

Article

Novel Distance Measures of q -Rung Orthopair Fuzzy Sets and Their Applications

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Abstract: The q -rung orthopair fuzzy sets (q -ROFSs), a novel concept for processing vague information, offer a more potent and all-encompassing method compared to traditional fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets. The inclusion of the parameter q allows for the q -rung orthopair fuzzy sets to capture a broader range of uncertainty of information. In this paper, we present two novel distance measures for q -ROFSs inspired by the Jensen–Shannon divergence, called D_{JS_2D} and D_{JS_3D} , and we analyze some properties they satisfy, such as non-degeneracy, symmetry, boundedness, and triangular inequality. Then, the normalized distance measures, called $\widetilde{D_{JS_2D}}$ and $\widetilde{D_{JS_3D}}$, are proposed and we verify their rationality through numerical experiments. Finally, we apply the proposed distance measures to practical scenarios, including pattern recognition and multicriteria decision-making, and the results demonstrate the effectiveness of the proposed distance measures.

Keywords: q -rung orthopair fuzzy sets (q -ROFSs); Jensen–Shannon divergence; distance measure; pattern recognition; multicriteria decision-making



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1. Introduction

The intricacies inherent in objective phenomena, combined with the constraints of human understanding, inevitably give rise to uncertainty [1,2]. A great number of theories have been proposed to model uncertainty, such as fuzzy sets [3,4], neutrosophic sets [5,6], rough sets [7,8], and evidence theory [2,9]. Fuzzy sets (FSs) [10], initially introduced by Zadeh, facilitate the handling of uncertainty by incorporating membership degree, offering a mathematical framework more adept at managing uncertainties. FSs have proven effective for addressing complexities and ambiguities. However, FSs represent only a certain level of uncertain information.

To encompass a broader spectrum and a higher degree of uncertainty, Atanassov [11] expanded upon this concept by introducing intuitionistic fuzzy sets (IFSs). IFSs include both membership (μ) and non-membership (θ) degrees, thereby providing a more detailed representation of uncertainty. Numerous researchers have explored various measures of distance or similarity between IFSs since the inception of the theory. Szmidt and Kacprzyk [12] introduced the Hamming distance and the Euclidean distance measures along with their normalized forms. Wang and Xin [13] formulated an expanded distance measure specifically for IFSs and effectively employed it to tackle pattern recognition challenges. Park et al. [14] introduced a new distance measure for IFSs, building upon the framework established by Wang and Xin [13]. Yang and Chiclana [15] introduced an innovative spherical distance measure specifically tailored for IFSs within a three-dimensional framework, effectively leveraging it in decision-making analysis for enhanced precision and accuracy. Further contributions include Son and Phong [16], who developed intuitionistic vector similarity

measures, applying these measures to medical diagnosis. Ye [17] introduced two novel measures, i.e., cosine similarity and weighted cosine similarity, based on the cosine function, which effectively incorporate fuzzy information within the framework of IFSs. Xu [18] developed the Minkowski distance measure for IFSs and utilized it in solving pattern recognition and medical diagnosis. More recently, Xiao [19] introduced a novel distance measure based on the Jensen–Shannon divergence between IFSs. Li et al. [20] presented an intuitionistic fuzzy distance measure based on the Hellinger distance and applied it to pattern classification. These measures, however, face challenges when the sum of the membership and non-membership degrees exceeds one, complicating their applications.

Later, Yager [21] introduced an extension to IFSs, known as Pythagorean fuzzy sets (PFSs), which are characterized by membership (μ) and non-membership (ν). In PFSs, the sum of the squares of these degrees does not exceed one, offering a richer representation of uncertainty. This development initiated numerous studies into the properties and applications of PFSs [22–24]. Liang and Xu [25] developed a measure for the TOPSIS within the context of PFSs. Wei and Lu [26] proposed a set of power-based aggregation functions tailored for PFSs. Wei [27] suggested some PF interaction aggregation functions with their utility in multicriteria decision-making (MCDM). Zhang and Xu developed a distance measure for PFSs that evaluates alternatives and introduced a modified TOPSIS method to tackle multicriteria decision-making. Zeng et al. [28] demonstrated the application of Pythagorean fuzzy measures to assess both distance and similarity in MCDM, underscoring the utility of these advanced fuzzy sets in complex decision-making scenarios. Farhadinia [29] introduced a variety of new similarity measures for PFSs. Xiao et al. [30] applied the distance measures based on the Jensen–Shannon divergence to the PFSs environment. Liu [31] proposed two Hellinger distance measures for PFSs. Senapati and Yager [32] further proposed Fermatean fuzzy sets (FFSs), which expand the constraint on the sum of membership and non-membership degrees to the cubic. A number of researchers have delved into FFSs to handle uncertain information. Zhou et al. [33] introduced a new distance measure based on the Jensen–Shannon divergence to the Fermatean fuzzy context. Deng and Wang [34] designed two distance measures for FFSs based on the Hellinger distance and the triangular divergence. Liu [35] proposed a new distance measure for FFSs to overcome the limitations of the previous triangular divergence. Recently, Liu [36] developed some Fermatean fuzzy similarity measures according to Tanimoto and Sørensen coefficients and applied them to various applications.

Expanding the field of fuzzy logic, Yager [37] introduced the concept of q -rung orthopair fuzzy sets (q -ROFSs), which enhance the representation of uncertainty by utilizing the adjustable parameter q . The key characteristics of q -ROFSs include membership (μ) and non-membership (ν) degrees, where the sum of their q powers does not exceed 1, expressed as $\mu^q + \nu^q \leq 1$. One significant benefit of q -ROFSs is their flexible and comprehensive depiction of uncertainty. By varying the q value, the granularity of the uncertainty representation can be adjusted over a wider spectrum, making q -ROFSs particularly suitable for more complex decision-making scenarios and handling a diverse range of information. Since their introduction, q -ROFSs have been extensively explored in the literature, with numerous studies contributing to their development. An increasing array of distance or similarity measures have been proposed, enhancing their utility. For example, Wang et al. [38] extended some similarity measure based on the cosine and cotangent functions from PFSs to q -ROFSs. Liu et al. [39] proposed a new similarity measure for q -ROFSs by combining cosine similarity and Euclidean distance. Singh [40] developed new correlation coefficients to assess the degree and nature of the correlation between q -ROFSs. Ali [41] proposed a new distance measure for q -ROFSs, rooted in the matrix norm and strictly monotonic functions. Rani et al. [42] developed a distance measure inspired by the Hausdorff distance for q -ROFSs. Du [43] introduced distance measures of the Minkowski type for q -ROFSs. Turkarslan [44] proposed a distance measure between q -ROFSs using the Choquet integral, and great results were achieved in pattern recognition.

Evidently, q -ROFSs possess a broader scope than both IFSs and PFSs, enabling them to encapsulate a richer array of fuzzy information and offer a more comprehensive representation of uncertainty and vagueness. However, there are challenges with some existing distance measures for q -ROFSs, such as producing identical results when assessing differences between distinct q -ROFSs, which can lead to unreasonable outcomes. This limits their discriminative capacity in more complex ambiguous environments. Therefore, it is logical to propose new solutions to address this limitation. Moreover, to date, the literature lacks examples of using the Jensen–Shannon divergence as a basis for defining a distance measure for q -ROFSs. Motivated by these gaps, we aim to introduce two novel distance measures for q -ROFSs that utilize the Jensen–Shannon divergence, potentially offering a more robust and effective tool for measuring differences within this framework.

The main contributions of this paper are displayed as follows:

1. We propose two new distance measures, i.e., a two-dimensional distance measure of a q -ROFS, which considers membership and non-membership degrees, and a three-dimensional distance measure of q -ROFSs, considering membership, non-membership, and hesitancy degrees.
2. We analyze some properties that the proposed distance measures satisfy, such as non-degeneracy, symmetry, boundedness, and triangle inequality.
3. We apply the proposed distance measures to pattern recognition and multicriteria decision-making issues, and excellent results are obtained to verify their performance.

The rest of this paper is structured as follows. In Section 2, we review some essential knowledge of IFSs, PFSs, FFSs, and q -ROFSs briefly. In Section 3, we first introduce the concept of the Jensen–Shannon divergence. Then, we define two novel distance measures for q -ROFSs based on the Jensen–Shannon divergence and prove some properties. In Section 4, the effectiveness of the proposed measures is verified by some numerical examples. Subsequently, we apply the proposed measures to practical applications including pattern recognition and multicriteria decision-making in Section 5. We conclude this paper in Section 6.

2. Preliminaries

In this section, we will review the fundamental concepts related to IFSs, PFSs, FFSs, and q -ROFSs and introduce some existing q -ROFS distance measures.

