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Symmetry, Symmetry Breaking and Topology

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Abstract: The ground state of a system with symmetry can be described by a group G . This symmetry group G can be discrete or continuous. Thus for a crystal G is a finite group while for the vacuum state of a grand unified theory G is a continuous Lie group. The ground state symmetry described by G can change spontaneously from G to one of its subgroups H as the external parameters of the system are modified. Such a macroscopic change of the ground state symmetry of a system from G to H correspond to a “phase transition”. Such phase transitions have been extensively studied within a framework due to Landau. A vast range of systems can be described using Landau’s approach, however there are also systems where the framework does not work. Recently there has been growing interest in looking at such non-Landau type of phase transitions. For instance there are several “quantum phase transitions” that are not of the Landau type. In this short review we first describe a refined version of Landau’s approach in which topological ideas are used together with group theory. The combined use of group theory and topological arguments allows us to determine selection rule which forbid transitions from G to certain of its subgroups. We end by making a few brief remarks about non-Landau type of phase transition.

Keywords: symmetry breaking; group theory; topology

1. Introduction

In many branches of physics the macroscopic symmetry properties of the ground state (vacuum state) of a system can be different from that of its Hamiltonian. When this happens we have a spontaneous symmetry breaking phase transitions. It is also observed that as the external conditions of a system change, such as its temperature and pressure, the ground state symmetry can get further modified. Typically if the Hamilton has symmetry group G then the ground state symmetry group is one of the subgroups H of G . It is of interest to understand how to model such transitions and to determine if a

broken ground state symmetry can be any one of the possible subgroups H_i of G or must obey selection rules. A very useful phenomenological approach for tackling this problem and for understanding how spontaneous symmetry breaking transitions can occur which we will use was proposed by Landau [1].

The idea of “spontaneous symmetry breaking” first appeared in solid state physics [2] but now forms a major pillar of fundamental high energy physics [2]. Landau’s approach was simple. It involved a model for the Free Energy of the system which seemed to be universally applicable, and led to experimentally testable predictions. There is, however, growing evidence [4,5] of systems which cannot be described within the framework of Landau [4–6].

In this brief survey our focus is on highlighting qualitative features of Landau theory obtained by adding on topological ideas to the group theoretic framework of the approach [7]. With topological input we will show how we can understand selection rules for “allowed” and “not allowed” symmetry breaking transitions that seem to be present. Three examples illustrating the topological approach are described. This is the main part of the review. We then briefly comment on a specific “Topological Phase transition” where the phase transition is not of the Landau type. A Landau description deals with a thermodynamic equilibrium situation, uses group theory and has a free quasiparticle picture for the excitations present in the broken symmetry phase. For the quantum hydrodynamic phase of graphene, which we consider, none of these conditions are present. Here the system is not in equilibrium and its excitations in the turbulent phase are not expected to be described by free quasiparticles but by more complicated excitations. Also a direct application of the topological ideas of [7] to this case is not possible as we explain later on. The example of graphene is included to show the richness of the concepts of a phase transition in the context of a currently active area of research. In this case the flow characteristics of the system change suddenly.

Our aim is to provide a broad picture which is self-contained. For this reason we provide intuitive justification of what is being done. Technical details are often not presented.

The first step taken by Landau in setting up his phenomenological model for the Free Energy was to introduce the idea of an “order parameter” for the system. The order parameter was to be zero when the ground state of the system did not break the symmetry of its Hamiltonian and was to be non-zero if the ground state broke the symmetry of the Hamiltonian. It was thus a measure of the “order” present in the system. Thus a zero value for the order parameter meant that the symmetry of the ground state was the same as the symmetry of the Hamiltonian while a non-zero value meant that symmetry breaking had occurred. A simple example illustrating this idea is provided by a model for ferromagnetism. A ferromagnetic system can be modeled by a Hamiltonian made out of elementary magnets which are placed at lattice sites. The elementary magnets interact in a rotationally invariant way. The Hamiltonian is thus rotationally invariant. The order parameter is taken to be the average magnetisation density. When this magnetisation is zero the ground state is not ferromagnetic and is rotationally invariant. When the magnetisation density is non-zero we have a ferromagnet which has directionality and is thus not rotationally invariant and breaks the rotational symmetry of the Hamiltonian. We have spontaneous symmetry breaking.

The next key idea introduced by Landau was to suppose that the order parameter was a smooth function of the external macroscopic parameters (such as the temperature and pressure) in the neighbourhood of a symmetry breaking transition. Such transitions are called second order phase transitions. For such transitions even though the symmetry changes abruptly for certain critical values of,

say the temperature, the order parameter does not change abruptly but is assumed to change smoothly from a non-zero value to zero. This also means that near a symmetry breaking transition the order parameter is small. Noting this fact Landau supposed that the Free Energy, near a phase transition, was a low order polynomial function of the order parameter. Instead of an arbitrary function of the order parameter the Free Energy was now reduced to be a low order polynomial in the order parameter. The coefficients of this Landau polynomial were taken to be smooth functions of the external macroscopic parameters.

Once a model for the Free Energy was constructed the equilibrium configuration of the system could be determined by minimising the Free Energy. The equilibrium value for the order parameter was then simply the value which minimised the Free Energy. These values were thus critical points of the Free Energy as a function of the order parameter. Each critical point, if it was a minimum, was a potential equilibrium state of the system with a value for the order parameter fixed in terms of the coefficients of the Landau polynomial. Thus a prediction regarding the way the order parameter would behave near a phase transition could be made and tested against experiment. It could also be checked that spontaneous symmetry breaking was allowed [1]. There is usually a natural system specific choice for the order parameter. For instance for a ferromagnet the natural order parameter was the average magnetisation density. A general framework for describing order parameters for a system with symmetry can be given, as we will see.

Determining equilibrium configurations for a system is thus reduced to the technical problem of determining the minima of the Free Energy. This in general is a difficult problem. The problem can be simplified by restricting the Free Energy, near a phase transition, to be quartic polynomials in the order parameter. This is reasonable as the order parameter is expected to be small near a transition point. Even after this simplification, for a general order parameter with group transformation properties, the problem is not easy. What we will show is that general qualitative features of the system can be found without ever actually finding these solutions! This is done by combining group theory methods with topological ideas. Such an approach does not give the exact functional form for the order parameter but gives the allowed symmetry breaking patterns the system can have. By a symmetry breaking pattern we mean that associated with a minimum state of a certain symmetry there must be saddle points and maxima with prescribed symmetries. Thus the qualitative shape of the allowed equilibrium configurations for Free Energy surface are determined by the approach.

Let us return to our simple example of spontaneous symmetry breaking, namely the case of a ferromagnet. Here as we said there is a natural order parameter namely the magnetisation density. It is non-zero in the ferromagnetic phase and is zero when the temperature is increased beyond the Curie temperature [1]. Let us be a bit more specific. In our example we introduced a lattice. This was to model a crystalline magnetic system in a simple way. We then introduced elementary magnets at each lattice point. These magnets could point in different directions. We now suppose that the Hamiltonian for such a system can be taken to be function of the spin variables which interact only with their nearest neighbours in a rotational invariant way. Since the system is a ferromagnet below the Curie temperature the ground state of the system is not rotationally invariant. The ferromagnet has a certain directionality given by the macroscopic average spin vector of the system while above the Curie temperature the system is no longer ferromagnetic and has rotational invariance. Thus, as we said before, we have an example of a

system with spontaneous symmetry breaking. The rotational symmetry of the Hamiltonian is broken by the ferromagnetic ground state when the temperature is below the Curie temperature. The point we want to stress is that in constructing the model we needed to make two general and one specific choice. The two general choices needed were to find a way to represent the crystalline material and to properly describe the physical variable (the elementary magnets) relevant for the system. The specific choice that had to be made was to construct a Hamiltonian. We next turn to setting up a general framework where these features are present.

