

Review

## Noether Symmetries and Covariant Conservation Laws in Classical, Relativistic and Quantum Physics

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**Abstract:** We review the Lagrangian formulation of (generalised) Noether symmetries in the framework of Calculus of Variations in Jet Bundles, with a special attention to so-called “Natural Theories” and “Gauge-Natural Theories” that include all relevant Field Theories and physical applications (from Mechanics to General Relativity, to Gauge Theories, Supersymmetric Theories, Spinors, *etc.*). It is discussed how the use of Poincaré–Cartan forms and decompositions of natural (or gauge-natural) variational operators give rise to notions such as “generators of Noether symmetries”, energy and reduced energy flow, Bianchi identities, weak and strong conservation laws, covariant conservation laws, Hamiltonian-like conservation laws (such as, e.g., so-called ADM laws in General Relativity) with emphasis on the physical interpretation of the quantities calculated in specific cases (energy, angular momentum, entropy, *etc.*). A few substantially new and very recent applications/examples are presented to better show the power of the methods introduced: one in Classical Mechanics (definition of strong conservation laws in a frame-independent setting and a discussion on the way in which conserved quantities depend on the choice of an observer); one in Classical Field Theories (energy and entropy in General Relativity, in its standard formulation, in its spin-frame formulation, in its first order formulation “à la Palatini” and in its extensions to Non-Linear Gravity Theories); one in Quantum Field Theories (applications to conservation laws in Loop Quantum Gravity via spin connections and Barbero–Immirzi connections).

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## 1. Introduction

Symmetries have acquired a central role in Physics. In Theoretical Physics discrete symmetries encode most of the intriguing structure of the Standard Model for particles, in Chemistry they encode for spectroscopic and physical properties of molecules.

Lie groups of transformations encode properties (often enhancing physical interpretation) of dynamical and Lagrangian systems. For example, theoretical Relativistic Cosmology is entirely based on the symmetry ansatz of homogeneity and isotropy (which of course can be later relaxed by using techniques of Perturbation Theory). Most applications of Quantum Gravity (either loopy or stringy) to Cosmology are entirely based on symmetries since the approach in full generality is still hindered by massive technical difficulties.

In the beginning of the 20th century, Emmy Noether discovered (see [1]) a relation between (continuous families of) symmetries of Lagrangian systems and their first integrals, *i.e.*, physical quantities which remain constant during the evolution of the system and are often related to fundamental physical quantities such as energy, momentum, angular momentum, *etc.*

Symmetries of mechanical systems (together with their associated conserved quantities) are the basis of the definition of *integrable systems* that form a class of systems for which dynamics can be determined in general by integration. Generally speaking, knowing a conserved quantity of a dynamical system allows to reduce the dimension in which the system is defined. General integration techniques, such as Hamilton–Jacobi equation, rely in fact entirely on existence of conserved quantities.

Very interesting issues arise when one extends the notion of *transformation* to encompass transformations depending on velocities (and more generally on accelerations, *etc.*). These can be seen as transformations on the infinite jet prolongation of the configuration space where the dynamical system can be seen to be prolonged. Here infinite dimensionality enters strategically to require new techniques to extend the procedures which are standard in a finite dimensional arena.

Another interesting setting where symmetries play their role is the framework of Lagrangian field theories, which are the current basis for any approach to fundamental interactions in Physics. This corresponds, loosely speaking, to consider a dynamical system with a continuous infinity of degrees of freedom. In this context, (second) Noether theorem still holds true, though conservation laws needs to be interpreted correctly and differently from what is usually done in Mechanics (in view of the so-called first Noether theorem); if the system lives on a manifold  $M$  of dimension  $m = \dim(M)$ —which usually is identified with the physical spacetime, or, mathematically speaking,  $m$  is the number of independent variables of the equations—Noether theorem implies the existence of a  $(m - 1)$ -form on  $M$  called the *Noether current*. Such a current results to be closed along solutions and hence implies a *continuity equation*. The *conserved quantities* are thence defined as the integrals of such currents on a  $(m - 1)$ -volume in  $M$ . The continuity equation holding for the Noether current relates the changes of conserved quantities to the flows at the boundary of the region (something entering or escaping the region) and some residual at singularities. This setting is particularly suitable for physical interpretation; it was in fact developed, e.g., to define electric charges in Electromagnetism. This setting results to go far beyond Electromagnetism and to be the typical situations for (at least) all fundamental interactions.

In our approach the difference between first and second Noether theorem that is often drawn in literature is quite evanescent. In fact, on one hand in field theory the so-called *global* transformations cannot in general be defined in a gauge covariant way and only so-called *local* transformations are intrinsically defined, unless configuration bundle is trivialized (as it happens in Mechanics). In any event, both *local* and *global* transformations are generated by a vector field on the configuration bundle of the system. On the other hand, Mechanics can be considered as a particular case of field theory. In fact, in standard (Newtonian) Mechanics  $m = 1$  (since  $M = \mathbb{R}$ ) and Noether currents are functions; being closed they are constant along solutions and the situation introduced above is obtained as a special case.

Of course, in Mechanics one can also consider *local* transformations (e.g., in relativistic point particle invariance with respect to reparametrizations is considered); in this case the Lagrangian is degenerate and constraints on momenta appear. Such constraints are known to be hidden in some (consequences) of the Lagrange equations though currently the most efficient tool to analyze the situation is based on canonical Hamiltonian formalism (see [2–7]). However, let us here only mention that these constraints are somehow trivial as long as conservation laws are considered since their associated currents vanish on-shell (once initial conditions obey constraints). The meaning of these conservation laws is very different depending if one is considering first or second Noether theorem but we shall discuss these differences below when discussing the hole argument. Hereafter, we are interested in conservation laws and hence we shall mainly consider general transformations and their relation to continuity equations.

In Field Theory the role of conservation laws, though certainly important as a support to the physical interpretation, appears to be weaker than in Mechanics since the mechanical geometrical picture is lost (we mean, the picture of first integrals each of which determines a level hypersurface of configuration space so that motion is constrained on the intersection of all these hypersurfaces and first integrals define a *reduced system* living on it). Noether currents are  $(m - 1)$ -forms and define no level hypersurface; on the other hand a Field Theory has infinitely many degrees of freedom and it is not obvious how to extend mechanical techniques to use conservation laws with the purpose of reducing the systems accordingly; however, though extra care is due to infinite dimensionality, such a reduction can be done; see [8].

On the other hand, the history of Field Theory is strongly entangled with symmetries. All the theory of *continuity equations* was developed to account in Field Theory of charge conservation. When Einstein proposed equations to describe the gravitational field in General Relativity (GR) their form was obtained on the basis of conservation of energy—momentum tensor describing matter (as well as, independently, by Hilbert on the basis of his variational principle).

GR itself is a source of fantastic examples of the role of symmetries in Field Theories. One can prove general theorems to show that Noether currents are not only closed forms, but even exact forms along solutions. This introduces a *superpotential*  $\mathcal{U}$  for each Noether current and conserved quantities are obtained by surface integrals of the superpotential. Generalized Stokes theorem in this context establishes a bridge between conserved quantities and (co)homology of forms (and hence with topology and global structure of the spacetime manifold) which happens to be central in the physical interpretation of the model.

Similar situations can be recognized in other areas of Physics such as in Gauge Theories; the same structures of superpotential forms can be generically recognized in Gauge Theories as well as back in

Mechanics when one wants to use (or needs to use, as it happens for the relativistic material points) homogeneous formalism (*i.e.*, treating time and space on equal footing).

Hereafter we shall show some of the relations among symmetries and conservation laws in different areas, preferring a unifying languages that stress similarities.

## 2. Geometrical Setting

We shall here present notation for a geometrical setting for Lagrangian systems which encompasses both Field Theories and a suitable setting for Mechanics. We refer the reader to [9] for further and deeper details.

Let  $M$  be an (orientable, connected, paracompact) manifold of dimension  $m = \dim(M)$  with local coordinates  $x^\mu$  that will be eventually considered as independent variables in the Variational Calculus.

Let  $\mathcal{C} = (C, M, \pi, F)$  be a bundle over  $M$  with projection  $\pi$  and standard fiber  $F$ ; see [9]. Let  $y^i$  be local coordinates on  $F$  that will eventually represent the dependent variables of the variational principle representing the dynamics of the Lagrangian system under consideration.

A vector  $w \in TC$  is *vertical* iff  $T\pi(w) = 0$ , *i.e.*, is expressed in any fibered coordinate system  $(x^\mu, y^i)$  as  $w = w^i \partial_i$ . The set of all vertical vectors at  $p \in C$  is denoted by  $V_p$ ; the union  $V(\pi) = \cup_p V_p \subset TC$  is a sub-bundle of  $TC$ .

This setting is general enough to encode Field Theories and Mechanics; in particular in Field Theory  $M$  is assumed to represent spacetime (hence usually  $m = 4$ ) while in Mechanics one has  $M = \mathbb{R}$  (hence  $m = 1$ ) and base coordinates  $x^\mu$  usually reduce to  $t$ . Configurations of the system are locally given by assigning the dependent variables as functions of the independent ones. In Field Theories this means  $y^i(x)$ , while in Mechanics  $y^i(t)$  locally represent curves in  $F$  (which in this case is called the *configuration space* of the system). Of course global structure of  $C$  (or  $F$ ) encodes the global properties of the system; in particular in Field Theories one considers transformation rules of fields with respect to changes of fibered coordinates; these transformation rules encode in particular the geometrical character of fields, *i.e.*, for a Riemannian metric one has for example

$$\begin{cases} x'^\mu = x'^\mu(x) \\ g'_{\mu\nu} = \bar{J}_\mu^\rho g_{\rho\sigma} \bar{J}_\nu^\sigma \end{cases} \quad (1)$$

where  $\bar{J}_\mu^\rho$  denotes the anti-Jacobian of the coordinate change on spacetime, namely of  $x'^\mu = x'^\mu(x)$ .

Moreover, these transformation rules encode also global properties, in the technical sense that once they have been fixed one can prove that there exists a unique bundle  $\mathcal{C}$  (modulo isomorphisms) having those functions as transition functions. This is quite satisfactory from the physical viewpoint since local descriptions given by local observers in terms of local fibered coordinates, together with transformation rules (which dictate how one can deduce the readings of an observer knowing the readings of any nearby observer) allow to uniquely describe the physical situation in an observer-independent and global way.

Global configurations are global sections of the bundle  $\mathcal{C}$ , *i.e.*, maps  $\sigma : M \rightarrow C$  such that  $\pi \circ \sigma = \text{id}_M$ . They are locally expressed by functions  $y^i(x)$  and transformation rules account for global properties. Usually only local sections exist (*i.e.*, sections on open subsets  $U \subset M$ ) and the existence of global sections of  $\mathcal{C}$  implies topological restrictions and/or obstructions.

