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Comparative Analysis of Bilinear Time Series Models with Time-Varying and Symmetric GARCH Coefficients: Estimation and Simulation

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Abstract: This paper makes a significant contribution by focusing on estimating the coefficients of a sample of non-linear time series, a subject well-established in the statistical literature, using bilinear time series. Specifically, this study delves into a subset of bilinear models where Generalized Autoregressive Conditional Heteroscedastic (GARCH) models serve as the white noise component. The methodology involves applying the Klimko–Nilsen theorem, which plays a crucial role in extracting the asymptotic behavior of the estimators. In this context, the Generalized Autoregressive Conditional Heteroscedastic model of order (1,1) noted that the GARCH (1,1) model is defined as the white noise for the coefficients of the example models. Notably, this GARCH model satisfies the condition of having time-varying coefficients. This study meticulously outlines the essential stationarity conditions required for these models. The estimation of coefficients is accomplished by applying the least squares method. One of the key contributions lies in utilizing the fundamental theorem of Klimko and Nilsen, to prove the asymptotic behavior of the estimators, particularly how they vary with changes in the sample size. This paper illuminates the impact of estimators and their approximations based on varying sample sizes. Extending our study to include the estimation of bilinear models alongside GARCH and GARCH symmetric coefficients adds depth to our analysis and provides valuable insights into modeling financial time series data. Furthermore, this study sheds light on the influence of the GARCH white noise trace on the estimation of model coefficients. The results establish a clear connection between the model characteristics and the nature of the white noise, contributing to a more profound understanding of the relationship between these elements.

Keywords: bilinear time series models; GARCH model; least squares approach; advanced in functional equations

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1. Introduction

In recent decades, a diverse range of time series models has emerged, thanks to the contributions of an expanding community of scholars. This interest has attracted not only mathematicians but also economists and sociologists, who recognize the broad applicability of these models in finance and economics. Nonlinear time series models, in particular, have gained prominence due to their ability to capture the nonlinear characteristics often observed in empirical finance models describing volatility and returns. These models find applications across various disciplines, including marketing, insurance, virology [1], and chemistry.

One of the two models introduced by Ganger and Andersen [2] in 1978 is a bilinear time series model. Applications of bilinear models can be found in the domains of engineering, medicine, and biology. Later research, like that carried out by Subba [2], concentrated on these models' statistical and probabilistic aspects.

Numerous approaches, such as moments and least squares (LS), have been used to address the estimating problem for bilinear models. Meanwhile, both Autoregressive Conditional Heteroskedasticity (ARCH) and Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models are important in financial time series analysis. They are used to estimate economic activities, analyze holding risk, and assess option pricing.

As a subclass of nonlinear models, bilinear time series models have attracted interest because of their capacity to capture complex relationships in data. These models find use in a variety of industries, including engineering, medicine, and biology. Bilinear models have been studied since Ganger and Andersen's 1978 paper [1], when curiosity about their statistical and probabilistic characteristics first arose.

In financial time series analysis, Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models have become essential tools. GARCH models were developed to capture time-varying variances and volatility clustering. They offer insights into economic forecasting, asset pricing, risk management, and option valuation. Prominent scholarly contributions, including those from the authors, highlighted in the introduction (references [3–6], for example), have demonstrated GARCH models' applicability in various financial contexts.

When it comes to bilinear time series models, the least squares method is a basic estimation technique. The least squares approach, which is popular due to its ease of use and effectiveness, attempts to reduce the sum of squared residuals to provide estimates for the model parameters. The least squares method is a useful technique for estimating parameters in the domain of bilinear models, where interactions between variables may display nonlinearity. Its use in bilinear models makes it easier to understand how variables relate to one another and makes it possible to simulate intricate, time-varying events.

This paper is organized as follows. Section 2 provides notifications and significant results on $GARCH$ with bilinear time series as preliminaries. Section 3, using the nature of the proposed model, presents the estimation approach for all coefficients of bilinear mixed models using white noise $GARCH$. Section 4 illustrates the asymptotic behavior of the estimators and their mechanisms through numerical examples and simulations, employing the *Klimko–Nilsen* theorem. Section 5 discusses the importance of the paper and its differences from other published works, defining the content of the new work related to this research. To enrich our study, a comparison was made between two bilinear models, the first followed by GARCH and the second by Symmetric GARCH.

2. Preliminaries

In 2004, Bibi studied, in reference [6], a sample of bilinear models with time-varying coefficients that we define in a probability space (Ω, Γ, P) according to the following general stochastic expression:

$$X_t = \sum_{i=1}^{p} u_{i,t}(a) X_{t-i} + \sum_{j=1}^{q} v_{t,j}(a)\varepsilon_{t-j} + \sum_{i=1}^{p_0}\sum_{j=1}^{q_0} \phi_{t,ij}(a) X_{t-i}\varepsilon_{t-j} + \varepsilon_t. \tag{1}$$

The process $(X_t)_{t\in\mathbb{Z}}$ denoted by $BL(p,q,p_0,q_0)$, where

$$\{u_{i,t}(a)t\}_{1\le i\le p}, \{v_j(t)\}_{1\le j\le q}, \{\phi_{ij}(t)\}, 1\le i\le p_0, 1\le j\le q_0$$

are usually constants, but there will be time-varying coefficients that take their values in $\mathbb{R}$; where a is a vector, the white noise sequence $(\varepsilon_t)_{t\in\mathbb{Z}}$ takes many forms and is generally distributed according to Gaussian law with zero mean and variance $\sigma^2 < \infty$. But we will aim at a specified sample of bilinear coefficient models provided by $GARCH(1,1)$ white noise.

$$\begin{cases} \varepsilon_t = & h_t\eta_t, \\ h_t^2 = & \gamma_0 + \alpha_t(\alpha)\varepsilon_{t-1}^2 + \beta_t(\beta)h_{t-1}^2. \end{cases} \tag{2}$$

where the general GARCH (p_0, q_0) model is defined as follows:

$$\begin{cases} \varepsilon^2(t) = h^2(t)\eta^2(t) \\ h^2(t) = \gamma_0 + \sum_{i=1}^{p_0} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q_0} \beta_i h_{t-j}^2 \end{cases} \tag{3}$$

Note that with the distribution $\eta_t \sim \mathcal{N}(0,1)$, the sequences $\alpha_t(\alpha)$ and $\beta_t(\beta)$ are time-varying coefficients while maintaining $\gamma_0 > 0$ as a constant. h_t is independent of the $\sigma-$field generated by $\{\eta_{t+k}, k \geq 0\}$. And ε_t is a measurable function of the variables $\eta_{t-\ell}, \ell \geq 0$. It is well known that the strong GARCH(1,1) model is strictly stationary if and only if

$$-\infty \leq E \log\Big|\alpha_t(\alpha)\eta_t^2 + \beta_t(\beta)\Big| < 0. \tag{4}$$

We define the function φ as following $\varphi(\eta_t^2) = \alpha_t(\alpha)\eta_t^2 + \beta_t(\beta)$, and through this function, we can write h_t^2 as the following form:

$$\begin{aligned} h_t^2 =& \gamma_0 + \alpha_t(\alpha)h_{t-1}^2\eta_{t-1}^2 + \beta_t(\beta)h_{t-1}^2 \\ =& \gamma_0 + \Big\{\alpha_t(\alpha)\eta_{t-1}^2 + \beta_t(\beta)\Big\}h_{t-1}^2 \\ =& \gamma_0 + \varphi(\eta_{t-1}^2)h_{t-1}^2. \end{aligned}$$

