Symmetry **2010**, *2*, 230-271; doi:10.3390/sym2010230

OPEN ACCESS

symmetry

ISSN 2073-8994

www.mdpi.com/journal/symmetry

Article

Doubly-Special Relativity: Facts, Myths and Some Key Open Issues

Giovanni Amelino-Camelia

Dipartimento di Fisica, Universit di Roma "La Sapienza" and Sez. Roma1 INFN, P.le A. Moro 2, 00185 Roma, Italy; E-Mail: giovanni.amelino-camelia@roma1.infn.it; Fax: +39-06-495-7697

Received: 26 January 2010 / Accepted: 3 March 2010 / Published: 8 March 2010

Abstract: I report, emphasizing some key open issues and some aspects that are particularly relevant for phenomenology, on the status of the development of "doubly-special" relativistic ("DSR") theories with both an observer-independent high-velocity scale and an observer-independent small-length/large-momentum scale, possibly relevant for the Planck-scale/quantum-gravity realm. I also give a true/false characterization of the structure of these theories. In particular, I discuss a DSR scenario without modification of the energy-momentum dispersion relation and without the κ-Poincaré Hopf algebra, a scenario with deformed Poincaré symmetries which is not a DSR scenario, some scenarios with both an invariant length scale and an invariant velocity scale which are not DSR scenarios, and a DSR scenario in which it is easy to verify that some observable relativistic (but non-special-relativistic) features are insensitive to possible nonlinear redefinitions of symmetry generators.

Keywords: Poincaré symmetries; quantum spacetime; quantum gravity

---

1. Introduction

The idea of "Doubly-Special-Relativity" (or "DSR") is now about 10 years old [1,2], and already a few hundred papers either fully focused on it or considered it alongside other possibilities (see, e.g., Refs. [1–64] and references therein). This allowed to achieve rapidly significant progress in exploring the DSR concept, but, as inevitable for such a large research effort over such a short time, it also affected negatively the process of establishing a common language and common conventions. It is clearly necessary at this point to devote some efforts to comparing different perspectives. The pace of development was such that it could not be organized "in series", with each new result obtained by one group being metabolized and used by other groups to produce the next significant result, but rather "in parallel", with research groups or clusters of research groups obtaining sequences of results from within one approach, and then attempting to compare to similar sequences of results obtained by other research groups when already a barrier of "local dialects", intuitions and prejudices has settled in.

It might be useful at this stage of the development of DSR to pause and try to find some unifying features among the encouraging results found adopting different approaches, and confront from a perspective that might combine the different approaches some of the most stubborn "unsolved issues" which often appear, although possibly differently disguised, in all approaches.

In the first part of these notes I follow my original discussion of Refs. [1,2] proposing the physics idea of a DSR theory. And to make the concept clearer I here also characterize it in the form of a "true/false exercise", whose entries are chosen on the basis of the experience of these past few years, in which some aspects of the DSR concept have been occasionally misunderstood. In particular, through the illustrative example of theories with "canonical" spacetime noncommutativity I discuss the possibility of a DSR scenario without modifications of the energy-momentum dispersion relation and without the κ-Poincaré Hopf algebra. Through the illustrative example of some results originally obtained by Fock, I discuss the possibility of a relativistic theory with both an invariant length scale and an invariant velocity scale, which is not a DSR scenario. And I use some recent results on the relation between the conserved charges and the algebraic properties of the generators of a Hopf algebra as an example of cases in which some characteristic relativistic (but non-special-relativistic) features of a theory are insensitive to possible nonlinear redefinitions of the symmetry generators.

Whenever possible I rely only on the physics concept of a DSR theory, without advocating one or another mathematical formalism, and I stress that this is at this stage still advisable since, although many encouraging results have been obtained, no mathematical formalism has been fully proven to be consistent with the DSR principles. (Actually for all of the candidate DSR formalisms that are under consideration the results obtained so far are not even sufficient to rigorously exclude the presence of a preferred frame, and it would therefore be rather dangerous to identify the DSR concept with one or another of these formalisms.)

When I do find useful to discuss some aspects of the DSR concept and some of the "open issues" in terms of a candidate mathematical formalism, I shall primarily resort to the mathematics of Hopf-algebra spacetime symmetries, which at present appear to provide the most promising candidate for a formalism able to accommodate the DSR principles. But I shall also (although more briefly) comment on other candidate DSR formalisms, and in doing so I shall stress the need to characterize these different approaches in terms of (at least in-principle) observable features. It is not implausible that some of these formally different approaches actually describe the same DSR physical theory.

I also offer some remarks (mainly in Section 6) on DSR phenomenology, primarily with the objective of showing that (in spite of the present, rather preliminary, stage of developments of DSR formalisms) we can already establish rather robustly certain general features of this phenomenology.

2. Relativity, Doubly Special

2.1. Motivation

My proposal [1,2] of the doubly-special-relativity scenario intended to provide an alternative perspective on the studies of quantum-gravity-induced Planck-scale departures from Lorentz symmetry which had been presented in numerous articles (see, e.g., Refs. [65–72]) between 1997 and 2000. These studies were advocating Planck-scale modifications of the energy-momentum dispersion relation, usually of the form $E^2 = p^2+m^2+\eta L_p^n p^2 E^n+O(L_p^{n+1}E^{n+3})$, on the basis of preliminary findings in the analysis of several formalisms in use in Planck-scale/quantum-gravity theoretical physics. The complexity of the formalisms is such that very little else was known about their physical consequences, but the evidence of a modification of the dispersion relation was becoming robust. In all of the relevant papers it had been assumed that such modifications of the dispersion relation would amount to a breakdown of Lorentz symmetry, with associated emergence of a preferred class of inertial observers (usually identified with the natural observer of the cosmic microwave background radiation). I was intrigued by a striking analogy between these developments and the developments which led to the emergence of Special Relativity, now more than a century ago. In Galilei Relativity there is no observer-independent scale, and in fact the energy-momentum relation is written as $E = p^2/(2m)$. When it became clear that electromagnetic phenomena could be formalized in terms of Maxwell equations, the fact that those equations involve a fundamental velocity scale appeared to require the introduction of a preferred class of inertial observers. But in the end we figured out that the situation was not demanding the introduction of a preferred frame, but rather a modification of the laws of transformation between inertial observers. Einstein's Special Relativity introduced the first observer-independent relativistic scale (the velocity scale c), its dispersion relation takes the form $E^2 = c^2p^2 + c^4m^2$ (in which c plays a crucial role for what concerns dimensional analysis), and the presence of c in Maxwell's equations is now understood as a manifestation of the necessity to deform the Galilei transformations.

I argued in Refs. [1,2] that it is not implausible that we might be presently confronted with an analogous scenario. Research in quantum gravity is increasingly providing reasons of interest in Planck-scale modifications of the dispersion relation, of the type mentioned above, and, while it was customary to assume that this would amount to the introduction of a preferred class of inertial frames (a "quantum-gravity ether"), the proper description of these new structures might require yet again a modification of the laws of transformation between inertial observers. The new transformation laws would have to be characterized by two scales (c and L_p) rather than the single one (c) of ordinary Special Relativity.

2.2. Defining the Concept

The "historical motivation" described above leads to a scenario for Planck-scale physics which is not intrinsically equipped with a mathematical formalism for its implementation, but still is rather well defined. With Doubly-Special Relativity one looks for a transition in the Relativity postulates, which should be largely analogous to the Galilei $\rightarrow$ Einstein transition. Just like it turned out to be necessary, in order to describe high-velocity particles, to set aside Galilei Relativity (with its lack of any characteristic invariant scale) and replace it with Special Relativity (characterized by the invariant velocity scale c), it is at least plausible that, in order to describe ultra-high-energy particles, we might have to set aside Special Relativity and replace it with a new relativity theory, a DSR, with two characteristic invariant scales, a new small-length/large-momentum scale in addition to the familiar velocity scale.

A theory will be compatible with the DSR principles if there is complete equivalence of inertial observers (Relativity Principle) and the laws of transformation between inertial observers are characterized by two scales, a high-velocity scale and a high-energy/short-length scale. Since in DSR one is proposing to modify the high-energy sector, it is safe to assume that the present operative characterization of the velocity scale c would be preserved: c is and should remain the speed of massless low-energy particles. Only experimental data could guide us toward the operative description of the second invariant scale L_{dsr}.

These characteristics can be summarized [1,2] of course in the form of some "DSR principles". First the statement that Galilei's relativity principle is valid:

- (RP): The laws of physics are the same in all inertial frames (for all inertial observers); in particular, the parameters that appear in the laws of physics take the same value in all inertial frames and, equivalently, if two inertial observers in relative motion setup the same experimental procedure they get exactly the same (possibly dimensionsful) numerical values for the measurement results

Then there must be a principle giving the operative definition of the length scale L_{dsr} (or a corresponding momentum/energy/frequency scale $1/L_{dsr}$). Since at present we have no data one can only describe the general form of this law [1,2]

- (La): The laws of physics, and in particular the laws of transformation between inertial observers, involve a fundamental/observer-independent small (possibly Planckian) length scale L_{dsr}, which can be measured by each inertial observer following the measurement procedure $\mathcal{M}_{L_{dsr}}$

And finally one must have the speed-of-light-scale principle [1,2]:

- (Lb): The laws of physics, and in particular the laws of transformation between inertial observers, involve a fundamental/observer-independent velocity scale c, which can be measured by each inertial observer as the speed of light with wavelength λ much larger than L_{dsr} (more rigorously, c is obtained as the infrared $\lambda/L_{dsr} \to \infty$ limit of the speed of light)

The postulate (La) is clearly incomplete: for the description of the $\mathcal{M}_{L_{dsr}}$ procedure for the measurement of L_{dsr} we do not have enough experimental information to even make an educated guess. There are many physical arguments and theoretical models that predict one or another physical role for the Planck length, but none of these scenarios has any experimental support. It is still plausible that the Planck length has no role in space-time structure and kinematics (which would render DSR research of mere academic interest). Even assuming DSR actually does play a role in the description of Nature it seems likely that the correct formulation of $\mathcal{M}_{L_{dsr}}$ would end up being different from any proposal we can contemplate presently, while we are still lacking the needed guidance of experimental information, but through the study of some specific examples we can already acquire some familiarity with the new elements required by a conceptual framework in which the Relativity Principle coexists with observer-independent high-velocity and small-length scales. In light of this situation in Ref. [1,2] I considered a specific illustrative example of the postulate (La), which was also inspired by the mentioned 1997-2000 studies of Planck-scale modification of the dispersion relation:

- (La*): The laws of physics, and in particular the laws of transformation between inertial observers, involve a fundamental/observer-independent small (possibly Planckian) length scale L_{dsr}, which can be measured by each inertial observer by determining the dispersion relation for photons. This relation has the form $E^2 - c^2p^2 + f(E, p; L_{dsr}) = 0$, where the function f is the same for all inertial observers and in particular all inertial observers agree on the leading L_{dsr} dependence of f: $f(E, p; L_{dsr}) \simeq L_{dsr}cp^2E$

In these past decade of course other examples of measurement procedure $\mathcal{M}_{L_{dsr}}$ have been considered by myself and others.

2.3. A Falsifiable Proposal

A key objective for these notes is to expose the physics content of the DSR proposal in ways that are unaffected by the fact that this proposal does not at present specifiy a mathematical formalism. Consistently with this objecties let me close this section on the "definition of doubly-special relativity", by making the elementary but significant observation that, even without specifying a mathematical formalism, the DSR idea is falsifiable. Many alternative mathematical formalisms could be tried for DSR, and probably should be tried, but any experimental evidence of a "preferred frame" would exclude DSR completely, without appeal, independently of the mathematical formalization. I am just stating the obvious: the idea that the laws of physics are relativistic is of course falsifiable. But this obvious fact carries some significance when DSR is viewed in the broader context of the quantum-gravity literature, affected as it is by the problem of proposals that perhaps may predict many things, but in fact, at least as presently understood, predict nothing and/or possibly could not be falsified.

The possibility of falsifying the DSR idea by finding evidence of a preferred frame is actually not of mere academic ("epistemological") interest: proposals put forward in the quantum-gravity literature that are related to (though alternative to) DSR are motivating experimental searches of "preferred-frame effects" in ways that could indeed falsify the whole DSR idea if successful. In fact, it has been recently realized (see e.g., Refs. [73–75]) that, when Lorentz symmetry is broken ("preferred-frame picture") at the Planck scale, there can be significant implications for certain decay processes. At the qualitative level the most significant novelty would be the possibility for massless particles to decay. Let us consider for example a photon decay into an electron-positron pair: $\gamma \to e^+e^-$. And let us analyze this process using the dispersion relation

$$m^2 \simeq E^2 - \vec{p}^2 + \eta\vec{p}^2\left(\frac{E}{E_p}\right) \tag{1}$$

where $E_p = 1/L_p$. Assuming then an unmodified law of energy-momentum conservation, one easily finds a relation between the energy E_γ of the incoming photon, the opening angle θ between the outgoing electron-positron pair, and the energy E_+ of the outgoing positron, which, for the region of phase space with $m_e \ll E_\gamma \ll E_p$, takes the form $\cos(\theta) = (A + B)/A$, with $A = E_+(E_\gamma - E_+)$ and $B = m_e^2 - \eta E_\gamma E_+(E_\gamma - E_+)/E_p$ (m_e denotes of course the electron mass). For $\eta < 0$ the process is still always forbidden, but for positive η and $E_\gamma \gg (m_e^2 E_p/|\eta|)^{1/3}$ one finds that $\cos(\theta) < 1$ in certain corresponding region of phase space.

The energy scale $(m_e^2 E_p)^{1/3} \sim 10^{13} eV$ is not too high for astrophysics. The fact that certain observations in astrophysics allow us to establish that photons of energies up to $\sim 10^{14} eV$ are not unstable (at least not noticeably unstable) could be used [73,75] to set valuable limits on η.

If following this strategy one did find photon decay then the idea of spacetime symmetries broken by Planck-scale effects would be strongly encouraged. The opposite is true of DSR, which essentially codifies a certain type of deformations of Special Relativity. Any theory compatible with the DSR principle must have stable massless particles. A threshold-energy requirement for massless-particle decay (such as the $E_\gamma \gg (m_e^2 E_p/|\eta|)^{1/3}$ mentioned above) cannot of course be introduced as an observer-independent law, and is therefore incompatible with the DSR principles. By establishing the existence of a threshold for photon decay one could therefore indeed falsify the DSR idea. And more generally any theory compatible with the DSR principle must not predict energy thresholds for the decay of particles. In fact, one could not state observer-independently a law setting a threshold energy for a certain particle decay, because different observers attribute different energy to a particle (so then the particle should be decaying according to some observers while being stable according to other observers).

3. More on the Concept (Some True/False Characterizations)

The definition of DSR given in Ref. [1,2], which (in order to render this review self-contained) I here repeated in the preceding section, is crisp enough not to require any further characterization. However, partly because it does not yet come with a specification of mathematical formalism, and partly because of some inconsistencies of terminology that characterized the rather large DSR literature produced in just a few years, has occasionally generated some misinterpretations. This motivates the redundant task to which I devote the present section: drawing from the experience of this past decade, I explicitly comment on some possible misconceptions concerning DSR.

3.1. Most Studied DSR Scenarios and Inequivalence to Special Relativity

The DSR concept, as introduced in Refs. [1,2], is very clearly alternative to Special Relativity. How could it then be possible that some authors (see, e.g., Ref. [49]) have instead naively argued the equivalence?