Landau Theory: Topological and Group Theoretical Ideas

We can proceed to set up a general framework for Landau theory. Let us again summarise what needs to be done: First we need to model the physical space corresponding to the system and then we need to model the specific order associated with symmetry of the system. Physical space could be taken, for instance, to be a smooth three dimensional manifold when we are modelling a high energy systems or it could be taken to be a lattice of points when we are modelling a crystal.

The second feature needed is to give a simple description of the specific physical variables relevant for describing the physical system of interest. Thus for a magnetic system a natural choice was to introduce elementary magnets at lattice points. We might further simplify the model by assuming that elementary magnet at all lattice sites are the same. Classically such a elementary magnetic system in three dimensions can be thought of as a collection of spin vectors (the elementary magnets), each located at a lattice site, of fixed length pointing in a certain direction of three dimensional space. The thermodynamic average of these spins then gives the magnetisation. This overall average is a natural macroscopic system order parameter while the spin at a lattice site can be taken to be a local order parameter. In this case the local order parameter can be described as a point on the surface of a sphere. The radius of the sphere represents the spin length while the join of the point on the sphere to the centre of the sphere gives the spin direction. Thus to describe this system we find we need a lattice, representing the crystal, and we need to introduce an order parameter space at each lattice point to describe the spin. The classical order parameter space we found was the surface of a sphere. We turn to the general situation of a system with group symmetry.

For a system with group symmetry we will take the order parameter to be a vector and the order parameter space to be a space associated with a linear representation of the group [8]. This immediately relates order parameter with symmetry and group theory. A linear representation of a group means that each group element is replaced by a matrix and the combination rule of the group is captured by the rules of matrix multiplication. These matrices are said to form a faithful representation of the group if the unit element of the group is uniquely represented by the diagonal unit matrix. If more than one matrix represents the unit element of the group we have a group homomorphism. If the matrices are $n \times n$ then the representation is said to be n dimensional. For such a representation the matrices act on vectors which are n dimensional. This is the vector space we choose for our order parameter space. It then allows us to consider systems which have symmetries described in terms of groups.

A group theory interpretation for the spin magnet system order parameter space we considered can be given. As we saw the order parameter space was the surface of a sphere. In terms of group theory the surface of a sphere corresponds to the coset space $\frac{SO(3)}{SO(2)}$. This can be understood as follows: the

rotational symmetry group is given by $SO(3)$. Rotating a classical spin vector fixed at one point leads to the end point of the vector to move on the surface of a sphere as rotations do not change the length of the vector. However not all rotations change the direction of the vector. Rotations of the vector about its own axis keep it fixed. These special rotations correspond to the group $SO(2)$. Hence points on the sphere are simply points of the coset $\frac{SO(3)}{SO(2)}$.

We now turn to the general situation using these ideas and proceed to show how to describe the order parameter of a crystal with symmetry group G and also describe how to symmetry breaking can be formulated. This means constructing a procedure for understanding how the model approach allows the group G to abruptly change to one of its subgroups H_i .

Let us then look at a crystal which has a certain symmetry group G , where G is a finite group of order $n(G)$. The order of the group is simply the number of elements present in G . The statement that the density function $\rho(x)$ of the crystal is invariant under transformations T_1, T_2, \dots belonging to the finite group G means that: $\rho(x) = \rho(T_k(x))$

We will show how the density function, due to its invariance under G , can be completely described as a vector with at most $n(G)$ components, where $n(G)$ is the order of the group. We also show how Landau theory constructed for this system allows us to understand why an abrupt change of the symmetry of the density function can happen. The density function change near such a transition is small and a quartic Landau polynomial for the system near a transition point is powerful enough to address symmetry breaking issues. It is also experimentally known that not all subgroups H of a given group G are realized, *i.e.*, selection rules seem to operate in these symmetry breaking transitions. Our goal is to understand this fact from a general point of view. To understand this problem we will need to use the elegant topological arguments given by Michel and Mozrzymas [7]. The topological idea used in [7] was topological Morse Theory.

To proceed, we need to introduce relevant ideas about topology and manifolds. We start with topology. We need to use Morse Theory. Morse Theory in its simplest form, places lower bounds on the number of critical point of index k that a real valued function on a manifold must have in terms of numbers called Betti numbers [9]. Betti numbers are easy to calculate and are fixed by the topological properties of the manifold. Intuitively the Betti numbers count the number of different dimensional holes present in a manifold. Critical points of a real valued function are points where a function has a maximum, minimum or a saddle points. The index k of a critical point tells us that in the neighbourhood of the critical point there are k Euclidean directions in which the function decreases in value. Thus Morse theory places constraints on the number and type of maxima, minima and saddle points that a real valued function (satisfying certain conditions) on a manifold can have. By choosing the real valued function to be, for instance, the Free Energy in thermodynamics or the Energy in mechanics consequences that of these topological constraints can be explored.

Next we recall that a manifold is a space which locally looks like Euclidean space. A simple example is the map of the earth in terms of an atlas. Each map in the atlas is a local description of a part of the earth.

A n dimensional manifold thus has a local description in terms of n Euclidean coordinates (x_1, \dots, x_n) and a real valued function can be written locally as $f(x_1, \dots, x_n)$. These Euclidean coordinates can be chosen in different ways. For a manifold of dimension n , if $k = n$ the critical point of a function

$f(x_1, \dots, x_n)$ is a maxima as the function decreases in value in all of its n directions while if $k = 0$ the corresponding critical point is a minima as the function increases in all of its n directions. In between values for k represent saddle points.

The use of Morse theory in Landau theory is rather natural as there is a natural real valued function, namely the Free Energy, whose critical points, if minima, represent possible equilibrium configurations of the system [9]. The thermodynamic Free Energy function $F(\rho(x), T, P)$ depends on T, P the temperature and pressure of the environment in which the crystal is placed and on $\rho(x)$ the density of the crystal. The equilibrium density $\rho(x)$ is determined by minimising F .

Even though $\rho(x)$ appears to be a function with uncountable degrees of freedom, the fact that it is invariant under a finite group G reduces its degrees of freedom as we now demonstrate. Indeed, as we now show, its degrees of freedom are the same as those of a Euclidean vector of dimension equal to the order of the group G . The equilibrium value of a density function with finite group symmetry G is thus fixed by determining the coordinates of a $n(G)$ dimensional vector where $n(G)$ is the order of the group G .

Let us formulate our symmetry breaking problem in group theoretical terms and then show how group theory reduces the degrees of freedom of the density function. Consider a crystal with finite symmetry group G for and $T > T_0$ and symmetry group H for $T < T_0$ where H is a subgroup of G . We want to determine if all subgroup of G are a possible “broken” symmetry group for the crystal or if selection rules operate. In view of our assumptions we write, for T close to T_0 , $\rho(x) = \rho_0(x) + \delta\rho(x)$ where $\rho_0(x)$ is invariant under G and $\delta\rho(x)$ is invariant under H . Thus is $\delta\rho(x) = 0$, the symmetry group of $\rho(x)$ is G . From group theory [8] we know that if G contains $n(G)$ elements $g_1, \dots, g_{n(G)}$ then a representation of G can be constructed by regarding the group elements as matrices $D(g)$ which describe transformations on a set of $n(G)$ linearly independent functions $\phi_1(x) \dots$ (characters of the group) so that $g\phi_k(x) = \sum D_{k,j}(g)\phi_j(x)$, $g \in G$. We also note that the set of functions $\phi(x)$ form a basis and we have a character expansion [8] $\delta\rho(x) = \sum C_i\phi_i(x)$.