Once the configuration bundle  $\mathcal{C}$  is given one can define a new bundle, canonically (*i.e.*, functorially) associated to  $\mathcal{C}$ , which accounts for derivatives of dependent variables with respect to independent variables up to some finite order  $k$ . This new bundle is called the  $k$ -order prolongation of  $\mathcal{C}$  and it is denoted by  $J^k\mathcal{C}$ . If  $(x^\mu, y^i)$  are fibered coordinates on  $\mathcal{C}$ , then  $(x^\mu, y^i, y_\mu^i, y_{\mu\nu}^i, \dots)$  are fibered coordinates on  $J^k\mathcal{C}$ , where  $y_\mu^i$  stands for the first derivatives,  $y_{\mu\nu}^i$  stands for the second derivatives (and are accordingly assumed to be symmetric in the lower indices  $(\mu\nu)$ ), and so on.

Any  $J^k\mathcal{C}$  is a bundle over any other  $J^{k-s}\mathcal{C}$  (for any integer  $s > 0$ ). All prolongations of a bundle define a projective family and one can define its inverse limit, that is denoted by  $J^\infty\mathcal{C}$ . (This is a bundle in a broader sense, since it is infinite dimensional).

A Lagrangian of order  $k$  is a bundle map  $L : J^k\mathcal{C} \rightarrow A_m(M)$  where  $A_m(M)$  is the bundle of  $m$ -forms over  $M$ . Equivalently, the Lagrangian can be seen as a horizontal  $m$ -form over  $J^k\mathcal{C}$ . The action is the functional defined as

$$A_D[\sigma] = \int_D L \circ j^k\sigma \quad (2)$$

for any  $m$ -region  $D \subset M$  and any (local) configuration (*i.e.*, section)  $\sigma$ . *Hamilton stationary action principle* is in this framework a definition for *critical sections*, *i.e.*, sections that are critical points of the action functional with respect to a canonical class of deformations. A *deformation* is a vertical vector field  $X$  on  $\mathcal{C}$ . Let us denote by  $\Phi_s$  its flow so that we can drag any configuration  $\sigma$  along  $X$  defining a 1-parameter family of configurations  $\sigma_s = \Phi_s \circ \sigma$ . The *variation* of the action along the deformation  $X$  is defined as

$$\delta_X A_D[\sigma] = \int_D \frac{d}{ds} (L \circ j^k\sigma_s) |_{s=0} \quad (3)$$

**Hamilton stationary action principle:** a configuration  $\sigma$  is critical iff for any compact  $m$ -region  $D \subset M$  and deformation  $X$  with  $\text{supp}(X) \subset D$  one has

$$\delta_X A_D[\sigma] = 0 \quad (4)$$

Equivalently, one can also consider more general deformations only requiring that  $X$  vanishes on the boundary  $\partial D$  together with its derivatives up to order  $k - 1$ . Critical sections can be shown to obey Euler-Lagrangian equations and physically represent allowed configurations (*i.e.*, configurations which satisfy field equations). This framework reduces locally to the usual Variational Calculus.

As we said, in Mechanics one has  $M = \mathbb{R}$  and  $m = 1$ . Being  $\mathbb{R}$  a contractible manifold the bundle  $\mathcal{C}$  is necessarily trivial, *i.e.*, diffeomorphic to a Cartesian product  $\mathbb{R} \times F$ ; see [10]. Nevertheless, this diffeomorphism is not canonical, but it depends on a *reference frame* which is realized mathematically by a connection on the bundle  $\mathcal{C}$ . A connection is a family of hyperplanes  $H_p \subset T_p\mathcal{C}$  such that  $H = \cup_p H_p$  is a sub-bundle in  $T\mathcal{C}$  and at each point  $T_p\mathcal{C} = V_p \oplus H_p$  (by the way, since the curvature of the connection is skew and in this case  $m = 1$ , any connection on  $\mathcal{C}$  is *flat*). A connection in this case is represented by a distribution  $H_p$  of rank 1 on  $\mathcal{C}$ ; being of rank 1 it is involutive and, by Fröbenius theorem, integrable. Hence one has defined a foliation in curves (trajectories, *i.e.* unparametrized curves, to be precise) of  $\mathcal{C}$ . The leaves  $\gamma_p$  are nowhere vertical and establish diffeomorphisms between any pair of fibers  $\pi^{-1}(t_0)$  and  $\pi^{-1}(t_1)$ ; hence they induce a particular diffeomorphism  $t : \mathcal{C} \rightarrow \mathbb{R} \times F$ , *i.e.*, a global trivialization, in which  $\gamma_p : t \mapsto (t, f_0)$  for some constant  $f_0 \in F$ . These sections represent *rest* for the global observer associated to the trivialization  $t$  defined above. One has thence two possible frameworks for Mechanics:

one on  $\mathbb{R} \times F$ , that is a framework for a fixed observer (or reference frame); and another on  $C$ , that is potentially independent of the observer. In both cases the framework is suitable for describing in particular any holonomic, possibly time-dependent, Lagrangian system.

Since the Lagrangian is used only to define the action (and then the action itself is used in Hamilton principle), in all Field Theories one could add to the Lagrangian terms which do not affect the value of the action functional. In fact there are forms, called *contact forms*, which vanish when computed along each configuration of the system; in fact, if a form is contact it factorizes terms such as  $\omega^i := dy^i - y_\lambda^i dx^\lambda$ ,  $\omega_\mu^i := dy_\mu^i - y_{\mu\lambda}^i dx^\lambda$ , and so on.

A *Poincaré–Cartan form* for a Lagrangian system is a form  $\Theta_L = L + \Omega$  on  $J^{2k-1}\mathcal{C}$  that differs from the Lagrangian  $L$  by a contact form  $\Omega$ . The action can be written also in terms of the Poincaré–Cartan form as

$$A_D[\sigma] = \int_D L \circ j^k \sigma = \int_D (j^k \sigma)^* \Theta_L \quad (5)$$

The contact term  $\Omega$ , which does not affect field equations, solutions and so on, can then be tuned to enhance the properties related to conservation laws and symmetries.

One requires (besides some other technical requirements; see [9]) that for all vertical fields  $X^{(2k-1)}$  of  $J^{2k-1}\mathcal{C}$  the form  $i_X d\Theta_L$  is contact. Here  $i_X$  denotes the usual duality between forms and vector fields. In the next Section we shall see how Poincaré–Cartan forms are adapted to conservation laws and symmetries. Below we shall see explicit examples for coordinate expressions of Poincaré–Cartan form.

The theory of Poincaré–Cartan form was fully developed in the late '70s–early '80s (see [11–15]). In Mechanics one can prove that there exists a unique Poincaré–Cartan form at each order. In Field Theory, there is a unique Poincaré–Cartan form in theories of order 1, there is a canonical choice for theories of order 2, while for Field Theories of order  $k \geq 3$  there is a Poincaré–Cartan form for each connection on the base manifold (one needs integration by parts to define it, and one needs *covariant* integration by parts to control globality; different connections define then different Poincaré–Cartan forms when  $k \geq 3$ ). This non-trivial (and to some extent *unexpected*) structure on uniqueness issue is very beautiful and inspiring, for example for issues related to Hamiltonian formalisms.

### 2.1. Field Equations

Most of Variational Calculus is in fact related (if not completely encoded) in how deformations defined on the configuration bundle  $\mathcal{C}$  prolong to higher order jet prolongations  $J^k\mathcal{C}$ .

A (projectable) vector field  $\Xi = \xi^\mu \partial_\mu + \xi^i \partial_i$  (here *projectable* refers to the fact that the components  $\xi^\mu$  are functions of the independent variables only, *i.e.*,  $\xi^\mu(x)$ ) can be prolonged to jet bundles to define vector fields  $j^k X$  on each  $J^k\mathcal{C}$ .

In general, let  $\Phi_s : C \rightarrow C$  be the flow of  $X$  which projects onto the flow  $\varphi_s : M \rightarrow M$  of the vector field  $\xi = \xi^\mu \partial_\mu \equiv (\pi)_* \Xi$  on  $M$ . For any configuration  $\sigma$  one can drag it along the flow by defining  $\sigma_s = \Phi_s \circ \sigma \circ \varphi_s^{-1}$ ; these can be easily checked to be configurations as well. Accordingly, one can define prolonged flows  $j^k \Phi_s : J^k C \rightarrow J^k \mathcal{C} : j_x^k \sigma \rightarrow j_{\varphi_s(x)}^k \sigma_s$  together with their infinitesimal generators  $j^k X = \frac{d}{ds} j^k \Phi_s|_{s=0}$ .

For example for  $k = 1$  one has that

$$j^1 X = \xi^\mu \partial_\mu + \xi^i \partial_i + (d_\mu \xi^i - d_\mu \xi^\lambda y_\lambda^i) \partial_i^\mu \quad (6)$$

is a good (*i.e.*, global) vector field on  $J^1\mathcal{C}$ . Here  $d_\mu = \partial_\mu + y_\mu^i \partial_i + y_{\mu\lambda}^i \partial_i^\lambda + \dots$  denotes the *total derivative* operator for (local) functions on  $J^k\mathcal{C}$ .

If one restricts to *deformations* (*i.e.*, vertical vector fields) prolongations are

$$\begin{aligned} X &= X^i \partial_i \\ j^1 X &= X^i \partial_i + d_\mu X^i \partial_i^\mu \\ j^2 X &= X^i \partial_i + d_\mu X^i \partial_i^\mu + d_{\mu\nu} X^i \partial_i^{\mu\nu} \\ &\dots \end{aligned} \quad (7)$$

Field equations of a ( $k$ -order) Lagrangian  $L$  are the differential equations which a configuration should obey in order to be critical. The ( $k$ -order) Lagrangian form is locally expressed as

$$L = \mathcal{L}(x^\lambda, y^i, y_\lambda^i, \dots) ds \quad (8)$$

where  $ds$  is the local canonical basis of  $m$ -forms on  $M$  induced by coordinates  $x^\mu$  (*i.e.*, local *volume*). The deformation is in the form  $X = X^i \partial_i = \delta y^i \partial_i$ .

By suitable integration by parts one can split the Lagrangian form in two parts, so that

$$\delta_X A_D[\sigma] = \int_D \mathbb{E}_i X^i ds + \int_{\partial D} \mathbb{F}_i^\mu X^i ds_\mu \quad (9)$$

where  $ds_\mu$  is the local canonical basis of  $(m - 1)$ -forms on  $M$  induced by coordinates  $x^\mu$ . This splitting is encoded in global bundle morphisms

$$\begin{cases} \mathbb{E} : J^{2k}C \rightarrow V^*(C) \otimes A_m(M) \\ \mathbb{F} : J^{2k-1}C \rightarrow V^*(J^{k-1}C) \otimes A_m(M) \end{cases} \quad (10)$$

where  $V^*(C)$  are dual to vertical vectors on  $C$ , while  $V^*(J^{k-1}C)$  are dual to vertical vectors on  $J^{k-1}C$ . Denoting the dualities by  $\langle | \rangle$  one has local expressions

$$\begin{cases} \langle \mathbb{E} | X \rangle = E_i X^i ds \\ \langle \mathbb{F} | j^{k-1} X \rangle = \mathbb{F}_i^\mu X^i ds_\mu \end{cases} \quad (11)$$

For a first order Lagrangian one has for instance

$$\begin{cases} \mathbb{E}_i = \frac{\partial \mathcal{L}}{\partial y^i} - d_\mu \frac{\partial \mathcal{L}}{\partial y_\mu^i} \\ \mathbb{F}_i^\mu = \frac{\partial \mathcal{L}}{\partial y_\mu^i} \end{cases} \quad (12)$$

While the *Euler-Lagrange morphism*  $\mathbb{E}$  is directly related to field equations (and it is unique at all orders) the *Poincaré–Cartan morphism*  $\mathbb{F}$ , that is not unique in general because of the non-uniqueness of the Poincaré–Cartan form, is more properly related to conservation laws.