What could happen is that some authors may propose a certain formalization of a DSR theory and then other authors show that the attempt was unsuccessful, meaning that the proposed formalization, while intended as a DSR candidate, actually turns out to be simply special relativistic. However, the cases that apparently are of this sort in the DSR literature (see, e.g., in Ref. [49]) have actually nothing to do with establishing whether or not one type of another of relativistic properties is found in a certain theory proposal, and rather are due to the mistake of reasoning on the basis of formal analogies rather than in terms of physical/observable/operatively-well-defined concepts.

The core misconceptions that generated claims of equivalence of some candidate DSR proposal with ordinary special relativity originates from the role played in these candidate DSR proposals by (one type or another, see later) modifications of the commutators of Poincaré generators. There is a certain (obviously incorrect, but surprisingly popular) frame of mind among some theorists according to which "finding a map between two theories establishes their equivalence". It is difficult for me to even pretend to put myself in that frame of mind, considering that, even just remaining within relativistic theories, one easily finds "some maps" that can convert Galileian relativity into Minkowski-space special relativity, and in turn Minkowski-space special relativity into deSitter-space special relativity. And specifically a map of type very similar to the one considered for example in Ref. [49] from a pseudo-DSR perspective can be used to convert the Galilei Lie algebra for 2D spacetimes into the Poincaré Lie algebra for 2D spacetimes.

However, rather than articulating in detail a critique of this "the map is the answer" methodology, I suppose it is more useful if I briefly characterize here two (of the many possible) scenarios for the formalization of the DSR concept and comment on the futility of seeking maps that would convert some symbols on the DSR side into special-relativistic symbols.

The DSRa scenario: nonlinear representations of the Poincaré Lie group

The most studied "toy-model scenarios" (see related comments in the section on phenomenology) for DSR essentially assume that the DSR principles could be implemented by seeking corresponding nonlinear representations of the standard Poincaré Lie group. The action of the generators of the Poincaré Lie group on such a representation is such that (when specified on that representation) one could effectively describe the Poincaré symmetries in terms of some deformed commutators between the generators.

So this is a case where the new observer-independent scale of DSR characterizes representations of a still classical/undeformed Poincaré Lie group. But one can effectively handle these proposals by replacing the standard Poincaré generators with new generators, specialized to the nonlinear representation: $T_a \rightarrow T^a_{dsr}$. Of course (since indeed one is merely adotping a possible scheme of analysis of a nonlinear representations of the standard Poincaré Lie group) it is easy to find in these instances some nonlinear map $\mathcal{F}$ such that $T_a = \mathcal{F}(\{T^b_{dsr}\})$.

Some authors (see e.g., in Ref. [49]) view the availability of such a map $\mathcal{F}$, converting the "effective generators" T^a_{dsr} of the DSR picture (specialized to a nonlinear representation) into ordinary generators $T_a = \mathcal{F}(\{T^b_{dsr}\})$, as evidence that the relevant DSR scenarios should actually be viewed as a description of completely standard physics within ordinary special relativity. But of course this is extremely naive since the case with nonlinear representations characterized by an observer-independent short-distance scale and the case without such an observer-independent scale produce different physics, different observable phenomena, and can be experimentally distinguished from the ordinarily special-relativistic case.

The DSRb scenario: Poincaré-like Hopf-algebra spacetime symmetrie

Perhaps the most intriguing among the scenarios that are being considered as candidates for a DSR theory are scenarios based on the mathematics of Poincaré-like Hopf algebras. I shall describe other aspects of this scenario later in these notes. In this subsection I just want to observe that also for the Hopf-algebra case the search of "some nonlinear maps from the DSR candidate to ordinary special relativity” appears to be futile, but for reasons that amusingly are complementary to the ones discussed in the previous subseciton for another popular candidate DSR scenario.

In this Hopf-algebra case dogmatic followers of “map is the answer” methodology have missed a key aspect of Hopf algebras: a satisfactory characterization of these algebras must describe for the generators not only the commutators but also the so-called cocommutators (the so-called coproduct rules), which affect the law of action of generators on products of functions. This is related to the fact that the classic application of Hopf-algebra symmetries is to frameowrks such that (unlike the ones treatable with Lie algebras) the action of symmetry transformations on products of functions is not obtainable by standard application of Leibniz rule to the action on a single function (as I shall discuss in more detail later in these notes). This fits naturally with the properties of some structures used in Planck-scale research, such as noncommutative spacetimes (since the action of operators on products of functions of the noncommutative spacetime coordinates is naturally not subject to Leibniz rule). Nonlinear redefinitions of generators for Hopf algebras are admitted, just because they are futile: the form of the commutators of course changes but there is a corresponding change in the cocommutators, and the combined effect amounts to no change at all for the overall description of the symmetries. In Refs. [76,77] this result, which can be shown already at the abstract algebra level, was worked out explicitly, for two nonlinear redefinitions of two examples of Hopf algebra deformations of the Poincaré algebra, finding that one obtains the same conserved charges independently of whether one analyzes the Hopf-algebra symmetries in terms of one set of generators or a nonlinear redefinition of that given set of generators. Essentially this comes from the fact that the Noether analysis of a theory with a given Lagrangian density will combine aspects described in terms of the action of generators on single functions and aspects which instead concern the action on product of functions.

Implications of the nonlinear maps for the action of generators

Let me close this subsection on the use of nonlinear maps in some candidate DSR frameworks with an observation that may apply to a rather broad spectrum of scenarios: when a map from one set of generators to another is analyzed one should not overlook the implications for the rule of action of those generators. For my purposes it suffices to raise this issue within a rather abstract/formal scheme of analysis, working in leading order in a deformation length scale λ, and focusing on the simplest case of the relativistic description of massless classical particles in 1+1D spacetimes. Let me start by introducing standard translation generators Π, Ω

$$[x, \Pi] = 1 \ , \ \ [t, \Omega] = 1 \ , \ \ [x, \Omega] = 0 \ , \ \ [t, \Pi] = 0 \ , \ \ [\Omega, \Pi] = 0 \tag{2}$$

A deformation of relativistic symmetries of DSR type may be obtained by then adopting a non-standard representation of the boost generator

$$N = x(\Omega - 2\lambda\Omega\Pi) + t(\Pi + \lambda\Omega^2) \tag{3}$$

from which it follows that

$$[N, \Omega] = \Pi + \lambda\Omega^2 \ , \ \ [N, \Pi] = \Omega - 2\lambda\Omega\Pi \tag{4}$$

which is a deformation of the commutators of the 1+1D Poincaré algebra, with mass Casimir $\mathcal{C}_{\Omega,\Pi} = \Omega^2 - \Pi^2 - 2\lambda\Omega^2\Pi$. Among the implications of this deformation most striking is the fact that the velocity of massless classical particles is not 1. This can be formally established by noticing that for an on-shell particle the generators take numerical value (conserved charges), and then differentiating (3) in dt one obtains

$$\frac{dx}{dt} \simeq 1 + 2\lambda\Omega_0 \tag{5}$$

where I denoted by Ω_0 the Ω-charge (energy) of the particle and also used the massless-shell condition inferred from the casimir $\mathcal{C}_{\Omega,\Pi}$.

Some authors are (erroneously) intrigued by the fact that the nontriviality (4) of the symmetry-algebra relations can be removed by adopting the following nonlinear map

$$\epsilon = \Omega \,, \quad \pi = \Pi + \lambda\Omega^2 \tag{6}$$

which indeed accoomplishes the task of producing from (4) the following outcome

$$[N, \epsilon] = \pi \,, \quad [N, \pi] = \epsilon \tag{7}$$

But it is crucial for such considerations not to miss that upon adopting ϵ, π such that $\epsilon = \Omega \,, \;\; \pi = \Pi + \lambda\Omega^2$ it then follows from (2) that

$$[x, \pi] = 1 \,, \quad [t, \epsilon] = 1 \,, \quad [x, \epsilon] = 0 \,, \quad [t, \pi] = 2\lambda\epsilon \tag{8}$$

Therefore one has accomplished a trivialization of the algebraic (commutator) relation among Poincare-sector generators (with mass Casimir $\mathcal{C}_{\epsilon,\pi} = \epsilon^2 - \pi^2$) only at the cost of introducing a nontrivial rule of action of the "would-be-translation generators" ϵ, π. It is then obvious that this nontriviality of the action of ϵ, π imposes a nontrivial representation of boosts, in spite of the fact that the Poincare-sector commutators are undeformed:

$$N = x(\epsilon - 2\lambda\epsilon\pi) + t\pi \tag{9}$$

And if one derives again formally the velocity of massless-shell classical particles the result is

$$\frac{dx}{dt} \simeq 1 + 2\lambda\epsilon \simeq 1 + 2\lambda\Omega \tag{10}$$

i.e., the velocity law is insensitive to the nonlinear map. This is an example of a nonlinear map which connects two different theories (e.g., the deBroglie relation implicit in these formal manipulations is different on the two sides of the map) which however have in common some nontrivial predictions, such as the one for the speed of massless particles.

3.2. Not Necessarily Involving the κ-Poincaré Hopf Algebra

The formalism of Hopf algebras which I just used to illustrate the weakness of some "equivalence claims" found in the literature, is also at present the most promising opportunity to find a formalism compatible with the DSR principles. However, it would be dangerous to identify the DSR idea with the mathematics of Hopf algebras, since at present we still know very little about the implications of

Hopf-algebra spacetime symmetries for observable properties of a theory, and of course the DSR concept refers to the observables of a theory and their properties under change of inertial observer.

Most attempts to construct a DSR theory using Hopf algebras focus on the so-called κ-Poincaré Hopf algebra [78–80], but if indeed Hopf algebras prove to be usable in DSR theories then other Hopf algebras might well also be considered. I will mention in various points of these notes the case of the twisted Hopf algebras of symmetries of observer-independent canonical noncommutative spacetime, which appears to be equally promising. And of course it is at present fully legitimate to look for realizations of the DSR concept that do not involve Hopf algebras.

Even if Hopf algebras do eventually turn out to be usable in the construction of DSR theories it would be inappropriate to identify the mathematics of (some) Hopf algebras with the DSR physics concept. This observation becomes obvious if one considers, for example, the nature of the works devoted to the κ-Poincaré Hopf algebra before the proposal of the DSR concept. In the pre-DSR κ-Poincaré literature one finds some warnings [81] against attempts to integrate the boost generators to obtain a candidate for finite boosts, which appeared to lead only to a "quasi-group" [81,82] structure, of unestablished applicability in physics. Similarly, the law of energy-momentum conservation which is advocated by some κ-Poincaré experts [8] is incompatible with the DSR principles [30]. These challenges of course do not imply that we should necessarily exclude the use of the κ-Poincaré formalism in attempts to construct DSR theories; it only suggests that such attempts should rely on some carefully devised interpretation of the symbolism.

3.3. Not Any Deformation, but A Certain Class of Deformations of Special Relativity

While, as I just stressed, it is clearly too early to associate with the DSR concept a specific mathematical formalism, it would also be disastrous to gradually transform (e.g., by gradual modifications of the definition of a DSR theory) the DSR concept into a large umbrella covering all scenarios for a length scale to enter spacetime symmetries. It is for this reason that I chose to do the redundant exercise of repeating here (in Section 2) exactly the definition of DSR theory originally given in Refs. [1,2]. It actually did happen over this past decade that occasionally the DSR proposal was confused as the proposal that we should have some deformation of Special Relativity, with the idea that any deformation would be DSR-acceptable. The definition given in Refs. [1,2] (here repeated in Section 2) provides a physics picture which amounts instead to a rather specific class of deformations of Special Relativity, alternative to other possibilities.

It is of course very easy to show that one could "deform" Special Relativity in ways that are not compatible with the DSR concept. One should think for example of deSitter relativity, which is of course a deformation of Special Relativity by the scale of curvature. And it is interesting that one could actually stumble upon deSitter space just in doing the exercise of looking for "some deformation" of Special Relativity. This actually happened: Fock, in an appendix of Ref. [83], explores the role that each of the postulated structures of Special Relativity plays in constraining the mathematics of the Special-Relativity framework. Unsurprisingly by removing one of the in-principle structures Fock obtains De Sitter spacetime instead of Minkowski spacetime. De Sitter spacetime is a deformation of Minkowski spacetime and the De Sitter algebra is a deformation of the Poincaré algebra (mediated by the Inonu-Wigner contraction procedure), but of course these standard formalisms do not provide examples of DSR theories. Indeed De Sitter spacetime is a deformation of Minkowski spacetime by a long-distance scale (Minkowski obtained from De Sitter as the deformation length scale is sent to infinity), whereas one of the requirements for a DSR theory is that the deformation scale be a short-distance (high-energy) scale (Special Relativity obtained from DSR as the deformation length scale is sent to zero).

Another example worth mentioning is the one of proposals in which one introduces a maximum-acceleration scale. Since already in Special Relativity acceleration is an invariant, such proposals do not *a priori* require a DSR formulation. Indeed often these proposals involve formalisms able to handle at once both inertial observers and essentially Rindler observers (of course acceleration changes when going from an inertial frame to a Rindler frame). At least in a generalized sense these are also "deformations" of Special Relativity, but typically they do not require a modification of the laws of transformation between inertial observers.

As these two examples illustrate clearly, not any "deformation" of Special Relativity provides a realization of the DSR concept. I guess the confusion some authors have on this point originates from a recent fashion to use "Deformed Special Relativity" as an equivalent name for Doubly-Special Relativity. Of course, there is no content in a name and one might consider it a free choice of the writer, but there are cases in which a certain choice of name may induce confusion, and this is certainly one of those cases, since "Deformed Special Relativity" invites a naive interpretation of the type "any deformation of Special Relativity will do". And an additional source of confusion originates from the fact that in the literature, already before the proposal of the Doubly-Special Relativity idea, there was a research programme named "deformed special relativity" (see e.g., Ref. [84] and references therein), which pursues physics motivation and physics objectives that are completely different from the ones of Doubly-Special Relativity.

3.4. A Physics Picture Leading to DSR and the Possibility of DSR Approximate Symmetries

One other point which I stressed in Refs. [1,2] and has been largely ignored concerns the possibility that "DSR symmetries" might actually be only approximate symmetries, even within the regime of physics observations where they do (hopefully) turn out to be relevant. In order to clarify this point let me first discuss a certain "vision" for the structure of spacetime at different scales.

While at present the status of development of research in quantum gravity allows of course some sort of "maximum freedom" to imagine the structure of spacetime at different distance scales, there are arguments (especially within the "emergent gravity" literature, but also beyond it) that would provide support for the following picture: (I) at superPlanckian distance scales the only proper description of spacetime degrees of freedom should be "strongly quantum", so much so that no meaningful concept of spacetime coordinates and continuous spacetime symmetries could be introduced, (II) then at subPlankian but nearly Planckian scales some sort of intelligible geometry of spacetime might emerge, possibly allowing the introduction of some spacetime coordinates, but the coordinates might well be "quantum coordinates" (e.g., noncommutative coordinates) and the symmetries of this type of spacetime geometry might well be DSR symmetries, (III) then finally in the infrared our familiar smooth classical description of spacetime becomes a sufficiently accurate description.

Besides providing a logical scheme for the emergence of DSR symmetries, this picture also explains in which sense I could argue that DSR symmetries might well be only approximate symmetries, even in the regime where they are relevant. Most authors (including myself) working on DSR insist on getting formulas that make sense all the way up to infinite particle energies. There is nothing wrong with that, but it might be too restrictive a criterion. If DSR is relevant only at energy scales that are subPlankian (but nearly Planckian), and if we actually expect to give up an intelligible picture of spacetime and its symmetries above the Plank scale, then perhaps we should be open to the possibility of using mathematics that provides an acceptable (closed) logical picture of DSR only at leading order (or some finite order) in the expansion of formulas in powers of the Planck length.