2.1. Intuitionistic Fuzzy Sets

Definition 1. [11] Let X be the universe of discourse (UOD), and an intuitionistic fuzzy set I in X is defined below:

$$I = \{ \langle x, \varkappa_I(x), \vartheta_I(x) \rangle | x \in X \} \quad (1)$$

where $\varkappa_I(x), \vartheta_I(x): X \rightarrow [0, 1]$ denote the membership degree and non-membership degree of $x \in X$, respectively. For any $x \in X$, we have the following constraint:

$$0 \leq \varkappa_I(x) + \vartheta_I(x) \leq 1 \quad (2)$$

The hesitancy degree $\pi_I(x)$ is as follows:

$$\pi_I(x) = 1 - \varkappa_I(x) - \vartheta_I(x) \quad (3)$$

2.2. Pythagorean Fuzzy Sets

Definition 2. [21] Let X be the UOD, and a Pythagorean fuzzy set P in X is defined below:

$$P = \{ \langle x, \mu_P(x), \nu_P(x) \rangle | x \in X \} \quad (4)$$

where $\mu_P(x): X \rightarrow [0, 1]$ represents the membership degree of $x \in X$ and $\nu_P(x): X \rightarrow [0, 1]$ expresses the non-membership degree of $x \in X$. For any $x \in X$, $\mu_P(x)$ and $\nu_P(x)$ meet the following:

$$0 \leq \mu_P^2(x) + \nu_P^2(x) \leq 1 \quad (5)$$

The hesitancy degree of $x \in X$ is determined below:

$$\pi_P(x) = \sqrt{1 - \mu_P^2(x) - \nu_P^2(x)} \quad (6)$$

2.3. Fermatean Fuzzy Sets

Definition 3. [32] Suppose that X is a finite UOD. The Fermatean fuzzy set F in X is defined below:

$$F = \{ \langle x, \mu_F(x), \nu_F(x) \rangle | x \in X \} \quad (7)$$

where $\mu_F(x): X \rightarrow [0, 1]$ represents the membership degree of $x \in X$ and $\nu_F(x): X \rightarrow [0, 1]$ expresses the non-membership degree of $x \in X$. For any $x \in X$, $\mu_F(x)$ and $\nu_F(x)$ meet the following:

$$0 \leq \mu_F^3(x) + \nu_F^3(x) \leq 1 \quad (8)$$

For any $x \in X$, the hesitancy degree of the element x is determined below:

$$\pi_F(x) = \sqrt[3]{1 - \mu_F^3(x) - \nu_F^3(x)} \quad (9)$$

2.4. Q-Rung Orthopair Fuzzy Sets

Definition 4. [37] Let X be a finite UOD. The q -rung orthopair fuzzy set Q in X can be denoted as:

$$Q = \{ \langle x, \mu_Q(x), \nu_Q(x) \rangle | x \in X \} \quad (10)$$

where $\mu_Q(x): X \rightarrow [0, 1]$ means the membership degree of $x \in X$ and $\nu_Q(x): X \rightarrow [0, 1]$ signifies the non-membership degree of $x \in X$. For any $x \in X$, $\mu_Q(x)$ and $\nu_Q(x)$ satisfy the following conditions:

$$0 \leq \mu_Q^q(x) + \nu_Q^q(x) \leq 1, \quad q \geq 1 \quad (11)$$

The hesitancy degree $\pi_Q(x)$ can be formulated as:

$$\pi_Q(x) = \sqrt[q]{1 - \mu_Q^q(x) - \nu_Q^q(x)} \quad (12)$$

2.5. Distance Measures For q -ROFSs

Let $Q_1 = \{ (x_j, \mu_{Q_1}(x_j), \nu_{Q_1}(x_j)) \}$ and $Q_2 = \{ (x_j, \mu_{Q_2}(x_j), \nu_{Q_2}(x_j)) \}$ be two q -rung orthopair fuzzy sets in the UOD X . $D(Q_1, Q_2)$ and $S(Q_1, Q_2)$ demonstrate a distance measure and a similarity measure between the q -ROFSs, respectively [38,45].

Definition 5. Let Q_1 and Q_2 be two q -ROFSs, and the distance measures $D_i (i = 1, 2, 3, 4, 5)$ are listed below:

$$\begin{aligned}
 D_1(Q_1, Q_2) &= \frac{1}{n} \sum_{j=1}^n |\kappa_{Q_1}^q(x_j) - \kappa_{Q_2}^q(x_j)| \vee |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)| \\
 D_2(Q_1, Q_2) &= \frac{2}{n} \sum_{j=1}^n \frac{|\kappa_{Q_1}^q(x_j) - \kappa_{Q_2}^q(x_j)| \vee |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)|}{1 + |\kappa_{Q_1}^q(x_j) - \kappa_{Q_2}^q(x_j)| \vee |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)|} \\
 D_3(Q_1, Q_2) &= \frac{2 \sum_{j=1}^n (|\kappa_{Q_1}^q(x_j) - \kappa_{Q_2}^q(x_j)| \vee |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)|)}{\sum_{j=1}^n (1 + |\kappa_{Q_1}^q(x_j) - \kappa_{Q_2}^q(x_j)| \vee |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)|)} \\
 D_4(Q_1, Q_2) &= 1 - \frac{\sum_{j=1}^n ((\kappa_{Q_1}^q(x_j) \wedge \kappa_{Q_2}^q(x_j)) + (\vartheta_{Q_1}^q(x_j) \wedge \vartheta_{Q_2}^q(x_j)))}{\sum_{j=1}^n ((\kappa_{Q_1}^q(x_j) \vee \kappa_{Q_2}^q(x_j)) + (\vartheta_{Q_1}^q(x_j) \vee \vartheta_{Q_2}^q(x_j)))} \\
 D_5(Q_1, Q_2) &= 1 - \frac{\sum_{j=1}^n ((\kappa_{Q_1}^q(x_j) \wedge \kappa_{Q_2}^q(x_j)) + (1 - \vartheta_{Q_1}^q(x_j)) \wedge (1 - \vartheta_{Q_2}^q(x_j)))}{\sum_{j=1}^n ((\kappa_{Q_1}^q(x_j) \vee \kappa_{Q_2}^q(x_j)) + (1 - \vartheta_{Q_1}^q(x_j)) \vee (1 - \vartheta_{Q_2}^q(x_j)))}
 \end{aligned}$$

Definition 6. Let Q_1 and Q_2 be two q -ROFSs, and the similarity measures $S_i (i = 1, 2, 3, 4, 5)$ are listed below:

$$\begin{aligned}
 S_1(Q_1, Q_2) &= \frac{1}{n} \sum_{j=1}^n \frac{\kappa_{Q_1}^q(x_j) \kappa_{Q_2}^q(x_j) + \vartheta_{Q_1}^q(x_j) \vartheta_{Q_2}^q(x_j)}{\sqrt{(\kappa_{Q_1}^q(x_j))^2 + (\vartheta_{Q_1}^q(x_j))^2} \sqrt{(\kappa_{Q_2}^q(x_j))^2 + (\vartheta_{Q_2}^q(x_j))^2}} \\
 S_2(Q_1, Q_2) &= \frac{1}{n} \sum_{j=1}^n \frac{\kappa_{Q_1}^q(x_j) \kappa_{Q_2}^q(x_j) + \vartheta_{Q_1}^q(x_j) \vartheta_{Q_2}^q(x_j) + \pi_{Q_1}^q(x_j) \pi_{Q_2}^q(x_j)}{\sqrt{(\kappa_{Q_1}^q(x_j))^2 + (\vartheta_{Q_1}^q(x_j))^2 + (\pi_{Q_1}^q(x_j))^2} \sqrt{(\kappa_{Q_2}^q(x_j))^2 + (\vartheta_{Q_2}^q(x_j))^2 + (\pi_{Q_2}^q(x_j))^2}} \\
 S_3(Q_1, Q_2) &= \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \left(\max(|\kappa_{Q_1}^q(x_j) - \kappa_{Q_2}^q(x_j)|, |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)|) \right) \right] \\
 S_4(Q_1, Q_2) &= \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{8} \left(|\kappa_{Q_1}^q(x_j) - \kappa_{Q_2}^q(x_j)|, |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)| \right) \right] \\
 S_5(Q_1, Q_2) &= \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \left(\max(|\kappa_{Q_1}^q(x_j) - \kappa_{Q_2}^q(x_j)|, |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)|, |\pi_{Q_1}^q(x_j) - \pi_{Q_2}^q(x_j)|) \right) \right] \\
 S_6(Q_1, Q_2) &= \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{8} \left(|\kappa_{Q_1}^q(x_j) - \kappa_{Q_2}^q(x_j)|, |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)|, |\pi_{Q_1}^q(x_j) - \pi_{Q_2}^q(x_j)| \right) \right]
 \end{aligned}$$