Globally one can characterize the morphisms  $\mathbb{E}$  and  $\mathbb{F}$  by the so-called *first variation formula*, that once integrated gives (9)

$$\langle \delta L | j^k X \rangle = \langle \mathbb{E} | X \rangle + \text{Div} \langle \mathbb{F} | j^{k-1} X \rangle \tag{13}$$

where Div is the *divergence operator* that corresponds, after evaluation along a configuration, to exterior differential on forms on  $M$ .

Let us remark that in Field Theory most Lagrangians are highly degenerate, which means that they do not depend on all derivatives of fields but just on some (suitable) combinations of them. Usually, these combinations are chosen to define some geometrical object with simple transformation laws. In this case it is often convenient to proceed by *covariant* integration by parts, in order to control globality of each single term in the equations instead of controlling only the globality of the whole equation. Examples will be considered below.

Before turning our attention to symmetries let us mention that the role played by flows above can be replaced by more general objects. Let us consider a family of maps  $\Phi_s : J^1 C \rightarrow C$  that are locally expressed by

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y^i = Y^i(x^\lambda, y^l, y'_\lambda) \end{cases} \tag{14}$$

and represent a sort of field transformation depending on the derivatives of field (as it happens in general for supersymmetries, Backlund transformations, etc.). Also in this more general case—in which the transformation can no longer be interpreted as a geometrical transformation on the manifold  $C$ —one can still define an *infinitesimal generator*

$$\Xi = \xi^\mu(x^\lambda) \partial_\mu + \xi^i(x^\lambda, y^l, y'_\lambda) \partial_i \tag{15}$$

However, this object is no longer a vector field on  $C$  (since its components are not local functions on  $C$ ). Objects like this are called *generalized vector fields* and can be seen as sections  $\Xi$  of the bundle  $(\pi_0^k)^*(TC) \rightarrow J^k C$  that is defined by pull-back as follows:

$$\begin{array}{ccc} (\pi_0^k)^*(TC) & \longrightarrow & TC \\ \downarrow & & \downarrow \tau \\ J^k C & \xrightarrow{\pi_0^k} & C \\ & & \downarrow \pi \\ & & M \end{array}$$

$\Xi$  (curved arrow from  $(\pi_0^k)^*(TC)$  to  $J^k C$ )

By the usual abuse of language the section  $\Xi$  has local expression (15).

One can then extend the prolongation to generalized vector fields by formally using the same formulae (6) and (7). In the projective limit these families generate well-defined vector fields on  $J^\infty C$ , while they are, although in a not yet formalized way, generalized at each finite level  $J^k C$ .

We shall see below examples of such generalized vector fields when they represent symmetries and hence they define conservation laws. Let us stress that Emmy Noether was in fact the first to consider these examples just in her early studies aimed at showing a sort of inverse theorem; each conservation law can be generated by a suitable generalized symmetry.

### 3. Noether Theorem

Since field equations are mainly encoded in the geometry of jet prolongations (see [16]) Noether theorem can be understood in terms of Lie derivatives (see [17]).

Let  $\Xi = \xi^\mu \partial_\mu + \xi^i \partial_i$  be a (projectable) vector field on the configuration bundle  $\mathcal{C}$  that projects onto a vector field  $\xi = \xi^\mu \partial_\mu$  on  $M$  and let  $\sigma$  be a configuration; let us define the *Lie derivative* of  $\sigma$  with respect to  $\Xi$  to be

$$\mathcal{L}_\Xi \sigma = (\xi^\mu(x) y_\mu^i - \xi^i(x, y)) \partial_i \quad (16)$$

This is a (generalized) vertical vector field and it accounts for the change of the configuration when dragged along the flow of  $\Xi$ . The same expression holds true when  $\Xi$  is a generalized vector field by itself.

This Lie derivative is *natural* in the sense that it preserves commutators, *i.e.*,

$$\mathcal{L}_{[\Xi_1, \Xi_2]} = [\mathcal{L}_{\Xi_1}, \mathcal{L}_{\Xi_2}] \quad (17)$$

Of course this is true when commutators are considered as the commutators of (possibly generalized) vector fields on  $\mathcal{C}$ . It is essential to notice here that configurations can be in principle dragged along the flow of vector fields on  $\mathcal{C}$  only. In general, there is no dragging in  $\mathcal{C}$  along vector fields on the base  $M$ , as one is instead “physically” used to expect in many physically relevant cases (e.g. in relativistic applications). Accordingly, there is no reason to expect that in general one could define Lie derivative of configurations in  $\mathcal{C}$  along vector fields on  $M$ , nor that this can be done in such a way that they are natural, *i.e.*, they preserve commutators.

There are however specific bundles in which one could naturally associate a vector field  $\hat{\xi}$  on  $\mathcal{C}$  to each vector field  $\xi$  on  $M$  (as, e.g., natural bundles or, in a sense, gauge-natural bundles; see [9].) For naturality one has that

$$[\xi, \zeta]^\wedge = [\hat{\xi}, \hat{\zeta}] \quad (18)$$

One classical example of this situation is on tangent bundles  $\mathcal{C} = TM$ ; on any tangent bundle one has in fact a tangent lift of vector fields which preserves commutators. We have to stress that the existence of such a natural lift is a property of the bundle  $\mathcal{C}$ . When such a lift is defined the bundle  $\mathcal{C}$  is in fact called *natural*; one could prove that natural bundles are associated to (some finite higher order) frame bundle  $L^s(M)$  on the base manifold; see [18]. The tangent bundles are in fact always associated to the standard frame bundle  $L(M)$ .

Only on natural bundles one can define dragging along  $\xi$  and the corresponding Lie derivative is defined as

$$\mathcal{L}_\xi \sigma := \mathcal{L}_{\hat{\xi}} \sigma \quad (19)$$

which is in turn defined as above. Since the lift  $\xi \rightarrow \hat{\xi}$  is natural one can prove that this Lie derivative is also natural, *i.e.*,

$$\mathcal{L}_{[\xi, \zeta]} \sigma = \mathcal{L}_{[\hat{\xi}, \hat{\zeta}]} \sigma = [\mathcal{L}_{\hat{\xi}}, \mathcal{L}_{\hat{\zeta}}] \sigma \equiv [\mathcal{L}_\xi, \mathcal{L}_\zeta] \sigma \quad (20)$$

We shall present below various examples of this and similar situations. We wish to stress here that the naturality (17) of Lie derivatives on  $\mathcal{C}$  is essential for applications to conservation laws, while the naturality of the lift (18)—as well as the naturality (20) of the Lie derivatives on  $M$ —is not. These

are “good news,” since not all Field Theories have a natural lift; examples will be presented in Gauge Theories and Spinor Theories, in which one could not define a gauge covariant lift that is natural while conservation laws are still perfectly defined.

Noether theorem asserts a correspondence between Lagrangian symmetries and conservation laws. A *Lagrangian symmetry* is a (possibly generalized) vector field  $\Xi$  on the configuration bundle  $\mathcal{C}$  such that

$$\langle \delta L | j^k \mathcal{L}_{\Xi} \sigma \rangle = \text{Div} (i_{\xi} L + \langle \alpha | j^r \mathcal{L}_{\Xi} \sigma \rangle) \quad (21)$$

for some bundle morphism  $\alpha : J^{k-1}\mathcal{C} \rightarrow V^*(J^r\mathcal{C}) \otimes A_{m-1}(M)$ .

By using the first variation formula on the l.h.s. one can easily prove Noether theorem in the form

$$\langle \mathbb{E} | \mathcal{L}_{\Xi} \sigma \rangle + \text{Div} (\langle \mathbb{F} | j^{k-1} \mathcal{L}_{\Xi} \sigma \rangle) = \text{Div} (i_{\xi} L + \langle \alpha | j^r \mathcal{L}_{\Xi} \sigma \rangle) \quad (22)$$

that can be recasted as

$$\text{Div} (\langle \mathbb{F} | j^{k-1} \mathcal{L}_{\Xi} \sigma \rangle - i_{\xi} L - \langle \alpha | j^r \mathcal{L}_{\Xi} \sigma \rangle) = - \langle \mathbb{E} | \mathcal{L}_{\Xi} \sigma \rangle \quad (23)$$

By direct inspection, the Noether current defined as  $\mathcal{E} = \langle \mathbb{F} | j^{k-1} \mathcal{L}_{\Xi} \sigma \rangle - i_{\xi} L - \langle \alpha | j^r \mathcal{L}_{\Xi} \sigma \rangle$  is closed along critical configurations, which manifestly annihilate the r.h.s. Of course the difficulty in finding conservation laws is here replaced by the difficulties in finding Lagrangian symmetries  $\Xi$ . Examples will also be given below.

Symmetries can be described and treated directly also in terms of the Poincaré–Cartan form. Condition (21) can be written in an equivalent form as follows

$$\mathcal{L}_{j^{2k-1}\Xi} \Theta_L = d\alpha \quad (24)$$

for some  $(m-1)$ -form  $\alpha$ ; *i.e.*, a Lagrangian symmetry leaves the Poincaré–Cartan form invariant. This characterization of symmetries is beautiful and geometric, and this is a motivation to define Poincaré–Cartan form.

Noether theorem is then simply obtained in the usual form for Lie derivatives of forms, namely

$$\mathcal{L}_{j^{2k-1}\Xi} \Theta_L = \text{Div}(i_{j^{2k-1}\Xi} \Theta_L) + i_{j^{2k-1}\Xi} \text{Div} \Theta_L \quad (25)$$

The term  $i_{j^{2k-1}\Xi} \text{Div} \Theta_L = 0$  is nothing but field equations and the Noether current is given as

$$\mathcal{E} = i_{j^{2k-1}\Xi} \Theta_L - \alpha \quad (26)$$

which, of course, is closed on-shell.

Conservation laws are thence expressed as (on-shell) closures of Noether currents. If one chooses coordinates  $x^\mu = (t, x^i)$  adapted to a spacelike foliation of  $M$ , to mimic what one is used to do when choosing Cartesian coordinates in Minkowski space, the conservation law is expressed as

$$\dot{\rho} + \text{div} j = 0 \quad (27)$$

where we set  $\rho = \mathcal{E}^0$  and  $j^i = \mathcal{E}^i$  and  $\cdot$  denotes derivative with respect to time  $t$ . These kind of equations are called *continuity equations* since  $\rho$  is a density and the quantity

$$Q = \int_D \rho = \int_D \mathcal{E} \quad (28)$$

(for any spacelike  $(m - 1)$ -region  $D$  in  $M$ ) is *conserved*, in the sense that its variations are controlled by the flow of  $j$  through the boundary of  $D$ . Having said that, conservation laws  $d\mathcal{E} = 0$  are nothing but the covariant form of continuity equations.