What I mean by this will be made somewhat clearer later in this notes, when I set up the discussion of DSR phenomenology. There a candidate DSR test theory will be based on formulas proposed exclusively as leading-order formulas.

3.5. Not Any Fundamental Length Scale

Inadequate description/investigation of the laws of transformation between inertial observers is, as stressed in part in the previous subsections, a very serious limitation for an analysis being proposed as a DSR study. However, there is worse: there are (fortunately only very few) self-proclaimed "DSR studies" which are only structured at the level of proposing some "fundamental length/energy scale", without any effort to establish whether this scale is of the type required by the DSR concept (in particular not providing any analysis of the role that this "fundamental scale" does or does not have in the laws of transformation between observers).

Unfortunately, even outside the DSR literature, it is not uncommon to find in the physics literature a rather sloppy use of terms such as "fundamental scale". In particular, "fundamental scales" are often discussed as if they were all naturally described within a single category. For DSR research it is instead rather important that these concepts be handled carefully.

In this respect the DSR concept is conveniently characterized through the presence of a short-distance (/high-energy) scale which is "relativistically fundamental" in the same sense already familiar for the role of the scale c in Special Relativity.

There are of course scales that are no less fundamental, but have properties that are very different from the ones of c in Special Relativity. Obvious examples are scales like the rest energy of the electron, which is of course a "fundamental" scale of Nature, but is relativistically trivial (a rest-frame property). Just a bit less obvious are cases like the one of the quantum-mechanics scale $\hbar$. Space-rotation symmetry is a classical continuous symmetry. One might, at first sight, be skeptical that some laws (quantum-mechanics laws) that discretize angular momentum could enjoy the continuous space-rotation symmetry, but more careful reasoning [85] will quickly lead to the conclusion that there is no *a priori* contradiction between discretization and a continuous symmetry. In fact, the type of discretization of angular momentum which emerges in ordinary non-relativistic quantum mechanics is fully consistent with classical space-rotation symmetry. All the measurements that quantum mechanics still allows (a subset of the measurements allowed in classical mechanics) are still subject to the rules imposed by rotation symmetry. Certain measurements that are allowed in classical mechanics are no longer allowed in quantum mechanics, but of course those measurements cannot be used to characterize rotation symmetry (they are not measurements in which rotation symmetry fails, they are just measurements which cannot be done). A more detailed discussion of this point can be found in Ref. [85]. Essentially one finds that $\hbar$ is not a scale pertaining to the structure of the rotation transformations. The rotation transformations can be described without any reference to the scale $\hbar$. The scale $\hbar$ sets, for example, the minimum non-zero value of angular momentum ($L^2_{min} = 3\hbar^2/4$), but this is done in a way that does not require modification of the action of rotation transformations. Galilei boosts are instead genuinely inconsistent with the introduction of c as observer-independent speed of massless particles (and maximum velocity attainable by massive particles). Lorentz/Poincaré transformations are genuinely different from Galilei transformations, and the scale c appears in the description of the action of Lorentz/Poincaré generators (it is indeed a scale of "deformation" of the Galilei transformations).

Both $\hbar$ and c are fundamental scales that establish properties of the results of the measurements of certain observables. In particular, $\hbar$ sets the minimum non-zero value of angular momentum and c sets the maximum value of speed. But $\hbar$ has no role in the structure of the transformation rules between observers, whereas the structure of the transformation rules between observers is affected by c. I am describing c as a "relativistically fundamental" scale, whereas $\hbar$ is a "relativistically trivial" scale, a fundamental scale that does not affect the transformation rules between observers.

A characterizing feature of the DSR proposal is that there should be one more scale playing a role analogous to c, and it should be a short-distance/high-energy scale.

One can try to introduce the Planck scale in analogy with $\hbar$ rather than with c. In particular, Snyder looked for a theory [86] in which spacetime coordinates would not commute, but Lorentz transformations would remain unmodified by the new commutation relations attributed to the coordinates. Such a scenario would of course not be a DSR scenario.

In closing this subsection I should comment on one more type of fundamental constants. The electron-mass scale m_e, the quantum-mechanics scale $\hbar$ and the (infrared-limit-of-the-) speed-of-light scale c are different types of fundamental constants, whose operative meaning is of course given through the measurements of certain corresponding observables. Another type of fundamental constants are the "coupling constants". For example, in our present description of physics the gravitational coupling G is a fundamental constant. It does not impose constraints on the measurements of a specific observable, but it governs the laws of dynamics for certain combinations of observables. Also G is observer independent, although a careful analysis (which goes beyond the scopes of this note) is needed to fully characterize this type of fundamental scales. One can define G operatively through the measurement of static force between planets. In modern language this amounts to stating that we could define G operatively as the low-energy limit of the gravitational coupling constant. All observers would find the same value for this (dimensionful!) constant.

This last remark on the nature of the fundamental constant G is particularly important for DSR theories. In our present description of physics the Planck length L_p is just the square root of G, rescaled through $\hbar$ and c. The idea of changing the status of G (*i.e.*, L_p) from the one of fundamental coupling scale to the one of relativistic fundamental scale might have deep implications [1,2,19].

3.6. And the Same Type of Scale may or may not be DSR Compatible

In the previous subsection I stressed the difference between scales that are potentially significant from a relativistic perspective (such as c) and scales that are not (such as $\hbar$ and the electron rest energy). It is important to also stress that even when dealing with a scale that is potentially significant from a relativistic perspective, without a description/analysis of the laws of transformation between inertial observers any claim of relevance from a DSR perspective is futile.

Consider for example the possibility of a "minimum wavelength". This is clearly a type of proposal that is potentially significant from a relativistic perspective, and indeed one of the objectives of DSR research is to find a meaningful framework for the implementation of a "minimum wavelength principle" in an observer-independent manner. However, the proposal of a minimum wavelength does not in itself provide us with a DSR proposal. In particular, the minimum-wavelength bound could be observer dependent. A calculation within a given theory providing evidence of a minimum-wavelength bound would of course not suffice to qualify the relevant theory as a DSR theory. One should at least also analyze what type of laws of transformation between observers apply in the relevant theory and verify that the minimum-wavelength bound (both the presence of a bound and the value of the bound) is observer-independent.

Similar considerations apply for proposals of modified energy-momentum dispersion relations and modified wavelength-momentum (deBroglie) relations. It is well understood that some frameworks in which one finds modified dispersion relations are actually not compatible with the DSR principles: in the relevant frameworks the modification of the dispersion relation is observer dependent (at least because of observer dependence of the scale of modification of the dispersion relation), and Poincaré symmetry is actually broken. Analogously in quantum-gravity scenarios (see e.g., Refs. [87,88]) in which one adopts a modified relation between momentum and wavelength (typically such that momentum is still allowed to go up to infinity but there is a finite minimum value of wavelength), before concluding in favour or against compatibility with the DSR principles an analysis of the laws of transformation between observers is of course necessary. One could imagine such a modification of the relation between momentum and wavelength to be a manifestation of a breakdown of Poincaré symmetry, but there is also no in-principle obstruction for trying to implement it in a DSR-compatible way.

4. Spacetime "Quantization" and Relativistic Paradoxes

Let me now comment on some issues concerning "spacetime quantization and relativistic paradoxes" in DSR, from a perspective which in part also contributes to a true/false characterization of the DSR concept: during this first decade of DSR research several authors appear to have been devoting a significant effort toward showing that there are formulations of DSR compatible with a classical-spacetime picture and such that no "relativistic paradox" is produced, but from the perspective I shall here advocate it appears that these efforts are based on incorrect assumptions.

At least on the basis of experience (let me postpone to another occasion the logical grounds for this) we should actually assume that the transition from a given relativistic theory to the next brings along (apparent) "paradoxes". This is simply because the relativistic framework in which the laws of physics are formulated affects deeply the nature of the observables that appear in those laws, first by characterizing the objectivity of physical processes (not withstandding a possible subectivity, *i.e.*, observer dependence, of the actual numerical results of measurement of some observables by different observers of the same system), and then in turn affecting our intuitive perception of those processes.

When Galileo introduced his relativistic theory, with anecdotal fame proportional to the fact that the some of the guardians of the previous (non-)relativistic theory where at the Vatican, a major "paradox" was that within Galieleian relativity (adopting modern contextualization) a basketball bouncing on a moving ship would keep landing on the same point of the floor of the ship: how could the basketball keep up with the ship when it is in the air, without any contact with the ship, and there is therefore no way for the ship to carry it along? The answer was perhaps very deep, but not truly paradoxical: it could, and it does.

With the advent of Einstenian relativity we were faced with the novelty of relative time and length, producing (among many other paradoxes) the famous "twin paradox" and "pole-in-a-barn paradox". We now know for a fact that some twins (at least twin particles) do age differently (if their histories of accelerations are different), but I sometime wonder whether Galileo could have come to terms with it, had he managed to stumble upon corresponding experimental evidence.

The fact that such a wide spectrum of tentative formalizations of DSR are being considered (and many more could clearly be considered) does not allow me to claim that necessarily the correct formulation(s) should have spacetime quantization and "relativistic paradoxes". But on the basis of analyses of some of the candidate DSR formalizations and on the basis of the history of the advent of previous relativistic theories I conjecture that this will be the case. Chances are there is actually no room for DSR in the laws of Nature, but if there was and it came without affeting deeply our intuition (thereby producing apparent paradoxes) it would be a major let down.

If I aimed for full generality I could not possibly go beyond intuition and desires, but I believe it is worth elaborating on these issues in at least one explicit context. And from this perspective it is particularly intriguing to contemplate DSR scenarios in which the speed of massless particles (e.g., photons) acquires a dependence on energy/wavelength. Because of the role that the speed of light has played in the history of physics it is certainly noteworthy that the DSR framework, by introducing an observer-independent energy/length scale, could provide room for a relativistic (observer-independent) law introducing a dependence on energy/wavelength for the speed of light. And indeed a large part of the DSR literature has considered this possibility, both because it is intriguing conceptually and because tests of energy dependence of the speed of photons are improving quickly, so that there could be opportunities for test/falsification of such DSR scenarios. But of course one can have DSR relativistic theories without such energy dependence of the speed of photons (by introducing the additional relativistic scale in other ways, such as, e.g., a "minimum length uncertainty" principle), and in deciding how much effort should be directed toward DSR scenarios with this energy dependence of the speed of photons and how much should be directed toward other DSR scenarios is may be valuable to fully appreciate the logical implications of an observer-independent law for the energy dependence of the speed of photons. In this section I will expose some of the relativistic paradoxes that are produced by an observer-independent law for the energy dependence of the speed of photons. At the level of contribution to the intuition of theorists these "paradoxes" can actually go both ways: some readers will find in them reason to focus on other DSR scenarios, others will perceive them as just the right "amount of paradoxality" that a truly new relativistic theory should introduce. But the technical implications of the observations reported in this section are unequivocal: it is futile to seek a classical-spacetime formulation of DSR theories with energy dependence of the speed of photons.

For the purposes of this section it suffices for me to contemplate a DSR scenario with the following properties:

- (i) for the reminder of this section it will be assumed that the speed of a “ultrarelativistic” ($E \gg m$) particle of mass m and energy E is given by

$$v(E) \simeq 1 - \frac{m^2}{2E^2} + \lambda E \tag{11}$$

- (ii) and also for the reminder of this section it will be assumed that the laws of transformation between inertial observers are deformed in such a way to leave invariant the relationship

$$E^2 \simeq p^2 + m^2 + \lambda p^2 E \tag{12}$$

and in particular the generators of boosts take the form

$$\mathcal{B}_j \simeq i p_j \frac{\partial}{\partial E} + i \left(E + \frac{\lambda}{2}\vec{p}^2 + \lambda E^2 \right) \frac{\partial}{\partial p_j} - i\lambda p_j \left(p_k \frac{\partial}{\partial p_k} \right) \tag{13}$$

In some DSR studies these 3 relations have been tentatively treated as parts of a consistent picture (different authors considering different frameworks, but sharing these 3 features) on the basis of the mentioned logical link between (12) and (13) (*i.e.*, the observation that adopting (13) one finds that (12) is a relativistically invariant relation) and observing that (11) follows from (12) if one preserves the validity of the (group-)velocity relation $v = dE/dp$.

However, as I shall show in the reminder of this section, (11) and (13) combine to lead to a description of distances and time intervals which is “paradoxical”, at least in the sense that it requires a very specific (and apparently rather “virulent”) non-classical/quantum structure for the description of spacetime. In light of these observations a legitimate choice is the one disregarding this possibility, and therefore focus DSR research on some of the many alternative possibilities. But it is also interesting to insist on the validity of these 3 ingredients and verify whether one actually runs into direct conflict with any established experimental facts. This is an exercise to which I devoted some effort, finding that there appears to be no such conflict with experiment, but rather the (scary but intriguing) need of revising very deeply our conceptualization of spacetime.

Through this exercise one also ends up putting in focus a plausible scenario for DSR, according to which one performs a genuine deformation of the special-relativistic properties of energy-momentum space, but one modifies the special-relativistic description of spacetime in a more virulent fashion, which might be not properly described by the term “deformation” (usually intended as a relatively soft modification, smoothly recovering the original theory as the deformation parameter is removed).

4.1. *Spacetime Fuzziness for Classical Particles*

The fact that assuming (11) and (13) one must immedietaly renounce to a classical spacetime picture was first established in Ref. [19]. The argument relies [19] on contemplating two classical massive particles, with different masses m_A and m_B but with the same velocity (upon adopting (11) and choosing appropriately their different energies) according to some observer O. So the starting point of the analysis is such that the observer O could see two particles with different masses m_A and m_B moving at the same speed and following the same trajectory (for O particles A and B are “near” at all times). It is then interesting to investigate how these two identical worldlines appear to a boosted observer O'. And one easily finds [19], using (11) and (13), that the two particles would have different velocities according to a second observer O' (boosted with respect to O). So according to O' the two particles could be "near" only for a limited amount of time. This observation establishes the need for a nonclassical spacetime, since in particular it implies that single points for O are mapped into pairs of (possibly sizeably distant) points for O'.

Similar conclusions are drawn by considering the implications of (11) and (13) for the analysis of "Einstein light clocks", the prototypical relativistic clocks, which are digital clocks counting the number of times a photon travels a known distance L between two ideal mirrors. Clearly in presence of energy dependence of the speed of photons, $v(E)$, such a clock measures time in units of $\tau = v(E)L$, and it is then straightforward to analyze the dilatation of this time interval τ under a boost acting in the direction orthogonal to the axis that connects the two mirrors [1,2]:

$$\tau' = \frac{v(E)}{\sqrt{v(E')^2 - V^2}}\,\tau \tag{14}$$

where V is the relative velocity of the reference frames connected by the boost, E is the energy of the photon in the rest frame of the clock, and E' is the energy of the photon in the boosted frame.

At first sight (14) appears to be describable as an energy-dependent but smooth deformation of the standard special-relativistic time-dilatation formula, but upon closer examination one quickly realizes that it has strikingly wild implications. For my purposes here it suffices to take into account this time-dilatation formula (14) in the analysis of a pair of Einstein light clocks, one using photons of energy E_1 and the other one, at rest with respect to the first, using photons of energy E_2. For simplicity, let me further specify that E_1 and E_2 be such that $v(E_2) = 2v(E_1)$, *i.e.*, the difference in energy is large enough to induce doubling of speed. In such a setup the two clocks could in principle share the same pair of mirrors, so that they provide a case with "clocks at the same point", which is of particular interest from a relativistic perspective.

