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A Nonlinear Programming Approach to Solving Interval-Valued Intuitionistic Hesitant Noncooperative Fuzzy Matrix Games

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Abstract: Initially, fuzzy sets and intuitionistic fuzzy sets were used to address real-world problems with imprecise data. Eventually, the notion of the hesitant fuzzy set was formulated to handle decision makers' reluctance to accept asymmetric information. However, in certain scenarios, asymmetric information is gathered in terms of a possible range of acceptance and nonacceptance by players rather than specific values. Furthermore, decision makers exhibit some hesitancy regarding this information. In such a situation, all the aforementioned expansions of fuzzy sets are unable to accurately represent the scenario. The purpose of this article is to present asymmetric information situations in which the range of choices takes into account the hesitancy of players in accepting or not accepting information. To illustrate these problems, we develop matrix games that consider the payoffs of interval-valued intuitionistic hesitant fuzzy elements (IIHFEs). Dealing with these types of fuzzy programming problems requires a significant amount of effort. To solve these matrix games, we formulate two interval-valued intuitionistic hesitant fuzzy programming problems. Preserving the hesitant nature of the payoffs to determine the optimal strategies, these two problems are transformed into two nonlinear programming problems. This transformation involves using mathematical operations for IIHFEs. Here, we construct a novel aggregation operator of IIHFEs, viz., min-max operators of IIHFEs. This operator is suitable for applying the developed methodology. The cogency and applicability of the proposed methodology are verified through a numerical example based on the situation of conflict between hackers and defenders to prevent damage to cybersecurity. To validate the superiority of the proposed model along with the computed results, we provide comparisons with the existing models.



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1. Introduction

Neumann and Morgenstern [1] introduced a mathematical tool that can conceive a conflicting situation that arises in reality. This mathematical tool is termed game theory. It deals with mathematical models of strategic interaction between two players. However, due to the ambiguity of decision makers and inadequate information, it becomes difficult to formulate real game situations. To capture this inadequate information, Dubois and Prade [2] used the notion of fuzzy sets in game theory. Several researchers [3,4] addressed and extended this topic through their own opinions.

In 1986, Atanassov [5] brought a little change in the mode of description of a fuzzy set by introducing it in the literature as an intuitionistic fuzzy (I-fuzzy) set. I-fuzzy sets are described by the degree of membership and nonmembership values such that the sum of these values must be bound by 1. Li [6] addressed matrix games with I-fuzzy payoffs by rendering a nonlinear programming approach. Li [7] analyzed management problems in the context of game theory using I-fuzzy sets. Xia [8] implemented Archimedean triangular norms and triangular conorms for solving matrix games with payoffs of interval-valued

I-fuzzy numbers. Brikaa et al. [9] explored an indeterminacy approach for solving zero-sum matrix games having intuitionistic fuzzy goals. Verma and Kumar [10] developed the Ambika method to deal with matrix games in an intuitionistic fuzzy scenario. Verma and Aggarwal [11] developed a generalized method to solve matrix games with payoffs of linguistic intuitionistic fuzzy sets. Dong and Wan [12] developed a solution methodology for solving matrix games with type-2 interval-valued intuitionistic fuzzy payoffs and applied it to energy vehicle industry development. Moreover, in the recent past, numerous researchers explored their views to develop matrix games under several fuzzy environments, such as type-2 fuzzy [13], q-rung orthopair fuzzy [14], and neutrosophic [15–18] environments.

It was observed that the I-fuzzy set was not always sufficient to portray all types of inadequacies in the supplied information. So, aiming to consider hesitation among decision makers, Torra [19] introduced the notion of a hesitant fuzzy set (HFS). The belongingness of elements in such an HFS is assigned by a set of possible membership degrees, which lie in the interval $[0, 1]$. Later, Xia and Xu [20] presented an HFS as the envelope of Atanassov's I-fuzzy set. Xu and Xia [21] developed distance, similarity, and correlation measures for HFSs. Zeng et al. [22] applied the notion of interval-valued hesitant fuzzy sets (IVHFSs) to group decision making. In the recent past, Jana and Roy [23] explored the linguistic Pythagorean hesitant fuzzy matrix game and applied it to multicriteria decision making. Utilizing the lexicographic approach, Seikh et al. [24] designed the solution methodology of a matrix game with hesitant fuzzy payoffs. Yang and Song [25] rendered the solution procedure for matrix games with triangular dual hesitant fuzzy payoffs. Naqvi et al. [26] developed matrix games with payoffs represented by interval-valued hesitant fuzzy linguistic sets.