For a wide class of Field Theories (including Natural and Gauge-Natural Theories) one can also show that Noether currents are not only closed forms on-shell, but they are also *exact*; see [9]. One can in fact show, by defining an explicit and algorithmic procedure of covariant integration by parts, that in those cases the Noether current can be (globally) recasted as

$$\mathcal{E} = \tilde{\mathcal{E}} + \text{Div}\mathcal{U} \quad (29)$$

The  $(m - 1)$ -form  $\tilde{\mathcal{E}}$  is called the *reduced current* and it vanishes on-shell, while the  $(m - 1)$ -form  $\mathcal{U}$  is called the *superpotential*. Examples will be presented below. Accordingly, the Noether current is written on-shell as the differential of the superpotential.

The corresponding conserved quantities are thence defined as surface integrals of the superpotential  $Q = \int_{\Omega} \mathcal{U}$  on  $(m - 2)$ -regions  $\Omega \subset M$ . This establishes a deep connection between conserved quantities and cohomology of the spacetime manifold  $M$ .

The conservation laws involving Noether current (*i.e.*,  $d\mathcal{E} = 0$ ) hold on on-shell, *i.e.*, along critical configurations, and therefore they are called *weak conservation laws*. When one defines the superpotential  $\mathcal{U}$  then an equivalent conservation law can be written as  $d(\mathcal{E} - \tilde{\mathcal{E}}) = d(\text{Div}\mathcal{U})$ ; this holds true for each single configuration (also non-critical). For such a reason these conservation laws induced by superpotentials are also called *strong conservation laws*.

In the case of Mechanics one can use the augmented de-Rham sequence

$$0 \rightarrow \mathbb{R} \rightarrow A_0(M) \rightarrow A_1(M) \rightarrow \dots \quad (30)$$

and the Noether current is an element in  $A_0(M)$  that is closed on-shell. In view of exactness at the relevant level, Noether currents in Mechanics are therefore *constant* on-shell and they are called *first integrals*.

Finally, let us remark that everything we said above applies also to Lagrangian symmetries  $\Xi$ ; equivalently, one has a flow of transformations that leaves the Poincaré–Cartan form invariant, namely the flow of  $\Xi$ , *i.e.*, a 1-parameter group of symmetries. In many interesting cases, some of which will be discussed below, Field Theories have a symmetry group of dimension higher than 1.

Noether current, which depends on the symmetry generator, can also be considered as a map from the Lie algebra of (infinitesimal) symmetries to (on-shell) closed  $(m - 1)$ -forms. Such a map is usually called *momentum map* (see [2]) and the naturality with respect to Lie derivation is the remnant of the group structure of symmetries and their preservation.

Let us stress that symmetries are required to form a group of transformations of configurations, not necessarily a group of transformations of spacetime. Sometimes a group of spacetime transformations naturally induces a group of transformations on configurations (e.g., when configurations are represented in terms of spacetime tensors or more generally by geometrical objects) but in general it does not. From this viewpoint, Lie derivatives of fields with respect to spacetime vector fields is an additional (and not necessary) structure; one somehow needs to require naturality (17) while the lift (19) and its naturality (18) is not always available (nor, in fact, necessary).

#### 4. Applications to Mechanics

As examples in Mechanics we shall briefly review some standard examples with the aim of fixing notation.

The standard framework for first order Mechanics is based on a configuration manifold  $Q$  with points that correspond to system configurations; local coordinates  $q^i$  are also known as Lagrangian coordinates for the system. The histories of the system are encoded by (parametrized) curves  $\gamma : \mathbb{R} \rightarrow Q$  in the configuration space  $Q$ .

Equivalently, one can define, as discussed above, the configuration bundle  $\mathcal{C}$ ; by means of a reference frame (*i.e.*, a connection on  $\mathcal{C}$ ) one can define a global trivialization  $\mathcal{C} \simeq \mathbb{R} \times Q$ . We shall denote fibered coordinates on  $\mathcal{C}$  by  $(t, q^i)$ .

In this setting histories are in 1-to-1 correspondence with sections of the configuration bundle, that will be denoted by an abuse of language again by  $\gamma$ . Since we are restricting to the first order Mechanics, the Lagrangian should be given on the first jet prolongation  $J^1(\mathbb{R} \times Q)$ . By means of the reference frame we have a special set of motions singled out in the configuration bundle; these denote “rest motions” and can be prolonged (as we saw for any motion) to first jet bundle. They define a global isomorphism  $J^1(\mathbb{R} \times Q) \simeq \mathbb{R} \times TQ$  since they define what one has to understand for “zero velocity” (of course depending on the reference frame).

Accordingly, we can use fibered coordinates  $(t, q^i, u^i)$  on  $\mathbb{R} \times TQ$ , where  $u^i$  denote the Lagrangian velocities. The dynamics is described by a Lagrangian

$$L = \mathcal{L}(t, q^i, u^i)dt \quad (31)$$

or equivalently by a Poincaré–Cartan form

$$\Theta_L = \mathcal{L}(t, q^i, u^i)dt + p_i \omega^i \quad (32)$$

where we set  $p_i = \frac{\partial \mathcal{L}}{\partial u^i}(t, q, u)$  and  $\omega^i = dq^i - u^i dt$  for the relevant contact 1-form on  $\mathbb{R} \times TQ$ .

The Noether theorem presented in general above is generalized in order to encompass the standard one for:

- i)* a symmetry that is a vertical vector field on the configuration bundle, *i.e.*,  $\Xi = \xi^i(q)\partial_i$  such that its tangent prolongation  $\hat{\Xi}$  leaves the Lagrangian invariant ( $\hat{\Xi}(L) = 0$ );
- ii)* a symmetry that is a (non-vertical) vector field  $\Xi$  on the configuration bundle, e.g., if the Lagrangian density is independent of time  $t$  then  $\Xi = \partial_t$  is in fact a symmetry;
- iii)* vector fields that leave *essentially* invariant the system (e.g., when in the notation introduced above one has  $\alpha \neq 0$ );
- iv)* generalized vector fields to obtain first integrals that are not simply linear in the momenta, as for example the Runge-Lenz vector in Kepler’s motion.

#### 4.1. Case (i)

One can always change fibered coordinates to new fibered coordinates  $(t, q^i)$  in which  $\Xi = \partial_t$ . That is a symmetry iff the Lagrangian does not depend on  $q^1$ , i.e., it is *cyclic*. In other words these cases correspond to all cases of ignorable coordinates, though  $\Xi$  is a symmetry in all coordinate systems, while  $q^1$  is cyclic only when  $\Xi = \partial_t$ . Thence these cases are equivalent to a coordinate free notation for cyclic coordinates.

The corresponding first integral is

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial u^i} \xi^i \quad (33)$$

that in adapted coordinates corresponds in fact to the momentum  $p_1$  conjugated to the cyclic coordinate  $q^1$ .

For example, a particle in a constant gravitational field is described by the Lagrangian

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \quad (34)$$

in Cartesian coordinates  $(x, y, z)$ .

The coordinate  $x$  ( $y$ , respectively) is cyclic and the corresponding momentum  $p_x = m\dot{x}$  ( $p_y = m\dot{y}$ , respectively) is a first integral that corresponds to *linear momentum*.

The vector field  $\Xi = x\partial_y - y\partial_x$  is a symmetry and it corresponds to the fact that in cylindrical coordinates  $(r, \theta, z)$  the angular coordinate  $\theta$  is cyclic. The corresponding first integral

$$\mathcal{E} = m(x\dot{y} - y\dot{x}) \quad (35)$$

corresponds to the  $z$ -component of *angular momentum*.

In cylindrical coordinates one has in fact  $\Xi = \partial_\theta$  and the corresponding first integral is given by  $\mathcal{E} = mr^2\dot{\theta}$ .

#### 4.2. Case (ii)

When the Lagrangian is independent of time  $t$  (e.g., whenever holonomic constraints are imposed with no explicit time dependence) the vector field  $\Xi = \partial_t$  is a symmetry and the corresponding first integral is given by

$$\mathcal{H} = p_i u^i - L \quad (36)$$

that corresponds to *mechanical energy*.

In the case of particles in a constant gravitational field one has

$$\mathcal{H} = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz \quad (37)$$

#### 4.3. Case (iii)

For the Lagrangian (34) the vector field  $\Xi = \partial_z$  fails to leave the Lagrangian invariant; in fact one has Lie derivatives  $\mathcal{L}_{\Xi}x = \mathcal{L}_{\Xi}y = 0$  and  $\mathcal{L}_{\Xi}z = -1$ . The l.h.s. of the covariance identity (21) reads then as

$$\langle \delta L | j^k \mathcal{L}_{\Xi} \sigma \rangle = -mg \equiv \frac{d}{dt} (-mgt) \quad (38)$$

that fails to vanish though it is easily recognized to be the total derivative of a quantity  $\alpha = -mgt$ . This is for sure physically expected since, in view of gravitational field constancy, the system is unchanged if everything is translated up along the  $z$ -axis. If this is physically trivial, one should notice that mathematically one needs to consider generalized symmetries even to encompass these simple examples.

Noether theorem hence applies and the corresponding first integral is

$$F = m(-\dot{z} + gt) \quad (39)$$

In this case we know the general solution of the equations of motion ( $x = x_0^1 + u_0^1 t$ ,  $y = x_0^2 + u_0^2 t$ ,  $z = x_0^3 + u_0^3 t - \frac{1}{2}gt^2$ ) and it is easy to show that the quantity  $F \equiv -mu_0^3$  is in fact constant on-shell.