Definition 7. Let Q_1 and Q_2 be two q -ROFSs. According to the similarity measures above, there are some distance measures defined as follows:

$$\begin{aligned}
D_6(Q_1, Q_2) &= 1 - \frac{1}{n} \sum_{j=1}^n \frac{\varkappa_{Q_1}^q(x_j) \varkappa_{Q_2}^q(x_j) + \vartheta_{Q_1}^q(x_j) \vartheta_{Q_2}^q(x_j)}{\sqrt{(\varkappa_{Q_1}^q(x_j))^2 + (\vartheta_{Q_1}^q(x_j))^2} \sqrt{(\varkappa_{Q_2}^q(x_j))^2 + (\vartheta_{Q_2}^q(x_j))^2}} \\
D_7(Q_1, Q_2) &= 1 - \frac{1}{n} \sum_{j=1}^n \frac{\varkappa_{Q_1}^q(x_j) \varkappa_{Q_2}^q(x_j) + \vartheta_{Q_1}^q(x_j) \vartheta_{Q_2}^q(x_j) + \pi_{Q_1}^q(x_j) \pi_{Q_2}^q(x_j)}{\sqrt{(\varkappa_{Q_1}^q(x_j))^2 + (\vartheta_{Q_1}^q(x_j))^2 + (\pi_{Q_1}^q(x_j))^2} \sqrt{(\varkappa_{Q_2}^q(x_j))^2 + (\vartheta_{Q_2}^q(x_j))^2 + (\pi_{Q_2}^q(x_j))^2}} \\
D_8(Q_1, Q_2) &= 1 - \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \left(\max \left(|\varkappa_{Q_1}^q(x_j) - \varkappa_{Q_2}^q(x_j)|, |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)| \right) \right) \right] \\
D_9(Q_1, Q_2) &= 1 - \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{8} \left(|\varkappa_{Q_1}^q(x_j) - \varkappa_{Q_2}^q(x_j)|, |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)| \right) \right] \\
D_{10}(Q_1, Q_2) &= 1 - \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \left(\max \left(|\varkappa_{Q_1}^q(x_j) - \varkappa_{Q_2}^q(x_j)|, |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)|, |\pi_{Q_1}^q(x_j) - \pi_{Q_2}^q(x_j)| \right) \right) \right] \\
D_{11}(Q_1, Q_2) &= 1 - \frac{1}{n} \sum_{j=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{8} \left(|\varkappa_{Q_1}^q(x_j) - \varkappa_{Q_2}^q(x_j)|, |\vartheta_{Q_1}^q(x_j) - \vartheta_{Q_2}^q(x_j)|, |\pi_{Q_1}^q(x_j) - \pi_{Q_2}^q(x_j)| \right) \right]
\end{aligned}$$

3. Two New Distance Measures of q -ROFSs

In this section, we propose two new distance measures specifically for q -ROFSs, drawing inspiration from the Jensen–Shannon divergence. Moreover, we deduce and prove the properties of these suggested measures, ensuring their accuracy and reliability.

3.1. Jensen–Shannon Divergence

Definition 8. [19] Assuming that $P = \{p_1, p_2, \dots, p_n\}$ and $Q = \{q_1, q_2, \dots, q_n\}$ are two probability distributions, the Jensen–Shannon divergence between them is formally defined in the following manner:

$$JS(P, Q) = \frac{1}{2} \left[KL(P, \frac{P+Q}{2}) + KL(Q, \frac{P+Q}{2}) \right] \quad (13)$$

with

$$KL(P, Q) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} \quad (14)$$

where $KL(P, Q)$ represents the Kullback–Leibler divergence.

$JS(P, Q)$ can alternatively be represented using the subsequent equation:

$$\begin{aligned}
JS(P, Q) &= H\left(\frac{P+Q}{2}\right) - \frac{1}{2}H(P) - \frac{1}{2}H(Q) \\
&= \frac{1}{2} \left(\sum_{i=1}^n p_i \log \frac{2p_i}{p_i + q_i} + \sum_{i=1}^n q_i \log \frac{2q_i}{p_i + q_i} \right)
\end{aligned} \quad (15)$$

with

$$H(P) = - \sum_{i=1}^n p_i \log p_i, \quad H(Q) = - \sum_{i=1}^n q_i \log q_i, \quad H\left(\frac{P+Q}{2}\right) = - \sum_{i=1}^n \frac{p_i + q_i}{2} \log \frac{p_i + q_i}{2} \quad (16)$$

where $H(\frac{P+Q}{2})$, $H(P)$, and $H(Q)$ represent the Shannon entropy.

3.2. Two-Dimensional (2D) Distance Measure of q -ROFSs

Definition 9. Suppose that $Q_1 = \langle x, \varkappa_{Q_1}(x), \vartheta_{Q_1}(x) \rangle$ and $Q_2 = \langle x, \varkappa_{Q_2}(x), \vartheta_{Q_2}(x) \rangle$ are two q -ROFSs in the UOD X . For any $x \in X$, then the distance between Q_1 and Q_2 is defined below:

$$D_{JS_2D}(Q_1, Q_2) = \left[\frac{1}{2} \left(\varkappa_{Q_1}^q(x) \log \frac{2\varkappa_{Q_1}^q(x)}{\varkappa_{Q_1}^q(x) + \varkappa_{Q_2}^q(x)} + \varkappa_{Q_2}^q(x) \log \frac{2\varkappa_{Q_2}^q(x)}{\varkappa_{Q_1}^q(x) + \varkappa_{Q_2}^q(x)} \right. \right. \\ \left. \left. + \vartheta_{Q_1}^q(x) \log \frac{2\vartheta_{Q_1}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} + \vartheta_{Q_2}^q(x) \log \frac{2\vartheta_{Q_2}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \right) \right]^{\frac{1}{2}} \quad (17)$$

Property 1. Let Q_1 , Q_2 , and Q_3 be three q -ROFSs in the UOD X , and the properties of D_{JS_2D} are displayed as follows:

1. Non-degeneracy: $D_{JS_2D}(Q_1, Q_2) = 0$, if and only if $Q_1 = Q_2$;
2. Symmetry: $D_{JS_2D}(Q_1, Q_2) = D_{JS_2D}(Q_2, Q_1)$;
3. Boundedness: $0 \leq D_{JS_2D}(Q_1, Q_2) \leq 1$;
4. Triangular inequality: $D_{JS_2D}(Q_3, Q_1) + D_{JS_2D}(Q_1, Q_2) \geq D_{JS_2D}(Q_3, Q_2)$.

Proof. $D_{JS_2D}(Q_1, Q_2) = 0$, if and only if $Q_1 = Q_2$.

Considering two q -ROFSs in the UOD X that meet the condition $Q_1 = Q_2$, we can obtain that:

$$\varkappa_{Q_1}^q(x) = \varkappa_{Q_2}^q(x) \quad \text{and} \quad \vartheta_{Q_1}^q(x) = \vartheta_{Q_2}^q(x)$$

Then, we have:

$$D_{JS_2D}(Q_1, Q_2) = \left[\frac{1}{2} \left(\varkappa_{Q_1}^q(x) \log \frac{2\varkappa_{Q_1}^q(x)}{\varkappa_{Q_1}^q(x) + \varkappa_{Q_2}^q(x)} + \varkappa_{Q_2}^q(x) \log \frac{2\varkappa_{Q_2}^q(x)}{\varkappa_{Q_1}^q(x) + \varkappa_{Q_2}^q(x)} \right. \right. \\ \left. \left. + \vartheta_{Q_1}^q(x) \log \frac{2\vartheta_{Q_1}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} + \vartheta_{Q_2}^q(x) \log \frac{2\vartheta_{Q_2}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \right) \right]^{\frac{1}{2}} = 0$$

Hence, we obtain

$$D_{JS_2D}(Q_1, Q_2) = 0.$$

When $D_{JS_2D}(Q_1, Q_2) = 0$, we have:

$$\varkappa_{Q_1}^q(x) = \varkappa_{Q_2}^q(x), \quad \vartheta_{Q_1}^q(x) = \vartheta_{Q_2}^q(x)$$

Thus,

$$\varkappa_{Q_1}(x) = \varkappa_{Q_2}(x), \quad \vartheta_{Q_1}(x) = \vartheta_{Q_2}(x)$$

Hence, we obtain

$$Q_1 = Q_2.$$

□

Proof. $D_{JS_2D}(Q_1, Q_2) = D_{JS_2D}(Q_2, Q_1)$.