Thus the coefficients C_1, \dots can be used as a finite set of coordinates to describe $\delta\rho(x)$ in the basis defined by the linearly independent functions $\phi(x)$. This demonstrates the enormous simplification that comes from the fact that the density has a finite symmetry group. Furthermore we can regard the action of the group G on $\delta\rho$ in two ways, namely $g\delta\rho = \delta\rho'(x)$ because the coordinates change under G or because the basis functions ϕ change under G . For the representation of a finite group the following theorems hold [8]

Theorem A Every representation of a finite group G is equivalent to a unitary representation.

Theorem B Every real representation of a finite group is equivalent to an orthogonal representation.

These theorems tell us that if we consider only real representations then the space of coordinates can be chosen to be an $n(G)$ dimensional Euclidean space. A theorem in group theory tells us that the representation of the group G of dimension $n(G)$ is reducible. This means that by suitable transformations each matrix representing a group element can be brought to a block diagonal form. The block diagonal entries can be used to represent the group: they are the irreducible representations. As a concrete example we consider a cubic crystal. This has symmetry group O_h which has order 48. The irreducible representations of this group have dimensions 1, 2 or 3. We will use this group to illustrate the general approach of combining group theoretical and topological ideas.

First we simplify the problem by using physical arguments. We start, following Landau, by restricting ourselves to fourth order polynomials in the order parameter, drop the cubic terms and further simplify our analysis by restricting ourselves to the three dimensional irreducible representation of the group O_h . The cubic term is dropped as we require the system to be invariant under sign change of density. The restriction to the three dimensional irreducible representation comes from the belief that the physical system settles down to a specific irreducible representation of a group and continues to be confined to the starting irreducible representations under small changes in its external environmental parameters. This means the order parameter space can be taken to be a three dimensional vector space.

These physics-based assumptions dramatically reduce the problem from one where the vector space of order parameter had dimension 48 to one where it has dimension 3! The mathematics of group theory tells us that one needs 48 bases functions (group characters) [8] to describe the group action. However the physics-based assumption suggests that it is enough to restrict base vectors to those belonging to the irreducible representations of the group assumed to be relevant. In our example this means working with only the three dimensions irreducible representation.

We also note the Free energy being invariant under O_h means that $F[D(g)C] = F[C]$ Here $D(g)$ is now a three dimensional representation of O_h and $g \in O_h$. The equilibrium density is determined by finding a three dimensional vector C_0 such that $F'[C_0] = 0$ Such a vector represents a critical point of $F[C]$. A further physical constraint we make is to require that $F[C]$ goes to zero at infinity—otherwise we would get an infinite value for the Free Energy. This converts the three dimensional space, topologically, to a three sphere.

Replacing the three dimensional Euclidean space by a three sphere has important consequences. It immediately leads to Morse constraints known as Morse inequalities which we now describe. Morse inequalities relate the number of critical points of index k a real valued function with non-degenerate, isolated critical points can have to the topological properties of the manifold encoded in numbers called Betti numbers. Betti numbers represent the dimension of Homology groups of the manifold. These groups intuitively count simple topological properties. Thus b_0 counts the number of connected pieces that make up the manifold, b_1 counts the number of two dimensional holes present in the manifold while b_3 counts the number of four dimensional holes present in the manifold and so on [9]. For our case the manifold is a three sphere and the Morse inequalities are:

$$\begin{aligned}n_0 &\geq b_0 \\n_1 - n_0 &\geq b_1 - b_0 \\n_2 - n_1 + n_0 &= b_2 - b_1 + b_0 \\n_3 - n_2 + n_1 + n_0 &= 0 \quad (\text{Euler Characteristic})\end{aligned}$$

where n_0 represent the number of minima, n_1 represent the number of saddle points and n_2 represent the number of maxima found by solving $F'[C_0] = 0$. Recall the general n_k represents the number of critical points of index k that we get by solving $F'[C_0] = 0$. The index k simply tells us that the real valued function whose critical points we are considering decreases in k Euclidean directions as one moves away from the critical point of index k . The right hand side of the Morse inequalities are fixed by topology. They are fixed by the Betti numbers b_3, b_2, b_1, b_0 which are topological invariants of the manifold, in our case the manifold is a three sphere [9]. The Betti numbers for a three sphere are:

$b_3 = 1, b_2 = 0, b_1 = 0, b_0 = 1$. Intuitively these numbers tell us that a three sphere has one component, that it is simply connected and that it encloses a four dimensional hole. The Betti number represents the dimension of the Homology group of the manifold [9]. The Homology groups capture topological features of a manifold in terms of groups. The study of Homology Groups is the subject matter of Algebraic Topology[9]. A manifold of dimension n has a maximum of n non-vanishing Homology groups. Calculating Homology groups is relatively simple.

Morse theory thus places constraints on the pattern of critical points a real valued function with non-degenerate, isolated critical points can have. Such a function is called a Morse function. At this stage we have a link between critical points and topology. Critical points of a function such as the Free Energy determine equilibrium properties. However we are still missing the symmetry and group theory element of our problem. The symmetry features are hidden in the number of critical point present. These numbers are constrained by group theory as we will show. Once this feature is properly introduced we can tackle our symmetry breaking problem as we will see.

We continue to work away in order to use the topological information contained in the Morse inequalities in our analysis of phase transitions. As stressed the equilibrium properties of a system are related to the critical points of the Free Energy function which we have assumed to be a quartic polynomial. We now assume the Landau polynomial is a Morse function. Our next step is to introduce group theoretical ideas relevant for our problem. We have stated that this information is hidden in the number of critical points present. We need to make this link explicit.

2. Symmetry Breaking Patterns and Selection Rules in a Crystal

Let us proceed to explain how group theory and the number of critical points of arbitrary index k are linked.

The key observation linking the number of critical points of index k to group theory is the following. Critical points correspond to solution of $F'[C_0] = 0$. Group theory enters at this stage. If C_0 is a solution, *i.e.*, it represents a critical point of certain index, then gC_0 is also a critical point with index k . This is because $F[C_0] = F[gC_0]$ so that if C_0 minimises so does gC_0 . Thus the number of critical points of index k present are exactly the number of distinct vectors we get by acting on C_0 with g . The maximum number of vectors that are possible are $n(G)$ the order of the group G . In this case the only element of G that does not change the starting vector is the identity element. Thus the broken symmetry group is, in this case, the identity group. This means the original symmetry G has been completely broken. Remember the symmetry of C_0 determines the symmetry of the density function. The minimum number is 1. In this case no elements of G gives a new vector, *i.e.*, the solution vector is invariant under G . This means the symmetry of the density function remains G , there is no symmetry breaking.