This setup with two Einstein clocks is such that, by construction/synchronization, in the rest frame of the clocks one always finds simultaneous "tick" of the E_2-clock whenever the E_1-clock "ticks", since we have arranged things in such a way that

$$\tau_1 = 2\tau_2$$

And it is particularly significant that, using (13) to relate E_1', E_2' to E_1, E_2, one finds

$$\tau_1' = \frac{v(E_1)}{\sqrt{v(E_1')^2 - V^2}}\,\tau_1 \neq 2\tau_2' = \frac{v(E_2)}{\sqrt{v(E_2')^2 - V^2}}\,2\tau_2 \tag{15}$$

for the same boost considered above (rapidity V, orthogonal to the axis of the mirrors). This implies that, while in their rest frame the ticks of the E_1-clock are always simultaneous to ticks of the E_2-clock, in the boosted frame the ticks of the E_1'-clock are not simultaneous to ticks of the E_2' clock. Ordinary special relativity (while removing the abstraction of an "absolute time") still affords us an objective, observer independent, concept of simultaneity, restricted to events occurring at the same spatial point, but in this candidate DSR framework there are events (e.g., ticks of our two Einstein clocks) which are simultaneous and at the same spatial point for observer O but are not simultaneous for observer O'. Using tentatively classical-spacetime reasoning one finds that an "event", a point of spacetime, for observer O is not mapped in an event for O', which in turn implies that the classical-spacetime description of an event (as a sharp point in spacetime) must be abandoned.

4.2. Spacetime Fuzziness for Quantum Particles

The previous subsection exposes the need for spacetime nonclassicality already for (formal) theories of classical particles. Let me now comment (following the line of reasoning adopted in Ref. [89]) on the even more virulent departures from spacetime classicality that are to be expected when a DSR framework predicts energy dependence of the speed of quantum photons.

The issue arises because in giving operative meaning to spacetime points (or, more rigorously, to distances between spacetime points) one inevitably resorts to the use of particle probes and spacetime can be meaningfully labelled "classical" only if the theory admits the possibility for such localization procedures to have absolutely sharp ("no uncertainty") outcome, at least as the endpoint of a well-defined limiting procedure. Within ordinary quantum mechanics one could still legitimately contemplate the ideal limit in which point particles have infinite inertial mass (so that the Heisenberg principle is ineffective) but that limit is not meaningful in theoretical frameworks, such as the DSR framework, in which the motivation originates from the quantum-gravity problem and therefore infinite inertial mass comes along with infinite gravitational charge (mass). In the opposite limit, the one of massless particles, one finds that the energy(/wavelength) dependence of the speed of light introduces an extra term in the balance of quantum uncertainties. Assume in fact that the measurement procedure requires some known time T_{obs} and therefore (in order to obtain measurement results compatible with the classical-spacetime idealization) we would like the particle probe to behave as a classical probe over that time. For that goal it is in particular necessary to keep under control two sources of quantum uncertainty, the one concerning the energy of the particle and the one concerning the time of emission of the particle. The uncertainty δx in the position of the massless probe when a time T_{obs} has lapsed since the observer (experimentalist) set off the measurement procedure will in general satisfy the following inequality

$$\delta x \geq \ \delta t + \delta v \, T_{obs} \simeq \delta t + \lambda \delta E T_{obs} \tag{16}$$

where δt is the uncertainty in the time of emission of the probe, and I used (11) to describe the uncertainty in the speed of the particle, δv, in terms of the uncertainty δE in its energy. Since the uncertainty in the time of emission of a particle and the uncertainty in its energy are related by $\delta t \, \delta E \geq 1$, Eq. (16) can be turned into an absolute bound on the uncertainty in the position of the massless probe when a time T_{obs} has lapsed since the observer set off the measurement procedure:

$$\delta x \geq \frac{1}{\delta E} + \lambda \delta E T_{obs} \geq \sqrt{\lambda T_{obs}} \tag{17}$$

The right-hand side of (17) does exploit the fact that in principle the observer can prepare the probe in a state with desired δE (so it is legitimate to minimize the uncertainty with respect to the free choice of δE), but the classical behaviour of the probe is not achieved in any case (in all cases δx is strictly greater than 0).

5. More on the Use of Hopf Algebras in DSR Research

I have already mentioned Hopf algebras as a candidate tool for the formalization of DSR relativistic theories. In this section I briefly summarize the technical reasons why this appears to be plausible, but also describe some technical challenges which still need to be addressed before drawing any conclusions on the applicability (or lack thereof) of Hopf algebras to DSR model building. Also on this topic of the relevance of Hopf algebras for DSR research some misconceptions are rather frequent, so I will find opportunities for additional true/false characterizations.

5.1. A Hopf-algebra Scenario with κ-Poincaré Etructure

In trying to give a brief summary of some key aspects of the possibility of DSR scenarios based on the mathematics of Hopf algebras, let me start with a rudimentary review of the most studied such scenario which is based on the κ-Poincaré Hopf algebra and the associated κ-Minkowski noncommutative spacetime.

The characteristic spacetime-coordinate noncommutativity of κ-Minkowski is given by

$$[x_j, x_0] = i\lambda x_j \tag{18}$$

$$[x_k, x_j] = 0 \tag{19}$$

where x_0 is the time coordinate, x_j are space coordinates ($j, k \in \{1, 2, 3\}$), and λ is a length scale, usually expected to be of the order of the Planck length. Functions of these noncommuting coordinates are usually conventionally taken to be of the form

$$f(x) = \int d^4k \tilde{f}(k) e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} \tag{20}$$

where the "Fourier parameters" $\{k_0, k_i\}$ are ordinary commutative variables.

I shall here be satisfied with a brief review of a frequently used characterization of symmetries of κ-Minkowski, in which generators for translations, space-rotations and boosts are introduced adopting the following definitions

$$P_\mu \left(e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0}\right) = k_\mu e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} \tag{21}$$

$$R_j \left(e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0}\right) = \epsilon_{jkl} x_k k_l e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} \tag{22}$$

$$N_j \left(e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0}\right) = -k_j e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} x_0 + \left[x_j \left(\frac{1 - e^{-2\lambda k_0}}{2\lambda} + \frac{\lambda}{2}|\vec{k}|^2\right) - \lambda x_l k_l k_j\right] e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} \tag{23}$$

The fact that we are here dealing with a Hopf algebra (indeed the κ-Poincaré Hopf algebra) is essentially seen by acting with these generators on products of functions ("coproduct"), observing for example that

$$\begin{aligned} P_\mu \left(e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} e^{i\vec{q}\cdot\vec{x}} e^{-iq_0x_0}\right) &= \left(k_\mu + e^{-\lambda k_0(1-\delta_{\mu 0})} q_\mu\right) \left(e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} e^{i\vec{q}\cdot\vec{x}} e^{-iq_0x_0}\right) \\ &\neq (k_\mu + q_\mu) \left(e^{i\vec{k}\cdot\vec{x}} e^{-ik_0x_0} e^{i\vec{q}\cdot\vec{x}} e^{-iq_0x_0}\right) \end{aligned} \tag{24}$$

Some authors naively focus their analyses exclusively on the properties of commutators of the generators of the κ-Poincaré algebra, which turn out to be deformations [90] of the commutators of the standard Poincaré Lie algebra. And several (mutually incompatible) naive arguments have been proposed on the basis of such naive characterizations: in some cases the nonlinearities present in these commutators are taken as a full characterization of the κ-Poincaré symmetries, while in other cases a large significance is incorrectly given to the fact that by nonlinear redefinition of the generators one can remove all anomalies of the commutators. However, a proper description of Hopf-algebra spacetime symmetries must clearly take into account both the commutators and the coproducts (and redefinitions of the generators affect simultaneously both commutators and coproducts).

Consider for example the case of translation transformations. A translation transformation for a function $f(x)$ should be described as

$$df(x) = i\epsilon^\mu P_\mu f(x) \tag{25}$$

in terms of the translation generators and some transformation parameters ϵ_μ. And it turns out that the transformation parameters must reflect the properties of the coproduct. In fact, the transformation parameters must ensure that (if x_μ is in κ-Minkowski) $x_\mu + \epsilon_\mu$ is still a point in κ-Minkowski:

$$[x_j + \epsilon_j, x_0 + \epsilon_0] = i\lambda(x_j + \epsilon_j)\ ,\ \ [x_i + \epsilon_i, x_j + \epsilon_j] = 0 \tag{26}$$

So the transformation parameters must not be simple numbers but should instead be endowed with nontrivial algebraic properties. And it is easy to see [76,91] that the form of these algebraic properties should reflect the properties of the coproduct in order to preserve Leibniz rule:

$$d(f \cdot g) = f \cdot dg + df \cdot g \tag{27}$$

Taking into account these observations it turned out to be possible [76,91] to obtain conserved charges associated to the Hopf symmetries for a theory with classical fields in the noncommutative κ-Minkowski spacetime (while all previous attempts, which had naively ignored the role of the coproduct in the full characterization of symmetry transformations, had failed).

Besides providing insight on the role on nonlinear redefinitions of generators within Hopf-algebra formulations (already here stressed in Subsection 3.1), these results on "noncommutative transformation parameters" might have other strong implications. As first stressed in Ref. [91], by examining in detail the nonocmmutativity properties of the transformation parameters one finds that pure boosts are forbidden: whenever the boost parameters are nonzero the noncommutativity properties are such that also some rotation parameters should necessarily be turned on [91]. This is rather significant for DSR scenarios which have been tentatively based on the κ-Poincaré/κ-Minkowski: it seems we should assume that, if eventually we do find a fully consistent DSR scenario based on the κ-Poincaré/κ-Minkowski formalism then the description of boosts should accommodate some nonclassical features even in the characterization of the differences between pairs of observers.

But let me stop here my brief summary of results obtained in the κ-Poincaré/κ-Minkowski framework, since it will suffice for my purposes. The interested reader can find a rather detailed description of what we presently know about the κ-Poincaré symmetries of κ-Minkowski in Refs. [76,91] and references therein. The idea that this mathematics might provide the basis for a DSR theory originates essentially in the observation that the spacetime noncommutativity of κ-Minkowski, as described in (19), is left invariant by the action of κ-Poincaré generators, and the scale λ appears to be a reasonable candidate for a DSR-type second relativistic scale. It has also long been conjectured that with this

κ-Minkowski/κ-Poincaré recipe one might end up having a DSR theory with modified dispersion relation, but originally this suggestion was only based on the observation that the "mass Casimir" of the κ-Poincaré Hopf algebra is a deformation of the mass Casimir of the standard Poincaré Lie algebra. A definite statement about the status of the dispersion relation in this framework will require a meaningfully physical identification and characterization of concepts such as energy, spatial momentum, frequency and wavelength, and this task is perhaps now finally within reach, since we can now rely on actual derivations of conserved charges using the techniques developed in Refs. [76,91]. However, a few residual issues must still be addressed [76,91–93] before we can safely identify energy, spatial momentum, frequency and wavelength.

And in general, as mentioned in earlier sections of these notes, some work remains to be done to fully establish that the κ-Minkowski/κ-Poincaré formalism can be really used to construct a DSR theory. It will require still some work of digging through κ-Minkowski/κ-Poincaré mathematics, guided by the DSR principles, looking for tools that are DSR compatible. But this is not trivial and not easily done without caution. For example, as mentioned, the κ-Poincaré mathematics can inspire (and has inspired) a description of the kinematics of particle-reaction processes which is manifestly in conflict with the DSR principles (it would select a preferred frame), but looking around in the κ-Minkowski/κ-Poincaré "zoo of mathematics", if we indeed look around using the DSR principles as guidance, we might find other (possibly DSR-compatible) structures on which the kinematics of particle-reaction processes could be based.

5.2. A Hopf-algebra Scenario without κ-Poincaré and without Modified Dispersion Relations

The fact that the κ-Poincaré/κ-Minkowski framework briefly discussed in the previous subsection provides a promising path toward a DSR theory is appreciated by most authors involved in DSR research. So much so that some do not appear to even see the possibility of alternatives. Actually, one does not need to look too far to find an alternative which is equally promising: even within the confines of approaches based on spacetime noncommutativity one finds a framework which is indeed equally promising from a DSR perspective and appears to motivate investigation of a different set of candidate "DSR effects". This is the case of the "canonical noncommutative spacetime" with characteristic spacetime-coordinate noncommutativity given by ($\mu, \nu \in \{0, 1, 2, 3\}$)

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{28}$$

These noncommutativity relations (28) can be meaningfully considered endowing $\theta^{\mu\nu}$ with the properties of a standard Lorentz tensor, in which case one of course ends up with a scenario where Lorentz symmetry is broken, but they can also be meaningfully considered endowing $\theta^{\mu\nu}$ with the properties of an observer-independent matrix, which would of course be the case of interest from a DSR perspective. [A third, even more ambitious, possibility has been developed in Refs. [94,95] by allowing $\theta^{\mu\nu}$ to acquire nontrivial algebraic properties.]

With observer-independent $\theta^{\mu\nu}$ the symmetries of this spacetime are described by a Hopf algebra which is significantly <u>different from κ-Poincaré</u>, whose generators can be described as follows:

$$P_\mu e^{ikx} \equiv P_\mu \Omega(e^{ikx^{(c)}}) \equiv \Omega(i\partial_\mu^{(c)} e^{ikx^{(c)}}) \tag{29}$$

$$M_{\mu\nu} e^{ikx} \equiv M_{\mu\nu} \Omega(e^{ikx^{(c)}}) \equiv \Omega(ix_{[\mu}^{(c)} \partial_{\nu]}^{(c)} e^{ikx^{(c)}}) \tag{30}$$

where $x_\mu^{(c)}$ are auxiliary commutative coordinates, $\partial_\nu^{(c)}$ are ordinary derivatives with respect to the auxiliary coordinates $x_\mu^{(c)}$, and (for any $\tilde{\Phi}(k)$)

$$\Omega\left(\int d^4k\,\tilde{\Phi}(k)e^{ikx^{(c)}}\right) = \int d^4k\,\tilde{\Phi}(k)e^{ikx} \tag{31}$$

The fact that these are generators of a Hopf algebra manifests itself as usual immediately upon noticing that the action of (Lorentz-sector) generators does not comply with Leibniz rule:

$$\begin{aligned} M_{\mu\nu}\left(e^{ikx}e^{iqx}\right) &= \left(M_{\mu\nu}e^{ikx}\right)e^{iqx} + e^{ikx}\left(M_{\mu\nu}e^{iqx}\right) + \\ &+\frac{1}{2}\theta^{\alpha\beta}\left[-\eta_{\alpha[\mu}\left(P_{\nu]}e^{ikx}\right)\left(P_\beta e^{iqx}\right) - \left(P_\alpha e^{ikx}\right)\eta_{\beta[\mu}\left(P_{\nu]}e^{iqx}\right)\right] \end{aligned} \tag{32}$$

As in the case of the κ-Poincaré/κ-Minkowski framework, the idea that this framework based on canonical noncommutativity might provide the basis for a DSR theory originates essentially in the observation that the spacetime noncommutativity of canonical form, (28), is left invariant by the action of the Hopf-algebra generators (29)-(30), so that any physical consequence of that noncommutativity (such as "spacetime fuzzyness") should be observer independent. To render explicit the presence of a short invariant length scale one can conveniently rewrite the observer-independent dimensionful matrix $\theta^{\mu\nu}$ in terms of an observer-independent length scale λ and an observer-independent dimensionless matrix $\tau^{\mu\nu}$ (with an extra restriction, e.g., unit determinant, to avoid apparent overcounting of parameters upon introducing λ): $\theta^{\mu\nu} = \lambda^2\tau^{\mu\nu}$. And the length scale λ should indeed be small in order to ensure that one introduces an acceptably small amount of coordinate noncommutativity (we clearly have robust experimental evidence against large noncommutativity).