The last expression can be written recurrently using the following formula:

$$h_t^2 = \gamma_0 + \gamma_0 \sum_{i=1}^{q} \left\{ \prod_{j=1}^{i} \varphi(\eta_{t-j}^2) \right\} + \left\{ \prod_{j=1}^{q+1} \varphi(\eta_{t-j}^2) \right\} h_{t-q-1}^2. \tag{5}$$

According to the stability conditions, we can demonstrate that $E(h_t^2) < \infty$; the proof will be checked, where it suffices to show that the following product is less than 1. Then, according to Jensen inequality, q tends to infinity, so

$$h_t^2 = \gamma_0 \left[1 + \sum_{i=1}^{\infty} \left\{ \prod_{j=1}^{i} \varphi(\eta_{t-j}^2) \right\} \right]. \tag{6}$$

We can prove that $\lim_{q\to\infty} \left\{ \prod_{j=1}^{q+1} \varphi(\eta_{t-j}^2) \right\} = 0$; it is clear that

$$\begin{aligned} \lim_{q\to\infty} E \left\{ \prod_{j=1}^{q+1} \varphi(\eta_{t-j}^2) \right\} \quad &\leq \quad \prod_{j=1}^{\infty} E\Big\{\varphi(\eta_{t-j}^2)\Big\} \\ &= \quad \prod_{j=1}^{\infty} \{\alpha_t(\alpha) + \beta_t(\beta)\}. \end{aligned}$$

By considering the following inequality $\alpha_t(\alpha) + \beta_t(\beta) = \delta < 1$, we will ensure that

$$\prod_{j=1}^{\infty} \delta_j, \delta_j = \delta = c, \tag{7}$$

where c is a constant, (7) tends towards zero, which demonstrates the following:

$$\lim_{q\to\infty} \left\{ \prod_{j=1}^{q+1} \varphi(\eta_{t-j}^2) \right\} = 0.$$

In another way, we can prove that the series is almost surely convergent, h_t^2, if we take the general term of the series and apply the Cauchy criterion of convergence:

$$\lim_{n\to\infty}\left\{\prod_{j=1}^{n}\varphi(\eta_{t-j}^2)\right\}^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\sum\limits_{j=1}^{n}\ln\varphi(\eta_{t-j}^2)}.$$

Also, if v_t is an i.i.d sequence of random variables admitting an expectation which can be infinite, then $\frac{1}{n}\sum\limits_{k=1}^{n} v_k$ is approximate towards $E(v_1)$. So, through this construction and using Jensen's inequality we obtain the following:

$$\begin{aligned}\lim_{n\to\infty} e^{\frac{1}{n}\sum\limits_{j=1}^{n}\ln\varphi(\eta_{t-j}^2)} &= \lim_{n\to\infty} e^{E(\ln\varphi(\eta_{t-1}^2)}\\ &\leq e^{\ln E\varphi(\eta_{t-1}^2)}\\ &= e^{\ln(\alpha_t(\alpha)+\beta_t(\beta))}\\ &< 1,\end{aligned}$$

which will show that the series is convergent.

2.1. Stationarity Study

In the following subsection, we will deal with the following bilinear model:

$$X_t = \phi_t(a)X_{t-s}\varepsilon_{t-1} + \varepsilon_t, \tag{8}$$

where $s \geq 1$, and ε_t is the white noise that follows the GARCH (1,1) model with time-varying coefficients, which was defined above using Expression (1); $\{\phi_t(a), t \in \mathbb{Z}\}$ is a sequence of time-varying coefficients, and a is defined as a vector $a = (a_1, a_2, \ldots, a_m)$ included in subset Θ of $\mathbb{R}^m$. It is well known that stationary solutions exist for this model if $\phi_t^2(a)E(\varepsilon_t^2) < 1$, where we can give some extensions of stability according to the white noise and, recurrently, ε_t is written as follows:

$$\varepsilon_t = X_t + \sum_{k=1}^{t-1}(-1)^k\left\{\prod_{i=0}^{k-1}\phi_{t-k}(a)\right\} \times \left(\prod_{i=0}^{k-1}X_{t-i-s}\right)X_{t-k}. \tag{9}$$

In the case where

$$\begin{aligned}E(\varepsilon_t^2) =&E(\eta_t^2)E(h_t^2) = E(h_t^2)\\ =&\gamma_0 + \alpha_t(\alpha)E(h_{t-1}^2)E(\eta_{t-1}^2) + \beta_t(\beta)E(h_{t-1}^2)\\ =&\gamma_0 + \alpha_t(\alpha)E(\varepsilon_{t-1}^2) + \beta_t(\beta)E(\varepsilon_{t-1}^2).\end{aligned}$$

And, where $E(\varepsilon_t^2) = E(\varepsilon_{t-1}^2)$, we find that $E(\varepsilon_t^2) = \frac{\gamma_0}{1-\alpha_t(\alpha)-\beta_t(\beta)}$, such as $\alpha_t(\alpha) + \beta_t(\beta) < 1$; then, the necessary conditions for model stability will be

$$\frac{\phi_t^2(a)\gamma_0}{1-\alpha_t(\alpha)-\beta_t(\beta)} < 1. \tag{10}$$

And the recurring expression of the model in case, where $s \neq 1$, will be

$$X_t = \varepsilon_t + \sum_{j=1}^{[t/s]-1}\left[\prod_{i=0}^{j-1}\{\phi_{t-is}(a)\varepsilon_{t-is-1}\}\right]\varepsilon_{t-sj}. \tag{11}$$

where $[y]$ denotes the integer part of value y, Model (3) under a condition of stability and the unique solution with recursive form when $s = 1$:

$$X_t = \varepsilon_t + \sum_{j=1}^{\infty} \left[\prod_{i=0}^{j-1} \{\phi_{t-i}(a)\varepsilon_{t-i-1}\} \right] \varepsilon_{t-j}. \tag{12}$$

Theorem 1. *The model solution series (12) under a condition of stability is almost surely convergent.*

Proof. Our method of proof will be based on showing that $E(X_t) < \infty$. So, by using the Schwartz inequality, we have

$$\begin{aligned} \mathbf{S} &= E\left|\varepsilon_{t-j} \prod_{i=0}^{j-1} \{\phi_{t-i}(a)\varepsilon_{t-i-1}\}\right| \\ &\leq \{E(\varepsilon_{t-j}^2)\}^{0.5} \prod_{i=0}^{j-1} |\phi_{t-i}(a)| \left\{E(\varepsilon_{t-i-1}^2)\right\}^{0.5} \\ &\leq \{E(h_{t-j}^2)\}^{0.5} \prod_{i=0}^{j-1} |\phi_{t-i}(a)| \left\{E(\varepsilon_{t-i-1}^2)\right\}^{0.5}. \end{aligned}$$

and as $|\phi_{t-i}(a)| \left\{E(\varepsilon_{t-i-1}^2)\right\}^{0.5} = \rho < 1$, $\{E(h_{t-j}^2)\}^{0.5} = M$ is bounded:

$$\begin{aligned} \sum_{j=1}^{\infty} E\left|\varepsilon_{t-j} \prod_{i=0}^{j-1} \{\phi_{t-i}(a)\varepsilon_{t-i-1}\}\right| &\leq M \sum_{j=1}^{\infty} \rho^j \\ &< \frac{M}{1-\rho}. \end{aligned}$$