Let us repeat this important point: If there was no residual symmetry, *i.e.*, the distortion due to environment changes have completely broken the original crystalline symmetry, then the number of critical points equivalent to C_0 would be equal to the order of the group O_h . However, assume there is residual broken symmetry of the crystal after distortion which is described by a subgroup H , of G . For instance if $G = O_h$ is the symmetry of a cubic crystal, then the number of critical points must be the integer $\frac{n(O_h)}{n(H)}$, where $n(O_h)$ is the order of the unbroken symmetry group O_h , $n(H)$ is the order of the broken symmetry group H and H is a subgroup of O_h . This follows from a theorem of Lagrange [8]

and is a consequence of the fact that any element h of the residual broken symmetry group H does not change the density (as it is a symmetry). This means h acting on the vector C_0 (which determines the crystal density) keeps it unchanged, *i.e.*, $hC_0 = C_0$, which means no new critical points are generated by elements of H . Thus the number of group elements $n(O_h)$ is partitioned into units (boxes) each of size $n(H)$. The number of boxes is $\frac{n(O_h)}{n(H)}$. The action of any element h of H simply permutes elements in a box while for elements of G that are not in the subgroup H when they act on elements of a box move them to a different box. Note the smaller the size of the residual subgroup the bigger is the ratio. Thus the number of critical points is determined by group theory. We repeat: the number of equivalent maxima, minima or saddle points cannot be arbitrary but are constrained by group theory. This also means that if we add up all the critical points with all allowed values for the index (*i.e.*, allow k to vary between n and 0) we would have counted the total number of solutions of the equation $F'[C_0] = 0$, since each solution is locally either a maximum, minimum or saddle point of $F[C]$. But there are also constraints on these numbers from topology. We thus have to solve the problem by producing lists of allowed values for the number of critical points which satisfy all these constraints. This list will contain not just allowed values for the number of minima but would have the number of maxima and saddle points which all together form an allowed set of integers that satisfy the topological constraints and are compatible with the number of solutions allowed for the equation $F'[C_0] = 0$. Since each integers has group theory information in them, each compatible set of integers will represent an allowed symmetry breaking pattern. Not only will we have determined the allowed symmetry breaking subgroups but also the number of unstable saddle points and maxima of prescribed symmetry as well. In getting this result we do not have to determine the explicit nature of solution vector C_0 !

Let us return to our example of a cubic crystal with symmetry group O_h of order 48. The subgroups of O_h have dimensions 24, 12, 8 or 6. Let us see if we can find a solution for which $n_0 = 24$. Such a solution would represent a minimum configuration with, we claim, residual symmetry group of order 2. Recall that if the system has residual symmetry group H then all points hC with $h \in H$ do not change C while points gC with $g \in G$ must be critical points as the original Landau polynomial was G invariant. A large number of equivalent critical minima means a small residual symmetry group. In this case the unbroken group is of order two since $24 = \frac{48}{2}$. We also note that the total number of critical points must reflect the type of Landau polynomial we choose. A polynomial of order n with m variables can have a maximum $(n - 1)^m$ critical points [9]. Following the suggestions of Landau we stick to polynomials of order four (a quartic) with vectors from a three dimensional irreducible representation of O_h . The total number of critical points (real and complex) for such a system is:

$$3^3 + 1 = 28$$

An extra one has been added. This represents the maxima we require the Landau polynomial to have at infinity. Thus we have the constraint $n_3 + n_2 + n_1 + n_0 \leq 28$. We also have the Morse inequalities for S^3

$$n_0 \geq 1$$

$$n_1 - n_0 \geq -1$$

$$n_2 - n_1 + n_0 \geq 1$$

$$n_3 - n_2 + n_1 - n_0 = 0$$

If we have $n_0 = 24$. The second Morse inequality requires that $n_1 \geq 23$. On the other hand group theory tells us that the list of possible values that a n_i can take must come from the list of numbers $\frac{n(O_h)}{n(H_i)}$. These numbers are 24, 12, 8, 6. Thus we must have $n_1 = 24$ as well. But $n_3 + n_2 + n_1 + n_0 \leq 28$. Hence having $n_0 = 24$ and $n_1 = 24$ is not allowed as $n_1 + n_0$ is the greater than 28. We have thus found a selection rule. The subgroup of order 2 is not allowed as the broken symmetry of the ground state. Carrying out this procedure we get possible patterns of symmetry breaking, *i.e.*, we get compatible numbers for n_0, n_1, n_2, n_3 . Each list of this kind represents a pattern of symmetry breaking. It tells us that associated with a minimum which represents an equilibrium state with a certain unbroken symmetry there are associated saddle points and maxima of specific symmetry.

Thomson Problem

We now apply these ideas to a rather unusual classical problem. We look at the classical Thomson problem. The Thomson problem is easy to state but no solutions other than numerical ones are known. The problem is to determine the minimum energy configurations for a arbitrary collection of N unit like-sign electric charges constrained to move on the surface of a two-sphere of radius R subject to mutual Coulomb repulsion [10]. The Coulomb energy of such a system is

$$E_C \sim \sum_{i < j} \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

where \vec{r}_i is the radial coordinate of the i^{th} charge on the sphere. The actual values of the electric charge and of the dielectric constant of the medium are inessential for the qualitative considerations we are going to make. This is an old—and largely unsolved—problem to which serious studies have been devoted over the years [12]. Recently, triggered by the seminal paper of reference [10], there has been a boost of interesting theoretical, numerical and experimental work on this subject [11]. It is worth to mention that, in its generalized version, *i.e.*, for more general repulsive potentials as well as for topological defects rather than unit electric charges, the Thomson problem finds applications ranging from superconductivity to biology. For an extensive list of such applications see, *e.g.*, the first reference in [11] and references therein.

In [10] the authors suggest a novel way of numerically determining minimum energy configurations. They do this by first looking at a very different problem which is to determine the minimum energy configuration of a system of disclination defects constrained to move on a sphere. They then show how an *effective* defect model, rather than the analysis of the actual elementary charges interaction, proves to be extremely reliable to numerically determine the ground state configurations in terms of the ratio of the defect core energies to the Young modulus for the Thomson problem. Using this approach they were able to find interesting empirical *solutions* to the Thomson problem and present patterns the defects form on the surface of the sphere at the various energy thresholds.

We would like to understand these patterns found numerically in [10], by using the ideas of symmetry and topology we have described. Surprisingly we will find that the mathematical problem we are led to consider is very similar to the one we looked at when analysing symmetry breaking patterns for a crystal!

We would expect the distribution to have the property that each charge has the same environment. If such an assumption is made the charges should be distributed in a way that represents a *tiling* of

the surface of the sphere. The general problem of tiling a genus g surface, say Σ_g , can be solved in a simple way. Suppose we want to tile Σ_g with regular p -sided polygons (p -gons) assembled in such a way that each vertex is shared by (is a common vertex for) 3 p -gons, and each edge is shared by (is a common edge for) 2 p -gons. If k_p is the number of p -gons used—there are, for instance, k_5 5-gons (pentagons), k_6 6-gons (hexagons), k_7 7-gons (heptagons), *etc.*—to tile Σ_g , the resulting *polyhedron* P has $V_P = 1/3 \sum_k k_p p$ vertices, $E_P = 1/2 \sum_k k_p p$ edges, and $F_P = \sum_k k_p$ faces, giving for the Euler characteristic $\chi(\Sigma_g) = V_P - E_P + F_P$, the following expression

$$\sum_k k_p (6 - p) = 6\chi(\Sigma_g) = 6(2 - 2g)$$

Generalization to the case of vertices shared by not always three p -gons can be easily constructed. For the sphere $g = 0$, and if only *hexagons* and *pentagons* are used, there can be an arbitrary number of hexagons, but there must be exactly 12 pentagons. It follows that $\sum_k k_p (6 - p) = k_5 (6 - 5) + k_6 (6 - 6) = 12$, hence $k_5 = 12$ and k_6 is arbitrary. If one also uses *heptagons* for the tiling of the sphere, then one has the interesting result: $k_5 - k_7 = 12$. Thus, one can start off by tiling the sphere with an arbitrary number of hexagons and exactly twelve pentagons. Then one can go on by adding an arbitrary number of *pairs* pentagon-heptagon, but not a pentagon or a heptagon separately. We notice *en passant* that other ways of tiling the sphere are, of course, possible. For instance, if only equilateral triangles (3-gons) are used, then $k_3 \cdot 3 = 12$ or $k_3 = 4$, that is the *tetrahedron*.