For a detailed description of what we presently know about this alternative Hopf-algebra-based DSR scenario readers can look at Ref. [77,91] and references therein. One key point is that all evidence gathered so far (which however only concerns classical particles and fields in this "quantum" geometry) suggests that the dispersion relation is not modified in this framework, and therefore also from this perspective we might already have a framework providing complementary DSR intuition with respect to the more popular κ-Poincaré/κ-Minkowski framework. Also for this canonical-noncommutativity framework it turned out to be possible [77] to derive conserved charges within an analysis *a la* Noether, and in doing this it turned out to be necessary to endow the transformation parameters with nontrivial algebraic properties reflecting the coproduct structure. And the choice of ordering implicitly introduced in (31) in this case can be shown very explicitly to be inessential: a variety of alternative ordering conventions are easily considered and found [77] to lead to the same conserved charges. Since by changing ordering convention for the spacetime coordinates one essentially ends up introducing nonlinear redefinitions of the generators of the Hopf algebra, the fact that the charges are independent of the choice of ordering convention also translates into yet another invitation for caution for those authors who naively assume that nonlinear redefinitions of the generators of a Hopf algebra might change the physical picture (in some extremely naive arguments it is stated that by nonlinear redefinition of the generators one might go from an incorrect to a correct picture, or that such a nonlinear redefinition of generators could eliminate the characteristic length scale of some relevant Hopf algebra, but here very explicitly one finds that such nonlinear redefinitions do not affect the conserved charges and their dependence on the characteristic scale of the Hopf algebra).

6. DSR Scenarios and DSR Phenomenology

Considering the very early stage of development of the Hopf-algebra scenarios for DSR, and the fact that the approach based on Hopf algebra is the best developed attempt of finding DSR compatible theories, it is clear that we are not ready to do any "real" DSR phenomenology. In order to claim one was doing real DSR phenomenology a minimum requisite would be the availability of a theoretical framework whose compatibility with the DSR principles was fully established and characterized in terms of genuinely observable features. Since we are not ready for that, one might perhaps consider postponing all reasoning about phenomenology to better times (ahead?). But, at least from some perspectives the development of (however incomplete and however *ad hoc*) "toy DSR test theories" can be valuable. The exercise of developing such "toy test theories", in the sense that will emerge from the next subsections, turns out to be valuable in providing a crisper physical characterization of the concept of a DSR theory (and in fact I found it useful to introduce one such "toy theory" already in my original papers in Refs. [1,2]) and allows to clarify some general arguments (as in the case of the incompatibility of a decay threshold with the DSR principles, see later).

6.1. A Toy-Model DSR-Scenario Test Theory, Confined to Leading Order

To illustrate what I qualify as a "toy DSR test theory" I can indeed use the example of such "toy theory" that I already used in my original papers in Refs. [1,2]. This is a "limited theory" in that it only concerns the laws of transformation of the energy-momentum observables. It assumes that the energy-momentum dispersion relation is observer independent and takes the following form in leading order in the Planck length:

$$E^2 \simeq \vec{p}^2 + m^2 + \lambda \vec{p}^2 E \tag{33}$$

This dispersion relation is clearly an invariant of space rotations, but it is not an invariant of ordinary boost transformations. Its invariance (to leading order) is ensured adopting standard space-rotation generators

$$R_j = -i\epsilon_{jkl} p_k \frac{\partial}{\partial p_l} \tag{34}$$

and a deformed action for boost generators

$$\mathcal{B}_j \simeq i p_j \frac{\partial}{\partial E} + i\left(E + \frac{\lambda}{2}\vec{p}^2 + \lambda E^2\right)\frac{\partial}{\partial p_j} - i\lambda p_j \left(p_k \frac{\partial}{\partial p_k}\right) \tag{35}$$

For the rudimentary "phenomenology of kinematics" which I intend to discuss within this limited "toy DSR test theory" the only remaining ingredient to be specified is the one linking incoming energymomenta to outgoing ones, as intended in a law of conservation of energymomentum. Let us start considering processes with two incoming particles, a and b, and two outgoing particles, c and d. The special-relativistic kinematic requirements for such processes are $E_a + E_b - E_c - E_d = 0$ and $p_a + p_b - p_c - p_d = 0$, but these clearly [1,2] would not be observer-independent laws in light of (35). Working in leading order actually one finds several [1,2] acceptable alternative possibilities for the deformation of the law of conservation of energymomentum. In the following I will adopt

$$E_a + E_b + \lambda p_a p_b \simeq E_c + E_d + \lambda p_c p_d \tag{36}$$

$$p_a + p_b + \lambda(E_a p_b + E_b p_a) \simeq p_c + p_d + \lambda(E_c p_d + E_d p_c) \tag{37}$$

Analogous formulas can be obtained for any process with n incoming particles and m outgoing particles. In particular, in the case of a two-body particle decay $a \to b + c$ the laws

$$E_a \simeq E_b + E_c + \lambda p_b p_c \tag{38}$$

$$p_a \simeq p_b + p_c + \lambda(E_b p_c + E_c p_b) \tag{39}$$

provide an acceptable (observer-independent, covariant according to (35)) possibility.

6.2. *On the Test Theory Viewed from An (unnecessary) All-Order Perspective*

As stressed earlier in these notes, the physical picture that motivates the proposal of Doubly-special Relativity allows one to contemplate DSR symmetries as exact symmetries (applicable in a certain corresponding regime), but it also invites one to consider the possibility that (even within the confines of the regime where DSR turns out to be applicable) DSR symmetries be just approximate symmetries. For example, formulas might not be exactly compatible with the DSR setup because of possibly including non-DSR terms that are negligible (but nonzero) in the DSR regime, but become large at some even higher energy scales, where quantum-gravity effects might become so virulent not to allow even a DSR description.

I should therefore correspondingly stress that it is not necessary (and not necessarily appropriate) to cast the "leading-order toy test theory" discussed in the previous subsection within some corresponding all-order DSR theory.

Let me nonetheless briefly review some evidence we have that there is an "all-order toy test theory" from which the leading-order one discussed in the previous subsection can be derived. A first ingredient for such a theory could be the following all-order dispersion relation

$$\frac{2}{\lambda^2}\left[\cosh(\lambda E) - \cosh(\lambda m)\right] = \vec{p}^2 e^{\lambda E} \tag{40}$$

in which case boost transformations could be generated by

$$\mathcal{B}_j = i p_j \frac{\partial}{\partial E} + i\left(\frac{\lambda}{2}\vec{p}^2 - \frac{1 - e^{2\lambda E}}{2\lambda}\right)\frac{\partial}{\partial p_j} - i\lambda p_j \left(p_k \frac{\partial}{\partial p_k}\right) \tag{41}$$

(whereas, even in the all-order formulation, space-rotation transformations do not require deformation).

Even for these "all-order boost generators" (and therefore, of course, also for their leading-order formulation) one manages to obtain explicit formulas [4] for the finite boost transformations that relate the observations of two observers. These are obtained by integrating the familiar differential equations

$$\frac{dE}{d\xi} = i[\mathcal{B}_j, E]\ , \quad \frac{dp_k}{d\xi} = i[\mathcal{B}_j, p_k] \tag{42}$$

which relate the variations of energy-moomentum with rapidity (ξ) to the commutators between the boost generator (along the direction j of the chosen boost) and energy-momentum.

The result is conveniently characterized by giving the formula expressing the amount of rapidity ξ needed to take a particle from its rest frame (where the energy is m) to a frame in which its energy is E:

$$\cosh(\xi) = \frac{e^{\lambda E} - \cosh(\lambda m)}{\sinh(\lambda m)}\ , \quad \sinh(\xi) = \frac{p e^{\lambda E}}{\lambda \sinh(\lambda m)} \tag{43}$$

Since I am here only setting up a discussion of DSR phenomenology I shall not make any further effort of characterization of this all-order setup. In fact, at least for teh foreseeable future, it appears likely that we could only attempt a leading order DSR phenomenology (sensitivities presently available do not appear to allow us to go beyond that). My discussion of an all-order DSR setup clearly was not aiming for an "all-order DSR phenomenology", but rather to stress that if one insists (and it is not necessary but possible to insist) on embedding the "leading order toy test theory" within an all-order setup, then a valuable constraint can emerge: while within the leading-order toy test theory nothing prevents the (dimensionful) parameter λ to be either positive or negative, within the specific all-order reformulation I just discussed λ must be positive. This comes about [28] because attempting to find real energy-momentum solutions for Eqs. (42) for all real values of ξ (which one ought to do in an all-order setup) and using the form (41) of the boost generators one actually finds the solutions codified in (43) only for positive λ, while the same equations for negative λ do not admit solution (not solutions for all values of ξ if the energy-momentum must be real [28]). Note however that the embedding of a "leading order toy test theory" within an "all-order toy test theory" is clearly not unique, and there may well be other such embeddings which instead are admissible only for negative λ. One can therefore tentatively conclude that we should not expect a definite sign prediction generically for the large class of possible DSR scenarios of this sort, but that within any given all-order formulation, with of course some associated leading-order formulation, the sign ambiguity can be studied and possibly removed (as it is removed in the specific all-order scenario discussed in this subsection).

6.3. Photon Stability

The first possibility that I want to consider in this phenomenology section is the one of energy thresholds for particle decay, such as the possibility of the decay of a photon into an electron-positron pair, $\gamma \rightarrow e^+ + e^-$. I already discussed this process earlier in these notes, as a way to show that the DSR concept makes definite predictions, in spite of not being at present attached to any specific mathematical formalism. Any experiment providing evidence of a preferred frame would rule out DSR, and this implies that any theory compatible with the DSR principle must not predict energy thresholds for the decay of particles. In fact, one could not state observer-independently a law setting a threshold energy for a certain particle decay, because different observers attribute different energy to a particle (so then the particle should be decaying according to some observers while being stable according to other observers).

This argument holds directly at the level of the logical structure of the DSR concept, but it is nonetheless useful to verify how our toy DSR test theory implements it. The key structure is the rigidity that the DSR concept introduces (for theories structured like our toy test theory) between the structure of the energy-momentum dispersion relation and the structure of the energy-momentum-conservation law. The boost transformations must leave invariant the dispersion relation, and under those same boost transformations the energy-momentum-conservation law must be covariant. This led me in particular to introduce the energy-momentum-conservation law

$$E_a \simeq E_b + E_c + \lambda p_b p_c \tag{44}$$

$$p_a = p_b + p_c + \lambda(E_b p_c + E_c p_b) \tag{45}$$

for two-body particle decays $a \to b + c$, in association with the dispersion relation

$$E^2 \simeq \vec{p}^2 + m^2 + \lambda \vec{p}^2 E \tag{46}$$

If one was to combine (46) with an unmodified law of energy-momentum conservation, as admissible in scenarios with broken Poincaré symmetry (but not allowed in scenarios in which Poincaré symmetry is deformed in the DSR sense), then it is well established that a threshold for the decay of a photon into electron-positron pairs can emerge. In the symmetry-breaking case the relation between the energy E_γ of the incoming photon, the opening angle θ between the outgoing electron-positron pair, and the energy E_+ of the outgoing positron, takes the form $\cos(\theta) = (A + B)/A$, where (for the region of phase space with $m_e \ll E_\gamma \ll E_p$) $A = E_+(E_\gamma - E_+)$ and $B = m_e^2 + \lambda E_\gamma E_+(E_\gamma - E_+)$ (m_e denotes of course the electron mass), and for $\lambda < 0$ (meaning λ/L_p negative real number) one finds that $\cos(\theta) < 1$ (and therefore the decay process is allowed) in certain corresponding region of phase space.

Instead if the same analysis is done in the DSR-compatible framework of our toy DSR test theory, and therefore one adopts the modified dispersion relation (46) and the modified energy-momentum-conservation law (44)-(45), one arrives at a result for $\cos(\theta)$ which is still of the form $(A + B)/A$ but now with $A = 2E_+(E_\gamma - E_+) + \lambda E_\gamma E_+(E_\gamma - E_+)$ and $B = 2m_e^2$. Evidently this formula is never compatible with $\cos(\theta) < 1$, consistently with the fact that $\gamma \to e^+e^-$ is always forbidden in our toy DSR test theory (so, in particular, there is no threshold for the decay).

This discussion also shows at least one way in which toy DSR test theories such as the one I am considering, in spite of all their limitations for what concerns applicability (the one I am considering only gives a rough description of some aspects of kinematics) and motivation (not being derived from, and not even inspired by, a DSR theory satisfactorily applied to a broad range of phenomena), can be valuable in DSR research. For example, by analyzing decay processes within the framework of a toy test theory one can see explicitly the DSR principles at work in preventing the emergence of particle-decay thresholds, and this would be particularly valuable if one had not noticed the general DSR argument that forbids such decay thresholds.

6.4. Weak Threshold Anomalies for Particle Reactions

In the recent wide literature on a possible Planck-scale breakdown of Lorentz symmetry there has been strong interest in the possibility of large "anomalies" in the evaluation of certain energy thresholds for particle reactions that are relevant in astrophysics. A simple way to see this is found in the analysis of collisions between a soft photon of energy ϵ and a high-energy photon of energy E that create an electron-positron pair: $\gamma\gamma \to e^+e^-$. For given soft-photon energy ϵ, the process is allowed only if E is greater than a certain threshold energy E_{th} which depends on ϵ and m_e^2. In a broken-Lorentz-symmetry scenario this threshold energy could be evaluated combining the dispersion relation (46) with ordinary energy-momentum conservation, and this leads to the result (assuming $\epsilon \ll m_e \ll E_{th} \ll 1/\lambda$)

$$E_{th}\epsilon + \lambda \frac{E_{th}^3}{8} = m_e^2 \tag{47}$$

The special-relativistic result $E_{th} = m_e^2/\epsilon$ corresponds of course to the $\lambda \to 0$ limit of (47). The Planck-scale (λ) correction can be safely neglected as long as $\epsilon/E_{th} > \lambda E_{th}$. But eventually, for sufficiently small values of ϵ and correspondingly large values of E_{th}, the Planck-scale correction cannot be ignored. This occurs for $\epsilon < (\lambda m_e^4)^{1/3}$ (when, correspondingly $E_{th} > (m_e^2/\lambda)^{1/3}$), and can be relevant for the analysis of observations of multi-TeV photons from certain Blazars [71,72]. Performing a similar analysis for the photo-pion-production process, in which one has a high-energy proton and a soft photon as incoming particles and a proton and a pion as outgoing particles, one ends up finding [67,70,72] the possibility of large effects for the analysis of the cosmic-ray spectrum in the neighborhood of the "GZK scale".

One might wonder whether something similar to these particle-reaction-threshold results that are so popular in the Lorentz-breaking literature could be found also in a DSR setup. In the case of photon stability we managed to draw a robust conclusion in spite of the present limited understanding and development of DSR theories, and our toy DSR test theory provided an explicit illustration of how the DSR principles intervene in the analysis. In the case of the analysis of particle-reaction thresholds the indications we can presently give from a DSR perspective are not as strong, but still rather valuable. The possibility of large particle-reaction threshold anomalies cannot be excluded simply on the basis of the DSR principles, since it does not conflict with any aspect of the structure of those principles, nonetheless, by trial and error, the DSR literature has provided evidence that large threshold anomalies are not naturally accommodated in a DSR framework. A key point to understand is that typically a DSR framework will produce smaller anomalies than a typical symmetry-breaking framework. This is to be expected simply because scenarios with deformation of symmetries produce of course softer departures from the original symmetries than scenarios in which those symmetries are broken (deforming is softer than breaking).