Given two q -ROFSs Q_1 and Q_2 in the UOD X , we have:

$$\begin{aligned}
D_{JS_2D}(Q_1, Q_2) &= \left[\frac{1}{2} \left(\kappa_{Q_1}^q(x) \log \frac{2\kappa_{Q_1}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} + \kappa_{Q_2}^q(x) \log \frac{2\kappa_{Q_2}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \right. \right. \\
&\quad \left. \left. + \vartheta_{Q_1}^q(x) \log \frac{2\vartheta_{Q_1}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} + \vartheta_{Q_2}^q(x) \log \frac{2\vartheta_{Q_2}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \right) \right]^{\frac{1}{2}} \\
&= \left[\frac{1}{2} \left(\left(\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x) \right) \left(\frac{\kappa_{Q_1}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \log \frac{2\kappa_{Q_1}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \right. \right. \right. \\
&\quad \left. \left. + \frac{\kappa_{Q_2}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \log \frac{2\kappa_{Q_2}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \right) \right. \\
&\quad \left. + \left(\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x) \right) \left(\frac{\vartheta_{Q_1}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \log \frac{2\vartheta_{Q_1}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \right. \right. \\
&\quad \left. \left. + \frac{\vartheta_{Q_2}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \log \frac{2\vartheta_{Q_2}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \right) \right]^{\frac{1}{2}} \\
&= D_{JS_2D}(Q_2, Q_1)
\end{aligned}$$

Thereby, we can prove that $D_{JS_2D}(Q_1, Q_2) = D_{JS_2D}(Q_2, Q_1)$. $\square$

Proof. $0 \leq D_{JS_2D}(Q_1, Q_2) \leq 1$.

Given two q -ROFSs Q_1 and Q_2 in the UOD X , we can obtain that:

$$\begin{aligned}
D_{JS_3D}(Q_1, Q_2) &= \left[\frac{1}{2} \left(\kappa_{Q_1}^q(x) \log \frac{2\kappa_{Q_1}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} + \kappa_{Q_2}^q(x) \log \frac{2\kappa_{Q_2}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \right. \right. \\
&\quad \left. \left. + \vartheta_{Q_1}^q(x) \log \frac{2\vartheta_{Q_1}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} + \vartheta_{Q_2}^q(x) \log \frac{2\vartheta_{Q_2}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \right) \right]^{\frac{1}{2}} \\
&= \left[\frac{1}{2} \left(\left(\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x) \right) \left(\frac{\kappa_{Q_1}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \log \frac{2\kappa_{Q_1}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \right. \right. \right. \\
&\quad \left. \left. + \frac{\kappa_{Q_2}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \log \frac{2\kappa_{Q_2}^q(x)}{\kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x)} \right) \right. \\
&\quad \left. + \left(\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x) \right) \left(\frac{\vartheta_{Q_1}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \log \frac{2\vartheta_{Q_1}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \right. \right. \\
&\quad \left. \left. + \frac{\vartheta_{Q_2}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \log \frac{2\vartheta_{Q_2}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \right) \right]^{\frac{1}{2}} \\
&= \left[\frac{1}{2} (Q_1 + Q_2) \left(1 - H \left(\frac{Q_1}{Q_1 + Q_2'} + \frac{Q_2}{Q_1 + Q_2'} \right) \right) \right]^{\frac{1}{2}}.
\end{aligned}$$

The proof for $0 \leq \xi \leq 1$ was established in [46].

$$1 - H(\xi, 1 - \xi) \leq |\xi - (1 - \xi)|.$$

Then, we have:

$$D_{JS_2D}(Q_1, Q_2) \leq \left[\frac{1}{2}(Q_1 + Q_2) \left| \frac{Q_1}{Q_1 + Q_2} - \frac{Q_2}{Q_1 + Q_2} \right| \right]^{\frac{1}{2}} = \left[\frac{1}{2}V(Q_1, Q_2) \right]^{\frac{1}{2}}$$

where $V(Q_1, Q_2)$ is the variational distance.

As proven in [47],

$$0 \leq V(Q_1, Q_2) \leq 2$$

Thereby, we can obtain:

$$0 \leq D_{JS_2D}(Q_1, Q_2) \leq 1.$$

Consequently, it is proved that $0 \leq D_{JS_2D}(Q_1, Q_2) \leq 1$.

□

Proof. $D_{JS_2D}(Q_3, Q_1) + D_{JS_2D}(Q_1, Q_2) \geq D_{JS_2D}(Q_3, Q_2)$.

There exists three q -ROFSs Q_1 , Q_2 , and Q_3 in X , considering assumptions $\mathcal{A}_i$, ($i = 1, 2, 3, 4$) for evaluation:

$$\begin{aligned} \mathcal{A}_1 : \kappa_{Q_3}^q(x) &\leq \kappa_{Q_1}^q(x) \leq \kappa_{Q_2}^q(x) \\ \mathcal{A}_2 : \kappa_{Q_2}^q(x) &\leq \kappa_{Q_1}^q(x) \leq \kappa_{Q_3}^q(x) \\ \mathcal{A}_3 : \kappa_{Q_1}^q(x) &\leq \min\{\kappa_{Q_3}^q(x), \kappa_{Q_2}^q(x)\} \\ \mathcal{A}_4 : \kappa_{Q_1}^q(x) &\geq \max\{\kappa_{Q_3}^q(x), \kappa_{Q_2}^q(x)\}. \end{aligned}$$

Given the assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$, it becomes evident that the triangle inequality holds true in the manner described below:

$$|\kappa_{Q_3}^q(x) - \kappa_{Q_2}^q(x)| \leq |\kappa_{Q_3}^q(x) - \kappa_{Q_1}^q(x)| + |\kappa_{Q_1}^q(x) - \kappa_{Q_2}^q(x)|.$$

With assumption $\mathcal{A}_3$, we obtain:

$$\kappa_{Q_3}^q(x) - \kappa_{Q_1}^q(x) \geq 0 \quad \text{and} \quad \kappa_{Q_2}^q(x) - \kappa_{Q_1}^q(x) \geq 0$$

Then, the calculation can be determined:

$$\begin{aligned} &|\kappa_{Q_3}^q(x) - \kappa_{Q_1}^q(x)| + |\kappa_{Q_1}^q(x) - \kappa_{Q_2}^q(x)| - |\kappa_{Q_3}^q(x) - \kappa_{Q_2}^q(x)| \\ &= \begin{cases} \kappa_{Q_3}^q(x) - \kappa_{Q_1}^q(x) + \kappa_{Q_2}^q(x) - \kappa_{Q_1}^q(x) - \kappa_{Q_3}^q(x) + \kappa_{Q_2}^q(x) & \text{if } \kappa_{Q_3}^q(x) \geq \kappa_{Q_2}^q(x) \\ \kappa_{Q_1}^q(x) - \kappa_{Q_3}^q(x) + \kappa_{Q_2}^q(x) - \kappa_{Q_1}^q(x) + \kappa_{Q_3}^q(x) - \kappa_{Q_2}^q(x) & \text{if } \kappa_{Q_3}^q(x) \leq \kappa_{Q_2}^q(x) \end{cases} \\ &= 2(\min\{\kappa_{Q_3}^q(x), \kappa_{Q_2}^q(x)\} - \kappa_{Q_1}^q(x)) \geq 0. \end{aligned}$$

With assumption $\mathcal{A}_4$ in place, it can be inferred that similarly:

$$\begin{aligned} & |\varkappa_{Q_3}^q(x) - \varkappa_{Q_1}^q(x)| + |\varkappa_{Q_1}^q(x) - \varkappa_{Q_2}^q(x)| - |\varkappa_{Q_3}^q(x) - \varkappa_{Q_2}^q(x)| \\ &= \begin{cases} \varkappa_{Q_1}^q(x) - \varkappa_{Q_3}^q(x) + \varkappa_{Q_1}^q(x) - \varkappa_{Q_2}^q(x) - \varkappa_{Q_3}^q(x) + \varkappa_{Q_2}^q(x) & \text{if } \varkappa_{Q_3}^q(x) \geq \varkappa_{Q_2}^q(x), \\ \varkappa_{Q_3}^q(x) - \varkappa_{Q_1}^q(x) + \varkappa_{Q_1}^q(x) - \varkappa_{Q_2}^q(x) + \varkappa_{Q_3}^q(x) - \varkappa_{Q_2}^q(x) & \text{if } \varkappa_{Q_3}^q(x) \leq \varkappa_{Q_2}^q(x), \end{cases} \\ &= 2(\varkappa_{Q_1}^q(x) - \max\{\varkappa_{Q_3}^q(x), \varkappa_{Q_2}^q(x)\}) \geq 0. \end{aligned}$$

As a result, the triangle inequality remains valid when founded on assumptions $\mathcal{A}_3$ and $\mathcal{A}_4$, in which case

$$|\varkappa_{Q_3}^q(x) - \varkappa_{Q_2}^q(x)| \leq |\varkappa_{Q_3}^q(x) - \varkappa_{Q_1}^q(x)| + |\varkappa_{Q_1}^q(x) - \varkappa_{Q_2}^q(x)|.$$

In a similar fashion, we can deduce that

$$|\vartheta_{Q_3}^q(x) - \vartheta_{Q_2}^q(x)| \leq |\vartheta_{Q_3}^q(x) - \vartheta_{Q_1}^q(x)| + |\vartheta_{Q_1}^q(x) - \vartheta_{Q_2}^q(x)|$$