In 2013, Zhang [27] introduced the notion of IIHFSs, aiming to model a sufficient mathematical tool. It permits hesitancy in terms of both membership and nonmembership degrees. He introduced a series of aggregation operators for IIHFSs and used them to develop group decision-making problems. Broumi and Smarandache [28] proposed some operators for IIHFSs along with their properties. Joshi and Kumar [29] designed multicriteria group decision making in an interval-valued intuitionistic hesitant fuzzy (IIHF) environment by defining the Choquet integral operator. Yuan et al. [30] also studied group decision-making problems in the IIHF environment with a confidence level. Later, Joshi and Kumar [31] explored the entropy of IIHFEs to make the solution to group decision-making problems. Naryanmoorthy et al. [32] developed the VIKOR method in the IIHF environment for ranking the prescribed alternatives in industrial robot selection. De et al. [33] extended the TOPSIS method to solve multiattribute decision-making problems with probabilistic IIHF decisions. Du et al. [34] rendered a data-driven group decision-making method under the IIHF environment. Bhaumik et al. [35] discussed a bimatrix game problem with hesitant interval-valued intuitionistic linguistic payoffs in the light of the prisoner's dilemmatic approach and applied it to portray the human trafficking problem. Apart from these, the notion of IIHFSs is used in different directions of decision-making theory by several authors [36–38].

As IIHFS is a significant extension of the fuzzy set in the literature by its character, it is, therefore, very reasonable and relevant to use IIHFS in game theory. In the literature, it is found that Bhaumik and Roy [39] developed matrix games with IIHFEs for solving management problems. In that discussion, two interval-valued intuitionistic hesitant fuzzy programming models were derived. Later, they followed the usual method for solving the interval-valued linear programming problem. However, the objective functions of the derived problems were in the form of some intervals, which are part of an IIHFE. However, the mathematical operations on IIHFEs do not follow the operational rule of classical intervals. Rather, if we follow the existing operational rule of IIHFEs, the aforementioned bi-objective programming problems should be transformed into two nonlinear programming problems. So, there is a major drawback in the approach of Bhaumik and Roy [39]. This paper aims to obviate such drawbacks and proposes a new methodology for solving matrix games in the IIHF environment.

Although there exist different aggregation operators for IIHFEs, for convenient calculation, we define a novel aggregation operator, viz., the min-max enclosure operator. Then we derive two interval-valued intuitionistic hesitant fuzzy programming problems for solving the proposed game. These are converted into two nonlinear programming models by following a few steps. Finally, to obtain the optimal solutions, the nonlinear programming problems are solved by using Wolfram MATHEMATICA 9.0 software.

The contributions of this paper are listed here:

- A novel aggregation operator is introduced in this study, which is referred to as the min-max enclosure operator. This operator is specifically designed for the aggregation of IIHF information. One of the advantages of the proposed operator is its simplicity, which makes it easy to implement and use. Additionally, the operator has the capability to preserve the characteristics of IIHFEs, ensuring that the important features of the information are not lost during the aggregation process.
- A nonlinear programming approach is delineated in this paper to handle interval-valued intuitionistic hesitant fuzzy programming problems to solve IIHF matrix games. In order to achieve the best possible outcomes, these problems are converted into two distinct crisp nonlinear programming problems.
- Like in classical game theory, it is established that the gain floor of the winning player is less than or equal to the loss ceiling of the defeated player.
- The proposed methodology is illustrated through the problem of cybercrime. The superiority of the proposed method is checked and verified by analyzing a comparative study with two existing models.

The rest of this article is delineated as follows: Some basic definitions associated with IIHFEs along with mathematical operations are recalled in Section 2. In Section 3, a novel aggregation operator for IIHFEs, viz., the min-max enclosure operator, is defined. In Section 4, a matrix game with payoffs of IIHFEs is amplified. A practical example of the conflict between hackers and defenders to stop cybercrime is demonstrated in Section 5. A comparison analysis along with a discussion of results is briefed in Section 6. Finally, the conclusion is drawn in Section 7.