#### 4.4. Case (iv)

For Kepler system

$$L = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{\kappa}{r} \quad (40)$$

one can consider the following generalized vector fields

$$\begin{aligned} \Xi_1 &= -r^2 \dot{\theta} \sin \theta \partial_r - (2r \dot{\theta} \cos \theta + \dot{r} \sin \theta) \partial_\theta \\ \Xi_2 &= r^2 \dot{\theta} \cos \theta \partial_r + (-2r \dot{\theta} \sin \theta + \dot{r} \cos \theta) \partial_\theta \end{aligned} \quad (41)$$

These are symmetries; in fact, the l.h.s. of the covariance identity for the first vector reads as

$$\langle \delta L | j^k \mathbf{f}_{\Xi_1} \sigma \rangle = \frac{d}{dt} (r^2 \sin \theta \dot{r} \dot{\theta} + r^3 \dot{\theta}^2 \cos \theta + \kappa \cos \theta) \quad (42)$$

while for the second vector it reads as

$$\langle \delta L | j^k \mathbf{f}_{\Xi_2} \sigma \rangle = -\frac{d}{dt} (r^2 \cos \theta \dot{r} \dot{\theta} - r^3 \dot{\theta}^2 \sin \theta - \kappa \cos \theta) \quad (43)$$

The corresponding first integrals  $R_A = -\frac{\partial \mathcal{L}}{\partial u^i} \xi_A^i - \alpha$  ( $A = 1, 2$ ) are

$$\begin{aligned} R_1 &= r^3 \dot{\theta}^2 \cos \theta + r^2 \dot{r} \dot{\theta} \sin \theta - \kappa \cos \theta \\ R_2 &= r^3 \dot{\theta}^2 \sin \theta - r^2 \dot{r} \dot{\theta} \cos \theta - \kappa \sin \theta \end{aligned} \quad (44)$$

One can easily check that these are the two components of the vector field

$$\vec{R} = R_1 \partial_x + R_2 \partial_y = \mathbf{v} \times (r \times \mathbf{v}) - \kappa \partial_r \quad (45)$$

which is called *Laplace vector* or *Runge-Lenz vector*. In other words the vector  $\vec{R}$  is constant on-shell. This vector was known in Kepler problems (or, equivalently, in Coulomb electrostatics) to be related to the fact that perihelia are fixed (and when perturbations are introduced it relates to the precession of perihelia).

Notice that the vector  $\vec{R}$  is quadratic in the Lagrangian velocities. Since momenta are linear in the Lagrangian velocities this means that  $\vec{R}$  is quadratic in the momenta. Hence, it could not be obtained by ordinary Noether theorem which produces only first integrals linear in the momenta.

## 5. GR and Natural Theories

General Relativity (GR) is based on the principle of general covariance (together with the equivalence principle). General covariance is a symmetry requirement; one assumes that spacetime diffeomorphisms  $\text{Diff}(M)$  act on configurations (*i.e.*, fields are *geometrical objects*, e.g., tensor fields) and all these transformations induced by spacetime diffeomorphisms are symmetries for the dynamics.

This assumption combined with Noether theorem has plenty of consequences. The configuration bundle is a *natural bundle*, the Lie derivatives are defined with respect to spacetime vector fields that are all symmetries and they all generate conservation laws. Moreover, one can show (see [9]) that Noether currents always admit superpotentials and conservation laws are always defined *à la* Gauss by surface integrals.

Moreover, in such kind of theories the whole set of conservation laws are equivalent to the dynamics of the system. In fact, one has a  $k$ -order Lagrangian  $L$  which defines field equations  $\mathbb{E} = 0$  via first variation formula

$$\langle \delta L|j^k X \rangle = \langle \mathbb{E}|X \rangle + \text{Div} \langle \mathbb{F}|j^{k-1} X \rangle \quad (46)$$

First variation formula holds in particular for the Lie derivative along each spacetime vector field  $X = L_\xi \sigma$  and gives conservation laws

$$\text{Div} \mathcal{E} = - \langle \mathbb{E}|L_\xi \sigma \rangle \quad (47)$$

for the Noether currents  $\mathcal{E} = \langle \mathbb{F}|j^{k-1} L_\xi \sigma \rangle - i_\xi L$ . Then one can define superpotentials  $\mathcal{U}$

$$\mathcal{E} = \tilde{\mathcal{E}} + \text{Div} \mathcal{U} \quad (48)$$

The reduced current is always a combination of field equations. Thus if one knows Noether currents of a Natural Theory and conserved quantities by means of their superpotentials, then one can compute the reduced currents (*i.e.*, field equations) purely in terms of conservation laws.

For example, let us consider “standard GR”, which is defined as a second order theory on the configuration bundle  $\text{Lor}(M)$  of Lorentian metrics on spacetime  $M$ , with coordinates  $(x^\mu, g_{\mu\nu})$ . Dynamics is defined by the Hilbert Lagrangian

$$L_H = \sqrt{g} R ds \quad (49)$$

where  $R$  is the scalar curvature of the metric  $g$  and  $\sqrt{g}$  denotes the square root of the absolute value of the determinant of the metric  $g_{\mu\nu}$ .

The variation of this Lagrangian along the deformation  $X = \delta g^{\alpha\beta} \partial_{\alpha\beta}$

$$\langle \delta L_H|j^2 X \rangle = \sqrt{g} \left( R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \right) \delta g^{\alpha\beta} ds + \nabla_\lambda (\sqrt{g} g^{\alpha\beta} \delta u_{\alpha\beta}^\lambda) ds \quad (50)$$

where  $u_{\alpha\beta}^\lambda = \{g\}_{\alpha\beta}^\lambda - \delta_{(\alpha}^\lambda \{g\}_{\beta)\epsilon}$  and  $\{g\}_{\alpha\beta}^\lambda$  denotes the Levi-Civita connection of the metric  $g$ , *i.e.* its Christoffel symbols.

Thence we have

$$\langle \mathbb{E}|X \rangle = \sqrt{g} \left( R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \right) \delta g^{\alpha\beta} ds \quad \langle \mathbb{F}|j^1 X \rangle = \sqrt{g} g^{\alpha\beta} \delta u_{\alpha\beta}^\lambda ds_\lambda \quad (51)$$

Let us stress that *each* spacetime vector field  $\xi$ —and not only Killing vectors as sometimes erroneously claimed in the literature—generates conservation laws. By expanding the Lie derivatives in terms of the spacetime vectors field one obtains Noether current as

$$\mathcal{E} = \sqrt{g} \left( \left( \frac{3}{2} R^{\alpha}_{\cdot\lambda} - R \delta^{\alpha}_{\lambda} \right) \xi^{\lambda} + \left( g^{\beta\gamma} \delta^{\alpha}_{\lambda} - g^{\alpha(\gamma} \delta^{\beta)}_{\lambda} \right) \nabla_{\beta\gamma} \xi^{\lambda} \right) ds_{\alpha} \quad (52)$$

which by suitable covariant integration by parts can be recasted as  $\mathcal{E} = \tilde{\mathcal{E}} + \text{Div}\mathcal{U}$ , where we set

$$\tilde{\mathcal{E}} = 2\sqrt{g} \left( R^{\alpha}_{\cdot\lambda} - \frac{1}{2} R \delta^{\alpha}_{\lambda} \right) \xi^{\lambda} ds_{\alpha} \quad \mathcal{U} = \sqrt{g} \nabla^{\beta} \xi^{\alpha} ds_{\alpha\beta} \quad (53)$$

The superpotential  $\mathcal{U}$  is called *Komar potential* in honor of Komar who first proposed it, though originally restricted to a timelike Killing vector  $\xi$ , while here  $\xi$  is instead *any* spacetime vector field.

As we discussed above, one can deduce field equations  $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0$  from Noether current  $\mathcal{E}$  and Komar superpotential  $\mathcal{U}$  by computing reduced current as

$$\tilde{\mathcal{E}} = \mathcal{E} - \text{Div}\mathcal{U} \quad (54)$$

just noticing that this quantity has to vanish for all solutions and all symmetry generators  $\xi$ .

This situation, in fact, is completely general for any Natural Theory of any order and any matter coupling, since it is based on general covariance principle only. Any Natural Theory comes with a huge symmetry group, namely  $\text{Diff}(M)$ , which identifies *intrinsically* (*i.e.*, in an observer independent way) a huge set of conservation laws and conserved quantities.

Problems start when one wants to identify some physically relevant quantity—e.g., the energy, the momentum, the angular momentum, ...—within this intrinsic set of conservation laws. This is already a problem in Newtonian Mechanics (see [19,20]) where it should be clear from the very beginning that there is no intrinsic notion as *the* energy of a system, in the sense that different observers (even inertial observers) do in fact measure different energies, momenta, ... for the same system. This obvious circumstance is often undervalued (or even ignored) in current literature on Mechanics, with the consequence of generating a number of misunderstandings about conservation laws that reverberate and amplify in Field Theories. This would indeed be a trivial remark if it were not used sometimes in the literature to argue that conserved quantities in GR must have a non-covariant genesis.

Such arguments come down to (at least) two different main points:

- i*) covariant conservation laws are not conservation laws;
- ii*) covariant conserved quantities would not depend on the observer as physically expected and as pseudotensorial prescriptions do.

Both these points have a long history and can be somehow traced back to Einstein himself. However, after almost one century of investigations they can be today shown to be flat wrong at least in some sense.

### 5.1. Covariant Conservation Laws

Item (*i*) comes historically from the observation that covariant conservation laws

$$\nabla_{\mu} \mathcal{E}^{\mu} = 0 \quad (55)$$

would not reduce to continuity equations (which is what one usually means for “conservation”) due to terms depending on the connection  $\{g\}_{\alpha\beta}^{\lambda}$  used to define the covariant derivatives. This would be certainly the case if  $\mathcal{E}^{\mu}$  were components of a vector field. Then *covariant conservation laws* would be *conservation laws tout court* only for those observers (here identified with coordinate systems) in which such terms vanish (*i.e.*, when  $\{g\}_{\alpha\beta}^{\lambda} = 0$ ). Only in these cases covariant conservation laws are genuine conservation laws and they are intrinsically non-covariant since they break down general covariance.

This argument is true for all currents  $\mathcal{E}$  except in one single case: the case in which  $\mathcal{E}^{\mu}$  is a vector density of weight 1. In this case there are two terms depending on the connection and they cancel out. In other words, when  $\mathcal{E}^{\mu}$  is a vector density of weight 1 the covariant divergence is automatically identical to the *ordinary divergence*, *i.e.*,

$$\nabla_{\mu}\mathcal{E}^{\mu} \equiv d_{\mu}\mathcal{E}^{\mu} \quad (56)$$

and they always define true continuity equations for any observer.

Of course, Noether theorem in the form presented here dictates for Noether currents to be  $(m - 1)$ -forms, *i.e.*, their components  $\mathcal{E}^{\mu}$  are in fact vector densities of weight 1. Hence it always produces authentic conservation laws which are at the same time *covariant*.

## 5.2. Observers

Item (*ii*) comes from the belief that covariant conservation laws would necessarily define conserved quantities independent of the observer. This is certainly true for the whole set of conservation quantities; it is defined covariantly and all observers agree on it.

However, each observer can then be asked to identify within the set of all conserved quantities which one represents a physically relevant quantity, e.g., the energy. It is not the notion of first integral to be observer dependent; a quantity either changes or not during the evolution of the system and all observers agree on it. But it is rather the *energy* that is related (in Newtonian Mechanics, but also in Special Relativity) to time translations; since in SR *time* depends on the observer, the notion of *energy* depends on the observer, accordingly.

In GR each observer (which is identified with a local coordinate system together with a protocol to synchronize clocks at different points, that foliates spacetime with leaves representing synchronous event sets; such foliations are known as ADM-foliations, see [21]) comes with a timelike vector field  $\xi$ . The Noether quantity associated to such a timelike vector field (no request about it being Killing; see [22]) is defined to be the *energy for the observer for which timeflow is given by  $\xi$* .