Consequently, the triangle inequality property for D_{JS_2D} has been proven, as shown by the following:

$$D_{JS_2D}(Q_3, Q_1) + D_{JS_2D}(Q_1, Q_2) \geq D_{JS_2D}(Q_3, Q_2)$$

□

3.3. Three-Dimensional (3D) Distance Measure of q -ROFSs

Definition 10. Let $Q_1 = \langle x, \varkappa_{Q_1}(x), \vartheta_{Q_1}(x) \rangle$ and $Q_2 = \langle x, \varkappa_{Q_2}(x), \vartheta_{Q_2}(x) \rangle$ be two q -ROFSs. The hesitancy grades of Q_1 and Q_2 are represented by π_{Q_1} and π_{Q_2} , respectively. For any $x \in X$, the distance measure between Q_1 and Q_2 is defined as follows:

$$\begin{aligned} D_{JS_3D}(Q_1, Q_2) = & \left[\frac{1}{2} \left(\varkappa_{Q_1}^q(x) \log \frac{2\varkappa_{Q_1}^q(x)}{\varkappa_{Q_1}^q(x) + \varkappa_{Q_2}^q(x)} + \varkappa_{Q_2}^q(x) \log \frac{2\varkappa_{Q_2}^q(x)}{\varkappa_{Q_1}^q(x) + \varkappa_{Q_2}^q(x)} \right. \right. \\ & + \vartheta_{Q_1}^q(x) \log \frac{2\vartheta_{Q_1}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} + \vartheta_{Q_2}^q(x) \log \frac{2\vartheta_{Q_2}^q(x)}{\vartheta_{Q_1}^q(x) + \vartheta_{Q_2}^q(x)} \\ & \left. \left. + \pi_{Q_1}^q(x) \log \frac{2\pi_{Q_1}^q(x)}{\pi_{Q_1}^q(x) + \pi_{Q_2}^q(x)} + \pi_{Q_2}^q(x) \log \frac{2\pi_{Q_2}^q(x)}{\pi_{Q_1}^q(x) + \pi_{Q_2}^q(x)} \right) \right]^{\frac{1}{2}} \quad (18) \end{aligned}$$

Property 2. Consider the three q -ROFSs Q_1 , Q_2 , and Q_3 in the UOD X , and the properties that D_{JS_3D} should satisfy are displayed by:

1. Non-degeneracy: $D_{JS_3D}(Q_1, Q_2) = 0$, if and only if $Q_1 = Q_2$;
2. Symmetry: $D_{JS_3D}(Q_1, Q_2) = D_{JS_3D}(Q_2, Q_1)$;
3. Boundedness: $0 \leq D_{JS_3D}(Q_1, Q_2) \leq 1$;
4. Triangular inequality: $D_{JS_3D}(Q_3, Q_1) + D_{JS_3D}(Q_1, Q_2) \geq D_{JS_3D}(Q_3, Q_2)$.

Proof. The proof of Property 2 is similar to Property 1. □

Definition 11. Assume that $Q_1 = \langle x, \varkappa_{Q_1}(x), \vartheta_{Q_1}(x) \rangle$ and $Q_2 = \langle x, \varkappa_{Q_2}(x), \vartheta_{Q_2}(x) \rangle$ are two q -ROFSs in the UOD $X = \{x_1, x_2, \dots, x_n\}$, and the normalized distance measures between Q_1 and Q_2 are defined as follows:

$$\begin{aligned} D_{\widetilde{JS_{2D}}}(Q_1, Q_2) &= \frac{1}{n} \sum_{i=1}^n D_{JS_{2D}}(Q_1, Q_2) \\ &= \frac{1}{n} \sum_{i=1}^n \left[\frac{1}{2} \left(\varkappa_{Q_1}^q(x_i) \log \frac{2\varkappa_{Q_1}^q(x_i)}{\varkappa_{Q_1}^q(x_i) + \varkappa_{Q_2}^q(x_i)} + \varkappa_{Q_2}^q(x_i) \log \frac{2\varkappa_{Q_2}^q(x_i)}{\varkappa_{Q_1}^q(x_i) + \varkappa_{Q_2}^q(x_i)} \right. \right. \\ &\quad \left. \left. + \vartheta_{Q_1}^q(x_i) \log \frac{2\vartheta_{Q_1}^q(x_i)}{\vartheta_{Q_1}^q(x_i) + \vartheta_{Q_2}^q(x_i)} + \vartheta_{Q_2}^q(x_i) \log \frac{2\vartheta_{Q_2}^q(x_i)}{\vartheta_{Q_1}^q(x_i) + \vartheta_{Q_2}^q(x_i)} \right) \right]^{\frac{1}{2}} \end{aligned} \quad (19)$$

$$\begin{aligned} D_{\widetilde{JS_{3D}}}(Q_1, Q_2) &= \frac{1}{n} \sum_{i=1}^n D_{JS_{3D}}(Q_1, Q_2) \\ &= \frac{1}{n} \sum_{i=1}^n \left[\frac{1}{2} \left(\varkappa_{Q_1}^q(x_i) \log \frac{2\varkappa_{Q_1}^q(x_i)}{\varkappa_{Q_1}^q(x_i) + \varkappa_{Q_2}^q(x_i)} + \varkappa_{Q_2}^q(x_i) \log \frac{2\varkappa_{Q_2}^q(x_i)}{\varkappa_{Q_1}^q(x_i) + \varkappa_{Q_2}^q(x_i)} \right. \right. \\ &\quad \left. \left. + \vartheta_{Q_1}^q(x_i) \log \frac{2\vartheta_{Q_1}^q(x_i)}{\vartheta_{Q_1}^q(x_i) + \vartheta_{Q_2}^q(x_i)} + \vartheta_{Q_2}^q(x_i) \log \frac{2\vartheta_{Q_2}^q(x_i)}{\vartheta_{Q_1}^q(x_i) + \vartheta_{Q_2}^q(x_i)} \right. \right. \\ &\quad \left. \left. + \pi_{Q_1}^q(x_i) \log \frac{2\pi_{Q_1}^q(x_i)}{\pi_{Q_1}^q(x_i) + \pi_{Q_2}^q(x_i)} + \pi_{Q_2}^q(x_i) \log \frac{2\pi_{Q_2}^q(x_i)}{\pi_{Q_1}^q(x_i) + \pi_{Q_2}^q(x_i)} \right) \right]^{\frac{1}{2}} \end{aligned} \quad (20)$$

Property 3. Consider the three q -ROFSs Q_1 , Q_2 , and Q_3 in the UOD X , and the properties that $D_{\widetilde{JS_{2D}}}$ and $D_{\widetilde{JS_{3D}}}$ should satisfy are displayed by:

1. Non-degeneracy: $D_{\widetilde{JS_{2D}}}(Q_1, Q_2) = 0$, and $D_{\widetilde{JS_{3D}}}(Q_1, Q_2) = 0$, if and only if $Q_1 = Q_2$;
2. Symmetry: $D_{\widetilde{JS_{2D}}}(Q_1, Q_2) = D_{\widetilde{JS_{2D}}}(Q_2, Q_1)$, and $D_{\widetilde{JS_{3D}}}(Q_1, Q_2) = D_{\widetilde{JS_{3D}}}(Q_2, Q_1)$;
3. Boundedness: $0 \leq D_{\widetilde{JS_{2D}}}(Q_1, Q_2) \leq 1$, and $0 \leq D_{\widetilde{JS_{3D}}}(Q_1, Q_2) \leq 1$;
4. Triangular inequality: $D_{\widetilde{JS_{2D}}}(Q_3, Q_1) + D_{\widetilde{JS_{2D}}}(Q_1, Q_2) \geq D_{\widetilde{JS_{2D}}}(Q_3, Q_2)$, and $D_{\widetilde{JS_{3D}}}(Q_3, Q_1) + D_{\widetilde{JS_{3D}}}(Q_1, Q_2) \geq D_{\widetilde{JS_{3D}}}(Q_3, Q_2)$.

Proof. The proof of Property 3 is similar to Property 1. $\square$

4. Numerical Examples

In the following section, we illustrate the properties of the two novel distance measures with some examples.