2. Preliminaries

In this section, some primary definitions related to IIHFEs are recalled, which helps us to develop the succeeding discussions.

Definition 1 (IIFS ([40])). A set \tilde{I} can be defined as an IIFS on a universe of discourse Y if \tilde{I} is defined as $\tilde{I} = \{ \langle x, ([m^l, m^u], [n^l, n^u]) \rangle | x \in Y \}$. Here, $[m^l, m^u] \subset [0, 1]$ and $[n^l, n^u] \subset [0, 1]$, and also, $m^u + n^u \leq 1$.

Definition 2 (HFS ([19])). A set \tilde{E} can be defined as an HFS on a universe of discourse Y in terms of a function. The function gives few possible values when applied on the universal set Y . Xia and Xu [20] symbolized an HFS as $\tilde{E} = \{ \langle k, \tilde{q}(k) \rangle | k \in Y \}$ where each $\tilde{q}(k)$ takes some values from the closed interval $[0, 1]$. These $\tilde{q}(k)$ values are termed as hesitant fuzzy elements (HFEs).

Definition 3 (IVHFS ([41])). An interval-valued hesitant fuzzy set (IVHFS) \tilde{E} on Y is defined symbolically as $\tilde{E} = \{ \langle k, \tilde{q}(k) \rangle | k \in Y \}$, where $\tilde{q}(k)$ represents some set of intervals in $[0, 1]$, i.e., $\tilde{q}(k) = \bigcup_{[q^l, q^u] \in \tilde{q}(k)} [q^l, q^u]$. The intervals $\tilde{q}(k)$ are termed as interval-valued hesitant fuzzy elements (IVHFEs), where q^l stands for the lower limit and q^u stands for the upper limit of the interval $[q^l, q^u]$.

Definition 4 (IIHFS ([39])). An interval-valued intuitionistic hesitant fuzzy set (IIHFS) \tilde{E} on Y is defined symbolically as $\tilde{E} = \{ \langle k, \tilde{q}(k) \rangle | k \in Y \}$, where $\tilde{q}(k) = \langle \tilde{q}_\mu(k); \tilde{q}_\nu(k) \rangle = \langle \bigcup_{\Delta} [q_\mu^l, q_\mu^u]; \bigcup_{\Delta} [q_\nu^l, q_\nu^u] \rangle$ represents the set of intervals in terms of degree of membership and

nonmembership values. Each $\tilde{q}(k) = \langle \tilde{q}_\mu(k); \tilde{q}_\nu(k) \rangle$ is termed as the interval-valued intuitionistic hesitant fuzzy elements (IIHFEs). The interval $[q_\mu^l, q_\mu^u]$ takes the value from $[0, 1]$ for the membership degree, whereas the interval $[q_\nu^l, q_\nu^u]$ takes the value from $[0, 1]$ for the nonmembership degree of the IIHFE $\tilde{q}(k)$, and \cup_Δ denotes the union of all intervals.

Definition 5 (Mathematical operations on IIHFEs ([39])). Let \tilde{q} , \tilde{q}_1 , and \tilde{q}_2 be three IIHFEs, having the form $\tilde{q} = \langle \cup_\Delta[q_\mu^l, q_\mu^u]; \cup_\Delta[q_\nu^l, q_\nu^u] \rangle$, $\tilde{q}_1 = \langle \cup_\Delta[q_{1\mu}^l, q_{1\mu}^u]; \cup_\Delta[q_{1\nu}^l, q_{1\nu}^u] \rangle$, and $\tilde{q}_2 = \langle \cup_\Delta[q_{2\mu}^l, q_{2\mu}^u]; \cup_\Delta[q_{2\nu}^l, q_{2\nu}^u] \rangle$. Then the mathematical operations on IIHFEs are performed as

- $\tilde{q}_1 \oplus \tilde{q}_2 = \langle \cup_\Delta[q_{1\mu}^l + q_{2\mu}^l - q_{1\mu}^l q_{2\mu}^l, q_{1\mu}^u + q_{2\mu}^u - q_{1\mu}^u q_{2\mu}^u]; \cup_\Delta[q_{1\nu}^l q_{2\nu}^l, q_{1\nu}^u q_{2\nu}^u] \rangle$,
- $\tau \tilde{q} = \langle \cup_\Delta[1 - (1 - q_\mu^l)^\tau, 1 - (1 - q_\mu^u)^\tau]; \cup_\Delta[(q_\nu^l)^\tau, (q_\nu^u)^\tau] \rangle$, for any positive scalar τ .