It is difficult to convey faithfully here in a few pages how the DSR literature provides support for this intuition, but I can quickly discuss how our toy DSR test theory deals with particle-reaction thresholds, and this will to some extent allow readers to start forming their own intuition. The laws (36)-(37) for the case of $\gamma\gamma \rightarrow e^+e^-$ take the form [1,2,27,28]

$$E + \epsilon - \lambda \vec{P}\cdot\vec{p} \simeq E_+ + E_- - \lambda \vec{p}_+\cdot\vec{p}_- \ , \quad \vec{P} + \vec{p} + \lambda E\vec{p} + \lambda\epsilon\vec{P} \simeq \vec{p}_+ + \vec{p}_- + \lambda E_+\vec{p}_- + \lambda E_-\vec{p}_+ \qquad (48)$$

where I denoted with $\vec{P}$ the momentum of the photon of energy E and I denoted with $\vec{p}$ the momentum of the photon of energy ϵ.

Using these (48) and the dispersion relation of our toy test theory (46) one obtains (keeping only terms that are meaningful for $\epsilon \ll m_e \ll E_{th} \ll 1/\lambda$)

$$E_{th} \simeq \frac{m_e^2}{\epsilon} \qquad (49)$$

i.e., one ends up with the same result as in the special-relativistic case.

This indicates that there is no large threshold anomaly in our toy DSR test theory. Actually this test theory does predict a small threshold anomaly, but truly much smaller than discussed in some symmetry-breaking scenarios. If, rather than working in leading order within the approximations allowed by the hierarchy $\epsilon \ll m_e \ll E_{th} \ll 1/\lambda$, one derives a DSR threshold formula of more general validity within our toy DSR test theory, one does find a result which is different from the special-relativistic one, but the differences are quantitatively much smaller than in some known symmetry-breaking frameworks (and in particular in the DSR case, unlike some symmetry-breaking scenarios, the differences are indeed negligible for $\epsilon \ll m_e \ll E_{th} \ll 1/\lambda$).

By trial and error one ends up figuring out that the same outcome is obtained in other toy DSR test theories, if they are all based on dispersion relations roughly of the type of (46). Note however that by ad hoc choice of the dispersion relation one can modify our toy test theory in such a way to produce a large threshold anomaly. This is for example accomplished [17] by adopting a dispersion relation which leads to the following energy-rapidity relation [17]

$$\cosh(\xi) = \frac{E}{m}(2\pi)^{-E^2 \tanh[m^2/(\lambda^4 E^6)]/(E^2+m/\lambda)} \tag{50}$$

but it might be inappropriate to attach much significance to such an ad hoc setup.

In closing this discussion of thresholds for particle-reaction thresholds let me just observe that in a theory that preserves the equivalence of inertial frames the threshold conditions must be written as a comparison of invariants. For example, it is no accident that in special relativity the threshold condition for $\gamma\gamma \to e^+e^-$ takes the form $E\epsilon \geq m_e^2$. In fact, m_e^2 is of course a special-relativistic invariant and $E\epsilon$ is also an invariant (for the head-on collision of a photon with four momentum P_μ, such that $P_0 = E$, and a photon with four momentum p_μ, such that $p_0 = \epsilon$, one finds, also using the special-relativistic dispersion relation, that $P_\mu p^\mu = 2E\epsilon$). The fact that in constructing a DSR framework, by definition, one must also insist on the equivalence of inertial frames, implies that in any genuine DSR framework the threshold conditions must also be written as a comparison of invariants.

6.5. *Wavelength Dependence of the Speed of Light*

The structure I have introduced so far in our toy test theory does not suffice to derive an energy dependence of the speed of photons, but this of course will emerge if one assumes that the standard formula $v = dE/dp$ applies. Assuming $v = dE/dp$ our test theory leads to the following velocity formula (for $m < E \ll E_p \sim 1/\lambda$):

$$v \simeq 1 - \frac{m^2}{2E^2} + \lambda E \tag{51}$$

and there is of course a rather direct way to investigate the possibility (51): whereas in ordinary special relativity two photons ($m = 0$) with different energies emitted simultaneously would reach simultaneously a far-away detector, those two photons should reach the detector at different times according to (51).

This type of effect emerging from an energy dependence of the speed of photons can be significant [65, 69] in the analysis of short-duration gamma-ray bursts that reach us from cosmological distances. For a gamma-ray burst it is not uncommon to find a time travelled before reaching our Earth detectors of order $T \sim 10^{17}s$. Microbursts within a burst can have very short duration, as short as $10^{-3}s$ (or even $10^{-4}s$), and this means that the photons that compose such a microburst are all emitted at the same time, up to an uncertainty of $10^{-3}s$. Some of the photons in these bursts have energies that extend at least up to the GeV range. For two photons with energy difference of order $\Delta E \sim 1GeV$ a $\lambda \Delta E$ speed difference over a time of travel of $10^{17}s$ would lead to a difference in times of arrival of order $\Delta t \sim \lambda T \Delta E/E_p \sim 10^{-2}s$, which is significant (the time-of-arrival differences would be larger than the time-of-emission differences within a single microburst).

So clearly observations of gamma-ray bursts should eventually provide valuable tests of DSR scenarios of this sort. At present however we must proceed cautiously, since all results so far reported in the DSR literature concern flat-space applications, whereas evidently spacetime curvature plays an important role in the analysis of gamma-ray bursts emitted from cosmologically distant sources. Of course, the ultimate goal of DSR research should be "DSR-compatible geometrodynamics" (a description of gravitational phenomena compatible with the DSR principles), but it seems that in order to profit from the opportunity that gamma-ray bursts provide it would suffice to make the next step on the way from DSR in flat spacetime to DSR geometrodynamics, *i.e.*, the formulation of scenarios for the DSR principle to be applied in a background (nondynamical) spacetime geometry with curvature.

6.6. Crab-nebula Synchrotron Radiation Data

For studies of scenarios in which Lorentz symmetry is broken by Planck-scale effects another valuable opportunity is provided by observations of the Crab nebula which are naturally interpreted as the result of synchrotron-radiation emission. This is part of the studies aimed at testing the possibility of energy dependence of the speed of particles of the type

$$v \simeq 1 - \frac{m^2}{2E^2} + \lambda E \tag{52}$$

within scenarios in which Lorentz symmetry is broken. Assuming that all other aspects of the analysis of synchrotron radiation remain unmodified at the Planck scale, one is led [96] to the conclusion that, if λ is negative ($|\lambda| \sim L_p$, $\lambda/L_p < 0$), the energy dependence of the Planck-scale (λ) term in (52) can severely affect the value of the cutoff energy for synchrotron radiation [97]. In fact, for negative λ, an electron on, say, a circular trajectory (which therefore could emit synchrotron radiation) cannot have a speed that exceeds the maximum value

$$v_e^{max} \simeq 1 - \frac{3}{2}\left(|\lambda| m_e\right)^{2/3} \tag{53}$$

whereas in special relativity of course $v_e^{max} = 1$ (although values of v_e that are close to 1 require a very large electron energy).

This may be used to argue that for negative λ the cutoff energy for synchrotron radiation should be lower than it appears to be suggested by certain observations of the Crab nebula [96].

Only very little of relevant for this phenomenology can be said at present from a DSR perspective. One key concern is that we have at present a very limited understanding of what should typically characterize interactions in a DSR framework, and a proper analysis of synchrotron radiation requires a description of interactions [98]. But clearly the implications for the cutoff energy for synchrotron radiation should be kept in focus by future work on (hopefully better developed) DSR phenomenology, and it may provide other opportunities to distinguish between DSR scenarios and Lorentz-symmetry-breaking scenarios: in fact, a proper analysis of synchrotron radiation requires, besides suitable handling of dynamics/interactions, also a description of the laws of energy-momentum conservation [98].

7. Some Other Results and Valuable Observations

I have so far discussed only points that, according to my perspective, are the core facts (and most "virulently incorrect" myths) about DSR. While this is not intended as a well-rounded review (and omissions, even of deserving works, will be inevitable), especially for the benefit of DSR newcomers (*DSR aficionados* know these and more) I think I should mention a few examples of other studies that are representative of the range of scenarios that are under consideration from a DSR perspective.

7.1. A 2+1D DSR Theory?

At least in some formulations of quantum gravity in 2+1 spacetime dimensions a q-deformed deSitter symmetry Hopf algebra $SO(3,1)_q$ emerges [99] for nonvanishing cosmological constant, and the relation between the cosmological constant Λ and the q deformation parameter takes the form $\ln q \sim \sqrt{\Lambda} L_p$ for small Λ. It was observed in Ref. [25] that this relation $\ln q \sim \sqrt{\Lambda} L_p$ implies (in the sense of the "Hopf-algebra contractions" already considered in Ref. [78,100]) that the $\Lambda \rightarrow 0$ limit is described by a κ-Poincaré Hopf algebra. However the analysis only shows that the κ-Poincaré Hopf algebra should have a role, without providing a fully physical picture. But, since, as mentioned earlier in these notes, there is some preliminary evidence that the mathematics of the κ-Poincaré Hopf algebra might be used to produce a DSR theory, the observation reported in Ref. [25] generated some justifiable interest: besides being an opportunity to perhaps find a genuine DSR toy theory, this would also provide a striking picture for how the DSR framework might emerge from a quantum-gravity theory.

It was then observed in Ref. [26] that some aspects of the formulation of Matschull *et al* [101–103] of classical gravity for point particles in 2+1 dimensions are compatible with the DSR idea. The key DSR-friendly ingredients are the presence of a maximum value of mass and a description of energy-momentum space with "deSitter type" geometry (see later). However, several additional results must be obtained in order to verify whether or not a DSR formulation is possible. One key point is that Matschull *et al* formulate [101,102] the theory by making explicit reference to the frame of the center of mass of the multiparticle system. It may therefore be illegitimate to assume that the features that emerge from the analysis are observer independent. Moreover, rather than a deformation of the translation/rotation/boost classical symmetries of 2+1D space, many aspects of the theory, because of an underlying conical geometry, appear to be characterized by only two symmetries: a rotation and a time translation. And this description in terms of conical geometry is also closely related to the fact that the "observers at infinity" in the framework of Matschull *et al* do not really decouple from the system under observation, and therefore might not be good examples for testing the Relativity Principle. Moreover, especially when considering particle collisions, Refs. [101–103] appear to describe frequently as total momentum of a multiparticle system simply the sum of the individual momenta of the particles composing the system, and, from a DSR perspective, such a linear-additivity law would be incompatible [1,2] with a deformed dispersion relation.

I should also stress that from a DSR perspective the 2+1 context might not provide the correct intuition for the 3+1 context. A key difference for what concerns the role of fundamental scales originates from the fact that in 3+1 dimensions both the Planck length and the Planck energy are related to the gravitational constant through the Planck constant ($L_p \equiv \sqrt{\hbar G/c^3}$, $E_p \equiv \sqrt{\hbar c^5/G}$), whereas in 2+1 dimensions the Planck energy is obtained only in terms of the speed-of-light scale and the gravitational constant: $E_p^{(2+1)} \equiv c^4/G^{(2+1)}$.

So there are "reasons for advancing cautiously", but still this should be considered one of the most exciting developments for DSR research, and indeed it continues to motivate several related studies (see, it e.g., Refs. [36,37]).

7.2. *A Path for DSR in Loop Quantum Gravity?*

Another exciting possibility that has received some attention in the DSR literature is the one of obtaining a DSR framework at some effective-theory level of description of Loop Quantum Gravity.

An early suggestion of this possibility was formulated already in Ref. [25], taking as starting point the fact that the Loop Quantum Gravity literature presents some support[104,105] for the presence of a q-deformation of the deSitter symmetry algebra when there is nonvanishing cosmological constant. As discussed in Ref. [106] (and preliminarily in Ref. [25]), in the 3+1D context one expects a renormalization of energy-momentum which is still not under control, and for the analysis of Ref. [25] this essentially translates into an inability to fully predict the relation between the q-deformation parameter and the cosmological constant, which in turn does not allow us to firmly establish the potentialities of this line of analysis to produce a genuine DSR framework.

While no fully robust derivation is available at present, the fact that over these past few years other arguments and lines of analysis have also suggested (see e.g., Refs [38,39,107,108]) that a DSR framework might emerge at some effective-theory level of description of Loop Quantum Gravity is certainly exciting.

7.3. *Maximum Momentum, Maximum Energy, Minimum Wavelength*

Going back to the more humble (but presently better controlled) context of toy DSR test theories, it perhaps deserves stressing (in spite of the limited scope and limited significance of such test theories) that these test theories have been shown to accommodated rather nicely some features that are appealing from the perspective of a popular quantum-gravity intuition. The toy test theory I discussed in the phenomenology section would clearly predict a maximum Planckian ($1/\lambda$) allowed value of spatial momentum. And clearly a correspondingly structured test theory for frequencies/wavelengths realizes a corresponding minimum-wavelength (λ) bound [1,2,19].

Also much studied is a toy test theory that was first proposed by Magueijo and Smolin [6] in which one arrives at a maximum Planckian ($1/\lambda$) allowed value of energy. This is a test theory with structure that is completely analogous to the one of the DSR test theory I am here focusing on as illustrative example, but based on the dispersion relation

$$\frac{E^2 - \vec{p}^{\,2}}{(1-\lambda E)^2} = \frac{m^2}{(1-\lambda m)^2} \tag{54}$$

rather than (40).

7.4. *Deformed Klein-Gordon/Dirac Equations*

Some progress has also been reached in formulating deformed Klein-Gordon and Dirac equations in ways that would be consistent with the structure of the toy DSR test theory I discussed in the phenomenology section. This is done primarily assuming that κ-Minkowski noncommutative spacetime provides an acceptable spacetime sector for our toy DSR test theory [11], but interestingly it can also be done fully within "energy-momentum space". In the case of the energy-momentum-space Dirac equation this latter possibility materializes [11,12] in the following form

$$(\gamma^\mu \mathcal{D}_\mu(E,p,m;\lambda) - I)\,\psi(\overrightarrow{p}) = 0 \tag{55}$$

where

$$\mathcal{D}_0 = \frac{e^{\lambda E} - \cosh(\lambda m)}{\sinh(\lambda m)} \tag{56}$$

$$\mathcal{D}_j = \frac{p_j}{p}\frac{\left(2e^{\lambda E}\left[\cosh(\lambda E) - \cosh(\lambda m)\right]\right)^{\frac{1}{2}}}{\sinh(\lambda m)} \tag{57}$$

I is the identity matrix, and the γ^μ are the familiar "γ matrices".

7.5. Looking beyond the "Soccerball Problem"

Within the specific setup of toy DSR test theories of the type I considered in the phenomenology section, there is a natural issue to be considered, which in the first days of development of such toy test theories Kowalski-Glikman and I lightly referred to as "the soccerball problem" (a picturesque characterization still widely used, for example, at workshops where DSR researchers meet). The point is that nonlinear deformations of the energy-momentum relation are certainly phenomenologically acceptable for fundamental particles (if the deformation scale is Planckian the effects are extremely small), but clearly the same modifications of the energy-momentum relation would be unacceptable for bodies with rest energy greater than the Planck energy, such as the moon or a soccerball.

This issue remains somehow surrounded by a "mystique" in some DSR circles, but it is probably not very significant. Even within the confines of that specific type of toy DSR test theories there probably is, as stressed already in Refs. [1,2], an easy solution. In fact, another challenge for those test theories is the introduction of a description of "total momentum" for a body composed of many particles, and the two difficulties might be linked: it does not appear to be unlikely [1,2] that the proper description of total momentum might be such that the nonlinear properties attributed to individual particles are automatically suppressed for a multiparticle body.