We are particularly interested in the 5-gon–6-gon–7-gon tiling because the authors of [10] consider the pentagons and heptagons as disclination defects in a sea of hexagons. They construct an effective theory for E_C in terms of interactions between the defects, the background charge distribution of hexagons merely providing the value of the effective Young modulus. They prove that for a sphere of large radius, distorting the curvature by the introduction of defects is not excessively expensive in terms of bending energy. A 5-gon defect makes the local curvature negative, while a 7-gon makes it positive. For a large spherical system, therefore, the introduction of a 5-gon–7-gon pair is an energetically reasonable way of joining a group of 12 charges. A calculation had to be carried out to see if such a charge configuration was energetically favored as compared to 12 charges organized on two additional hexagons. They find that when the total number N of charges on the sphere exceeds a certain critical value of $\mathcal{O}(500)$, the system prefers to have a collection of 5-gons and 7-gons arranged in the form of a “scar”, *i.e.*, a linear chain of alternating 5-gon–7-gon sequence. For even larger values of N , the preferred form is what they name “pentagonal buttons”. These are configurations of two nested circles, with five 5-gons placed on the outer circle, five 7-gons on the inner circle, and finally a 5-gon in the common center. As N further increases, the defect system forms more intricate patterns. A C_3 symmetric configuration of defects is noted.

We now proceed to give qualitative topological arguments suggesting why these different defect configurations, such as pentagonal buttons or C_3 symmetric configurations discovered by numerical minimization of E_C , can appear using the topological ideas we have introduced. We shall now extensively rely on the model built up in [10], which deals with a system of topological defects on the surface of a spherical elastic material. Our considerations will be qualitative in nature, hence we shall skip many of the details of that analysis and will focus only on the topological and group-theoretical aspects of the problem. This way we shall provide an explanation of (some) of the empirical results

of that work with the hope that our analysis could provide useful insight for that problem and, more generally, for the Thomson problem.

The authors of [10] deal with a sphere, originally tiled with an arbitrary number of hexagons and exactly twelve pentagons. The topology of the sphere requires that the defects, emerging as *disclinations* in the underlying structure of hexagons, must be: first the initial 12 pentagons, then the pairs pentagons-heptagons. On this we commented earlier. For interactions among the defects that are strong as compared to the interactions among the particles forming the material, as collectively described by the Young modulus (identified as the “large core” energy regime in [10]), the twelve pentagons are the only defects in the system and are at the vertices of an *icosahedron*, because this configuration minimizes the energy E_C . In the “small core” energy regimes, new defects are energetically allowed as 12-plets of pentagon-heptagon pairs, and again they prefer to arrange themselves at the vertices of icosahedra. The patterns these additional defects describe on the surface of the sphere is our concern. We shall apply the above presented analysis by considering the energy as a functional of the density function. Thus we shall regard E_C as a map

$$E_C : \vec{\eta} \in S^n \rightarrow R$$

where, as explained, n is related to the symmetry group, the icosahedral group I in this case, the point at infinity is added for stability as a maximum, and the notation is the same as before. In the large core energy regime E_C and its minimum (vacuum) are I -symmetric. In the small core energy regimes we shall make the approximation that E_C is still I -symmetric but the vacuum will only be invariant under a subgroup of I . This way we are dealing with the spontaneous breaking of the icosahedral symmetry into smaller symmetries we want to determine.

The icosahedron is a polyhedron with twelve vertices, twenty faces and thirty edges. Its symmetry group I has 60 elements arranged into 5 conjugacy classes [8]: Identity (1), C_2 (15), C_3 (20), C_5 (12), C_5^2 (12). (The number of elements in the class is in parentheses.) The subgroups of I are C_2 , C_3 , C_5 —of order 2, 3, 5, respectively—and those obtained by the coset decomposition of I with respect to them. The list of the *orders* of the subgroups is

$$\{2, 3, 5, 12, 20, 30\}$$

Of course, by definition of polyhedron, the vertices of the icosahedron lie on the surface of a sphere. Nonetheless, we want to stress this here because, together with the topological constraints on the tiling and the choice of the irreducible representation of I relevant for the problem in point—*i.e.* the *three-dimensional* one—it makes clear the role of the two-sphere in configuration space. These comments are also in order because the topology of a *different* sphere, the $S^n = S^3$ will now play an important role.

Near the phase transition we suppose that E_C is a polynomial of order four, just as for the thermodynamic potential Φ of crystals. Hence the number of real solutions of $E'_C = 0$ is bounded from above as we saw by

$$c_0 + c_1 + c_2 + c_3 \leq 3^3 + 1 = 28$$

where the point at infinity is also added. We can now use the relevant Morse inequalities for the function with $n = 3$

$$\begin{aligned}
 c_0 &\geq 1 \\
 c_0 &\leq 1 + c_1 \\
 c_0 + c_2 &\geq 1 + c_1 \\
 c_0 + c_2 &= c_1 + c_3
 \end{aligned}$$

These, the request that $c_3 \geq 2$ and the Morse constraint are what we need to identify the order of the allowed subgroups H^0 . We just have to insert the given order allowed by group theory into

$$c_0 = 60/\text{ord}(H^0)$$

The resulting list of allowed subgroups consistent with the Morse inequalities and with the Morse constraint is

$$\{5, 12, 20, 30\}$$

Let us analyze the case $\text{ord}(H^0) = 12$. In this case the I -symmetry has been broken in such a way that there are 5 distinct minima ($c_0 = 5$) connected by a C_5 transformation (recall that 12 is the order of the coset of I with respect to C_5). Each minimum corresponds to a new icosahedron of defects, thus, according to this argument, there would be 5 new icosahedra. But, as shown, the new defects cannot appear unless they are in pairs of 5-gons–7-gons, thus, being 5 odd, we *must* have 10 new icosahedra: 5 icosahedra of pentagons next to 5 icosahedra of heptagons. These are the pentagonal buttons found in [10]. It would be interesting to see how the residual symmetry of order twelve is related to the way these buttons arrange themselves on the sphere which, according to [10], is at certain vertices of the rhombic tricontahedron. Similarly the case $\text{ord}(H^0) = 20$ refers to 3 distinct minima for the new symmetry, connected by a C_3 transformation. By means of the same constraint to have 5-gons–7-gons pairs, we would expect to see 6 new icosahedra at the time, while the numerical results show 7 such clusters. This discrepancy could be due to the approximations we are introducing, either for the invariance of the energy or of the order of the polynomial near the phase transition. We find also the cases: $\text{ord}(H^0) = 30$, which refers to 2 distinct minima connected by a C_2 transformation, and the case $\text{ord}(H^0) = 5$ which means $H = C_5$.

We conclude that we are able to describe the pentagonal buttons and the C_3 symmetry found in [10], although this last symmetry is only to an approximated extent. We also notice residual symmetries of orders not seen in that numerical work. This does not necessarily mean that these new symmetries are to be found exactly in the form given here, as our results are qualitative and the analysis might require further care in order to be applied to these cases. For instance, within this approach it is not possible to address the important issue of the various energy thresholds involved in the phase transitions. Thus we cannot say *when* the transitions happen. Nonetheless, we can address the question *why* these transitions occur and in such a general fashion that it is suitable, in principle, for a wide range of applications. Hence the results presented here are interesting especially in the view of stimulating the discussion for a deeper understanding of the Thomson problem in general. We also notice that, having clarified the topological origin of the pentagonal buttons and C_3 symmetric configurations, it is reasonable to argue that these configurations share some features of *solitons*, in the sense that they cannot be undone by a continuous transformation.

