) and (71), we conclude that M has the meaning of the two-body mass and therefore $M(\mathbf{q}_{12}, \mathbf{r}_{12})$ is the internal two-body Hamiltonian. As a consequence, in quasiclassical approximation the relative acceleration is given by the same expression (3) but now $\mathbf{a}$ is the relative acceleration and $\mathbf{r}$ is the relative radius vector.

The fact that two free particles have a relative acceleration is well known for cosmologists who consider the dS symmetry on classical level. This effect is called the dS antigravity. The term antigravity in this context means that the particles repulse rather than attract each other. In the case of the dS antigravity the relative acceleration of two free particles is proportional (not inversely proportional!) to the distance between them. This classical result (which in our approach has been obtained without involving dS space and Riemannian geometry) is a special case of the dS symmetry on quantum level when quasiclassical approximation works with a good accuracy.

For a system of two antiparticles the result is obviously the same since Equation (66) can be formally obtained from Equation (65) if R is replaced by $-R$. At the same time, in the case of a particle-antiparticle system a problem with the separation of external and internal variables arises. In any case the standard result can be obtained by using Equation (67).

Another problem discussed in the literature (see e.g., Reference [48] and references therein) is that composite particles in the dS theory are unstable. As shown in References [25,26], if we assume that non-relativistic approximation is valid but quasiclassical approximation is not necessarily valid then the result (71) can be generalized as

$$H_{nr} = \frac{\mathbf{q}^2}{2m_{12}} + V_{dS}, \quad V_{dS} = \frac{i}{R}(\mathbf{q}\frac{\partial}{\partial\mathbf{q}} + \frac{3}{2}) \tag{72}$$

where H_{nr} is the non-relativistic internal Hamiltonian and $\mathbf{q} = \mathbf{q}_{12}$. In spherical coordinates this expression reads

$$H_{nr} = \frac{q^2}{2m_{12}} + \frac{i}{R}(q\frac{\partial}{\partial q} + \frac{3}{2}) \tag{73}$$

where $q = |\mathbf{q}|$. The operator (73) acts in the space of functions $\psi(q)$ such that $\int_0^\infty |\psi(q)|^2 q^2 dq < \infty$ and the eigenfunction ψ_K of E_{nr} with the eigenvalue K satisfies the equation

$$q\frac{d\psi_K}{dq} = \frac{iRq^2}{m_{12}}\psi_K - (\frac{3}{2} + 2iRK)\psi_K \tag{74}$$

The solution of this equation is

$$\psi_K = \sqrt{\frac{R}{\pi}}q^{-3/2}exp(\frac{iRq^2}{2m_{12}} - 2iRKlnq) \tag{75}$$

and the normalization condition is $(\psi_K, \psi_{K'}) = \delta(K - K')$. The spectrum of the operator E_{nr} obviously belongs to the interval $(-\infty, \infty)$ and one might think that this is unacceptable. Suppose however that $f(q)$ is a wave function of some state. As follows from Equation (75), the probability to have the value of the energy K in this state is defined by the coefficient $c(K)$ such that

$$c(K) = \sqrt{\frac{R}{\pi}} \int_0^\infty exp(-\frac{iRq^2}{2m_{12}} + 2iRKlnq)f(q)\sqrt{q}dq \tag{76}$$

If $f(q)$ does not depend on R and R is very large then $c(K)$ will practically be different from zero only if the integrand in Equation (76) has a stationary point q_0, which is defined by the condition $K = q_0^2/2m_{12}$. Therefore, for negative K, when the stationary point is absent, the value of $c(K)$ will be exponentially small.

This result confirms that, as one might expect from Equation (71), the dS antigravity is not important for local physics when $r \ll R$. At the same time, at cosmological distances the dS antigravity is much stronger than any other interaction (gravitational, electromagnetic *etc.*). Since the spectrum of the energy operator is defined by its behavior at large distances, this means that in the dS theory there are no bound states. This does not mean that the theory is unphysical since stationary bound states in standard theory become quasistationary with a very large lifetime if R is large. For example, as shown in Equations (14) and (19) of Reference [48], a quasiclassical calculation of the probability of the decay of the two-body composite system gives that the probability equals $w = exp(-\pi\epsilon/H)$ where ϵ is the binding energy and H is the Hubble constant. If we replace H by $1/R$ and assume that $R = 10^{28} cm$ then for the probability of the decay of the ground state of the hydrogen atom we get that w is of order $exp(-10^{35})$ *i.e.*, an extremely small value. This result is in agreement with our remark after Equation (76).