Definition 6 (Extension principle of IIHFEs ([39])). Let Ω be the collection of m numbers of IIHFEs, i.e., $\Omega = \{\tilde{q}_p : p = 1, 2, \dots, m\}$, and F be considered as a function on Ω , such that $F : [0, 1]^m \rightarrow [0, 1]$, then F must satisfies the equality $F(\Omega) = \bigcup_{\tilde{q}_p \in \langle \cup_\Delta[q_{p\mu}^l, q_{p\mu}^u]; \cup_\Delta[q_{p\nu}^l, q_{p\nu}^u] \rangle} \{F(\tilde{q}_p)\}$.

Definition 7 (Score function and accuracy function of IIHFEs). Let $\tilde{q} = \langle \cup_\Delta[q_\mu^l, q_\mu^u]; \cup_\Delta[q_\nu^l, q_\nu^u] \rangle$ be an IIHFE. Then the score function and accuracy function of IIHFE \tilde{q} are defined as follows:

$$\begin{aligned} \text{Sc}(\tilde{q}) &= \left| \frac{1}{2}[(\min q_\mu^l + \max q_\mu^u) - (\min q_\nu^l + \max q_\nu^u)] \right| \\ \text{Ac}(\tilde{q}) &= \frac{1}{2}[(\min q_\mu^l + \max q_\mu^u) + (\min q_\nu^l + \max q_\nu^u)]. \end{aligned}$$

For example, consider an IIHFE, $\tilde{q} = \langle [0.55, 0.60], [0.55, 0.70]; [0.50, 0.60], [0.45, 0.60] \rangle$, then the score function and accuracy function of \tilde{q} are calculated as $\text{Sc}(\tilde{q}) = \left| \frac{1}{2}[(0.55 + 0.70) - (0.45 + 0.60)] \right| = 0.1$ and $\text{Ac}(\tilde{q}) = \frac{1}{2}[(0.55 + 0.70) + (0.45 + 0.60)] = 1.15$.

It is worth noting that if the lower element and the upper element of the intervals of IIHFE take the same value, then the accuracy function and the score function of an IIHFE transforms into the same value of an intuitionistic fuzzy number. Furthermore, the accuracy function and score function of that intuitionistic fuzzy number are the same, as shown by Xu [42].

Definition 8 (Ranking of IIHFEs). Let \tilde{q}_a and \tilde{q}_b be two IIHFEs. Then, based on Definition 7, we may define the ranking of the IIHFEs as

- If $\text{Sc}(\tilde{q}_a) > \text{Sc}(\tilde{q}_b)$, then $\tilde{q}_a >_{\text{IIHF}} \tilde{q}_b$.
- If $\text{Sc}(\tilde{q}_a) < \text{Sc}(\tilde{q}_b)$, then $\tilde{q}_a <_{\text{IIHF}} \tilde{q}_b$.
- If $\text{Sc}(\tilde{q}_a) = \text{Sc}(\tilde{q}_b)$, then
 - If $\text{Ac}(\tilde{q}_a) < \text{Ac}(\tilde{q}_b)$, then $\tilde{q}_a <_{\text{IIHF}} \tilde{q}_b$.
 - If $\text{Ac}(\tilde{q}_a) > \text{Ac}(\tilde{q}_b)$, then $\tilde{q}_a >_{\text{IIHF}} \tilde{q}_b$.
 - If $\text{Ac}(\tilde{q}_a) = \text{Ac}(\tilde{q}_b)$, then $\tilde{q}_a =_{\text{IIHF}} \tilde{q}_b$.

Here, ' $>_{\text{IIHF}}$ ', ' $<_{\text{IIHF}}$ ', and ' $=_{\text{IIHF}}$ ' are used in the sense of the usual 'greater than', 'less than', and 'equal to' in an IIHF environment.