By showing that the series is a converging geometric series, our proof is complete here. □

Theorem 2. *Consider the stochastic process defined by Model* (3)*, driven by the noise ARCH(1). If the following conditions hold,*

1. *For all t, $\alpha_t(\alpha) \in [0, 1)$,*
2. $|\phi_t(a)| \sqrt{\frac{\gamma_0}{1-\alpha_t(\alpha)}} < 1$,

   then the series

$$X_t = \varepsilon_t + \sum_{j=1}^{\infty} \left\{ \prod_{i=0}^{j-1} \phi_t(a)\varepsilon_{t-i-1} \right\} \varepsilon_{t-j} = f \tag{13}$$

almost surely converges, and the defined X_t is the unique, strictly stationary solution.

Proof. Firstly, we show that $\mathbb{E}\left[\left|\prod_{i=0}^{j-1} \phi_t(a)\varepsilon_{t-i-1}\right|\right] < \infty$, where j is any integer, by using the Schwartz inequality and the condition of stability:

$$\begin{aligned} \mathbb{E}\left[\left|\prod_{i=0}^{j-1} \phi_t(a)\varepsilon_{t-ik-1}\right|\right] &\leq \left(\prod_{i=0}^{j-1} \mathbb{E}\left[|\phi_t(a)\varepsilon_{t-ik-1}|^2\right]\right)^{0.5} \\ &= \left(\mathbb{E}\left[(\phi_t(a)\varepsilon_{t-ik-1})^2\right]\right)^{0.5} \\ &= |\phi_t(a)| \sqrt{\frac{\gamma_0}{1-\alpha_t(\alpha)}} \\ &\leq \zeta < 1, \end{aligned}$$

then

$$\sum_{j=1}^{\infty}\prod_{i=0}^{j-1}\mathbb{E}[|\phi_t(a)\varepsilon_{t-ik-1}|] \leq \sum_{j=1}^{\infty}\xi^i < \infty,$$

where $s \neq 1$, because of the recurrent model

$$X_t = \varepsilon_t + \sum_{j=1}^{\lfloor t/s\rfloor-1}\left[\prod_{i=0}^{j-1}(\phi_{t-is}(a)\varepsilon_{t-is-1})\right]\varepsilon_{t-sj} = g.$$

Here, $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. It is worth noticing that $|g| \leq |f|$ implies $\mathbb{E}[|g|] \leq \mathbb{E}[|f|]$. $\square$

Theorem 3. *Let $s = 1$ in Model (3) be driven by the noise ARCH(1). If the condition $E\ln|\phi_t(a)\varepsilon_t| \in [-\infty, 0]$ holds, then it implies that, for $\alpha_t(\alpha) \in [0,1]$,*

$$|\phi_t(a)|\sqrt{\frac{\gamma_0}{1-\alpha_t(\alpha)}} < 1,$$

where $\alpha_t(\alpha)$ represents the parameter in the range $[0,1]$.

Proof. Using Jensen's inequality, we have

$$E\ln|\phi_t(a)\varepsilon_t| \leq \ln E|\phi_t(a)\varepsilon_t|.$$

Now, by applying the Schwartz inequality, we obtain

$$\begin{aligned}\ln E|\phi_t(a)\varepsilon_t| &\leq \ln\left\{E|\phi_t(a)\varepsilon_t|^2\right\}^{0.5}\\ &= \ln\left|\phi_t(a)\left\{\frac{\gamma_0}{1-\alpha_t(\alpha)}\right\}^{0.5}\right| < 0,\end{aligned}$$

where we observe that $\left|\phi_t(a)\left\{\frac{\gamma_0}{1-\alpha_t(\alpha)}\right\}^{0.5}\right| < 1$, ensuring the logarithm is negative.

Thus, we have shown that the condition $E\ln|\phi_t(a)\varepsilon_t| \in [-\infty, 0]$ implies that $|\phi_t(a)|\sqrt{\frac{\gamma_0}{1-\alpha_t(\alpha)}} < 1$, for $\alpha_t(\alpha) \in [0,1]$, which completes the proof. $\square$

Theorem 4. *For the ARCH(1) model, under the condition $|\alpha_t(\alpha)| < 1$ and $\gamma_0 = 0$, the process is bounded.*

Proof. Since $\varepsilon_t = \eta_t h_t$ and $h_t = \alpha_t(\alpha)\varepsilon_{t-1}$, we find that

$$h_{t-i} = \prod_{k=1}^{i}\alpha_t(\alpha)\varepsilon_{t-i-1},$$

then, $\lim_{i\to\infty} h_{t-i} = 0$, demonstrating that there exists M such that $|\varepsilon_t| \leq M$. $\square$

2.2. Klimko–Nilsen Theorem and Estimation Approach

This theorem played a fundamental role in proving the existence and uniqueness of the estimators, as well as their asymptotic behaviors according to the approach of square worlds. Here is its text in the form of the hypotheses.

Firstly, let $H_N(X_t)$ be the σ-field generated by the set of observations $\{X_t,\ t = 1,\ldots,N\}$ and the vectors $a = (a_1, a_2, \ldots, a_{n_1})$; as for the white noise coefficients, we consider here the vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n_2})$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{n_3})$, where the image of these vectors with their functions exists in $\mathbb{R}$. Then, the parameter that we are going to estimate will be $\omega = (a, \gamma_0, \alpha, \beta)$, and we assume ω_0 is a true value of ω included in an open Ω of $\mathbb{R}^{n_1+n_2+n_3+1}$. We introduce the predictor or orthogonal projection $\rho_{t|t-1}$ on the observations up to time $t-1$ by the following difference: $\rho_{t|t-1}(\omega) = X_t - \varepsilon_t(\omega)$.

So, the principle of the least squares method is based on looking for the parameter ω, such that the penalty function will be defined with the following standard algorithm of estimation:

$$
\begin{aligned}
&w_1 = a_1, \ldots, \omega_{n_1} = a_{n_1}, \\
&\vdots \\
&\omega_{n_1+1} = \gamma_0, \omega_{n_1+2} = \alpha_1, \ldots, \omega_{n_1+n_2+1} = \alpha_{n_2}, \\
&\omega_{n_1+n_2+2} = \beta_1, \ldots, \omega_{n_1+n_2+n_3+1} = \beta_{n_3}.
\end{aligned}
$$

Theorem 5. *Under stable conditions for $(X_t)_{t\in\mathbb{Z}}$ generated by Equation (8), and by adding $|\phi(a)| < 1$, where its white noise follows GARCH(1,1), we suppose that $\rho_{t|t-1}(\omega)$ is almost surely doubly continuously differentiable in an open subset containing the true value ω_0 of the vector ω. We assume two constants, K_0 and K_1, such that*

(a) $E_{\omega_0}\left\{\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega_i}\right\}^4 \le K_0$, *$i = n_1+n_2+n_3+1$.*