In Reference [25] we discussed the following question. In standard quantum mechanics the free Hamiltonian H_0 and the full Hamiltonian H are not always unitarily equivalent since in the presence of bound states they have different spectra. However, in the dS theory there are no bound states, the free and full Hamiltonians have the same spectra and it is possible to show that they are unitarily equivalent. Therefore one can work in the formalism when interaction is introduced not by adding an interaction operator to the free Hamiltonian but by a unitary transformation of this operator. Such a formalism might shed light on understanding of interactions in quantum theory.

10. Discussion and Conclusions

The experimental fact that $\Lambda > 0$ might be an indication that for some reasons nature prefers dS invariance vs. AdS invariance ($\Lambda < 0$) and Poincare invariance ($\Lambda = 0$). A question arises whether there exist theoretical arguments explaining this fact. However, the majority of authors treat $\Lambda > 0$ as an anomaly since in their opinion AdS invariance or Poincare invariance are more preferable than dS invariance. One of the arguments is that dS symmetry does not have a supersymmetric generalization in contrast with the other two symmetries. Also, as argued by many authors (see e.g., Reference [49]), in QFT and its generalizations (string theory, M-theory *etc.*) a theory based on the dS algebra encounters serious difficulties. One of the reasons is that IRs of the Poincare and AdS algebras describing elementary particles are the lowest weight representations where the Hamiltonian is positive definite. On the other hand, as noted in the literature on IRs of the dS algebra, the spectrum of any representation operator of this algebra is symmetric relative to zero, *i.e.*, if $\lambda > 0$ is an eigenvalue then $-\lambda$ also is an eigenvalue. Polyakov [50] believes that for this reason "Nature seems to abhor positive curvature and is trying to get rid of it as fast as it can".

Let us discuss this objection in greater details. IRs of the Poincare and AdS algebras with the lowest weight are implemented on the upper Lorentz hyperboloid in the velocity space. Then the following question arises. If nature likes Poincare and AdS symmetries then how should one treat the fact that for any IR with the lowest weight $E_0 > 0$ there exists an IR with the highest weight $-E_0$ on the lower

hyperboloid? Should one declare such IRs with negative energies unphysical and throw them away? It is well known that the answer is "no" since local covariant objects needed for constructing QFT (e.g., Dirac fields) necessarily contain IRs with lowest and highest weight on equal footing. IRs with the lowest weight are associated with particles while IRs with the highest weight (and negative energies) are associated with antiparticles. Then the problem of negative energies is solved by second quantization after which both, the energies of particles and antiparticles become positive. So, Polyakov's objection should be understood such that only secondly quantized IRs with positive energies are physical. If this is true then IRs of the dS algebra are indeed unphysical.

In AdS and Poincare invariant theories, neutral particles are described as follows. One first construct a covariant field containing both IRs, with positive and negative energies. Therefore the number of states is doubled in comparison with the IR. However, to satisfy the requirement that neutral particles should be described by real (not complex) fields, one has to impose a relation between the creation and annihilation operators for states with positive and negative energies. Then the number of states describing a neutral field again becomes equal to the number of states in the IR.

As shown in Reference [27] and other papers, IRs of the dS algebra are implemented on both, upper and lower Lorentz hyperboloids and therefore the number of states in IRs is twice as big as for IRs of the Poincare and AdS algebras. A question arises whether such a description is physical since the dS theory should become the Poincare one when $R \to \infty$. Another question is how one should distinguish particles and antiparticles in the dS theory. In Reference [5] we argued that the only possible physical interpretation of IRs is such that they describe an object such that a particle and its antiparticle are different states of this object in cases when the wave function in the velocity space has a carrier on the upper and lower Lorentz hyperboloids, respectively. In Section 8 of the present paper we have shown that in dS invariant theory

- Each state is either a particle or antiparticle only when one does not consider transformations mixing states on the upper and lower hyperboloids. Only in this case additive quantum numbers such as electric, baryon and lepton charges are conserved. In particular, they are conserved if Poincare approximation works with a high accuracy.

In general, there is no superselection rule prohibiting states which are superpositions of a particle and its antiparticle. This shows that dS invariant theory implies a considerably new understanding of the notion of particles and antiparticles. In contrast with Poincare or AdS theories, for combining a particle and its antiparticle together, there is no need to construct a local covariant object since they are already combined at the level of IRs.