Example 1. Given the three q -ROFSs Q_1 , Q_2 , and Q_3 in $X = \{x_1, x_2\}$, they are listed as follows:

$$\begin{aligned} Q_1 &= \{\langle x_1, 0.7, 0.5 \rangle, \langle x_2, 0.4, 0.8 \rangle\}; \\ Q_2 &= \{\langle x_1, 0.7, 0.5 \rangle, \langle x_2, 0.4, 0.8 \rangle\}; \\ Q_3 &= \{\langle x_1, 0.48, 0.74 \rangle, \langle x_2, 0.82, 0.6 \rangle\}. \end{aligned}$$

If $q = 3$, according to Equations (19) and (20), the distance between the q -ROFSs Q_1 , Q_2 , and Q_3 can be calculated as:

The values of $D_{\widetilde{JS_{2D}}}$ are depicted as follows:

$$\begin{aligned} D_{\widetilde{JS_{2D}}}(Q_1, Q_2) &= 0, & D_{\widetilde{JS_{2D}}}(Q_2, Q_1) &= 0 \\ D_{\widetilde{JS_{2D}}}(Q_1, Q_3) &= 0.3851, & D_{\widetilde{JS_{2D}}}(Q_3, Q_1) &= 0.3851 \\ D_{\widetilde{JS_{2D}}}(Q_2, Q_3) &= 0.3851, & D_{\widetilde{JS_{2D}}}(Q_3, Q_2) &= 0.3851 \end{aligned}$$

The values of $D_{\widetilde{JS_3D}}$ are depicted as follows:

$$\begin{aligned} D_{\widetilde{JS_3D}}(Q_1, Q_2) &= 0, & D_{\widetilde{JS_3D}}(Q_2, Q_1) &= 0 \\ D_{\widetilde{JS_3D}}(Q_1, Q_3) &= 0.3967, & D_{\widetilde{JS_3D}}(Q_3, Q_1) &= 0.3967 \\ D_{\widetilde{JS_3D}}(Q_2, Q_3) &= 0.3967, & D_{\widetilde{JS_3D}}(Q_3, Q_2) &= 0.3967 \end{aligned}$$

We can verify that the proposed measures satisfy the first two properties according to the calculated results.

Example 2. There are three q -ROFSs Q_1 , Q_2 , and Q_3 in $X = \{x_1, x_2\}$, which are expressed as follows:

$$\begin{aligned} Q_1 &= \{\langle x_1, 0.7, 0.5 \rangle, \langle x_2, 0.7, 0.4 \rangle\}; \\ Q_2 &= \{\langle x_1, 0.6, 0.4 \rangle, \langle x_2, 0.8, 0.5 \rangle\}; \\ Q_3 &= \{\langle x_1, 0.9, 0.3 \rangle, \langle x_2, 0.8, 0.2 \rangle\}; \end{aligned}$$

If $q = 3$, the values of the $D_{\widetilde{JS_2D}}(Q_3, Q_1)$ and $D_{\widetilde{JS_3D}}(Q_3, Q_1)$ distance measures are depicted as follows:

$$\begin{aligned} D_{\widetilde{JS_2D}}(Q_3, Q_1) &= 0.2244, D_{\widetilde{JS_2D}}(Q_1, Q_2) = 0.1361, D_{\widetilde{JS_2D}}(Q_3, Q_2) = 0.2728 \\ D_{\widetilde{JS_3D}}(Q_3, Q_1) &= 0.2625, D_{\widetilde{JS_3D}}(Q_1, Q_2) = 0.1830, D_{\widetilde{JS_3D}}(Q_3, Q_2) = 0.3362 \end{aligned}$$

Then, it is calculated that

$$\begin{aligned} D_{\widetilde{JS_3D}}(Q_3, Q_1) + D_{\widetilde{JS_3D}}(Q_1, Q_2) &= 0.5455 \\ D_{\widetilde{JS_2D}}(Q_3, Q_1) + D_{\widetilde{JS_2D}}(Q_1, Q_2) &= 0.3605 \end{aligned}$$

Thereby, this result verifies the property:

$$D_{JS}(Q_3, Q_1) + D_{JS}(Q_1, Q_2) \geq D_{JS}(Q_3, Q_2)$$

Example 3. Consider two q -ROFSs Q_1 and Q_2 in X , where

$$Q_1 = \{\langle x, \varkappa, \vartheta \rangle\}; Q_2 = \{\langle x, \vartheta, \varkappa \rangle\}$$

The range of the parameters $\varkappa$ and ϑ are $[0, 1]$ and meet the requirement that $0 \leq \varkappa^q + \vartheta^q \leq 1$. Equation (19) is used to compute the distance between Q_1 and Q_2 . (Assume $q = 3$.)

Figure 1a shows the variations of $\varkappa$ and ϑ . Figure 1b demonstrates that when $\varkappa$ and ϑ vary from 0 to 1, the values of $D_{\widetilde{JS_2D}}$ vary within $[0, 1]$. When $\varkappa = \vartheta$, the distance between Q_1 and Q_2 is 0. Moreover, when $\varkappa = 0, \vartheta = 1$ (or $\varkappa = 1, \vartheta = 0$), $D_{\widetilde{JS_2D}}$ has a maximum of 1. With these results, we can verify the property of the boundedness of $D_{\widetilde{JS_2D}}$.

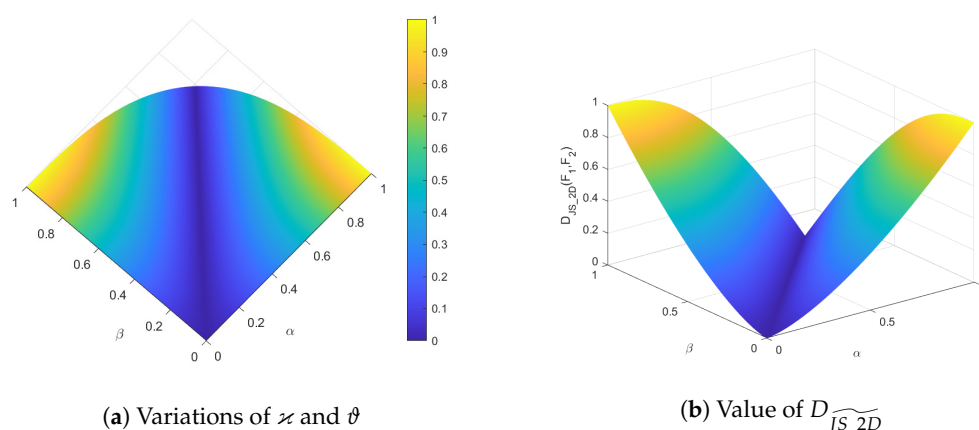


Figure 1. $D_{\widetilde{JS_2D}}$ distance measure between Q_1 and Q_2 .

Example 4. Let Q_1 and Q_2 be two q -ROFSs in X . To compare the experimental results of the proposed distance measures with the existing measures, two cases of q -ROFSs are employed, and the findings are presented in Table 1 (suppose $q = 4$).

From Table 1, we find that Case 1 and Case 2 are two different q -ROFSs. However, the values calculated by D_1 , D_2 , D_3 , D_8 , and D_{10} are equal, which indicates that these existing distance measures will lead to counter-intuitive results. Additionally, D_4 , D_5 , D_6 , D_7 , D_9 , D_{11} , and our proposed distance measures $D_{\widetilde{JS_2D}}$ and $D_{\widetilde{JS_3D}}$ can generate accurate results. This suggests that our proposed measures demonstrate remarkable efficacy in discriminating between distinct q -ROFSs.

Table 1. Comparisons of various similarity measures in Example 4.

Distance	Case 1	Case 2
	$Q_1 = \langle 0.2, 0.4 \rangle$ $Q_2 = \langle 0.5, 0.2 \rangle$	$Q_1 = \langle 0.2, 0.4 \rangle$ $Q_2 = \langle 0.5, 0.4 \rangle$
D_1	0.0609	0.0609
D_2	0.1148	0.1148
D_3	0.1148	0.1148
D_4	0.9637	0.6913
D_5	0.0800	0.0587
D_6	0.9121	0.5640
D_7	0.4378	0.2427
D_8	0.0914	0.0914
D_9	0.0646	0.0467
D_{10}	0.0914	0.0914
D_{11}	0.0913	0.0902
$D_{\widetilde{JS_2D}}$	0.1894	0.1633
$D_{\widetilde{JS_3D}}$	0.1900	0.1654

5. Applications

In this section, we will present two examples involving the developed distance measures within a q -rung orthopair fuzzy environment. The measures introduced in this study are utilized in pattern recognition and multicriteria decision-making, showcasing the efficiency of these measures.

5.1. Pattern Recognition

Problem description: Given a finite UOD $X = \{x_1, x_2, \dots, x_n\}$. There are k patterns $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k\}$. Every pattern is depicted by a q -ROFS denoted as $\mathcal{P}_j = \{(x_i, \kappa_{\mathcal{P}_j}(x_i), \vartheta_{\mathcal{P}_j}(x_i))\}$ ($j = 1, 2, \dots, k$). For m test samples, $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m\}$ are represented by q -ROFSs as $\mathcal{S}_l = \{(x_i, \kappa_{\mathcal{S}_l}(x_i), \vartheta_{\mathcal{S}_l}(x_i))\}$ ($l = 1, 2, \dots, m$). The objective is to

classify the test sample according to the designated pattern, ensuring accurate categorization.

Step 1: We use the suggested measures for measuring the distance $D_{\tilde{JS}}(\mathcal{P}_j, S_l)$ between the known pattern $\mathcal{P}_j$ and the test sample S_l .