(b) $E_{\omega_0}\left\{\left(\frac{\partial^2 \varepsilon_t^2(\omega)}{\partial \omega_i \partial \omega_j} - E_{\omega_0}\left\{\frac{\partial^2 \varepsilon_t^2(\omega)}{\partial \omega_i \partial \omega_j} \mid H_{t-1}\right\}\right)^2\right\} \le K_1$, *$i, j = n_1+n_2+n_3+1$.*

(c) $\frac{1}{2N}\sum\limits_{t=1}^{N} E_{\omega_0}\left\{\frac{\partial^2 \varepsilon_t^2(\omega)}{\partial \omega_i \partial \omega_j} \mid H_{N-1}\right\}$ *converges almost surely to the matrix $M(\omega)$ which is strictly positive.*

(d) $\lim_{N\to\infty}\left\{\sup_{\delta\to 0}\left|\frac{1}{\delta N}\sum_{t=1}^{N}\left[\left(\frac{\partial^2 \varepsilon_t^2(\omega)}{\partial \omega_i \partial \omega_j}\right)_{\omega=\tilde{\omega}} - \left(\frac{\partial^2 \varepsilon_t^2(\omega)}{\partial \omega_i \partial \omega_j}\right)_{\omega=\omega_0}\right]\right|\right\} < \infty$, *where $\|\omega - \omega_0\| < \delta$, $\delta > 0$, where $\tilde{\omega}$ represents the intermediate value between ω and ω_0.*

Then, there exists an estimator $\hat{\omega}_N$ such that $\hat{\omega}_N \to \omega_0$ as $N \to \infty$, if these conditions are satisfied as well as the following assumption:

(e) $\frac{1}{N}\sum\limits_{t=1}^{N}\left[E_{\omega_0}\left\{\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega}\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega^T} \mid H_{t-1}\right\} - E_{\omega_0}\left\{\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega}\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega^T}\right\}\right] \to 0$ *as $N \to \infty$.*

Proof. See [7]. □

We will prove some elements of this theorem according to the model proposed in this paper. First, we will give some techniques to calculate the derivatives. Let

$$\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega_i} = -2\varepsilon_t(\omega)\frac{\partial \varepsilon_t(\omega)}{\partial \omega_i} = -2\varepsilon_t(\omega)\frac{\partial \rho_{t|t-1}(\omega)}{\partial \omega_i}.$$

Then, $8E_{\omega_0}\{\varepsilon_t^4(\omega)\} = E_{\omega_0}\left\{\left(\frac{\partial \rho_{t|t-1}(\omega)}{\partial \omega_i}\right)^4\right\}$.

So, in case $s = 1$, we obtain

$$\rho_{t|t-1}(\omega) = \sum_{j=1}^{\infty}\left[\prod_{i=0}^{j-1}\{\phi_{t-i}(a)\varepsilon_{t-i-1}\}\right]\varepsilon_{t-j}. \tag{14}$$

In the case where $\omega_p = a_p,\ p = 1, \ldots, n$, we will use

$$
\begin{aligned}
\frac{\partial \rho_{t|t-1}(\omega)}{\partial \omega_i} &= \sum_{k=1}^{t-1}(-1)^k\left\{\frac{\partial}{\partial a_i}\prod_{i=0}^{k-1}\phi_{t-k}(a)\right\} \\
&\quad \times \left(\prod_{i=0}^{k-1} X_{t-i-s}\right) X_{t-k}.
\end{aligned} \tag{15}
$$

And

$$Z_1 = \frac{\partial}{\partial a_p} \prod_{i=0}^{k-1} \phi_{t-k}(a) = \sum_{p=0}^{k-1} \left\{ \prod_{\substack{i=0 \\ i \neq p}}^{k-1} \phi_{t-i}(a) \right\} \frac{\partial \phi_{t-p}(a)}{\partial a_p}, \tag{16}$$

where we put $q = \max\left\{ \frac{\partial \phi_{t-i}(a)}{\partial a_p} \right\}$, and $|\phi_{t-i}(a)| \leq \theta < 1$; then, it will be

$$|Z_1| \leq q(k)\theta^{k-1}.$$

We will obtain

$$\left| \frac{\partial \rho_{t|t-1}(\omega)}{\partial \omega_i} \right| \leq \sum_{k=1}^{t-1} (-1)^k q(k) \theta^{k-1} \left(\prod_{i=0}^{k-1} |X_{t-i-s}| \right) |X_{t-k}| < \infty. \tag{17}$$

In addition, we establish $E\left| \frac{\partial \rho_{t|t-1}(\omega)}{\partial \omega_i} \right| < \infty$. According to previous analysis, given $E\{\varepsilon_t^2(\omega)\} = E\{\eta^2\}E(h_t^2)$, where $E(h_t^2)$ is bounded, it follows that $E\{\varepsilon_t^4(\omega)\} < \infty$. Consequently, we deduce $E_{\omega_0}\left\{ \frac{\partial \varepsilon_t^2(\omega)}{\partial \omega_i} \right\}^4 \leq K_0$, thus validating the hypothesis. Further details are provided in the subsequent sections (see [7]).

3. Least Squares Approach

The least squares estimators for ARCH models are asymptotically normal, though less efficient than methods such as the generalized method of moments (GMM) and maximum likelihood estimation (MLE) [7–9]. However, in this section, we will focus on the least squares method, particularly in situations where the coefficients are time-varying and driven by white noise with time-varying coefficients.

3.1. Algorithm

The primary algorithm of the least squares method aims to find the estimator $\hat{\omega}_N$ by minimizing the penalty function $q_N(\omega)$, defined as follows:

$$\arg\min_{\omega \in \Omega} q_N(\omega), \tag{18}$$

where

$$q_N(\omega) = \frac{1}{N} \sum_{t=1}^{N} \varepsilon_t^2(\omega). \tag{19}$$

3.2. Derivation Techniques

The least squares approach relies on Taylor's second-degree formula. For any $i, j \in \{1, \ldots, n_1 + n_2 + n_3 + 1\}$, we have the following:

$$q_N(\omega) = q_N(\omega_0) + (\omega - \omega_0) \frac{\partial q_N}{\partial \omega^T}(\omega_0) + \frac{1}{2}(\omega - \omega_0) \frac{\partial^2 q_N}{\partial \omega^T \partial \omega}(\tilde{\omega})(\omega - \omega_0)^T, \tag{20}$$

where $\tilde{\omega}$ is an intermediate point between ω and ω_0 with $\|\omega - \tilde{\omega}\| \leq \delta$, $\delta > 0$. The derivatives $\frac{\partial q_N(\omega)}{\partial \omega_i}$ form a vector with $n_1 + n_2 + n_3 + 1$ coordinates, and $\frac{\partial^2 q_N(\omega)}{\partial \omega_i \partial \omega_j}$ presents a matrix of size $(n_1 + n_2 + n_3 + 1)^2$.