Since the prescription is covariant, any other observer agrees that this energy is the Noether quantity associated to that timeflow  $\xi$ ; however, different observers might disagree on  $\xi$  being the timeflow and consequently that such a conserved quantity has to be considered to be *the energy* of the system. On the contrary, they use *their* own timeflow  $\xi'$  and define *their own energy*. In this way, which is the only reasonable way to extend to GR what one is used to do in SR, despite the prescription is covariant still the *energy* is not and still it depends on the observer.

It is therefore an open issue whether we really need to know what is the *energy* or we could have a fundamentally good description of the physical world by just using the set of conserved quantities. In other words, if energy is just something we are used from Newtonian Physics, or rather one can produce

a fundamental description of the physical world without singling out special conserved quantities to be given special meanings.

One could also ask whether it is useful to have covariant prescriptions for conserved quantities, since sooner or later covariance have to be broken in favor of observer-dependent quantities. Why one should not be satisfied with coordinate-depending prescriptions such as the ones based on pseudotensors? Let us counter-argue that on manifolds there is nothing defined as coordinate dependent integrals to define conserved quantities from pseudotensors. Pseudotensors are coordinate dependent quantities, often their genesis from geometric objects is not clear, they are known in some coordinate system but they are often unspecified in other coordinates. Moreover—and for obvious reasons—they lack of any coherent and (mathematically and physically) intrinsic meaning, if any may even exist!

We do not claim that pseudotensors should be completely forbidden in Physics, since they appear to be often “useful”; but we believe that for a reasonable, global and covariant interpretation their genesis from some geometric object must be always made explicit and, in any case, they have to be treated *cum grano salis*. Pseudotensors should be defined starting from geometric objects by fixing coordinates in order to neglect some term. This scenario has two good features to be noticed: it implicitly defines the coordinate systems in which pseudotensors can be used and it defines its integrals by means of the integral of the original geometric object. Let us refer to [22] for an example of this coherent strategy.

### 5.3. Extended Theories of Gravitation

As we said above, the structure of GR is not peculiar of the Hilbert Lagrangian (49); most of this structure is determined by the symmetry group, *i.e.*, spacetime diffeomorphisms, and it is a feature of any generally covariant theory.

Let us here briefly consider the class of *Extended Theories of Gravitation* (ETG); see [23–25]. This class of theories is used in Cosmology and Astrophysics in order to model phenomena and observations that are usually related to dark matter and dark energy; [26].

Let us first consider a Lorentzian metric  $g_{\mu\nu}$  and a connection  $\Gamma_{\mu\nu}^\lambda$  as fundamental fields. The configuration bundle is the product  $\text{Lor}(M) \times C(M)$  of the bundle of Lorentzian metrics and the bundle of connections on  $M$  (here assumed for simplicity with no torsion). Here connections are in principle independent of the metric field.

Let us denote by  $R^\alpha_{\beta\mu\nu}$  ( $R_{\beta\nu}$ ) the Riemann (Ricci) tensor of the connection  $\Gamma$  and by  $R = g^{\beta\nu} R_{\beta\nu}$  the scalar curvature which depends on both the metric and the connection. These *curvatures* are not to be confused with the curvatures  ${}^{(g)}R^\alpha_{\beta\mu\nu}$ ,  ${}^{(g)}R_{\beta\nu}$ ,  ${}^{(g)}R$  of the metric  $g$  alone. This double affine structure (the one induced by  $\Gamma$  and the one induced by  $g$ ) opens a number of issues about physical interpretation; see [27].

In the notation here introduced let us consider the Lagrangian

$$L_f = \sqrt{g} f(R) ds \quad (57)$$

where  $f$  is a *generic* analytic function of the scalar curvature. For the specific case  $f(R) = R$  this reduces to standard Hilbert-Einstein Gravity (49).

When varying, the metric and connection are deformed independently and there are field equations from variation of both fields:

$$\begin{cases} f'(R)R_{(\mu\nu)} - \frac{1}{2}f g_{\mu\nu} = 0 \\ \nabla_\lambda (\sqrt{g}f'g^{\mu\nu}) = 0 \end{cases} \quad (58)$$

The second equation is solved defining a new metric field  $h_{\mu\nu} = f'(R)g_{\mu\nu}$  conformal to the original one; the connection is then forced to be the Levi-Civita connection of the metric  $h$ , *i.e.*,  $\Gamma_{\beta\mu}^\alpha = \{h\}_{\beta\mu}^\alpha$ .

The trace (by  $g^{\mu\nu}$ ) of the first equation  $f'(R)R - 2R = 0$  is called the *master equation* and (once  $f$  is considered a fixed function) it is *algebraic* in the scalar curvature  $R$ . In general the master equation allows to solve the scalar curvature as a function of the matter sources; here we are considering the vacuum case and  $R$  is constant (on  $M$ ) and determined by the zeroes of the master equation (which are finite and generically simple). When this information is plugged into the first field equation one can show that the metric  $h$  obeys *modified* Einstein equations; the function  $f$  can be chosen so that the modified dynamics accounts for dark matter/dark energy phenomenology with no need to introduce exotic matter sources, which currently cannot be observed directly.

Again these models are generally covariant and any spacetime diffeomorphism is a symmetry of the theory; they are Natural Theories. Any spacetime vector field  $\xi$  is a symmetry generator and its Noether current is given by

$$\begin{aligned} \mathcal{E} &= \sqrt{g} (f'(R) g^{\alpha\beta} \mathcal{L}_\xi u_{\alpha\beta}^\lambda - \xi^\lambda f(R)) ds_\lambda \\ &= 2\sqrt{h} \left( {}^{(h)}R^\lambda{}_\epsilon - \frac{1}{4} {}^{(h)}R \delta_\epsilon^\lambda \right) \xi^\epsilon ds_\lambda + \nabla_\epsilon \left( \sqrt{h} \nabla^{[\epsilon} \xi^{\lambda]} \right) ds_\lambda \end{aligned} \quad (59)$$

where we used the master equation,  $\Gamma = \{h\}$  and the curvature tensors of the conformal metric  $h$  are denoted by  ${}^{(h)}R^\alpha{}_{\beta\mu\nu}$ ,  ${}^{(h)}R_{\beta\nu}$  and  ${}^{(h)}R$ .

This corresponds to define the reduced current and the superpotential as:

$$\tilde{\mathcal{E}} = 2\sqrt{h} \left( {}^{(h)}R^\lambda{}_\epsilon - \frac{1}{4} {}^{(h)}R \delta_\epsilon^\lambda \right) \xi^\epsilon ds_\lambda \quad \mathcal{U} = \sqrt{h} \nabla^{[\epsilon} \xi^{\lambda]} ds_{\lambda\epsilon} \quad (60)$$

Notice once again that the reduced current  $\tilde{\mathcal{E}}$  identically vanishes on-shell and it is equivalent to field equations.

ETG are also considered in the purely metric formalism, in which one considers the connection  $\Gamma$  to be  $\Gamma = \{g\}$  from the beginning, *i.e.*, at the kinematical level. The configuration bundle is simply  $\text{Lor}(M)$  and the Lagrangian (57) (now considered as a function of the metric  $g$  alone) is second order. This leads to forth order equations with modified effects with respect to standard GR.

From this point of view standard GR is shown to be degenerate (it has second order field equations) with respect to the extensions considered here (which generically have forth order field equations). Let us remark how the structure of symmetry group and conservation laws is instead similar in standard GR and ETG. It does not depend in fact on the particular dynamics considered but just on the principle of general covariance.

## 6. Gauge Theories

Natural Theories are defined starting from a spacetime manifold  $M$  together with its diffeomorphisms that encode symmetries. It is well-known that this is suitable for some physical systems (specifically,

Relativistic Theories) while there are more general physical systems (e.g., Gauge Theories) which need bigger symmetry groups. Gauge Theories are needed today to describe some fundamental interactions and together with GR they are enough for Fundamental Physics.

A Gauge Theory depends on a principal bundle  $P$  with a structure Lie group  $G$  over spacetime  $M$ . Principal automorphisms  $\text{Aut}(P)$  form a transformation group; this group projects onto  $\text{Diff}(M)$ , but  $\text{Diff}(M)$  is not in general realised as a subgroup of  $\text{Aut}(P)$ . Automorphisms  $\Phi \in \text{Aut}(P)$  are realised locally as changes of coordinates together with an element of  $G$  acting at all spacetime points (which in Physics is called a *local action* of  $G$ ).

A Gauge-Natural Theory is a Field Theory in which  $\text{Aut}(P)$  acts on configurations generating Noether symmetries of the dynamics. This request forces the configuration bundle to be a gauge-natural bundle (see [9,28,29]); all fundamental theories of Physics can be cast in this form.

As in Natural Theories, also in Gauge-Natural Theories one can associate a Noether current to any right-invariant vector field on  $P$  (i.e., to each generator of principal automorphisms). Noether currents always allow superpotentials (see [9]) and field equations can be obtained purely from conservation laws. Yang–Mills theories will be discussed below.

### 6.1. Yang–Mills Theory

Let us briefly present Yang–Mills theories as a paradigm for Gauge(-Natural) Theories. Also Dirac spinors share most of the characteristics of Gauge Theories and can in fact be formulated as a Gauge-Natural Theory; see [9].

Let us consider a semisimple Lie group  $G$  and denote by  $\delta$  the Cartan–Killing (ad-invariant) metric on its Lie algebra  $\mathfrak{g}$ . Let  $P$  be a principal bundle with structure group  $G$  and  $C(P)$  be the bundle of principal connections on  $P$ . Automorphisms act on connections so that  $C(P)$  is a gauge-natural bundle. Once one fixes a  $\delta$ -orthonormal basis  $T_A$  in the Lie algebra  $\mathfrak{g}$  a connection is expressed by coefficients  $\omega_\mu^A(x)$ ; the curvature is defined by  $F_{\mu\nu}^A = d_\mu\omega_\nu^A - d_\nu\omega_\mu^A + c^A_{BC}\omega_\mu^B\omega_\nu^C$ , where  $c^A_{BC}$  are “structure constants” of the group  $G$ . An automorphism on  $P$  is locally expressed as

$$\begin{cases} x'^\mu = x^\mu(x) \\ g' = \varphi(x) \cdot g \end{cases} \quad (61)$$

for some local map  $\varphi : M \rightarrow G$ ,  $\cdot$  being the product in the group; it acts on the curvature as

$$F'^A_{\mu\nu} = \text{ad}_B^A(\varphi) F^B_{\rho\sigma} \bar{J}_\mu^\rho \bar{J}_\nu^\sigma \quad (62)$$

where  $\bar{J}_\mu^\rho$  is the anti-Jacobian of the coordinate change and  $\text{ad}_B^A$  is the adjoint action of the group  $G$  on its Lie algebra  $\mathfrak{g}$ .

The Yang–Mills Lagrangian

$$L_{YM} = -\frac{1}{4}\sqrt{g}\delta_{AB}F_{\mu\nu}^A F_{\rho\sigma}^B g^{\mu\rho} g^{\nu\sigma} ds \quad (63)$$

defines the dynamics and gauge transformations (i.e., automorphisms in  $P$ ) are symmetries.