We believe that this is an important argument in favor of dS symmetry. Indeed, the fact that in AdS and Poincare invariant theories a particle and its antiparticle are described by different IRs means that they are different objects. Then a problem arises why they have the same masses and spins but opposite charges. In QFT this follows from the CPT theorem which is a consequence of locality since *we construct* local covariant fields from a particle and its antiparticle with equal masses. A question arises what happens if locality is only an approximation: In that case the equality of masses, spins *etc.*, is exact or approximate? Consider a simple model when electromagnetic and weak interactions are absent. Then the fact that the proton and the neutron have the same masses and spins has nothing to do with locality; it is only a

consequence of the fact that the proton and the neutron belong to the same isotopic multiplet. In other words, they are simply different states of the same object—the nucleon. We see, that in dS invariant theories the situation is analogous. The fact that a particle and its antiparticle have the same masses and spins but opposite charges (in the approximation when the notions of particles, antiparticles and charges are valid) has nothing to do with locality or non-locality and is simply a consequence of the fact that they are different states of the same object since they belong to the same IR.

The non-conservation of the baryon and lepton quantum numbers has been already considered in models of Grand Unification but the electric charge has been always believed to be a strictly conserved quantum number. In our approach all those quantum numbers are not strictly conserved because in the case of de Sitter symmetry transitions between a particle and its antiparticle are not prohibited. The experimental data that these quantum numbers are conserved reflect the fact that at present Poincare approximation works with a very high accuracy. As noted in Section 2, the cosmological constant is not a fundamental physical quantity and if the quantity R is very large now, there is no reason to think that it was large always. This completely changes the status of the problem known as "baryon asymmetry of the Universe".

Another consequence of our consideration is that *in dS invariant theory only fermions can be elementary and there are no neutral elementary particles*. The latter is obvious from the fact that there is no way to reduce the number of states in the IR. One might think that theories where the photon (and also the graviton and the Higgs boson, if they exist) is not elementary, cannot be physical. However, several authors discussed models where the photon is composite; in particular, in AdS theory it might be a composite state of Dirac's singletons [51–53]. An indirect confirmation of our conclusions is that all known neutral particles are bosons.

One might say that a possibility that only fermions can be elementary is not attractive since such a possibility would imply that supersymmetry is not fundamental. There is no doubt that supersymmetry is a beautiful idea. On the other hand, one might say that there is no reason for nature to have both, elementary fermions and elementary bosons since the latter can be constructed from the former. A well know historical analogy is that the simplest covariant equation is not the Klein-Gordon equation for spinless fields but the Dirac and Weyl equations for the spin 1/2 fields since the former is the equation of the second order while the latter are the equations of the first order.

We see that theories based on dS symmetry on one hand and on AdS and Poincare symmetries on the other, are considerably different. The problem with the interpretation of IRs of the dS algebra has a clear analogy with the fact that the Dirac equation has solutions with both, positive and negative energies. As already noted, in modern quantum theory the latter problem has been solved by second quantization. This is possible because IRs of the Poincare and AdS algebras with positive and negative energies are independent of each other. On the contrary, one IR of the dS algebra contains states which can be treated as particles and antiparticles only in some approximations while in general, as we have seen in Section 8, the second quantization formalism is not sufficient for splitting all possible states into particle and antiparticle ones.

Depending on their preferences, physicists may have different opinions on this situation. As noted above, many physicists treat the dS theory as unphysical and the fact that $\Lambda > 0$ as an anomaly. However, as noted in Section 1, the accuracy of experimental data is such that the possibilities $\Lambda = 0$ or $\Lambda < 0$

are practically excluded and this is an indication that the dS theory is more relevant than the Poincare and AdS ones. We believe that for understanding which of those possibilities are "better" (if any), other approaches to quantum theory should be investigated. For example, in a quantum theory over a Galois field [54–56], Poincare symmetry is not possible and even in the AdS case one IR describes a particle and its antiparticle simultaneously. In Galois fields the notions of "less than", "greater than" and "positive and negative numbers" can be only approximate. In particular, there can be no IRs over a Galois field with the lowest weight or highest weight. As a consequence, in this theory, as well as in standard dS theory, such quantum numbers as the electric, baryon and lepton charges can be conserved only in some approximations and there are no neutral elementary particles. However, we did not succeed in proving that only fermions can be elementary since in Galois fields not only Equation (54) is possible but $\eta(j)\eta(j)^* = -1$ is possible too.

A possible approach for seeking new theories might be based on finding new symmetries such that known symmetries are special cases of the new ones when a contraction parameter goes to zero or infinity (see e.g., the famous paper [57] entitled "Missed Opportunities"). For example, classical theory is a special case of quantum one when $\hbar \to 0$ and non-relativistic theory is a special case of relativistic one when $c \to \infty$. From this point of view, dS and AdS symmetries are "better" than Poincare one since the latter is a special case of the former when $R \to \infty$. A question arises whether there exists a ten-dimensional algebra, which is more general than the dS or AdS one, *i.e.*, the dS or AdS algebra is a special case of this hypothetical new algebra when some parameter goes to zero or infinity. As noted in Reference [57], the answer is "no" since the dS and AdS algebras are semisimple. So one might think that the only way to extend the de Sitter symmetries is to consider higher dimensions and this is in the spirit of modern trend.