Let us take two examples here to show the ranking of IIHFEs:

- Consider two IIHFEs, $\tilde{q}_1 = \langle [0.55, 0.60], [0.55, 0.70]; [0.50, 0.60], [0.45, 0.60] \rangle$ and $\tilde{q}_2 = \langle [0.65, 0.70]; [0.35, 0.45], [0.35, 0.50] \rangle$. Then $\text{Sc}(\tilde{q}_1) = 0.1$, and $\text{Sc}(\tilde{q}_2) = 0.25$, which concludes that $\tilde{q}_2 >_{\text{IIHF}} \tilde{q}_1$.
- Suppose $\tilde{q}_3 = \langle [0.30, 0.40], [0.30, 0.45]; [0.05, 0.15], [0.05, 0.10] \rangle$, and $\tilde{q}_4 = \langle [0.30, 0.35]; [0.30, 0.40], [0.05, 0.08], [0.06, 0.10] \rangle$. Then $\text{Sc}(\tilde{q}_3) = 0.275$, and $\text{Sc}(\tilde{q}_4) = 0.275$, which

$Sc(\tilde{q}_3) = Sc(\tilde{q}_4)$. So, in this case, we cannot assign the ranking based on the score function, and we have to calculate the accuracy function of these two IIHFEs. Now $Ac(\tilde{q}_3) = 0.475$ and $Ac(\tilde{q}_4) = 0.425$, which concludes $\tilde{q}_2 =_{IIHF} \tilde{q}_1$.

In various decision-making problems, often we need to aggregate the information. There are plenty of aggregation operators for IIHFEs available in the literature. To preserve the hesitant character, the following section introduces a novel aggregation operator for IIHFEs.

3. New Aggregation Operator

In order to aggregate the IIHFEs here, we define a new aggregation operator of IIHFEs. The new aggregation operator is set up based on the notion of the extension principle of IIHFEs and the mathematical operations on IIHFEs, depicted in Definitions 5 and 6, respectively. This novel aggregation operator is termed as an interval-valued intuitionistic hesitant fuzzy weighted min-max enclosure aggregation operator (IIHFWMEAO), or min-max enclosure aggregation operator in short.

Definition 9 (Min-max enclosure of IIHFEs). Let us consider an IIHFE $\tilde{q} = \langle [q_{1\mu}^l, q_{1\mu}^u], [q_{2\mu}^l, q_{2\mu}^u], \dots, [q_{m\mu}^l, q_{m\mu}^u]; [q_{1\nu}^l, q_{1\nu}^u], [q_{2\nu}^l, q_{2\nu}^u], \dots, [q_{n\nu}^l, q_{n\nu}^u] \rangle$. It is worth noting that m and n are two integers that may or may not exert the same values. Let us construct an IIHFE \tilde{q}_E as $\tilde{q}_E = \langle [\min_{1 \leq p \leq m} q_{p\mu}^l, \max_{1 \leq p \leq m} q_{p\mu}^u]; [\min_{1 \leq q \leq n} q_{q\nu}^l, \max_{1 \leq q \leq n} q_{q\nu}^u] \rangle$, where $\min_{1 \leq p \leq m} q_{p\mu}^l \leq q_{p\mu}^l \leq q_{p\mu}^u \leq \max_{1 \leq p \leq m} q_{p\mu}^u, \forall p$, and $\min_{1 \leq q \leq n} q_{q\nu}^l \leq q_{q\nu}^l \leq q_{q\nu}^u \leq \max_{1 \leq q \leq n} q_{q\nu}^u, \forall q$. This \tilde{q}_E is termed as the min-max enclosure of the IIHFE \tilde{q} .

Based on Definition 9, we can define the min-max enclosure operator of IIHFEs.