Another possibility is the one of reformulating the test theory for frequencies/wavelengths rather than for energy/momentum, in which case the "soccerball problem" essentially disappears [19].

7.6. Curvature in Energy-momentum Space

While not adding to the scientific content of toy DSR test theories of the type I considered in the phenomenology section, it is conceptually intriguing and possibly valuable to notice that the nonlinearities that these test theories implement can be viewed as a manifestation of curvature in energy-momentum space. Indeed, as primarily stressed by Kowalski-Glikman [39] (applying in the DSR arena a line of reasoning previously advocated, from a wider quantum-gravity perspective, by Majid [109,110]), in such test theories the energy-momentum variables can be viewed as the coordinates of a DeSitter-like geometry.

Applying this viewpoint has proven valuable to quickly seeing some properties of a given toy DSR test theory of the type I considered in the phenomenology section. One must however keep in mind the physics of the variables one is handling: it would be for example erroneous to assume that diffeomorphism transformations of energy-momentum variables could be handled/viewed in exactly the same way as diffeomorphism transformations of spacetime coordinates.

I should also stress that, while it might be valuable to view (when possible) in terms of curvature in energy-momentum space a given framework for which compatibility with the DSR principles has already been independently established, the availability of a natural-looking map from the energy-momentum variables to some coordinates over a curved (e.g., deSitter) geometry does not suffice to guarantee the availability of a DSR formulation of the relevant framework. In order to make this remark more concrete let me propose a simple analogy. The propagation of light in a water-pool is (to very good approximation) described by a dispersion relation $E = \sqrt{c_{water}^2 p^2 + c_{water}^4 m^2}$ ($m = 0$ for photons), which of course allows a map from the energy-momentum variables to some coordinates on a Minkowski geometry, but we know that the scale c_{water} is not observer independent, and the laws of propagation of light in water do not admit a special-relativistic formulation.

7.7. A Rainbow Metric?

In a sense similar to the usefulness of the observation concerning curvature in energy-momentum space it may also be useful to look at toy DSR test theories of the type I considered in the phenomenology section as theories which, at least to some extent, are characterized by an energy-dependent metric. This was already the intuition behind some parts of my first DSR papers [1,2], especially when I considered *gedanken* procedures for the measurement of distances in theories with observer-independent modifications of the dispersion relation (because of the dependence on the energy of the probes found in the analysis of measurements of the length of a physical/material object [1,2]). More recently Magueijo and Smolin [22] took these arguments to a higher level of abstraction, and also introduced the well-chosen label "rainbow metric". The point being raised by Magueijo and Smolin could be very powerful, since it would clearly be (if for no other reasons, at least practically/computationally) advantageous to find a collection of features of a DSR theory that could all be codified under the common unifying umbrella of a given energy-dependent metric. However, in pursuing this objective it is of course necessary to proceed cautiously: if, rather than rephrasing established DSR features of a given framework in terms of a rainbow metric, one simply took a formal/technical approach, introducing here and there (wherever feasible) an energy-dependent metric, the end result might well not have DSR-compatible structure. Of course, just like a postulate of "curvature in energy-momentum space" would not in itself suffice to qualify the corresponding framework as DSR-compatible, the introduction of some energy-dependent metric does not automatically lead to a DSR theory. For example, the propagation of light in dispersive materials often leads to a dispersion relation that could be formally arranged in the form $p_\mu g(p_0; L_p)^{\mu\nu} p_\nu = 0$, thereby introducing an energy-dependent metric, but of course the presence of a dispersion-inducing material actually selects a preferred frame, and therefore is incompatible with any relativistic formulation.

Among the examples of concepts that are discussed in the DSR literature and clearly do admit rainbow-metric description let me also mention, in addition to the aspects of distance/length measurement discussed in Ref. [1,2], also the case of modified energy-momentum (dispersion) relations: indeed at least for massless particles some of the modified energy-momentum relations that have most attracted attention in the relevant literature, which can always be cast in the form $f(p_\mu; L_p) = 0$, could be rewritten (and be accordingly reinterpreted) in the form $f(p_\mu; L_p) = p_\mu g(p_0; L_p)^{\mu\nu} p_\nu = 0$.

A robust rainbow-metric description might be more challenging (if at all available) in multiparticle contexts where there is no obvious characteristic energy scale on which to link the metric dependence. For formulas involving the energy-momentum of several particles, with large hierarchies between the energies of different particles in the system, it would seem that there might be no natural choice of energy scale on which to anchor the rainbow metric. An intriguing possibility in relation to these challenges for the rainbow-metric description is perhaps found in the observations I offered in Section 4: the concept of a energy-dependent metric is problematic in a classical spacetime but may well be a valuable charterization at the level of the quantum-spacetime picture.

8. Some Recent Proposals

8.1. A Phase-Space-Algebra Approach

Already in some of the first investigations of the DSR idea a few authors (perhaps most notably Kowalski-Glikman (see, it e.g., Ref. [9]) had explored the possibility of obtaining a DSR-compatible framework by postulating a noncommutativity of spacetime coordinates that would be merged within a single algebraic structure with the symmetry-transformation generators, thereby forming a 14-generator "phase-space algebra". Recently the development of a new approach essentially based on that perspective was started in Refs. [40–43], with the objective of constructing point-particle models that possess a noncommutative (and non-canonical) simplectic structure and satisfy a modified dispersion relation, also hoping that such a setup might facilitate the description of interactions and the introduction of quantum properties for the point particles. A valuable tool that has been developed [41] for this approach is a map connecting the novel phase-space structures to ordinary canonical phase space. And among the features that already emerged from this approach I should at least also mention an alternative derivation of the Dirac equation (both in the case of the maximum-momentum toy DSR test theory that I used as example in the phenomenology section, and for the maximum-energy toy DSR test theory that was first considered by Magueijo and Smolin), and a proposal of a Nambu-goto setup which is expected [41] to be suitable for a DSR-compatible description of point particles.

8.2. Finsler Geometry

¿From various perspectives one can look at the structure of the DSR principles as suggesting that spacetime should be described in terms of some "exotic" geometry, and indeed, as mentioned, the majority of DSR studies assume a noncommutative spacetime geometry. There has been recently some exploration [53] (also see Ref. [41]) from the DSR perspective of another candidate as exotic spacetime geometry: Finsler geometry.

This idea is in its very early stage of exploration, and in particular for the Finsler geometries considered in Refs. [53] the presence of 10-generator Poincaré-like symmetries (a minimum prerequisite to even contemplate a formalism as DSR candidate) has not yet been established. It may be significant that the Finsler line element is not invariant [61] under DSR-type transformations.

8.3. A 5D Perspective

Several DSR studies, while aiming for a proposal for 4D physics, finds useful to adopt a formalism or a perspective which is of 5D nature. For those exploring the possibility of obtaining a DSR-compatible picture with theories formulated in κ-Minkowski noncommutative spacetime, an invitation toward a 5D perspective comes from the fact that for 4D κ-Minkowski spacetime it is not unnatural to consider a 5D differential calculus [92,93]. This mainly has its roots in the fact that the κ-Poincaré Hopf algebra is most naturally introduced [78] as a contraction of the q-deSitter Hopf algebra (whose dual spacetime picture is naturally described in terms of a 5D embedding environment). And even some authors who are not advocating the noncommutative spacetime formulation (and are probably unaware of the peculiarities of the 5D differential calculus) have independently argued [56,57] that it might be beneficial to set up the construction of DSR theories from a sort of embedding-space 5D picture of spacetime variables and/or energy-momentum variables. It is probably fair to say that at present it is still unclear whether the type of intuition generated by these 5D perspectives can be truly valuable, but it is certainly striking that different areas of DSR research, working independently, ended up advocating a 5D perspective.

8.4. Everything Rainbow

Taking off from the "rainbow metric" perspective, which I briefly discussed in the previous section, one can find arguments to advocate also [54] that other scales, including the quantum-mechanics scale $\hbar$, should acquire an energy dependence. This is too recent a proposal for me to comment robustly on it, but I should stress that the challenges for interpretation of some applications of the "rainbow metric" that I mentioned in the previous section of course also apply to this "everything rainbow" approach, and probably apply in more severe way. There are formulas in physics that refer simultaneously to the metric, to $\hbar$ and to a multitude of energy scales (formulas that apply to a large system, composed of many subsystems, for which one can meaningfully introduce a "subsystem energy" concept), and in such cases it seems that one should find a perfect recipe in order to constructively obtain an effective metric and effective $\hbar$.

In partly related work [55] it has been argued that rather than concepts such as maximum space momentum, maximum energy and minimum wavelengths (that I had so far mentioned in describing work on toy DSR test theories), one could perhaps consider the development of a toy DSR theory in which the new relativistic scale actually sets a maximum value for energy density. This in an intriguing proposal, which might gain supporters if a satisfactory relativistic formulation is found and explicitly articulated in terms of operatively well-defined entities. The construction of such a relativistic formulation may prove however rather challenging, even more challenging than it has been for the type of toy DSR test theories I considered in the phenomenology section. In fact, the laws of transformation of energy density should require from the onset a description of symmetry transformations that acts simultaneously on energy-momentum variables (energy) and spacetime variables (the volume where the relevant energy is "contained" which is of course needed for the energy-density considerations).

9. Closing Remarks

Doubly-Special Relativity is maturing quickly, as a result of the interest it is attracting from various research groups, each bringing its relevant expertise to the programme. A key factor for the future success of this large multi-perspective effort is the adoption of a common language, used to give and gain full access to the different bits of progress achieved toward the development of the DSR idea. I have here attempted to contribute to this goal, advocating the usefulness of characterizing results from a physics-content perspective (rather than technical/mathematical), consistently with the spirit of my first DSR papers [1,2].

Clearly the key objective at this point should be the one of constructing a full theory (with spacetime observables, energy-momentum observables, frequencies, wavelengths, cross sections...) compatible with the DSR requirements. I have mentioned here some partial results which might suggest that this ambitious objective is now not far. In the meantime the use of "toy DSR test theories" (in the sense I have here clarified) can be valuable, both to develop some intuition for what type of effects could be predicted within a DSR framework, and to anchor on some formulas (of however limited scope) the debate on conceptual aspects of DSR research.

Acknowledgments

My work on this manuscript was supported in part by grant RFP2-08-02 from The Foundational Questions Institute (fqxi.org). The preparation of this manuscript was diluted over a very long time, and conversations with several colleagues were valuable for my work, in one way or another. I am particularly grateful to F. Fiore, S. Ghosh, G. Gubitosi, S. Hossenfelder, U. Jacob, J. Kowalski-Glikman, A. Marcianó, M. Matassa, F. Mercati, T. Piran, G. Rosati, and L. Smolin.

References

1. Amelino-Camelia, G. Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale. *Int. J. Mod. Phys.* **2002**, *D11*, 35–60.
2. Amelino-Camelia, G. Testable scenario for Relativity with minimum length. *Phys. Lett.* **2001**, *B510*, 255–263.
3. Kowalski-Glikman, J. Observer independent quantum of mass. *Phys. Lett.* **2001**, *A286*, 391–394.
4. Bruno, N.R.; Amelino-Camelia, G.; Kowalski-Glikman, J. Deformed boost transformations that saturate at the Planck scale. *Phys. Lett.* **2001**, *B522*, 133–138.
5. Alexander, S.; Brandenberger, R.; Magueijo, J. Noncommutative inflation. *Phys. Rev.* **2003**, *D67*, 081301.
6. Magueijo, J.; Smolin, L. Lorentz invariance with an invariant energy scale. *Phys. Rev. Lett.* **2002**, *88*, 190403.
7. Amelino-Camelia, G.; Benedetti, D.; D'Andrea, F.; Procaccini, A. Comparison of relativity theories with observer independent scales of both velocity and length/mass. *Class. Quant. Grav.* **2003**, *20*, 5353–5370.
8. Lukierski J.; A. Nowicki, A. Doubly special relativity versus kappa deformation of relativistic kinematics. *Int. J. Mod. Phys.* **2003**, *A18*, 7–18.

9. Kowalski-Glikman, J.; Nowak, S. Noncommutative space-time of doubly special relativity theories. *Int. J. Mod. Phys.* **2003**, *D12*, 299–316.
10. Judes, S.; Visser, M. Conservation laws in doubly special relativity. *Phys.Rev.* **2003**, *D68*, 045001.
11. Agostini, A.; Amelino-Camelia, G.; Arzano, M. Dirac spinors for doubly special relativity and kappa Minkowski noncummutative space-time. *Class. Quant. Grav.* **2004**, *21*, 2179–2202.
12. Ahluwalia, D.V. Fermions, bosons, and locality in special relativity with two invariant scales. 2002, arXiv:gr-qc/0207004v4. arXiv.org, e-Print Arch. High Energy Phys.Theor. http://arxiv.org/abs/gr-qc/0207004 (accessed 3 March 2010).
13. Amelino-Camelia, G. Relativity: Special treatment. *Nature* **2002**, *418*, 34–35.
14. Rembielinski, J.; Smolinski, K.A. Unphysical predictions of some doubly special relativity theories. *Bull. Soc. Sci. Lett. Lodz* **2003**, *53*, 57–63.
15. Magueijo, J.; Smolin, L. Generalized Lorentz invariance with an invariant energy scale. *Phys.Rev.* **2003**, *D67*, 044017.
16. Bruno, N.R. Group of boost and rotation transformations with two observer independent scales. *Phys. Lett.* **2002**, *B547*, 109–115.
17. Amelino-Camelia, G. Kinematical solution of the UHE-cosmic-ray puzzle without a preferred class of inertial observers. *Int. J. Mod. Phys.* **2003**, *D12*, 1211–1226.
18. Feoli, A. Maximal acceleration or maximal accelerations? *Int. J. Mod. Phys.* **2003**, *D12*, 271–280.
19. Amelino-Camelia, G. Doubly special relativity: First results and key open problems. *Int. J. Mod. Phys.* **2002**, *D11*, 1643–1669.
20. Chakrabarti, A. Nonlinear transforms of momenta and Planck scale limit. *J. Math. Phys.* **2003**, *44*, 3800.
21. Blaut, A.; Daszkiewicz, M.; Kowalski-Glikman, J. Doubly special relativity with light-cone deformation. *Mod. Phys. Lett.* **2003**, *A18*, 1711–1719.
22. Magueijo, J.; Smolin, L. Gravity's rainbow. *Class. Quant. Grav.* **2004**, *21*, 1725–1736.
23. Kimberly, D.; Magueijo, J.; Medeiros, J. Nonlinear relativity in position space. *Phys. Rev.* **2004**, *D70*, 084007.
24. Ballestero, A.; Bruno, N.R.; Herranz, F.J. A New doubly special relativity theory from a quantum conformal algebra. *J. Phys.* **2003**, *A36*, 10493–10503.
25. Amelino-Camelia, G.; Smolin, L.; Starodubtsev, A. Quantum symmetry, the cosmological constant and Planck scale phenomenology. *Class. Quant. Grav.* **2004**, *21*, 3095–3110.
26. Freidel, L.; Kowalski-Glikman, J.; Smolin, L. 2+1 gravity and doubly special relativity. *Phys.Rev.* **2004**, *D69*, 044001.
27. Heyman, D.; Hinteleitner, F.; Major, S. On reaction thresholds in DSR theories. *Phys. Rev.* **2004**, *D69*, 105016.
28. Amelino-Camelia, G.; Kowalski-Glikman, J.; Mandanici, G.; Procaccini, A. Phenomenology of doubly special relativity. *Int. J. Mod. Phys.* **2005**, *A20*, 6007.
29. Magueijo, J.; Smolin, L. String theories with deformed energy momentum relations, and a possible nontachyonic bosonic string. *Phys. Rev.* **2005**, *D71*, 026010.