Step 2: The minimum distance $D_{\tilde{JS}}(\mathcal{P}_j, S_l)$ among all the calculated distances $D_{\tilde{JS}}(\mathcal{P}_j, S_l)$ between the pattern $\mathcal{P}_j$ and the sample S_l will be selected by using the equation below:

$$D_{\tilde{JS}}(\mathcal{P}_j, S_l) = \min_{1 \leq j \leq k} D_{\tilde{JS}}(\mathcal{P}_j, S_l) \quad (21)$$

Step 3: The result of classifying the sample S_k is determined as follows:

$$j = \operatorname{argmin}_{1 \leq j \leq k} \{D_{\tilde{JS}}(\mathcal{P}_j, S_l)\}, \mathcal{P}_j \rightarrow S_l \quad (22)$$

The pseudocode of the proposed algorithm is shown in Algorithm 1.

Algorithm 1 Pattern classification algorithm.

Require: $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_j, \dots, \mathcal{P}_t\}$, $\mathcal{S} = \{S_1, S_2, \dots, S_k, \dots, S_m\}$

Ensure: Classification of sample S_k

```

1: for  $j = 1; j \leq t$  do
2:   for  $k = 1; k \leq m$  do
3:     Compute the distance  $D_{\tilde{JS}}(\mathcal{P}_j, S_k)$  using Equations (19) and (20);
4:   end for
5: end for
6: for  $k = 1; k \leq m$  do
7:   Choose the smallest distance  $D_{\tilde{JS}}(\mathcal{P}_j, S_k)$  using Equation (21);
8: end for
9: for  $k = 1; k \leq m$  do
10:  Classify the sample  $S_k$  according to Equation (22);
11: end for

```

Example 5. Assume that Q_1, Q_2, Q_3, Q_4 are four known patterns in $Z = \{z_1, z_2, z_3\}$. Each pattern is represented by a q -ROFS as $Q_j = \{\langle z_i, \kappa_{Q_j}(z_i), \vartheta_{Q_j}(z_i) \rangle\} (j = 1, 2, 3, 4)$ and the sample Q is expressed as $Q = \{\langle z_i, \kappa_Q(z_i), \vartheta_Q(z_i) \rangle\}$. The purpose is to categorize the sample Q based on the known patterns. The distance between Q and Q_j computed by different distance measures is displayed in Table 2, where $q = 3$.

Aiming to probe deeply into the flexibility and sensitivity characteristics of the parameter q , we choose a different value q to measure the distance between Q_j and Q , and the result is shown in Table 3.

$$\begin{aligned}
Q_1 &= \{(z_1, 0.32, 0.36), (z_2, 0.41, 0.47), (z_3, 0.54, 0.48)\} \\
Q_2 &= \{(z_1, 0.41, 0.43), (z_2, 0.52, 0.51), (z_3, 0.60, 0.32)\} \\
Q_3 &= \{(z_1, 0.34, 0.57), (z_2, 0.56, 0.61), (z_3, 0.86, 0.71)\} \\
Q_4 &= \{(z_1, 0.23, 0.11), (z_2, 0.31, 0.30), (z_3, 0.45, 0.35)\} \\
Q &= \{(z_1, 0.25, 0.22), (z_2, 0.46, 0.34), (z_3, 0.47, 0.54)\}
\end{aligned}$$

Step 1: Different distance measures are employed to compute the distances between Q and Q_j . Table 2 shows the results.

Step 2: The minimum distances between Q and Q_j based on $D_{\widetilde{JS_2D}}$ and $D_{\widetilde{JS_3D}}$ are depicted as follows:

$$D_{\widetilde{JS_2D}}(Q_1, Q) = 0.1003$$

$$D_{\widetilde{JS_3D}}(Q_1, Q) = 0.1016$$

Step 3: The classification outcome of Q is determined as follows:

$$Q_1 \rightarrow Q$$

As shown in Table 2, it is obvious that there is a minimum distance between Q_1 and Q calculated by the majority of the q -ROFS distance measures, including the suggested q -ROFS measures. Thus, the sample Q should be assigned to Q_1 . Furthermore, we notice that D_5 and D_9 classify Q as Q_4 . This does not align with our initial expectations. Therefore, the existing measures D_5 and D_9 make it difficult to obtain satisfactory results in practical application.

To assess the impact of altering the parameter q on the outcomes, we incorporate various values into the proposed distance measures. From Tables 3 and 4, we can find that as the value of q rises, the distance between Q_1 and Q remains consistently the smallest, which indicates that the sample Q should be allocated to the known pattern Q_1 . Hence, we conclude that the distance measures we introduced are capable of achieving identical results as the existing measures and avoiding counter-intuitive results in the application of pattern recognition.

Table 2. Compare with various distance measures for Example 5.

Method	(Q_1, Q)	(Q_2, Q)	(Q_3, Q)	(Q_4, Q)	Classification
D_1	0.0514	0.0956	0.2982	0.0638	Q_1
D_2	0.0975	0.1738	0.4360	0.1169	Q_1
D_3	0.0978	0.1746	0.4593	0.1200	Q_1
D_4	0.4141	0.6233	0.7383	0.5184	Q_1
D_5	0.0789	0.1483	0.3285	0.0699	Q_4
D_6	0.1030	0.1441	0.1750	0.1084	Q_1
D_7	0.0379	0.0586	0.0719	0.0434	Q_1
D_8	0.0775	0.1395	0.3721	0.0936	Q_1
D_9	0.0624	0.1214	0.2637	0.0554	Q_4
D_{10}	0.0852	0.1742	0.3955	0.0998	Q_1
D_{11}	0.0845	0.1766	0.4316	0.1033	Q_1
$D_{\widetilde{JS_2D}}$	0.1003	0.1835	0.3140	0.1119	Q_1
$D_{\widetilde{JS_3D}}$	0.1016	0.1907	0.4302	0.1174	Q_1

Table 3. $D_{\widetilde{JS_2D}}$ between Q and Q_j with different q value in Example 5.

q	(Q_1, Q)	(Q_2, Q)	(Q_3, Q)	(Q_4, Q)	Classification
3	0.1003	0.1835	0.3140	0.1119	Q_1
4	0.0831	0.1544	0.2999	0.0894	Q_1
5	0.0654	0.1233	0.2743	0.0691	Q_1
6	0.0501	0.0958	0.2458	0.0521	Q_1
7	0.0378	0.0733	0.2182	0.0387	Q_1
8	0.0282	0.0556	0.1930	0.0285	Q_1

Table 4. $D_{\widetilde{JS_3D}}$ between Q and Q_j with different q value in Example 5.

q	(Q_1, Q)	(Q_2, Q)	(Q_3, Q)	(Q_4, Q)	Classification
3	0.1016	0.1907	0.4302	0.1174	Q_1
4	0.0834	0.1568	0.3575	0.0916	Q_1
5	0.0655	0.1242	0.3097	0.0698	Q_1
6	0.0501	0.0961	0.2692	0.0523	Q_1
7	0.0378	0.0734	0.2343	0.0388	Q_1
8	0.0282	0.0556	0.2045	0.0285	Q_1

5.2. Multicriteria Decision-Making

In this study, we showcase the effectiveness of the proposed q -ROFS distance measures in tackling multicriteria decision-making issues that are inherently uncertain and ambiguous.

Problem description: We have m feasible alternatives expressed as $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m\}$ and n criteria expressed as $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_n\}$. The goal is to determine the best alternative.

Step 1: Construct the decision matrix $D = [\langle \varkappa_{jk}, \vartheta_{jk} \rangle]_{n \times m}$ that includes the data pertaining to the available options in relation to the criteria. In this matrix, $\varkappa_{jk}$ signifies the extent to which option $\mathcal{L}_j$ fulfills the criteria $\mathcal{I}_k$, whereas ϑ_{jk} denotes the extent to which it falls short of satisfying the same criteria.

Step 2: Formulate the normalized decision matrix using Equation (23).

$$\langle \varkappa'_{jk}, \vartheta'_{jk} \rangle = \begin{cases} \langle \varkappa_{jk}, \vartheta_{jk} \rangle, & \text{if } \mathcal{I}_k \text{ is benefit criteria} \\ \langle \vartheta_{jk}, \varkappa_{jk} \rangle, & \text{if } \mathcal{I}_k \text{ is cost criteria} \end{cases} \quad (23)$$

Step 3: Determine the q -ROFSs's ideal opinion $\mathcal{P}^* = \langle \varkappa_1^*, \vartheta_1^* \rangle, \langle \varkappa_2^*, \vartheta_2^* \rangle, \dots, \langle \varkappa_5^*, \vartheta_5^* \rangle$ where $\varkappa_k^* = \max_j \varkappa_{jk}$ and $\vartheta_k^* = \min_j \vartheta_{jk}, k = 1, 2, \dots, 5$.

Step 4: Compute the distance of the alternative $\mathcal{L}_j$ from the q -ROFSs's ideal solution $\mathcal{P}^*$ utilizing the proposed q -ROFS distance measures $D_{\widetilde{JS_2D}}$ and $D_{\widetilde{JS_2D}}$.