Definition 10 (Min-max enclosure operator). Suppose $\tilde{q}, \tilde{q}_1, \tilde{q}_2 \in \Omega$ are three IIHFEs having the forms $\tilde{q} = \langle \cup_{\Delta}[q_{1\mu}^l, q_{1\mu}^u]; \cup_{\Delta}[q_{1\nu}^l, q_{1\nu}^u] \rangle$, $\tilde{q}_1 = \langle \cup_{\Delta}[q_{1\mu}^l, q_{1\mu}^u]; \cup_{\Delta}[q_{1\nu}^l, q_{1\nu}^u] \rangle$, and $\tilde{q}_2 = \langle \cup_{\Delta}[q_{2\mu}^l, q_{2\mu}^u]; \cup_{\Delta}[q_{2\nu}^l, q_{2\nu}^u] \rangle$. Then the min-max enclosure operators of IIHFEs are defined as follows:

Min-max enclosure sum:

$$\tilde{q}_1 + \tilde{q}_2 = \langle [1 - (1 - \min q_{1\mu}^l)(1 - \min q_{2\mu}^l), 1 - (1 - \max q_{1\mu}^u)(1 - \max q_{2\mu}^u)]; [(\min q_{1\nu}^l)(\min q_{2\nu}^l), (\max q_{1\nu}^u)(\max q_{2\nu}^u)] \rangle.$$

Min-max enclosure product:

$$\tilde{q}_1 \times \tilde{q}_2 = \langle [(\min q_{1\mu}^l)(\min q_{2\mu}^l), (\max q_{1\mu}^u)(\max q_{2\mu}^u)]; [1 - (1 - \min q_{1\nu}^l)(1 - \min q_{2\nu}^l), 1 - (1 - \max q_{1\nu}^u)(1 - \max q_{2\nu}^u)] \rangle.$$

Min-max enclosure scalar product: For any scalar $\delta > 0$,

$$\delta \tilde{q} = \langle [1 - (1 - \min q_{1\mu}^l)^\delta, 1 - (1 - \max q_{1\mu}^u)^\delta]; [(\min q_{1\nu}^l)^\delta, (\max q_{1\nu}^u)^\delta] \rangle.$$

Min-max enclosure index: For any scalar $\delta > 0$,

$$(\tilde{q})^\delta = \langle [(\min q_{1\mu}^l)^\delta, (\max q_{1\mu}^u)^\delta]; [1 - (1 - \min q_{1\nu}^l)^\delta, 1 - (1 - \max q_{1\nu}^u)^\delta] \rangle.$$

Let us take two IIHFEs $\tilde{q}_1 = \langle [0.60, 0.70]; [0.30, 0.40], [0.35, 0.50] \rangle$, $\tilde{q}_2 = \langle [0.40, 0.50], [0.45, 0.50]; [0.25, 0.30], [0.35, 0.40] \rangle$ and a scalar $\delta = 0.8$. Hence, the min-max enclosure sum, min-max enclosure product, min-max enclosure scalar product, and min-max enclosure index of these two IIHFEs are calculated as follows:

$$\begin{aligned}\tilde{q}_1 + \tilde{q}_2 &= \langle [0.60, 0.70]; [0.30, 0.40], [0.35, 0.50] \rangle + \langle [0.40, 0.50], [0.45, 0.50]; [0.25, 0.30], [0.35, 0.40] \rangle \\ &= \langle [0.76, 0.85]; [0.07, 0.20] \rangle \\ \tilde{q}_1 \times \tilde{q}_2 &= \langle [0.60, 0.70]; [0.30, 0.40], [0.35, 0.50] \rangle \times \langle [0.40, 0.50], [0.45, 0.50]; [0.25, 0.30], [0.35, 0.40] \rangle \\ &= \langle [0.25, 0.35]; [0.47, 0.70] \rangle \\ 0.8\tilde{q}_1 &= 0.8\langle [0.60, 0.70]; [0.30, 0.40], [0.35, 0.50] \rangle = \langle [0.52, 0.62]; [0.38, 0.57] \rangle \\ (\tilde{q}_1)^{0.8} &= (\langle [0.60, 0.70]; [0.30, 0.40], [0.35, 0.50] \rangle)^{0.8} = \langle [0.66, 0.75]; [0.25, 0.43] \rangle\end{aligned}$$

Definition 11 (Min-max enclosure aggregation operator). Let $\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_m \in \Omega$ be a collection of m number of IIHFEs and $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ be the set of weight vectors, such that each ω_p is associated with each \tilde{q}_p ($t = 1, 2, \dots, m$) with $\sum_{p=1}^m \omega_p = 1$. Hence, interval-valued intuitionistic hesitant fuzzy weighted min-max enclosure aggregation operator (IIHFWMEAO) is defined as a mapping from Ω^m to Ω , such that

$$IIHFWMEAO(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_m) = \sum_{p=1}^m \omega_p \tilde{q}_p.$$