3.3. Derivatives Calculation

We derive expressions for the first and second derivatives concerning the parameters ω_i and ω_j. Firstly, for $\omega_\ell = a_\ell$, we have

$$\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega_\ell} = 2\varepsilon_t(\omega)\frac{\partial \varepsilon_t(\omega)}{\partial \omega_\ell} = 2\varepsilon_t(\omega)\frac{\partial \rho_{t|t-1}(\omega)}{\partial \omega_\ell}, \tag{21}$$

where

$$\frac{\partial \rho_{t|t-1}(\omega)}{\partial \omega_\ell} = \sum_{k=1}^{t-1}(-1)^k\left\{\frac{\partial}{\partial a_\ell}\prod_{i=0}^{k-1}\phi_{t-k}(a)\right\}\left(\prod_{i=0}^{k-1}X_{t-i-s}\right)X_{t-k}. \tag{22}$$

Similarly, for $\omega_\ell = \alpha_\ell$, the expression is given by

$$\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega_\ell} = 4\eta_t^2\frac{\alpha_t(\alpha)}{1-\beta_t(\beta)}\varepsilon_{t-1}\frac{\partial \varepsilon_{t-1}}{\partial \omega_\ell}, \tag{23}$$

with

$$\frac{\partial \varepsilon_{t-1}}{\partial \omega_i} = \sum_{k=1}^{t-2}(-1)^k\left\{\frac{\partial}{\partial \omega_i}\prod_{i=0}^{k-1}\phi_{t-1-k}(a)\right\}\left(\prod_{i=0}^{k-1}X_{t-1-i-s}\right)X_{t-1-k}. \tag{24}$$

For the second derivative, considering $\omega_i = a_i$ and $\omega_j = \alpha_j$, we have

$$\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega_i \partial \omega_j} = 4\eta_t^3\frac{\partial \varepsilon_{t-1}}{\partial \omega_i}\times\left[\frac{\partial}{\partial \omega_j}\left\{\frac{\alpha_t(\alpha)}{1-\beta_t(\beta)}\right\}h_t + \frac{\partial h_t}{\partial \omega_j}\frac{\alpha_t(\alpha)}{1-\beta_t(\beta)}\right]. \tag{25}$$

Additionally, when $\omega_i = \alpha_i$ and $\omega_j = \alpha_j$, where $i, j \in \{1, \ldots, n_2\}$, the second derivative is given by

$$\frac{\partial \varepsilon_t^2(\omega)}{\partial \omega_i \partial \omega_j} = 4\eta_t^3\frac{\partial \varepsilon_{t-1}}{\partial \omega_i}\times\left[\frac{\partial}{\partial \omega_j}\left\{\frac{\alpha_t(\alpha)}{1-\beta_t(\beta)}\right\}h_t + \frac{\partial h_t}{\partial \omega_j}\frac{\alpha_t(\alpha)}{1-\beta_t(\beta)}\right]. \tag{26}$$

These derivative expressions facilitate the computation of second-order terms in the least squares optimization process.

The derived equations lay the foundation for implementing the least squares approach, enabling the estimation of time-varying coefficients within the framework of ARCH models. This method offers a practical means of modeling volatility dynamics, particularly in financial time series data, where coefficients may exhibit temporal variations.

In summary, the least squares approach outlined here provides a systematic methodology for estimating parameters in ARCH models, leveraging optimization principles and derivative calculus to refine parameter estimates iteratively. Through careful consideration of the model structure and the properties of the underlying data, this approach offers valuable insights into the dynamics of volatility and enables effective forecasting in various domains.

4. Simulation and Graphic Illustrations

This section presents a simulation study and graphical illustrations for the model under consideration. We begin by defining a vector of parameters $a = (a_1, a_2, a_3)$ belonging to an open subset $I_1 \times I_2 \times I_3$ within $\mathbb{R}^3$. We impose the constraint $a_2 = 1 - a_1$ and consider $a_3 = \max(a_1, a_2)$. The time-varying coefficients are expressed as follows:

$$\phi_t(a) = \begin{cases} a_1, & t \equiv 1[3], \\ a_2, & t \equiv 2[3], \\ a_3, & t \equiv 0[3]. \end{cases} \tag{27}$$

The objective of this section is to simulate the model $X_t = \phi_t(a)X_{t-s}\varepsilon_{t-1} + \varepsilon_t$ with GARCH(1,1) white noise, where $\gamma_0 = 0.005$, $\alpha = (\alpha_1, \alpha_2)$, and $\beta = (\beta_1, \beta_2)$, such that

$$\alpha_t(\alpha_1, \alpha_2) = \begin{cases} \alpha_1, & t \in 2\mathbb{N} + 1, \\ \alpha_2, & t \in 2\mathbb{N}. \end{cases}$$

$$\beta_t(\beta_1, \beta_2) = \begin{cases} \beta_1, & t \in 2\mathbb{N} + 1, \\ \beta_2, & t \in 2\mathbb{N}. \end{cases}$$

We employ the notation of N_s for the number of simulations, N for the sample size, the true parameter values (a_1, a_2, a_3), and their estimated counterparts $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$. In our simulations, we set $s = 2$ and adopt the GARCH white noise formulation:

$$\begin{cases} \varepsilon_t = \eta_t h_t \\ h_t^2 = 0.005 + \alpha_t(0.3, 0.4)\varepsilon_t^2 + \beta_t(0.003, 0.001)h_t^2. \end{cases} \tag{28}$$

where $\omega = (0.005, 0.3, 0.4, 0.003, 0.001)$ is the real value proposed for the model and $\hat{\omega}$ is its estimated value. In a situation where $\beta_t(\beta) = 0$, we have the following simulation.

By considering Tables 1–3, we can take the same model proposed but with constant coefficients and with the same GARCH proposed in our paper to extend our study of this model on the side of simulations and get the following Table 4.

$$X_t = aX_{t-s}\varepsilon_{t-1} + \varepsilon_t. \tag{29}$$

Table 1. Estimation of model coefficients according to different sizes, where $s = 1$.

N_s	N	Real Values (a_1, a_2, a_3)	Estimates $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$
250	300		(0.0097, 0.3230, 0.3706)
	600		(0.0182, 0.2872, 0.3797)
	900	(0.02, 0.25, 0.45)	(0.0129, 0.2879, 0.3802)
500	300		(0.0166, 0.2704, 0.4119)
	600		(0.0225, 0.2787, 0.4152)
	900		(0.0220, 0.2795, 0.4456)

Table 2. Estimation of model coefficients and ARCH white noise coefficients, where $s = 1$.

N_s	N	Real Value	Estimates $\hat{\omega}$
250	300		0.015, 0.098, 0.235, 0.082, 0.033
	600		0.009, 0.165, 0.385, 0.007, 0.014
	900	ω	0.003, 0.263, 0.388, 0.007, 0.014
500	300		0.003, 0.369, 0.387, 0.006, 0.003
	600		0.003, 0.369, 0.387, 0.006, 0.002
	900		0.004, 0.318, 0.407, 0.010, 0.002

Table 3. Estimation of the model according to other coordinates, for $(a_1, a_2, a_3, \gamma_0, \alpha_1, \alpha_2)$.

N_s	N	$(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{\gamma}_0, \hat{\alpha}_1, \hat{\alpha}_2)$
250	300	(0.022, 0.234, 0.453, 0.005, 0.289, 0.406)
500	600	(0.0198, 0.252, 0.398, 0.005, 0.301, 0.411)

where $(a_1, a_2, a_3, \gamma_0, \alpha_1, \alpha_2) = (0.02, 0.25, 0.45, 0.005, 0.3, 0.4)$.