We now turn to systems which have continuous symmetry. Such systems are of interest in high energy physics. In order to tackle this problem we need to describe generalisations of the Morse inequality and also to define variants of the Landau polynomial which reflect the symmetry of the system.

3. Symmetry Breaking Selection Rules: Continuous Group

We start by outlining general features present when a continuous symmetry group symmetry is broken spontaneously within a gauge theory framework. Gauge symmetry leads to many zero mass excitations which are not observed. There is a standard procedure for dealing with this problem. It involves introducing a polynomial function of a spacetime scalar field $\phi(x)$ called a Higgs field. This is to be added to the Lagrangian describing the system. A procedure for introducing interactions between the original fields and the Higgs field is also well established. Such a modified Lagrangian eliminates unwanted zero mass excitations and makes specific predictions regarding the way the physical system behaves near the minima of the Higgs polynomial. For the qualitative analysis which we are carrying out we do not need to know anything more about the system other than the continuous Lie invariance group of the Higgs polynomial. The critical points of the Higgs polynomial give the broken symmetry ground state (vacuum state) of the system. To explicitly determine the observed excitations in this spontaneously broken phase, one needs to have explicit details of the Higgs field which minimises the Higgs potential, but we do not need these details for our qualitative description of possible symmetry breaking patterns the system can have. Thus even though the systems in high energy physics are mathematically very different, the Higgs polynomial and the Higgs field fit into the general picture of Landau. We will therefore refer to the Higgs polynomial as the Higgs-Landau polynomial [2].

We can summarise the mathematical setup as follows. We have a compact connected Lie group G representing the unbroken symmetry of the system. The group G is given in terms of a specific linear matrix representation which acts on a real vector space V . Symmetry breaking is implemented by the Higgs field [2] which is zero when the symmetry is not broken and non-zero when the symmetry is broken. It is thus the order parameter field of the problem. The Higgs field $\phi(x)$ is related to the group G by choosing it to be a vector in V which is a representation space for G . To allow symmetry breaking to happen a Higgs-Landau polynomial $P(\phi)$ on V with the following properties is introduced:

1. $P(\phi)$ is invariant under the action of G
2. $P(\phi)$ has a local maximum at $\phi = 0$
3. $P(\phi)$ is bounded below and $P(\phi) \rightarrow \infty$ as $(\phi, \phi) \rightarrow \infty$
4. $P(\phi)$ is of maximum degree four
5. $P(\phi) = P(-\phi)$

The minimum of the Higgs-Landau polynomial determines the symmetry of the ground state and, once found, can be used to determine the associated excitations of the system by expanding the Lagrangian describing the system about the critical value of the Higgs field. Thus the first step is to find the critical points of the Higgs-Landau polynomial. Such points give a critical vector which, if non-zero, will not have G as its invariant group. The smaller symmetry group of this vector, called its isotropy group, represents the reduced symmetry group of the ground state. Since the symmetry groups in this case

are all Lie groups, a critical point is no longer isolated but is, as we will sketch, a manifold. Instead of critical points we thus now have critical manifolds. This means that the Morse inequality we used for finite group symmetry which required critical points to be isolated is no longer applicable. We thus have to deal with two problems. We need to generalise Morse inequalities so that critical manifolds can occur and we need to introduce a convenient G invariant set of variables to replace the Higgs fields. This is because the Higgs fields change under G . The first step requires us to use an Extended form of the Morse Inequality where critical manifolds rather than critical points are allowed. The appropriate setup is well established [3]. For the next step we rewrite the Higgs polynomial in terms of a set of independent homogeneous elementary G invariant polynomials. Such a sets of G invariant polynomials for Lie groups are well known [3]. They provide rather natural coordinates to describe the geometry of the system. We can have, in general, homogeneous invariants of different degrees, *i.e.*, quadratic, quartic, ... elementary G invariant polynomials. The list of such polynomials will depend on the symmetry group we consider [3]. In terms of these invariants the Higgs-Landau polynomial can be written as:

$$P(\phi) = \sum_i a_i^{(4)} I_i^{(4)} + \sum_{ij} a_{ij}^{(2,2)} I_i^{(2)} I_j^{(2)} + \sum_i a_i^{(2)} I_i^{(2)}$$

Here $I_i^{(4)}$ represents a G invariant homogeneous polynomials of degree four while $I_i^{(2)}$ are G invariant homogeneous polynomial of degree two.

We stress again that the critical “points” of $P(\phi)$ are no longer points. They are manifolds as we now sketch. If ϕ is a solution then an element of G acting on it will either keep it unchanged or change it. If the certain group elements keep $\phi(x)$ unchanged they belong to the isotropy group of $\phi(x)$. For group elements that do not belong to the isotropy group ϕ changes in a continuous way. But each new $\phi(x)$ is an equivalent critical point. This is because the Higgs-Landau polynomial is G invariant so that $P\phi) = P(g\phi)$. The action of such Lie group elements thus generate a continuously connected set of equivalent critical points: they form a critical manifold. If a local Euclidean description of a $(n - k)$ dimensional critical manifold sitting in an n manifold is given then we see the Euclidean coordinates split into ones which describe the manifold, say $(y_{k+1}, ..y_n)$ and Euclidean coordinates that are orthogonal to these manifold coordinates, say $(x_1, ..x_k)$ which describe the way the critical manifold sits in the original manifold. In terms of these coordinates we can now talk about the index of a critical manifold for a real valued function. Let $f(x_1, ...x_k, y_{k+1}, ...y_n)$ be a local description of a real valued function in the neighbourhood of such a critical manifold. If the function decreases in value for l directions from the set $(x_1, ..., x_k)$ then the critical manifold has index l . Thus l ranges between 0 and k [7]. The Extended Morse theory will now be presented in a slightly more abstract way. To do this we need to introduce two polynomial functions: First we have the Poincare polynomial associated with a given manifold M written as $H_t(M)$ and secondly we have the Morse polynomial $M_t(P)$ associated with the Higgs-Landau polynomial $P(\phi)$. We have $H_t(M) = \sum t^i b_i(M)$ $M_t(P) = \sum t^\lambda(N)H_t(N)$ Here $b_i(M)$ is the *ith* Betti number of the manifold M while $\lambda(N)$ describes the way the critical submanifold N sits in the manifold M . It sits so that there are $\lambda(N)$ directions in which the Higgs-Landau polynomial decreases in value as the critical point as one moves away from the critical manifold: it is the index of the critical manifold. We can now state the Extended Morse inequality

Theorem $M_t(P) - H_t(M) = (1 + t)Q(P)$

where $Q(P)$ is a polynomial in t of degree $m - 1$ with positive coefficients where the dimension of the manifold M is m . To apply this formula it is best to regroup manifolds with the same Betti numbers. Let us denote a typical element of each class by N^μ and let $d - N^\mu, j$ be the number of critical manifolds in the same class as N^μ with critical index $j, j = 0, 1, \dots, m$. The Morse function can now be written as

$$M_t(P) = \sum c_i(P)t^i$$

$$c_i(P) = \sum \sum b_i(N^\mu)d_{N^\mu, j-i}$$

The number of minimal (maximal) critical manifolds in the same class as N^μ is given by $d_{N^\mu, 0}$. The Morse inequalities now take the form

$$\sum (-1)^l c_l(P) \geq \sum (-1)^l b_l(M), \quad \text{for } l \leq (m - 1)$$

$$\sum (-1)^m c_m(P) = \sum (-1)^m b_m(M)$$

These inequalities on their own are not very restrictive but they can become restrictive if the Higgs-Landau polynomials is taken to be of low order. Then $\sum d_N^\mu, j \leq T_P + 1$ Here T_P denotes the upper bound on the number of critical manifolds. It is easy to estimate T_P . The example of the symmetry breaking patterns possible for the group $SU(4)$ is worked out in [13] using this approach. Setting $T_P = 4$, for instance, only one symmetry breaking pattern is allowed, namely $SU(4)$ breaks to $S(U(1)XU(3))$ while setting $T_P = 5$ gives many possibilities. This brief overview illustrates the way topological ideas combined with physics can lead to predictions. The physical systems where these ideas can be used is vast [3].