30. Amelino-Camelia, G. Some encouraging and some cautionary remarks on doubly special relativity in quantum gravity. 2004, arXiv:gr-qc/0402092v1. arXiv.org, e-Print Arch. Gen. Relativ. Quantum Cosmol. http://arxiv.org/abs/gr-qc/0402092 (accessed 3 March 2010).
31. Mignemi, S. Hamiltonian formalism and space-time symmetries in generic DSR models. *Int. J. Mod. Phys.* **2006**, *D15*, 925–936.
32. Aloisio, R.; Carmona, J.M.; Cortes, J.L.; Galante, A.; Grillo, A.F.; Mendez, F. Particle and antiparticle sectors in DSR1 and kappa-Minkowski space-time. *JHEP* **2004**, *0405*, 028.
33. Livine, E.R.; Oriti, D. About Lorentz invariance in a discrete quantum setting. *JHEP* **2004**, *0406*, 050.
34. Cortes, J.L.; Gamboa, J. Quantum uncertainty in doubly special relativity. *Phys. Rev.* **2005**, *D71*, 065015.
35. Kowalski-Glikman, J.; Smolin, L. Triply special relativity. *Phys. Rev.* **2004**, *D70*, 065020.
36. Freidel, L.; Livine, E.R. Ponzano-Regge model revisited III: Feynman diagrams and effective field theory. *Class. Quant. Grav.* **2006**, *23*, 2021–2062.
37. Livine, E.R.; Oriti, D. Coherent states for 3-D deformed special relativity: Semi-classical points in a quantum flat spacetime. *JHEP* **2006**, *0511*, 050.
38. Smolin, L. Falsifiable predictions from semiclassical quantum gravity. *Nucl. Phys.* **2006**, *B742*, 142–157.
39. Imilkowska, K.; Kowalski-Glikman, J. Doubly special relativity as a limit of gravity. *Lect. Notes Phys.* **2006**, *702*, 279–298.
40. Ghosh, S.; Pal, P. Kappa-Minkowski spacetime through exotic 'oscillator'. *Phys. Lett.* **2005**, *B618*, 243–251.
41. Ghosh, S.; Pal, P. Deformed special relativity and deformed symmetries in a canonical framework. *Phys. Rev.* **2007**, *D75*, 105021.
42. Ghosh, S. DSR relativistic particle in a Lagrangian formulation and non-commutative spacetime: A gauge independent analysis. *Phys. Lett.* **2007**, *B648*, 262–265.
43. Gosselin, P.; Berard, A.; Mohrbach, H.; Ghosh, S. Berry phase effects in the dynamics of Dirac electrons in doubly special relativity framework. *Phys. Lett.* **2008**, *B660*, 267–274.
44. Calisto, H.; Leiva, C. Generalized commutation relations and DSR theories, a close relationship. *Int. J. Mod. Phys.* **2007**, *D16*, 927–940.
45. Aloisio, R.; Galante, A.; Grillo, A.; Liberati, S.; Luzio, E.; Mendez, F. Deformed special relativity as an effective theory of measurements on quantum gravitational backgrounds. *Phys. Rev.* **2006**, *D73*, 045020.
46. Girelli, F.; Konopka, T.; Kowalski-Glikman, J.; Livine, E.R. The Free particle in deformed special relativity. *Phys. Rev.* **2006**, *D73*, 045009.
47. Ling, Y.; Li, X.; Zhang, H.-b. Thermodynamics of modified black holes from gravity's rainbow. *Mod. Phys. Lett.* **2007**, *A22*, 2749–2756.
48. Konopka, T. A field theory model with a new Lorentz-invariant energy scale. *Mod. Phys. Lett.* **2008**, *A23*, 319–335.
49. Jafari, N.; Shariati, A. Doubly special relativity: A New relativity or not? *AIP Conf. Proc.* **2006**, *841*, 462–465.

50. Das, A.; Kong, O.C.W. Physics of quantum relativity through a linear realization. *Phys. Rev.* **2006**, *D73*, 124029.
51. Aldrovandi, R.; Beltran Almeida, J.P.; Pereira, J.G. de Sitter special relativity. *Class. Quant. Grav.* **2007**, *24*, 1385-1404.
52. Aloisio, R.; Galante, A.; Grillo, A.F.; Liberati, S.; Luzio, E.; Mendez, F. Modified special relativity on a fluctuating spacetime. *Phys. Rev.* **2006**, *D74*, 085017.
53. Girelli, F.; Liberati, S.; Sindoni, L. Planck-scale modified dispersion relations and Finsler geometry. *Phys. Rev.* **2007**, *D75*, 064015.
54. Hossenfelder, S. Deformed special relativity in position space. *Phys. Lett.* **2007**, *B649*, 310–316.
55. Hossenfelder, S. Multi-Particle states in deformed special relativity. *Phys. Rev.* **2007**, *D75*, 105005.
56. Freidel, L.; Girelli, F.; Livine, E.R. The relativistic particle: Dirac observables and feynman propagator. *Phys. Rev.* **2007**, *D75*, 105016.
57. Girelli, F.; Livine, E.R. Non-Commutativity of effective space-time coordinates and the minimal length. 2007, arXiv:0708.3813v1. arXiv.org, e-Print Arch. High Energy Phys.Theor. http://arxiv.org/abs/0708.3813 (accessed 3 March 2010).
58. Galan, P.; Mena Marugan, G.A. Canonical realizations of doubly special relativity. *Int. J. Mod. Phys.* **2007**, *D16*, 1133–1147.
59. Kresic-Juric, S.; Meljanac, S.; Stojic, M. Covariant realizations of kappa-deformed space. *Eur. Phys. J.* **2007**, *C51*, 229–240.
60. Liu, C.-Z.; Zhu, J.-Y. Asymptotic quasinormal modes of scalar field in a gravity's rainbow. *Chin. Phys.* **2009**, *B18*, 4161–4168.
61. Mignemi, S. Doubly special relativity and Finsler geometry. *Phys. Rev.* **2007**, *D76*, 047702.
62. Kim, H.-C.; Rim, C.; Yee, J.H. Blackbody radiation in kappa-Minkowski spacetime. *Phys. Rev.* **2007**, *D76*, 105012.
63. Hinterleitner, F. Remarks on DSR and gravity. *Class. Quant. Grav.* **2008**, *25*, 075018
64. Mandanici, G. Undeformed (additive) energy conservation law in Doubly Special Relativity. *Mod. Phys. Lett.* **2009**, *A24*, 739–745.
65. Amelino-Camelia, G.; Ellis, J.; Mavromatos, N.E.; Nanopoulos, D.V.; Sarkar, S. Tests of quantum gravity from observations of γ-ray bursts. *Nature* **1998**, *393*, 763–765.
66. Gambini, R.; Pullin, J. Nonstandard optics from quantum space-time. *Phys. Rev.* **1999**, *D59*, 124021.
67. Kifune, T. Invariance violation extends the cosmic ray horizon? *Astrophys. J. Lett.* **1999**, *518*, L21–L24.
68. Alfaro, J.; Morales-Tecotl, H.A.; Urrutia, L.F. Quantum gravity corrections to neutrino propagation. *Phys. Rev. Lett.* **2000**, *84*, 2318–2321.
69. Schaefer B.E.; Severe Limits on Variations of the Speed of Light with Frequency *Phys. Rev. Lett.* **1999**, *82*, 4964–4966.
70. Aloisio, R.; Blasi, P.; Ghia, P.L.; Grillo, A.F. Probing The Structure of Space-Time with Cosmic Rays. *Phys. Rev.* **2000**, *D62*, 053010.

71. Protheroe, R.J.; Meyer, H. An infrared background TeV gamma ray crisis? *Phys. Lett.* **2000**, *B493*, 1–6.
72. Amelino-Camelia, G.; Piran, T. Planck scale deformation of Lorentz symmetry as a solution to the UHECR and the TeV gamma paradoxes. *Phys. Rev.* **2001**, *D64*, 036005.
73. Jacobson, T.; Liberati, S.; Mattingly, D. TeV astrophysics constraints on Planck scale Lorentz violation. *Phys. Rev.* **2002**, *D66*, 081302.
74. Amelino-Camelia, G. Space-time quantum solves three experimental paradoxes. *Phys. Lett.* **2002**, *B528*, 181–187.
75. Konopka, T.J.; Major, S.A. Observational limits on quantum geometry effects. *New J. Phys.* **2002**, *4*, 57.
76. Agostini, A.; Amelino-Camelia, G.; Arzano, M.; Marcianò, A.; Tacchi, R.A. Generalizing the Noether theorem for Hopf-algebra spacetime symmetries. *Mod. Phys. Lett.* **2007**, *A22*, 1779–1786.
77. Amelino-Camelia, G.; Briscese, F.; Gubitosi, G.; Marcianò, A.; Martinetti, P.; Mercati, F. Noether analysis of the twisted Hopf symmetries of canonical noncommutative spacetimes. *Phys. Rev.* **2008**, *D78*, 025005.
78. Lukierski, J.; Ruegg, H.; Nowicki, A.; Tolstoi, V.N. Q deformation of Poincare algebra. *Phys. Lett.* **1991**, *B264*, 331–338.
79. Majid, S.; Ruegg, H. Bicrossproduct structure of kappa Poincaré group and noncommutative geometry. *Phys. Lett.* **1994**, *B334*, 348–354.
80. Lukierski, J.; Ruegg, H.; Zakrzewski, W.J. Classical and quantum-mechanics of free κ-relativistic systems. *Ann. Phys.* **1995**, *243*, 90–116.
81. Lukierski, J.; Ruegg, H.; Ruhl, W. ¿From kappa Poincare algebra to kappa Lorentz quasigroup: A Deformation of relativistic symmetry. *Phys. Lett.* **1993**, *B313*, 357-366.
82. Batalin, I.A. Quasigroup construction and first class constraints. *J. Math. Phys.* **1981**, *22*, 1837–1850.
83. Fock V. *The theory of space-time and gravitation*; Pergamon Press: Oxford, UK, 1964.
84. Cardone, F.; Marrani, A.; Mignani, R. Boosts in an Arbitrary Direction and Maximal Causal Velocities in a Deformed Minkowski Space. *Found. Phys. Lett.* **2003**, *16*, 163–181.
85. Amelino-Camelia, G. On the fate of Lorentz symmetry in loop quantum gravity and noncommutative space-times. 2002, arXiv:gr-qc/0205125v1. arXiv.org, e-Print Arch. Gen. Relativ. Quantum Cosmol. http://arxiv.org/abs/gr-qc/0205125 (accessed 3 March 2010).
86. Snyder, H.S. Quantized space-time. *Phys. Rev.* **1947**, *71*, 38–41.
87. Kempf, A.; Mangano, G.; Mann, R.B. Hilbert space representation of the minimal length uncertainty relation. *Phys. Rev.* **1995**, *D52*, 1108–1118.
88. Ahluwalia, D.V. Wave particle duality at the Planck scale: Freezing of neutrino oscillations. *Phys. Lett.* **2000**, *A275*, 31–35.
89. Amelino-Camelia, G. Gravity-wave interferometers as probes of a low-energy effective quantum gravity. *Phys. Rev.* **2000**, *D62*, 024015.
90. Agostini, A.; Amelino-Camelia, G.; D'Andrea, F. Hopf algebra description of noncommutative-spacetime symmetries. *Int. J. Mod. Phys.* **2004**, *A19*, 5187–5219.

91. Amelino-Camelia, G.; Gubitosi, G.; Marcianò, A.; Martinetti, P.; Mercati, F. A no-pure-boost uncertainty principle from spacetime noncommutativity. *Phys. Lett.* **2000**, *B671*, 298–302.
92. Freidel, L.; Kowalski-Glikman, J.; Nowak, S. Field theory on κ–Minkowski space revisited: Noether charges and breaking of Lorentz symmetry. *Int. J. Mod. Phys.* **2008**, *A23*, 2687–2718.
93. Amelino-Camelia, G.; Marcianò, A.; Pranzetti, D. On the 5D differential calculus and translation transformations in 4D kappa-Minkowski noncommutative spacetime. *Int. J. Mod. Phys.* **2009**, *A24*, 5445–5463.
94. Doplicher S.; Fredenhagen, K.; Roberts, J.E.; The Quantum structure of space-time at the Planck scale and quantum fields. *Commun. Math. Phys.* **1995**, *172*, 187–220.
95. Bahns D.; Doplicher, S.; Fredenhagen, K.; Piacitelli, G.; Field theory on noncommutative spacetimes: Quasiplanar Wick products. *Phys. Rev.* **2005**, *D71*, 025022.
96. Jacobson, T.; Liberati, S.; Mattingly, D. A strong astrophysical constraint on the violation of special relativity by quantum gravity. *Nature* **2003**, *424*, 1019–1021.
97. Jackson, J.D. *Classical Electrodynamics*, 3rd Edition; J. Wiley & Sons, New York, NY, USA, 1999.
98. Amelino-Camelia, G. Phenomenology of Planck-scale Lorentz-symmetry test theories. *New J. Phys.* **2004**, *6*, 188.
99. Buffenoir, E.; Noui, K.; Roche, P. Hamiltonian quantization of Chern-Simons theory with SL(2,C) group. *Class. Quant. Grav.* **2002**, *19*, 4953–5015.
100. Celeghini, E.; Giachetti, R.; Sorace, E.; Tarlini, M. Three-dimensional quantum groups from contractions of SU(2)q. *J. Math. Phys.* **1990**, *31*, 2548-2551.
101. Matschull, H.-J.; Welling, M. Quantum mechanics of a point particle in (2+1)-dimensional gravity. *Class. Quant. Grav.* **1998**, *15*, 2981–3030.
102. Louko, J.; Matschull, H.-J. (2+1)-dimensional Einstein-Kepler problem in the center-of-mass frame. *Class. Quant. Grav.* **2000**, *17*, 1847–1873.
103. Matschull, H.-J. The Phase space structure of multi particle models in 2+1 gravity. *Class. Quant. Grav.* **2001**, *18*, 3497–3560.
104. Kodama, H. Holomorphic wave function of the Universe. *Phys. Rev.* **1990**, *D42*, 2548–2565.
105. Starodubtsev, A. Topological excitations around the vacuum of quantum gravity I: The symmetries of the vacuum. 2003, arXiv:hep-th/0306135v2. arXiv.org, e-Print Arch. High Energy Phys.Theor. http://arxiv.org/abs/hep-th/0306135 (accessed 3 March 2010).
106. Smolin, L. Quantum gravity with a positive cosmological constant. 2002, arXiv:hep-th/0209079v1. arXiv.org, e-Print Arch. High Energy Phys. Theor. http://arxiv.org/abs/hep-th/0209079 (accessed 3 March 2010).
107. Rovelli, C. A Note on DSR. 2008, arXiv:0808.3505v2 arXiv.org. arXiv.org. e-Print Arch. Gen. Relativ. Quantum Cosmol. http://arxiv.org/abs/0808.3505 (accessed 3 March 2010).
108. Smolin, L. Could deformed special relativity naturally arise from the semiclassical limit of quantum gravity? 2008, arXiv:0808.3765v1 arXiv.org, e-Print Arch. Gen. Relativ. Quantum Cosmol. http://arxiv.org/abs/0808.3765 (accessed 3 March 2010).
109. Majid, S. *Foundations of Quantum Group Theory*; Cambridge University Press: Cambridge, UK, 1995.

110. Amelino-Camelia, G.; Majid, S. Waves on noncommutative spacetime and gamma ray bursts. *Int. J. Mod. Phys.* **2000**, *A15*, 4301–4324.