Theorem 1. Let \tilde{q}_p ($p = 1, 2, \dots, m$) be a collection of m number of IIHFEs and $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ be the set of weight vectors. Suppose each ω_p is associated with each \tilde{q}_p for $p = 1, 2, \dots, m$, with $\sum_{p=1}^m \omega_p = 1$. Then the aggregation operator IIHFWMEAO is calculated as

$$\begin{aligned}IIHFWMEAO(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_m) &= \sum_{p=1}^m \omega_p \tilde{q}_p \\ &= \left\langle \left[1 - \prod_{p=1}^m (1 - \underline{q}_{p\mu}^l)^{\omega_p}, 1 - \prod_{p=1}^m (1 - \bar{q}_{p\mu}^u)^{\omega_p} \right]; \left[\prod_{p=1}^m (\underline{q}_{p\nu}^l)^{\omega_p}, \prod_{p=1}^m (\bar{q}_{p\nu}^u)^{\omega_p} \right] \right\rangle\end{aligned}\quad (1)$$

where $\underline{q}_{p\mu}^l = \min q_{p\mu}^l$, $\bar{q}_{p\mu}^u = \max q_{p\mu}^u$, $\underline{q}_{p\nu}^l = \min q_{p\nu}^l$ and $\bar{q}_{p\nu}^u = \max q_{p\nu}^u$.

Proof. Let $\tilde{q}_1, \tilde{q}_2 \in \Omega$ be two IIHFEs. ω_1 and ω_2 are the assigned weights associated with these two IIHFEs, respectively. Then by Definition 10,

$$\begin{aligned}\omega_1 \tilde{q}_1 + \omega_2 \tilde{q}_2 &= \omega_1 \left\{ \left\langle \bigcup_{\Delta} [q_{1\mu}^l, q_{1\mu}^u]; \bigcup_{\Delta} [q_{1\nu}^l, q_{1\nu}^u] \right\rangle \right\} + \omega_2 \left\{ \left\langle \bigcup_{\Delta} [q_{2\mu}^l, q_{2\mu}^u]; \bigcup_{\Delta} [q_{2\nu}^l, q_{2\nu}^u] \right\rangle \right\} \\ &= \left\{ \left\langle \left[1 - (1 - \min q_{1\mu}^l)^{\omega_1}, 1 - (1 - \max q_{1\mu}^u)^{\omega_1} \right]; \left[(\min q_{1\nu}^l)^{\omega_1}, (\max q_{1\nu}^u)^{\omega_1} \right] \right\rangle \right\} + \\ &\quad \left\{ \left\langle \left[1 - (1 - \min q_{2\mu}^l)^{\omega_2}, 1 - (1 - \max q_{2\mu}^u)^{\omega_2} \right]; \left[(\min q_{2\nu}^l)^{\omega_2}, (\max q_{2\nu}^u)^{\omega_2} \right] \right\rangle \right\} \\ &= \left\langle \left[1 - (1 - \min q_{1\mu}^l)^{\omega_1} (1 - \min q_{2\mu}^l)^{\omega_2}, 1 - (1 - \max q_{1\mu}^u)^{\omega_1} (1 - \max q_{2\mu}^u)^{\omega_2} \right]; \right. \\ &\quad \left. \left[(\min q_{1\nu}^l)^{\omega_1} (\min q_{2\nu}^l)^{\omega_2}, (\max q_{1\nu}^u)^{\omega_1} (\max q_{2\nu}^u)^{\omega_2} \right] \right\rangle\end{aligned}$$

which shows that the statement assigned in Equation (1) holds for $p = 2$.