4. Topological Phase Transitions: Graphene

We finally turn to an example where the Landau approach does not work. We look at the hydrodynamic region of graphene and explain why this phase does not fit into the Landau picture. In the Landau approach we saw there was a symmetry group present which was broken by the ground state. The excitations of the system in the broken phase were the quasiparticles. These gave a good description of the physical excitations observed. It was, however, recognised that there are systems where such a symmetry group, quasiparticle approach are not applicable. This is the case for some “quantum topological phase transitions” [4,5]. In these situations there were two distinct phases at $T = 0$. The phases depend, schematically, on the ratio $\frac{U}{T}$ where U represents the strength of interactions and T the kinetic energy. The point of separation was a “quantum critical point”. For temperatures above zero the quantum critical point region opened up to form a new “topological phase”. In this phase the quantum system had a hydrodynamic description.

Ideas for describing certain non-Landau phases in terms of the condensation of a network of strings into a degenerate ground have recently been proposed [6]. The suggested analogue of the groups present in the Landau approach was the abstract notion of a “tensor category”. This mathematical structure is rather natural for describing networks as it allows the fusion and braiding of strings. Furthermore the theoretical framework needed two topological field theories with opposite chiralities to be present. This was to make the system parity and time reversal invariant. The introduction of a topological field

theory is again rather natural in the context of string networks as it allows topological ideas of equivalent configurations of strings to be incorporated. Such equivalence ideas are expected to be present in the mathematical framework in order to properly capture essential features of a string network. Examples where this approach can be used was shown to include all discrete lattice gauge theories, and all doubled Chern-Simons theories which had non-Landau phase transitions. The approach was capable of providing exactly soluble Hamiltonians and ground state wave functions for each topological phase. In this approach, for example, a lattice $U(1)$ gauge theory can be formulated as a theory of flux lines. In a dual description more general string networks where strings can join is required. The deconfined phase then corresponds to a highly fluctuating string net, namely a string net condensate. However it should be noted that non-Landau topological phases similar to those considered by [6] may be recast into a Landau theory framework in a variety of ways such as by using duality transforms or dimensional reduction [14,15]. If dimensional reduction occurs, the physical higher dimensional system may have features captured by a lower dimensional Landau type system.

Here we look at a theoretically possible non-Landau phase, namely the quantum hydrodynamic phase of graphene, using standard tools. The system considered is an example where a theoretically possible hydrodynamic non-Landau phase with universal features [16] can appear. The system is non-Landau as it is a non-equilibrium transition with excitations that are expected to be collective modes, not free quasiparticle. Further more the topology methods we described cannot be applied as we do not have the analogue of critical points, or critical manifolds for modeling an out of equilibrium situation. Thus there is no natural underlying real valued function whose critical points are associated with the new non-equilibrium phases that are theoretically possible. The phase change for this case is characterised by a sudden change in the electrical flow properties of the system.

We start by reviewing the general setup and then make a few remarks about ideas that are emerging. Let us start by describing the basic tight binding model for graphene that we need [16]. Graphene, a monolayer of carbon atoms arranged in a hexagonal array was fabricated in 2004 [17]. From the moment it was fabricated it was clear that it was a system with strange properties, for instance, the low energy effective theory for graphene was a mass zero Dirac equation where the spin had nothing to do with the angular momentum but represented a two-valuedness which came from the hexagonal geometry. Thus an equation which was important for highly relativistic electrons shows up as an effective low energy description for a many body system near certain Fermi points [18]. Let us sketch how this happens [19]. The hexagonal array of carbon atoms had the geometrical property that the neighbourhood of one atom had a “Y” shape while the neighbourhood of the atom opposite to it had an “inverted Y” shape. The system was thus made up of two triangular non-equivalent lattices. These two sub lattices are conveniently called the A and B sublattices. Each atom of the A lattice had three atoms of the B lattice as neighbours. Taking the lattice size to be unity the three nearest neighbour vectors of a A lattice could be taken to be: $s_1 = (0, -1)$, $s_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $s_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. In the tight binding approximation the hexagonal system was described by the Hamiltonian

$$H = t \sum [\hat{a}_r b_{r+s_i}] + \text{hermitian conjugate}$$

where \hat{a}_r was a creation operator for an electron at the A lattice site r while b_{r+s_i} was a destruction operator for an electron at one of the three neighbouring B lattice sites introduced. The electron has

could hop from a A site to a neighbouring B lattice site. The coefficient t represented the strength of this transition. The summation was over r and s_i . Moving to a Fourier transform representation we get

$$H = \sum [\Phi(k)\hat{a}_k b_k] + \text{hermitian conjugate}$$

Here $\Phi(k) = \sum \exp(k.s_i)$. Using the specific geometrical values for s_i given it is possible to show that $\Phi(k) = 0$ for two values of k . Expanding about either of these two values leads to a mass less two dimensional Dirac equation. The two “spin” states correspond to the two lattice sites A, B . In recent work [16] the two dimensional Dirac Hamiltonian was enlarged by including electron electron interaction and it was shown that there was a hydrodynamic regime present in the system with universal properties. It was further shown that the shear viscosity of this quantum fluid was small [22]. Finally it was shown [23] that turbulent flow with scaling behaviour could be present in the system. Let us outline the way this is established. Turbulent flows are characterized by a probability distribution function with scaling properties and an associated non-vanishing energy flux. Quantum turbulent flows have been studied using the methods of quantum field theory, as such systems have many degrees of freedom exhibiting stochastic behaviour. Systems where the wave motion is a recognized feature and turbulent flows are generated due to non-linear interactions are known as weak wave turbulent systems.

Graphene near the Fermi points with an electron-electron interaction [22] is described by a Hamiltonian given by

$$\begin{aligned} H &= H_0 + H_1 \\ H_0 &= \sum_{\lambda,i} \int \frac{d^2k}{(2\pi)^2} \lambda v_F k a_{\lambda i}^\dagger(\mathbf{k}) a_{\lambda i}(\mathbf{k}) \\ H_1 &= \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4; i, j} \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2k_3}{(2\pi)^2} \frac{d^2k_4}{(2\pi)^2} \delta^2(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \\ &\quad \times T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) a_{\lambda_4 j}^\dagger(\mathbf{k}_4) a_{\lambda_3 i}^\dagger(\mathbf{k}_3) a_{\lambda_2 i}(\mathbf{k}_2) a_{\lambda_1 j}(\mathbf{k}_1) \end{aligned}$$

where a_{+i}, a_{-i} are the Fourier mode operators, $i, j = 1, \dots, 4$ for graphene corresponding to two valleys and two spins and $\lambda = (+, -)$. We also define $k = |\mathbf{k}|$, which is the modulus of the two momentum \mathbf{k} . The explicit expression for $T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ is given in Reference [22]. For our discussion, we need to note that under scaling,

$$T(\mu \mathbf{k}_1, \mu \mathbf{k}_2, \mu \mathbf{k}_3, \mu \mathbf{k}_4) = \frac{1}{\mu} T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$$

where μ is a scalar.