Step 5: Rank the alternatives in ascending order according to their distance, with the one having the minimum distance deemed as the most optimal alternative.

Example 6. Considering the problem for choosing one among five different houses $\mathcal{T}_j (j = 1, 2, 3, 4, 5)$ to purchase, a prospective homebuyer will take into account the following criteria $\mathcal{M}_k$ when making a purchase decision.

$\mathcal{M}_1$: Ceiling height, $\mathcal{M}_2$: Design, $\mathcal{M}_3$: Location, $\mathcal{M}_4$: Purchase price, and $\mathcal{M}_5$: Ventilation.

Step 1: The details regarding the five houses based on the previously mentioned criteria are represented by q -ROFSs within Table 5.

Step 2: Given that attribute $\mathcal{M}_4$ is the only cost attribute, we utilize Equation (23) to establish the normalized decision matrix, as depicted in Table 6.

Step 3: Assume that all the attributes have the same weight. Then, we will determine the q -ROFSs's ideal opinion $\mathcal{P}^* = \langle \varkappa_1^*, \vartheta_1^* \rangle, \langle \varkappa_2^*, \vartheta_2^* \rangle, \dots, \langle \varkappa_5^*, \vartheta_5^* \rangle$ where $\varkappa_k^* = \max_j \varkappa_{jk}$ and $\vartheta_k^* = \min_j \vartheta_{jk}, k = 1, 2, \dots, 5$. So, the q -ROFSs's ideal solution A^* is expressed as:

$$\mathcal{P}^* = \{ \langle 0.7, 0.5 \rangle, \langle 0.7, 0.3 \rangle, \langle 0.6, 0.6 \rangle, \langle 0.6, 0.4 \rangle, \langle 0.6, 0.3 \rangle \}$$

Step 4: We compute the distance of each alternative from the q -ROFSs's ideal solution with the novel distance measures given in Equations (19) and (20). The result is shown in Table 7 (suppose $q = 3$).

Step 5: In ascending order of distance, the final ranking of alternatives is depicted in Table 8.

Table 8 demonstrates that while the ranking outcomes derived from various distance measures may vary, the optimal alternative consistently remains $\mathcal{T}_2$, and the second most desirable choice is $\mathcal{T}_1$. Thus, we conclude that $\mathcal{T}_2$ is the most feasible option considering all the proposed q -ROFS

distance measures. This demonstrates that our proposed measures can attain consistent outcomes with the established measures when addressing multicriteria decision-making challenges. In other words, the two proposed q -ROFS distance measures are effective for evaluation.

Table 5. Decision matrix.

	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$
$\mathcal{T}_1$	$\langle 0.7, 0.5 \rangle$	$\langle 0.6, 0.8 \rangle$	$\langle 0.4, 0.7 \rangle$	$\langle 0.8, 0.3 \rangle$	$\langle 0.6, 0.5 \rangle$
$\mathcal{T}_2$	$\langle 0.6, 0.6 \rangle$	$\langle 0.7, 0.3 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.4, 0.6 \rangle$	$\langle 0.1, 0.7 \rangle$
$\mathcal{T}_3$	$\langle 0.29, 0.8 \rangle$	$\langle 0.21, 0.9 \rangle$	$\langle 0.6, 0.8 \rangle$	$\langle 0.71, 0.3 \rangle$	$\langle 0.1, 0.3 \rangle$
$\mathcal{T}_4$	$\langle 0.2, 0.9 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.1, 0.6 \rangle$	$\langle 0.5, 0.6 \rangle$	$\langle 0.4, 0.7 \rangle$
$\mathcal{T}_5$	$\langle 0.3, 0.9 \rangle$	$\langle 0.32, 0.9 \rangle$	$\langle 0.4, 0.8 \rangle$	$\langle 0.6, 0.6 \rangle$	$\langle 0.3, 0.4 \rangle$

Table 6. Normalized decision matrix.

	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$
$\mathcal{T}_1$	$\langle 0.7, 0.5 \rangle$	$\langle 0.6, 0.8 \rangle$	$\langle 0.4, 0.7 \rangle$	$\langle 0.3, 0.8 \rangle$	$\langle 0.6, 0.5 \rangle$
$\mathcal{T}_2$	$\langle 0.6, 0.6 \rangle$	$\langle 0.7, 0.3 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.6, 0.4 \rangle$	$\langle 0.1, 0.7 \rangle$
$\mathcal{T}_3$	$\langle 0.29, 0.8 \rangle$	$\langle 0.21, 0.9 \rangle$	$\langle 0.6, 0.8 \rangle$	$\langle 0.3, 0.71 \rangle$	$\langle 0.1, 0.3 \rangle$
$\mathcal{T}_4$	$\langle 0.2, 0.9 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.1, 0.6 \rangle$	$\langle 0.6, 0.5 \rangle$	$\langle 0.4, 0.7 \rangle$
$\mathcal{T}_5$	$\langle 0.3, 0.9 \rangle$	$\langle 0.32, 0.9 \rangle$	$\langle 0.4, 0.8 \rangle$	$\langle 0.6, 0.6 \rangle$	$\langle 0.3, 0.4 \rangle$

Table 7. The distance of each alternative from the q -ROFS ideal solution.

	$(\mathcal{T}_1, \mathcal{P}^*)$	$(\mathcal{T}_2, \mathcal{P}^*)$	$(\mathcal{T}_3, \mathcal{P}^*)$	$(\mathcal{T}_4, \mathcal{P}^*)$	$(\mathcal{T}_5, \mathcal{P}^*)$
D_1	0.2366	0.1302	0.3788	0.3362	0.3886
D_2	0.3429	0.2100	0.5296	0.4711	0.5233
D_3	0.3824	0.2304	0.5494	0.5032	0.5597
D_4	0.5510	0.4658	0.7878	0.7680	0.7696
D_5	0.2768	0.1485	0.4656	0.4260	0.4695
D_6	0.2920	0.2486	0.6159	0.5134	0.4663
D_7	0.1328	0.1166	0.2883	0.2405	0.2136
D_8	0.2933	0.1739	0.4588	0.4075	0.4595
D_9	0.2146	0.1482	0.3464	0.3148	0.3436
D_{10}	0.2933	0.1739	0.4588	0.4075	0.4595
D_{11}	0.2776	0.1697	0.4547	0.4020	0.4571
$D_{\widetilde{JS_2D}}$	0.2526	0.1839	0.4062	0.3876	0.3757
$D_{\widetilde{JS_3D}}$	0.2698	0.1854	0.4340	0.4013	0.4009

Table 8. Ranking of the alternatives.

	Ranking
D_1	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_4 > \mathcal{T}_5 > \mathcal{T}_3$
D_2	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_4 > \mathcal{T}_5 > \mathcal{T}_3$
D_3	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_4 > \mathcal{T}_3 > \mathcal{T}_5$
D_4	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_4 > \mathcal{T}_5 > \mathcal{T}_3$
D_5	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_4 > \mathcal{T}_3 > \mathcal{T}_5$
D_6	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_5 > \mathcal{T}_4 > \mathcal{T}_3$
D_7	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_5 > \mathcal{T}_4 > \mathcal{T}_3$
D_8	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_4 > \mathcal{T}_3 > \mathcal{T}_5$
D_9	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_4 > \mathcal{T}_5 > \mathcal{T}_3$
D_{10}	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_4 > \mathcal{T}_3 > \mathcal{T}_5$
D_{11}	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_4 > \mathcal{T}_3 > \mathcal{T}_5$
$D_{\widetilde{JS_2D}}$	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_5 > \mathcal{T}_4 > \mathcal{T}_3$
$D_{\widetilde{JS_3D}}$	$\mathcal{T}_2 > \mathcal{T}_1 > \mathcal{T}_5 > \mathcal{T}_4 > \mathcal{T}_3$

6. Conclusions

In this study, we proposed two new distance measures based on the Jensen–Shannon divergence, called $D_{\widetilde{JS_{2D}}}$ and $D_{\widetilde{JS_{3D}}}$. The proposed measures comply with the fundamental definition of a distance measure, demonstrating its capability to accurately measure the variance between q -rung orthopair fuzzy sets. Furthermore, the normalized forms for the distance measure are also introduced. Numerical experiments show that the proposed measures can satisfy the axioms for distance measures. Finally, the application of the proposed measures across various fields has yielded accurate outcomes. By comparing with the existing methods, it verifies the feasibility and rationality of the proposed measures. Additionally, as the value of the rung q rises, q -ROFSs can be utilized to encompass a broader spectrum of decision-making information.

It is important to note that the proposed distance measures encounter several challenges. For instance, in practical applications, various attributes or criteria often carry differing levels of importance in decision-making processes. However, the current distance measures treat each attribute equally, which may not reflect the nuanced realities of these decisions. In the future, integrating these measures with weighted information could enhance their practicality and effectiveness in solving real-world problems. Additionally, there is an opportunity to extend the application of these distance measures to more complex decision-making environments.

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