Table 4. Estimation of GARCH model coefficients for a bilinear model with constant coefficients.

True Values: $\theta^0 = (a, \gamma_0, \alpha_1, \alpha_2, \beta_1, \beta_2) = (0.07, 0.01, 0.1, 0.3, 0.6, 0.7)$		
N	N_s	$\hat{\theta} = (\hat{a}, \hat{\gamma}_0, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2)$
300	250	$(0.0694, 0.0124, 0.0999, 0.4179, 0.5825, 0.5125)$
⋮	⋮	⋮
600	500	$(0.0642, 0.0115, 0.1004, 0.4145, 0.5997, 0.5122)$

Notably, models featuring estimated coefficients and bilinear models guided by the GARCH(1,1) model with true coefficients exhibit compatibility as shown in Figures 1–3. This compatibility effectively demonstrates that the estimation of these proposed models delivers efficient outcomes. Furthermore, these models validate the asymptotic behaviors of the estimators.

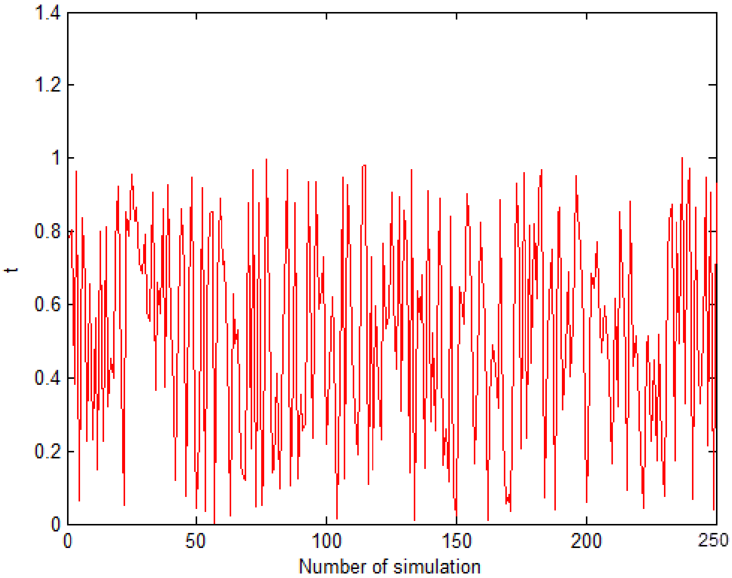

Figure 1. Simulation for bilinear time series model with its true values $N = 900, N_s = 250$.

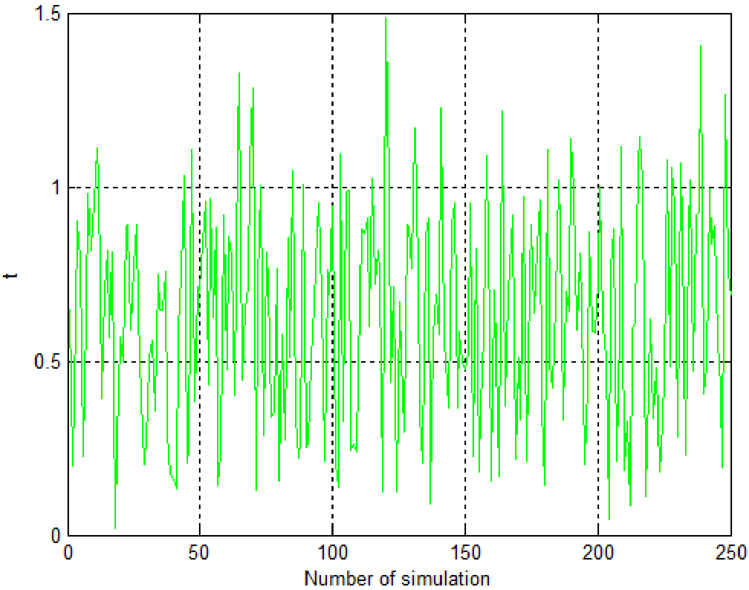

Figure 2. Simulation for bilinear time series model replaced by its estimated values $N = 900,\ N_s = 250$.

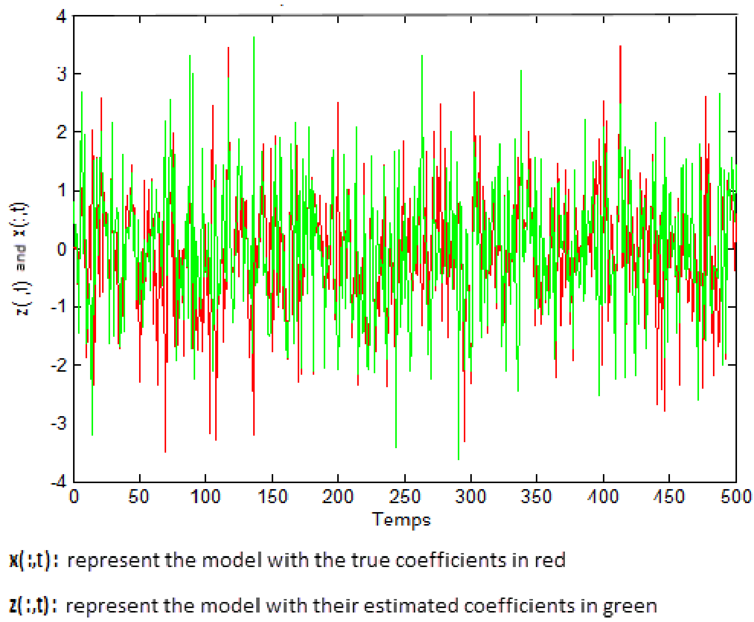

Figure 3. Comparisonbetween the two models.

4.1. Asymmetric and Symmetric GARCH Models

The significance and impact of symmetric and asymmetric GARCH models in financial econometrics cannot be overstated. These models serve as indispensable tools for capturing the intricate dynamics of volatility in financial time series data.

Symmetric and asymmetric GARCH models offer distinct methodologies for volatility modeling. Symmetric GARCH models assume equal effects of positive and negative shocks on volatility, while asymmetric models such as EGARCH or GJR-GARCH allow for varying responses to such shocks. Depending on the characteristics of the data, one model type may yield more precise estimates of volatility dynamics than the other.

The choice of a GARCH model profoundly influences the accuracy of results in simulating future scenarios or conducting Monte Carlo simulations in financial modeling. Asymmetric GARCH models, adept at capturing the asymmetry and skewness commonly observed in financial returns data, often provide more realistic simulations of future volatility and asset prices. Moreover, the selection between symmetric and asymmetric GARCH models carries implications for risk management practices. Asymmetric GARCH models excel in capturing volatility clustering, a critical aspect for assessing and managing financial risk. However, it is essential to note that asymmetric GARCH models may require more computational resources for estimation and simulation, particularly when dealing with large datasets or extensive simulations. Researchers and practitioners must carefully consider computational efficiency alongside model accuracy when choosing between symmetric and asymmetric GARCH models.