The first variation formula of the Lagrangian with respect to the deformation  $X = \delta g^{\alpha\beta} \partial_{\alpha\beta} + \delta\omega_\mu^A \partial_\mu^A$  defines the morphisms

$$\langle \mathbb{E}|X \rangle = (\nabla_\mu (\sqrt{g}F_A^{\mu\nu}) \delta\omega_\nu^B + \sqrt{g}H_{\mu\nu} \delta g^{\mu\nu}) ds \quad \langle \mathbb{F}|X \rangle = -\sqrt{g}F_A^{\mu\nu} \delta\omega_\nu^B ds_\mu \quad (64)$$

where we set  $F_A^{\mu\nu} := \delta_{AB} F_{\rho\sigma}^B g^{\mu\rho} g^{\nu\sigma}$  and  $H_{\mu\nu} := F_{\mu\lambda}^A F_{A\nu}^\lambda - \frac{1}{4} F_{\lambda\sigma}^A F_A^{\lambda\sigma} g_{\mu\nu}$ . The tensor  $H_{\mu\nu}$  is also known as the energy-momentum tensor of the Yang–Mills field. If the metric is frozen, then field equations are  $\nabla_\mu (\sqrt{g} F_A^{\mu\nu}) = 0$ , *i.e.*, they are the Yang–Mills equations. In general, the Yang–Mills Lagrangian is coupled to gravity and  $H_{\mu\nu}$  acts as a source for the gravitational field.

Given a pointwise basis  $\rho_A$  of vertical right-invariant vector fields on  $P$ , symmetry generators are in the form  $\Xi = \xi^\mu(x)\partial_\mu + \xi^A(x)\rho_A$ . The corresponding Noether current is

$$\mathcal{E} = -\sqrt{g} (H^\nu{}_\mu \xi^\mu + F_A^{\nu\mu} \nabla_\mu \xi_V^A) ds_\nu \quad (65)$$

where we set  $\xi_V^A := \xi^A + \omega_\mu^A \xi^\mu$  for the vertical part of  $\Xi$  with respect to the dynamical connection  $\omega$ .

The superpotential is given by

$$\mathcal{U} = -\frac{1}{2} \sqrt{g} F_A^{\nu\mu} \xi_V^A ds_{\mu\nu} \quad (66)$$

In the special case  $G = U(1)$  (*i.e.*, Maxwell's Electromagnetism) the group is commutative and the adjoint representation is trivial; the curvature is then a 2-form on spacetime and it is invariant with respect to (proper) gauge transformations (*i.e.*, vertical automorphisms). In this case Yang–Mills theory reduces to standard Electromagnetism and the connection  $\omega_\mu$  represents the quadripotential of electromagnetic field. Field equations reduce to Maxwell equations and gauge charges reduce to electric charges.

## 6.2. Hole Argument

There is another important crosspoint among GR, Gauge Theories and their symmetries. It is called the *hole argument* and it was first discovered in GR by Einstein, though it persists in Gauge Theories as well. It is originated when among symmetries of the system there are compact supported symmetries, *i.e.*, symmetries that vanish (or become the identity) out of a compact sub-domain of spacetime  $M$ .

In GR one can easily define compact supported spacetime diffeomorphisms (and vector fields) on any paracompact spacetime manifold  $M$ . In Gauge Theories one can define compact supported automorphisms. Of course in Mechanics it is difficult to find examples of compact supported symmetries (e.g., there are no compact supported rotations in space). The only well-known exception is curve reparametrization in SR; there Physics is totally represented in terms of trajectories in spacetime (the so-called *worldlines*) and their parametrizations are introduced for technical reasons only, since they are not endowed with any physical meaning. Equations of motion should however induce equations on trajectories; hence they must be invariant with respect to general reparametrizations, which are thence required to be symmetries of the system. In this case, it is easy to produce compact support reparametrizations since the parameter lives in  $\mathbb{R}$  which is paracompact.

The hole argument is based on the fact that symmetries transform solutions into solutions (since they leave the action invariant). One can show in an elementary way that compact supported symmetries contradict directly and essentially Cauchy theorem (*i.e.*, determinism). In fact, given a solution  $\sigma$  and acting on it by a compact supported symmetry  $\Phi$  one obtains a new solution  $\sigma' = \Phi_*\sigma$ . The two solutions  $\sigma$  and  $\sigma'$  are identical everywhere except within the compact support  $D$  of  $\Phi$ . One has thence two different solutions which are exactly the same far in the past (in particular on a spacelike hypersurface where initial conditions can be imposed and which does not intersect  $D \subset M$ ) while they differ at some point (in  $D$ ).

This is a very general property of possible dynamics with symmetry groups that contain compact supported symmetries. Since in Physics one does not want to give up determinism too easily the only other possible way out is to assume that configurations which differ by compact supported symmetries (e.g.,  $\sigma$  and  $\sigma'$  above) represent in fact the same physical situation. One physical situation can be represented by many different mathematical representations, *i.e.*, physical situations are not in 1-to-1 correspondence with configurations but with equivalence classes of configurations. In other words, physical situations are realized by some suitable quotient of configuration space.

Dynamics must therefore respect the quotient (that is equivalent to assume that a change of representative in equivalence class is a dynamical symmetry). On the other way, just because of the structure of the symmetry group, one cannot hope to observe any difference among different representatives of the same class of equivalence. In other words, symmetries not only constrain physically reasonable dynamics but they also impose strong constraints on what can be physically observed. Accordingly, in all cases that are relevant to fundamental interaction Physics, the dynamics is degenerate.

This is, to the best of our knowledge, the widest definition of what one could mean by *Gauge Theories*. A Gauge Theory could be defined as a Field Theory with a symmetry group that contains enough compact supported symmetries so to be able to act non-trivially on any compact hole in the base manifold  $M$  (being trivial outside the holes, as above). [Of course if  $M$  is compact then *any* symmetry is compactly supported; here we mean the support is *small*, *i.e.*, disjoint from the Cauchy surface where initial conditions are set.]. Accordingly, its dynamics is degenerate and it induces a deterministic dynamics on the space of gauge classes of configurations that are identified with physical situations. Of course, one cannot obtain every feature of standard gauge theories under these very mild hypotheses. For example, one has to ensure the symmetry group to be *infinite dimensional* (or, equivalently, parametrized by functions) to allow reduction of the gauge degrees of freedom. However, these are the mildest conditions for which field equations are underdetermined and hole argument apply. If one wants to keep be compliant to the standard nomenclature these could be called *Underdetermined Field Theories* or *Generalized Gauge Theories*.

For this reason physical gravitational fields in GR are not described in terms of “metrics” but rather in terms of “geometries” (*i.e.*, equivalence classes of metrics with respect to spacetime diffeomorphisms). In Gauge Theories Yang-Mills fields are identified with gauge classes of connections on  $P$ . In SR motions are unparametrized worldlines.

## 7. Frame-Affine Formalism for GR

In order to provide an example of concrete application of our formalism here introduced in action we shall here consider the application to the so called *Holst's action principle* (see [30]) which is used as an equivalent formulation of GR suitable for developing LQG through the use of the Barbero–Immirzi connection (see [31–34] as well as references quoted therein).

Let  $M$  be a  $m$ -dimensional manifold (which will be required to allow global metrics of signature  $\eta = (r, s)$ , with  $m = r + s$ ). Let us denote by  $x^\mu$  local coordinates on  $M$ , which induce a basis  $\partial_\mu$  of

tangent spaces; let  $L(M)$  denote the *general frame bundle* of  $M$  and set  $(x^\mu, V_a^\mu)$  for fibered coordinates on  $L(M)$ . We can define a right-invariant basis for vertical vectors on  $L(M)$

$$\rho_\nu^\mu = V_a^\mu \frac{\partial}{\partial V_a^\nu} \tag{67}$$

The general frame bundle is natural (see [28]), hence any spacetime vector field  $\xi = \xi^\mu \partial_\mu$  defines a natural lift on  $L(M)$

$$\hat{\xi} = \xi^\mu \partial_\mu + \partial_\mu \xi^\nu \rho_\nu^\mu \tag{68}$$

We stress that the lift vector field  $\hat{\xi}$  is global whenever  $\xi$  is global.

A connection on  $L(M)$  is denoted by  $\Gamma_{\beta\mu}^\alpha$  and it defines a lift

$$\Gamma : TM \rightarrow TL(M) : \xi^\mu \partial_\mu \mapsto \xi^\mu (\partial_\mu - \Gamma_{\beta\mu}^\alpha \rho_\alpha^\beta) \tag{69}$$

This lift does not in general preserve commutators, unless the connection is flat.

Let now  $(\Sigma, M, \pi, \text{SO}(\eta))$  be a principal bundle over the manifold  $M$  and let  $(x^\mu, S_b^a)$  be (overdetermined) fibered “coordinates” on the principal bundle  $\Sigma$ . We can define a right-invariant pointwise basis  $\sigma_{ab}$  for vertical vectors on  $\Sigma$  by setting

$$\sigma_{ab} = \eta_{d[a} \rho_{b]}^d \qquad \rho_b^d = S_c^d \frac{\partial}{\partial S_c^b} \tag{70}$$

where  $\eta_{ab}$  is the canonical diagonal matrix of signature  $\eta = (r, s)$  and square brackets denote skew-symmetrization over indices.

A connection on  $\Sigma$  is in the form

$$\omega = dx^\mu \otimes (\partial_\mu - \omega_\mu^{ab} \sigma_{ab}) \tag{71}$$

Also in this case the connection on  $\Sigma$  induces connections on any associated bundle and there defines covariant derivatives of sections.

A *frame* is a bundle map  $e : \Sigma \rightarrow L(M)$  which preserves the right action, *i.e.*, such that

$$\begin{array}{ccc} \Sigma & \xrightarrow{e} & L(M) \\ \downarrow & & \downarrow \\ M & \xlongequal{\quad} & M \end{array} \qquad \begin{array}{ccc} \Sigma & \xrightarrow{e} & L(M) \\ \downarrow R_S & & \downarrow R_{i(S)} \\ \Sigma & \xrightarrow{e} & L(M) \end{array} \tag{72}$$

*i.e.*,  $e \circ R_S = R_{i(S)} \circ e$ , where  $R$  denotes the relevant canonical right actions defined on both principal bundles  $\Sigma$  and  $L(M)$  and where  $i : \text{SO}(\eta) \rightarrow \text{GL}(m)$  is the canonical group inclusion. We stress that on any  $M$  which allows global metrics of signature  $\eta$  the bundle  $\Sigma$  can always be chosen so that there exist global frames; see [35]. Locally the frame is represented by invertible matrices  $e_a^\mu$  and it defines a spacetime metric  $g_{\mu\nu} = e_a^\mu \eta_{ab} e_b^\nu$  which is called the *induced metric*.