A turbulent flow represents a steady state of the system in which, unlike a thermal equilibrium state, there exists a transport of a charge current. Such flows are expected to have scaling properties. To calculate the probability distribution function (PDF) for such a flow, we follow the method described in [21], modified to deal with quantum massless Dirac system. The basic idea is to calculate

$$\frac{d}{dt} n_{k,\lambda} = \frac{d}{dt} \langle a_{k,\lambda}^\dagger a_{k,\lambda} \rangle$$

between two “in-states” [20], which are eigenstates of H_0 . Next we look for scaling solutions which have the property that

$$\frac{d}{dt}n_{k,\lambda} = 0$$

and have a non-vanishing associated flux, to be discussed later. This gives,

$$0 = \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4; i, j} \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2k_3}{(2\pi)^2} \delta^2(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}) |T_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k})|^2 n_{k_1} n_{k_2} n_{k_3} n_k \times \left[-16 \left(\frac{1}{n_k} + \frac{1}{n_{k_3}} - \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} \right) + 8 \left(\frac{1}{n_{k_3} n_k} + \frac{1}{n_{k_1} n_k} - \frac{1}{n_{k_2} n_{k_3}} - \frac{1}{n_{k_1} n_{k_2}} \right) \right]$$

We would like to obtain the stationary solutions in terms of the energy ϵ_i instead of the two momenta \mathbf{k}_i . We shall further assume that the solutions are isotropic, so that both $n_{k,\lambda}$ and the energy ϵ depend only on the magnitude k of the momentum vector \mathbf{k} . Following the procedure of Zakharov [21] we change variables from k to $\epsilon(k)$ where the latter denotes the energy and replace T by

$$U(\epsilon_1 \epsilon_2 \epsilon_3 \epsilon) = (k_1 k_2 k_3 k) \left| \frac{d\epsilon_1}{dk_1} \frac{d\epsilon_2}{dk_2} \frac{d\epsilon_3}{dk_3} \frac{d\epsilon}{dk} \right|^{-1} \int |T_{k_1 k_2 k_3 k_4}|^2 \delta^2(k_1 + k_2 - k_3 - k) d\Omega_1 d\Omega_2 d\Omega_3$$

where we have used the relation $d^2k_i = k_i dk_i d\Omega_i$. From Equation 6 it can be easily seen that the quantity U is invariant under scaling.

From these two expressions and using the free particle density, we get

$$0 = \int_0^\infty d\epsilon_1 d\epsilon_2 d\epsilon_3 U(\epsilon_1 \epsilon_2 \epsilon_3 \epsilon) \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon) n_{k_1} n_{k_2} n_{k_3} n_k \times \left[-16 \left(\frac{1}{n_k} + \frac{1}{n_{k_3}} - \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} \right) + 8 \left(\frac{1}{n_{k_3} n_k} + \frac{1}{n_{k_1} n_k} - \frac{1}{n_{k_2} n_{k_3}} - \frac{1}{n_{k_1} n_{k_2}} \right) \right]$$

We now analyze Equation 7. The δ -function can be used to carry out the ϵ_3 integral. What remains is a region D of the (ϵ_1, ϵ_2) plane. In addition we have a constraint that $\epsilon_3 = \epsilon_1 + \epsilon_2 - \epsilon \geq 0$. Following [21], we divide D into four sectors as follows

$$\begin{aligned} D_1 &= \{(\epsilon_1, \epsilon_2) \in D \mid \epsilon_1 < \epsilon, \epsilon_2 < \epsilon\} \\ D_2 &= \{(\epsilon_1, \epsilon_2) \in D \mid \epsilon_1 > \epsilon, \epsilon_2 > \epsilon\} \\ D_3 &= \{(\epsilon_1, \epsilon_2) \in D \mid \epsilon_1 < \epsilon, \epsilon_2 < \epsilon\} \\ D_4 &= \{(\epsilon_1, \epsilon_2) \in D \mid \epsilon_1 > \epsilon, \epsilon_2 < \epsilon\} \end{aligned}$$

It was shown in [21] that the regions D_2, D_3, D_4 can be mapped into D_1 using the Zakharov transformations. For example D_2 with variables $(\epsilon'_1, \epsilon'_2)$ can be mapped to D_1 with variables (ϵ_1, ϵ_2) using

$$\begin{aligned} \epsilon'_1 &= \frac{\epsilon \epsilon_1}{\epsilon_1 + \epsilon_2 - \epsilon} \\ \epsilon'_2 &= \frac{\epsilon \epsilon_1^2}{\epsilon_1 + \epsilon_2 - \epsilon} \end{aligned}$$

and so on. Using these transformations and the ansatz $n(\epsilon) = C\epsilon^{-x}$, we get

$$0 = \int_{D_1} d\epsilon_1 d\epsilon_2 U(\epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2 - \epsilon)(\epsilon_1 \epsilon_2 (\epsilon_1 + \epsilon_2 - \epsilon) \epsilon)^x \left[1 + \left(\frac{\epsilon_1 + \epsilon_2 - \epsilon}{\epsilon} \right)^y - \left(\frac{\epsilon_2}{\epsilon} \right)^y - \left(\frac{\epsilon_1}{\epsilon} \right)^y \right] \\ \times \text{other factors}$$

where $y = 3x - 3$. The factor within the square brackets in Equation 10 can be made to vanish by choosing $y = 0$ or $y = 1$, irrespective of other factors in the equation. It is possible to show [24] that only the solution corresponding to $y = 0$ or $x = 1$ has an associated non-zero flux and hence represents a turbulent state. This is the solution that we shall use.

Our solution for the TPDF is thus given by

$$n(k, \lambda) = \frac{C(\lambda)}{k}$$

We now need to determine C . To do this, we observe that the total number of turbulent particles per unit area, that is the turbulent number density N_T is given by

$$N_T = \sum_{\lambda} \int \frac{d^2k}{(2\pi\hbar)^3} n(k, \lambda)$$

where we have used the free particle density of states appropriate for our calculation. We find

$$C\epsilon_{max} = N_T \left[\frac{(2\pi\hbar)^2}{4} \frac{1}{2\pi} \right]^{-1}$$

where $\epsilon_{max} = v_F k_{max} = \alpha kT$, where T is the background temperature of the system before the turbulent flow starts and $\alpha \ll 1$, which is required for the hydrodynamic regime to be relevant.

The excitations of the system in this regime were not particle (e.g., quasiparticles) but hydrodynamic waves whose nature was determined by basic conservation laws and non-linear interactions. There was no symmetry breakdown but the spontaneous emergence of a new phase, namely a quantum coherent state with fluid like properties. The turbulent flow is a non-equilibrium driven phase. It is not the minimum of a Free Energy, hence the state found is non-Landau. Thus there is no Morse type constraint that can be used. This novel system does not fit into the general scheme considered in earlier sections of this review, however it shows the richness of the concept of a phase transitions. Unraveling and understanding the physics of the fascinating graphene system guided by experimental results is a challenge for the future.

5. Conclusions

We have briefly sketched old ideas for understanding phase transitions refined by topology. In this area there is room for imaginative applications in different areas. We found that qualitative statements about the ground state could be made without explicit knowledge about the solutions. This is truly remarkable. We ended with a few brief comments about the emerging field of “quantum criticality”. Here there are major conceptual challenges to find a general approach for understanding and dealing with such systems. One approach was very briefly sketched and one theoretically possible example of a new non-Landau, non-equilibrium state of graphene was outlined.

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