Extending the analysis, GARCH models are recognized for their superiority in capturing large shocks and volatility clustering compared to ARCH models, as highlighted by Patton and Sheppard (2015). Nevertheless, GARCH models have notable limitations, particularly their assumption of symmetric volatility regardless of the nature of shocks, as emphasized by Lubrano (2001) [10]. In contrast, ARCH and GARCH models are considered more suitable for risk measurement than descriptive statistics, especially in scenarios where stock returns undergo frequent fluctuations and the distribution of returns is markedly non-symmetric, according to Ang and Bekaert (2007) [11]. Despite this, variance remains a trusted measure for positive volatility, an aspect often overlooked by stockholders, underscoring the issue of instability and the uneven distribution of stock returns. Also, see [12–14].

To address the challenge of asymmetric instability, the present study explores various models for measuring GARCH effects, with a specific focus on GARCH symmetry.

$$\begin{cases} \varepsilon(t) = z(t)\eta(t) \\ z^2(t) = |\theta_0| + |\theta_1|\varepsilon^2(t-1) + |\theta_2|z^2(t-1) \end{cases}. \tag{30}$$

And in another way, the following model explains the Asymmetric GARCH model:

$$\begin{cases} \varepsilon(t) = z(t)\eta(t) \\ z(t) = \theta_0 + \theta_1|\varepsilon(t-1)| + \theta_2 z(t-1) \end{cases}. \tag{31}$$

To construct the asymmetry in a model driven by GARCH, we can write our model using the following expression:

$$\begin{cases} X(t) = \theta_0 X(t-s)\varepsilon(t-1) + \varepsilon(t) \\ \varepsilon(t) \rightsquigarrow GARCH(1,1) \end{cases}. \tag{32}$$

The asymmetric or symmetric nature of the white noise GARCH component imparts its characteristic properties to the bilinear models of the time series. This influence is demonstrated through the resulting curves, which depict the intricate interplay between volatility dynamics and the underlying characteristics of the GARCH model. By incorporating asymmetric or symmetric GARCH into bilinear models, we can capture the nuanced features of volatility, including asymmetry, clustering, and the impact of shocks, thereby enhancing our ability to model and forecast financial time series data accurately. The curves serve as visual representations of how the properties of GARCH reverberate through the bilinear models, providing valuable insights into the complex interactions within the data and informing our understanding of financial market behavior. See [15–18].

The estimation of model coefficients oriented by a GARCH model gives us the estimated data with the following Table 5, where a is the true value of model (32), $\hat{a}$ is its estimators, and ns represents the number of simulations

Table 5. The estimation of model coefficients oriented by a GARCH model.

ns	N	θ_0	$\hat{\theta}_0$ **Using Symmetric GARCH**	$\hat{\theta}_0$ **Using Asymmetric GARCH**
100	120	0.09	0.0879	0.1672
100	240	0.09	0.0902	0.1677
100	300	0.09	0.0898	0.1399
300	120	0.2	0.1944	0.1944
300	240	0.2	0.1967	0.1967
300	300	0.2	0.2013	0.2013
900	120	0.7	0.5972	0.5972
900	240	0.7	0.6897	0.6897
900	300	0.7	0.7011	0.7011
1200	1200	0.9	0.9002	0.9038

It can be observed that estimators of variable values in the bilinear model, as defined by its expression, yield efficient results when driven by Symmetric GARCH compared to Asymmetric GARCH, indicating that symmetry plays a crucial role in enhancing the accuracy of estimation for the proposed model [19–25].

4.2. Graphic Illustration

It is evident from the graphs that the impact of GARCH white noise on the bilinear model is pronounced. The asymmetry and symmetry inherent in GARCH manifest clearly in the original bilinear models depicted in the following Figures 4–9. Additionally, some deviations from symmetry are observable in the models, indicating that the time-varying coefficients, specifically the constant coefficients of the models, contribute to maintaining the properties of white noise. The final graph illuminates the overall symmetry of graphs driven by symmetric GARC.

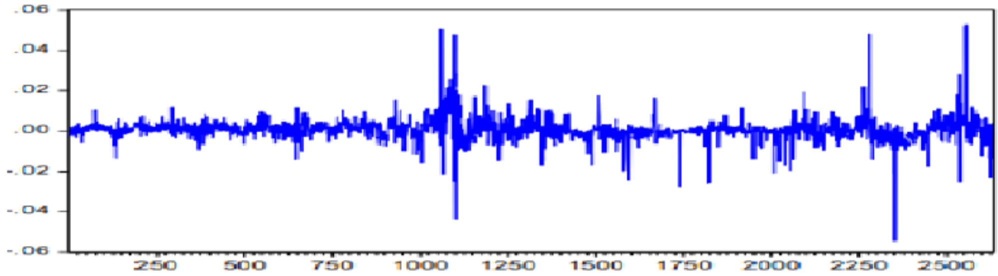

Figure 4. The time plot of daily Iran stock returns.

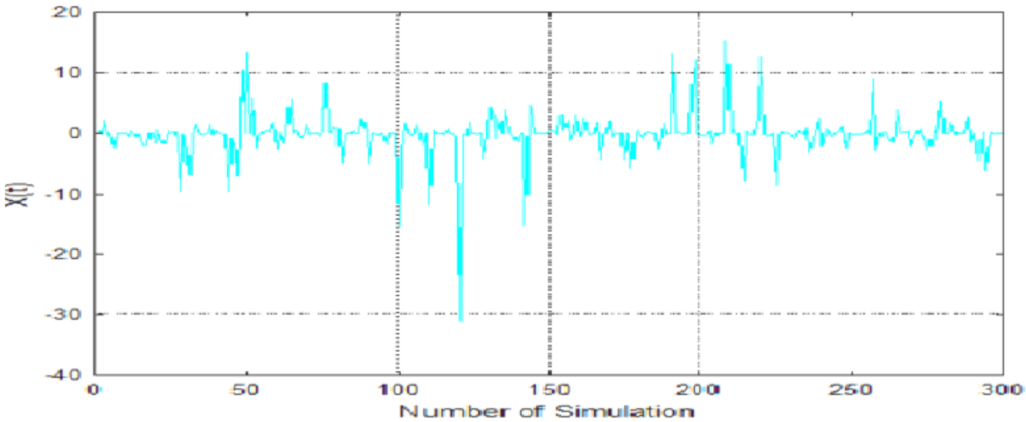

Figure 5. The number of simulations.

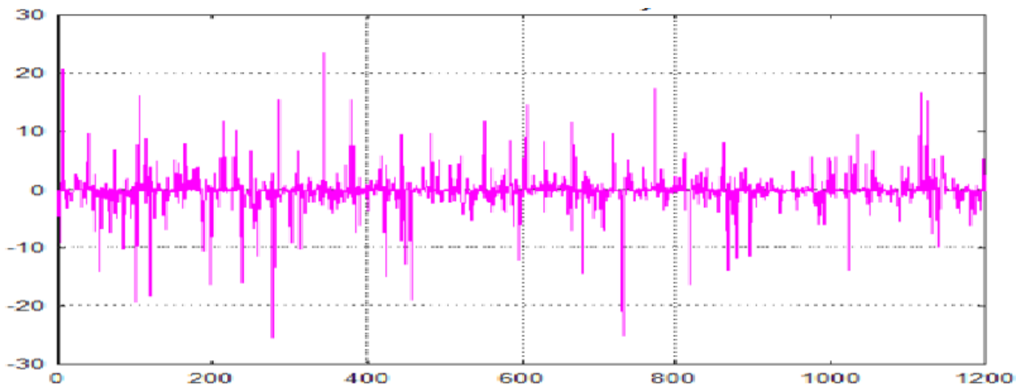

Figure 6. Bilinear time series model with constant coefficients driven by Asymmetric GARCH, where $N_s = 1200$ and $N = 900$.