However, if we consider a quantum theory not over complex numbers but over a Galois field of characteristic p then standard dS and AdS symmetries can be extended as follows. We require that the operators M^{ab} satisfy the same commutation relations but those operators are considered in spaces over a Galois field. Such operators implicitly depend on p but they still do not depend on R. This approach, which we call quantum theory over a Galois field (GFQT), has been discussed in details in References [26,36,54–56]. GFQT is a more general theory than standard one since the latter is a special case of the former when $p \to \infty$. In the approximation when p is very large, GFQT can reproduce all the standard results of quantum theory. At the same time, GFQT is well defined mathematically since it does not contain infinities. While in standard theory the dS and AdS algebras are "better" than the Poincare algebra from aesthetic considerations (see the discussion in Section 4), in GFQT there is no choice since the Poincare algebra over a Galois field is unphysical (see the discussion in References [26,54–56]).

In view of the above discussion, it seems natural to express all dimensionful quantities in terms of $(c, \hbar, R)$ rather than $(c, \hbar, G)$ since the former is a set of parameters characterizing transitions from higher symmetries to lower ones. Then a reasonable question is why the quantity G is so small. Indeed, in units $\hbar = c = 1$, G has the dimension $length^2$ and so one might expect that it should be of order $R^2 = 3/\Lambda$. So again the disagreement is more that 120 orders of magnitude and one might call this the gravitational constant problem rather than the cosmological constant problem. As noted above, in standard theory a reasonable possibility is that $G\Lambda$ is of order unity. However, in GFQT we have a parameter p. In

Reference [58] we have described our hypothesis that G contains a factor $1/lnp$ and that is why it is so small.

In the present paper we have shown that the well known classical result about the cosmological repulsion in the dS space is a special case of quantum theory with the dS algebra as the symmetry algebra when no interaction between particles is introduced and quasiclassical approximation is valid. Our result has been obtained without using the notions of spacetime background, Riemannian geometry and dS QFT. This result shows that for explaining the fact that $\Lambda > 0$ there is no need to involve dark energy or other fields.

The main achievements of modern theory have been obtained in the approach proceeding from spacetime background. In quantum theory this approach is not based on a solid mathematical basis and, as a consequence, the problem of infinities arises. While in QED and other renormalizable theories this problem can be somehow circumvented, in quantum gravity this is not possible even in lowest orders of perturbation theory. Mathematical problems of quantum theory are discussed in a wide literature. For example, in the well known textbook [59] it is explained in details that interacting quantized fields can be treated only as operatorial distributions and hence their product at the same point is not well defined. One of ideas of the string theory is that if a point (a zero-dimensional object) is replaced by a string (a one-dimensional object) then there is hope that infinities will be less singular.

For the majority of physicists the fact that GR and quantum theory describe many experimental data with an unprecedented accuracy is much more important than a lack of mathematical rigor, existence of infinities and that the notion of spacetime background is not physical. For this reason physicists do not wish to abandon this notion. As one of the consequences, the cosmological constant problem arises and it is now believed that dark energy accounts for more than 70% of the total energy of the Universe. There exists a vast literature where different authors propose different approaches and some of the authors claim that they have found the solution of the problem. Meanwhile the above discussion clearly demonstrates that the cosmological constant problem (which is often called the dark energy problem) is a purely artificial problem arising as a result of using the notion of spacetime background while this notion is not physical.

The conclusion that the cosmological constant problem does not exist has been made earlier by different authors from different considerations but probably all those authors accepted approaches based on spacetime background. For example, Bianchi and Rovelli in their paper [6] entitled "Why all these prejudices against a constant?" discussed this problem in the framework of classical GR. They argue that since the most general form of Einstein equations contains both, G and Λ, there is no reason to believe that nature prefers a special case $\Lambda = 0$. In their approach, both G and Λ are fundamental physical quantities. In our paper we argue that none of this quantity is fundamental and this is in agreement with recent intensive investigation of a possibility that gravity in not fundamental but emergent.

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only angular momenta are fundamental physical quantities [60]. I am also grateful to Volodya Netchitailo and Teodor Shtilkind for numerous discussions and to Jim Bogan for an interesting correspondence and explaining me the experimental status of the cosmological constant problem.

References and Notes

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