As for the Levi-Civita connection, a frame defines a connection on  $\Sigma$  (called the spin-connection of the frame) given by

$$\omega_\mu^{ab} = e_\alpha^a (\Gamma_{\beta\mu}^\alpha e^{b\beta} + d_\mu e^{b\alpha}) \tag{73}$$

where  $\{g\}_{\beta\mu}^\alpha$  denote Christoffel symbols of the induced metric. The spin-connection is compatible with the frame in the sense that

$$\nabla_\mu e_a^\nu = d_\mu e_a^\nu + \{g\}_{\lambda\mu}^\nu e_a^\lambda - \omega^c_{a\mu} e_c^\nu \equiv 0 \quad (74)$$

In general the (natural) lift  $\hat{\xi}$  to  $L(M)$  of a spacetime vector field  $\xi$  is not adapted to the image  $e(\Sigma) \subset L(M)$  and thence it does not define a vector field on  $\Sigma$ . With this notation the Kosmann lift of  $\xi = \xi^\mu \partial_\mu$  is defined by  $\hat{\xi}_K = \xi^\mu \partial_\mu + \hat{\xi}^{ab} \sigma_{ab}$  (see [36]) where we set:

$$\hat{\xi}^{ab} = e_\nu^{[a} \nabla_\mu \xi^\nu e^{b]\mu} - \omega_\mu^{ab} \xi^\mu \quad (75)$$

and where  $e^{a\mu} = \eta^{ac} e_c^\mu$  while  $e_\nu^b$  denote the inverse frame matrix.

Let us stress that despite appearing so, the Kosmann lift (75) does not in fact depend on the connection, but just on the frame and its first derivatives. The same lift can be written as  $\hat{\xi}^{ab} = \nabla^{[b} \xi^{a]} - \omega_\mu^{ab} \xi^\mu$  where we set  $\xi^a = \xi^\mu e_\mu^a$  since one can prove that

$$\nabla_b \xi^a = e_\nu^a \nabla_\mu \xi^\nu e_b^\mu \quad (76)$$

Another useful equivalent expression for the Kosmann lift is giving the vertical part of the lift with respect to the spin connection (see [9], pp. 288–290), namely

$$\hat{\xi}^{ab}_{(V)} := \hat{\xi}^{ab} + \omega_\mu^{ab} \xi^\mu = e_\nu^{[a} \nabla_\mu \xi^\nu e^{b]\mu} = \nabla^{[b} \xi^{a]} \quad (77)$$

This last expression is useful since it expresses a manifestly covariant quantity.

We have to stress that the Kosmann lift does not preserve commutators. In fact if one considers two spacetime vectors  $\xi$  and  $\zeta$  and computes the Kosmann lift of the commutator  $[\xi, \zeta]$  one can easily prove that

$$[\xi, \zeta] \hat{K} = [\hat{\xi}_K, \hat{\zeta}_K] + \frac{1}{2} e_\alpha^a \mathcal{L}_\zeta g^{\alpha\lambda} \mathcal{L}_\xi g_{\lambda\beta} e^{b\beta} \sigma_{ab} \quad (78)$$

Thence only if one restricts to Killing vectors (*i.e.*,  $\mathcal{L}_\xi g = 0$ ) one recovers that the lift preserves commutators.

Let us now consider tetrad-affine formulation of GR: the fundamental fields are a Lorentz connection  $\Gamma_\mu^{ab}$  and a vielbein  $e^a = e_\mu^a dx^\mu$ . The connection defines the curvature form  $R^{ab} = \frac{1}{2} R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu$ . Let us also set  $e = \det |e_\mu^a|$ ,  $R^a_{\mu} = R^{ab}_{\mu\nu} e_b^\nu$  and  $R = R^{ab}_{\mu\nu} e_a^\mu e_b^\nu$ ; here  $e_b^\nu$  denotes the inverse frame matrix of  $e_\nu^b$ . The frame also defines a metric  $g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b$  which in turn defines its Levi-Civita spacetime connection  $\Gamma_{\beta\mu}^\alpha$ .

On a spacetime of dimension 4 let us consider the Lagrangian

$$L_{tA} = R^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} \quad (79)$$

By variation we obtain

$$\begin{aligned} \delta L_{tA} = & -2e e_a^\sigma \left( R^a_{\mu} - \frac{1}{2} R e_\mu^a \right) e_d^\mu \delta e_\mu^d - \epsilon_{abcd} \nabla_\mu (e_\rho^c e_\sigma^d) \epsilon^{\mu\nu\rho\sigma} \delta \Gamma_\mu^{ab} \\ & + \epsilon_{abcd} \nabla_\mu (e_\rho^c e_\sigma^d \delta \Gamma_\mu^{ab}) \epsilon^{\mu\nu\rho\sigma} \end{aligned} \quad (80)$$

Thus one obtains field equations in the form

$$\begin{cases} R^a{}_{\mu} - \frac{1}{2} R e_{\mu}^a = 0 \\ \nabla_{[\mu} \left( e_{\rho}^{[c} e_{\sigma]}^d \right) = 0 \end{cases} \quad (81)$$

The second field equation forces the connection to be the connection induced by the frame, *i.e.*,  $\Gamma_{\mu}^{ab} = \omega_{\mu}^{ab}$ ; then the first equation forces the induced metric to obey Einstein equations.

This Field Theory is dynamically equivalent to standard GR, in the sense that it obeys equivalent field equations. However, the theory is in fact richer in its physical interpretation, since the use of different variables and action principles generate larger symmetry and extra conservation laws. In fact, this theory has a bigger symmetry group being both generally covariant and Lorentz covariant.

Noether theorem implies then conservation of the current

$$\mathcal{E}^{\mu} = 4e e_a^{\mu} e_b^{\nu} \mathcal{L}_{\Xi} \Gamma_{\nu}^{ab} - \xi^{\mu} L_{tA} \quad (82)$$

along any Lorentz gauge generator  $\Xi = \xi^{\mu} \partial_{\mu} + \xi^{ab} \sigma_{ab}$ . The Lie derivative of a connection is given by

$$\mathcal{L}_{\Xi} \Gamma_{\nu}^{ab} = \xi^{\lambda} R^{ab}{}_{\lambda\nu} + \nabla_{\nu} \hat{\xi}^{ab} \quad (83)$$

where we set  $\hat{\xi}^{ab} = \xi^{ab} + \xi^{\lambda} \Gamma_{\lambda}^{ab}$ .

Hence one obtains

$$\mathcal{E}^{\mu} = 4e e_a^{\mu} \left( R^a{}_{\mu} - \frac{1}{2} R e_{\mu}^a \right) \xi^{\lambda} - 4 \nabla_{\nu} (e e_a^{\mu} e_b^{\nu}) \hat{\xi}^{ab} + 4 \nabla_{\nu} (e e_a^{\mu} e_b^{\nu} \hat{\xi}^{ab}) \quad (84)$$

The first and second terms vanish on-shell; hence one obtains

$$\mathcal{E}^{\mu} = 4 \nabla_{\nu} (e e_a^{\mu} e_b^{\nu} \hat{\xi}^{ab}) \quad (85)$$

Let us stress that this current depends only on the Lorentz generator  $\hat{\xi}^{ab}$ .

Here is the issue with physical interpretation: we have two equivalent formulations of Einstein GR. Noether currents in one case depend on spacetime vector fields, while in tetrad-affine formulation Noether currents depend on Lorentz generators that *a priori* have nothing to do with spacetime transformations. Let us stress of course that unless the spacetime is Minkowski, there is no class of (global) spacetime diffeomorphisms representing *Lorentz transformations*.

Considering the dynamical equivalence at the level of field equations and solution space, one would like this equivalence to be extended at the level of conservation laws. Moreover, some of the conserved quantities in standard GR are known to be related to physical quantities such as energy, momentum and angular momentum, while one would wish to be able to identify the corresponding quantities also in the second formulation. Kosmann lift is in fact essential to relate Lorentz generators to spacetime diffeomorphisms and the corresponding conservation laws.

The Noether current (85) can be restricted by setting  $\Xi = \hat{\xi}_K$ , so that one obtains

$$\mathcal{E}_{tA}^{\mu} = 4 \nabla_{\nu} (e \nabla^{\mu} \xi^{\nu}) \quad (86)$$

which corresponds to the standard conserved quantity associated to spacetime diffeomorphisms in GR written in terms of Komar superpotential. This (and only this) restores the equivalence between standard GR and tetrad-affine formulation at the level of conservation laws.

As a further example let us consider the covariant Lagrangian:

$$L_H = L_{tA} + \beta R^{ab} \wedge e_a \wedge e_b \quad (87)$$

which is known as Holst's Lagrangian.

By variations one obtains equations

$$\left\{ \begin{array}{l} e_d^\mu \left( R_\mu^a - \frac{1}{2} R e_\mu^a \right) e_a^\sigma - \beta R_{d\rho\mu\nu} \epsilon^{\mu\nu\rho\sigma} = 0 \\ \nabla_{[\mu} \left( e_{\rho]}^{[c} e_{\sigma]}^{d]} \right) = 0 \end{array} \right. \quad (88)$$

The second equation still imposes  $\{g\}_\mu^{ab} = \omega_\mu^{ab}$ ; this in turns implies  $R^a_{[\rho\mu\nu]} = 0$  (first Bianchi identity) and hence Einstein equations. This shows how Holst's Lagrangian provides an equivalent formulation of standard GR as well.

It is interesting to check if also in this case the equivalence is preserved also at level of conservation laws. The Noether current is

$$\mathcal{E}_H^\mu = 4e e_a^\mu e_b^\nu \mathfrak{L}_\Xi \Gamma_\nu^{ab} + e e_c^\mu e_d^\nu \epsilon_{ab}^{cd} \cdot \mathfrak{L}_\Xi \Gamma_\nu^{ab} - \xi^\mu L_H \quad (89)$$

As in the previous case this can be recasted modulo terms vanishing on-shell as follows

$$\mathcal{E}_H^\mu - \mathcal{E}_{tA}^\mu = \nabla_\nu \left( e e_c^\mu e_d^\nu \epsilon_{ab}^{cd} \cdot \hat{\xi}^{ab} \right) \quad (90)$$

Again this has nothing to do with spacetime symmetries and in general it would affect conserved quantities. When Kosmann lift is again inserted into these conservation laws one obtains

$$\mathcal{E}_H^\mu - \mathcal{E}_{tA}^\mu = \nabla_\nu \left( \nabla^\rho \xi^\sigma \epsilon^{\mu\nu\rho\sigma} \right) \quad (91)$$

that vanishes being the divergence of a divergence. Hence once again the correspondence at the level of conservation laws is preserved when the Kosmann lift is used.

## 8. Conclusions and Perspectives

The kinematics and dynamics of most fundamental interaction Physics can be defined purely in terms of symmetries. Noether theorem and conservation laws define a considerable part of its physical interpretation.

The role of strong conservation laws is still unclear. One can imagine some role in the Hamiltonian framework that is however, still unclear and it deserves further investigations.

From a fundamental viewpoint one should also investigate whether one could describe Nature in terms of the whole set of conservation laws, without selecting special conservation laws to be endowed with a special meanings. In this direction we can mention the approach to Noether theorem in terms of *momentum maps*; see [2].

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