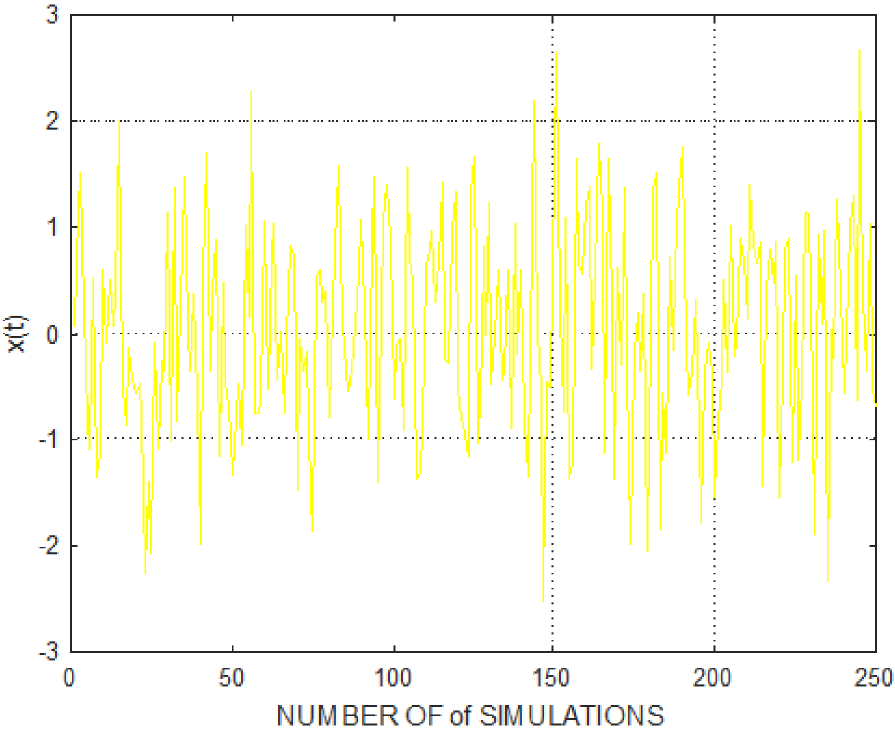

Figure 7. The number of simulations.

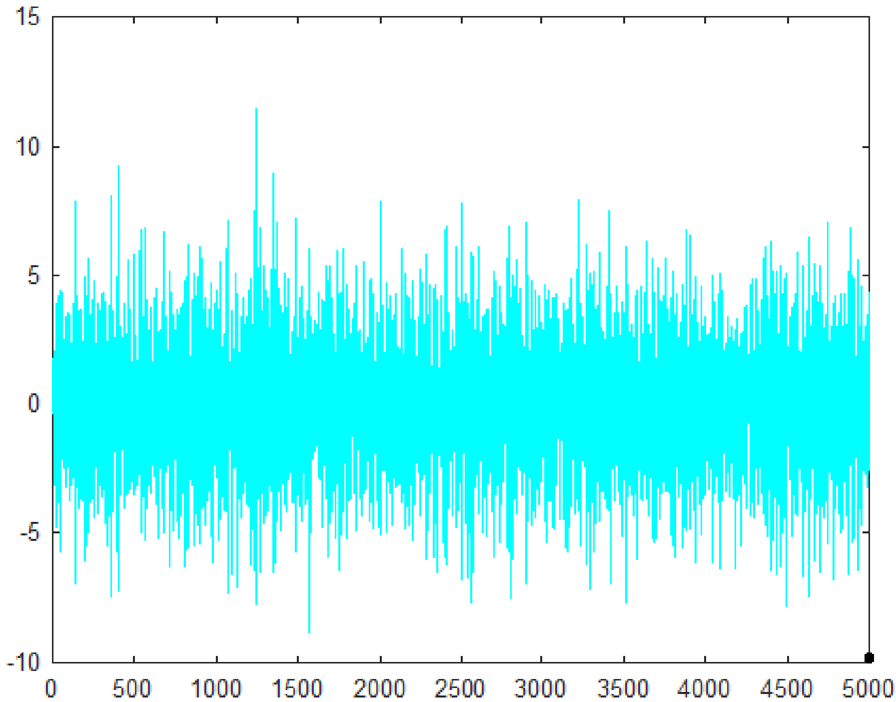

Figure 8. Bilinear time series model with constant coefficients driven by Symmetric GARCH, where $N_s = 5000$ and $N = 1000$.

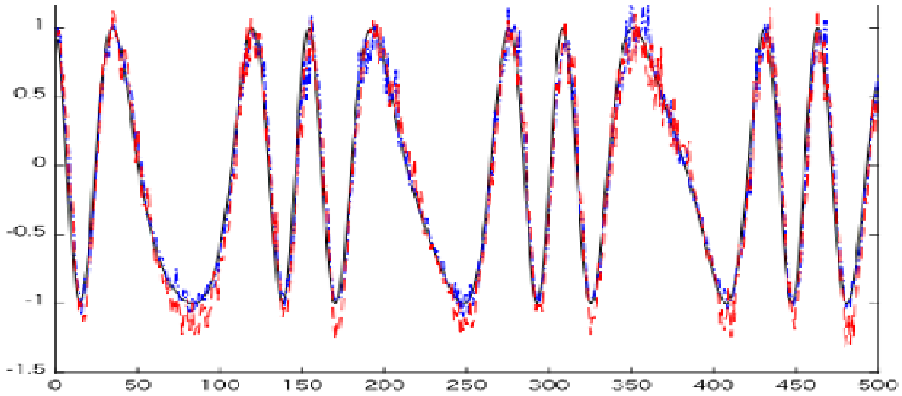

Figure 9. Curve illustrates the symmetry of the bilinear model driven by symmetric GRACH compared with several lines.

5. Concluding Comments

The simulations conducted on the bilinear model outlined in our paper illustrate a notable trend: as the sample size increases, the estimators converge toward their true values. This underscores the critical role of sample size in achieving accurate approximations. Additionally, we observe that increasing the number of simulations enhances the proximity between the estimated values and their true counterparts.

Interestingly, the white noise ARCH(1) model demonstrates commendable accuracy in approximating the estimators compared to the GARCH model for the coefficients. Moreover, we consistently observe a close resemblance between the two graphs, indicating robustness in the estimation process.

It is noteworthy that there exists a critical sample size, beyond which slight perturbations may arise in our simulations. However, the asymptotic behaviors of the estimators remain unaffected, suggesting the efficiency of our estimation approach in capturing temporal variations in coefficients.

In summary, our estimation approach, incorporating time-varying coefficients in the bilinear model along with a GARCH white noise component, yields significant theoretical and numerical insights, as evidenced by the simulations. Future research endeavors will delve into exploring various models in economics that integrate a combination of GARCH models to better capture and model realistic data patterns.

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