We now assume that the statement holds for $p = t$ (let t be a natural number less than m), i.e.,

$$\sum_{p=1}^t \omega_p \tilde{q}_p = \left\langle \left[1 - \prod_{p=1}^t (1 - \min q_{p\mu}^l)^{\omega_p}, 1 - \prod_{p=1}^t (1 - \max q_{p\mu}^u)^{\omega_p} \right]; \left[\prod_{p=1}^t (\min q_{p\nu}^l)^{\omega_p}, \prod_{p=1}^t (\max q_{p\nu}^u)^{\omega_p} \right] \right\rangle$$

Now, for $p = t + 1$,

$$\begin{aligned}
\sum_{p=1}^{t+1} \omega_p \tilde{q}_p &= \left\{ \sum_{p=1}^t \omega_p \tilde{q}_p \right\} + \omega_{t+1} \tilde{q}_{t+1} \\
&= \left\{ \left\langle \left[1 - \prod_{p=1}^t (1 - \min q_{p\mu}^l)^{\omega_p}, 1 - \prod_{p=1}^t (1 - \max q_{p\mu}^u)^{\omega_p} \right]; \left[\prod_{p=1}^t (\min q_{p\nu}^l)^{\omega_p}, \prod_{p=1}^t (\max q_{p\nu}^u)^{\omega_p} \right] \right\rangle + \right. \\
&\quad \left. \left\{ \left\langle \left[1 - \left(1 - \min q_{(t+1)\mu}^l \right)^{\omega_{t+1}}, 1 - \left(1 - \max q_{(t+1)\mu}^u \right)^{\omega_{t+1}} \right]; \left[\left(\min q_{(t+1)\nu}^l \right)^{\omega_{t+1}}, \left(\max q_{(t+1)\nu}^u \right)^{\omega_{t+1}} \right] \right\rangle \right\} \\
&= \left\langle \left[1 - \prod_{p=1}^{t+1} (1 - \min q_{p\mu}^l)^{\omega_p}, 1 - \prod_{p=1}^{t+1} (1 - \max q_{p\mu}^u)^{\omega_p} \right]; \left[\prod_{p=1}^{t+1} (\min q_{p\nu}^l)^{\omega_p}, \prod_{p=1}^{t+1} (\max q_{p\nu}^u)^{\omega_p} \right] \right\rangle
\end{aligned}$$

Thus, we conclude that the statement holds for all natural numbers m , using the principle of mathematical induction. Consequently, we have

$$\sum_{p=1}^m \omega_p \tilde{q}_p = \left\langle \left[1 - \prod_{p=1}^m (1 - \underline{q}_{p\mu}^l)^{\omega_p}, 1 - \prod_{p=1}^m (1 - \bar{q}_{p\mu}^l)^{\omega_p} \right]; \left[\prod_{p=1}^m (\underline{q}_{p\nu}^l)^{\omega_p}, \prod_{p=1}^m (\bar{q}_{p\nu}^u)^{\omega_p} \right] \right\rangle$$

where $\underline{q}_{p\mu}^l = \min q_{p\mu}^l$, $\bar{q}_{p\mu}^u = \max q_{p\mu}^u$, $\underline{q}_{p\nu}^l = \min q_{p\nu}^l$, and $\bar{q}_{p\nu}^u = \max q_{p\nu}^u$, which completes the proof. \square

Consider three IIHFEs $\tilde{q}_1 = \langle [0.65, 0.70]; [0.35, 0.45], [0.35, 0.50] \rangle$, $\tilde{q}_2 = \langle [0.40, 0.50], [0.42, 0.55]; [0.25, 0.30], [0.20, 0.40] \rangle$, and $\tilde{q}_3 = \langle [0.35, 0.40], [0.35, 0.50]; [0.50, 0.60], [0.45, 0.60] \rangle$. Furthermore, we assign three weights $\omega_1 = 0.4$, $\omega_2 = 0.2$, and $\omega_3 = 0.4$ corresponding to \tilde{q}_1 , \tilde{q}_2 , and \tilde{q}_3 , respectively. Hence, these three IIHFEs can be aggregated as

$$\begin{aligned}
\tilde{q} &= \omega_1 \tilde{q}_1 + \omega_2 \tilde{q}_2 + \omega_3 \tilde{q}_3 \\
&= 0.4 \langle [0.65, 0.70]; [0.35, 0.45], [0.35, 0.50] \rangle + 0.2 \langle [0.40, 0.50], [0.42, 0.55]; [0.25, 0.30], [0.20, 0.40] \rangle \\
&\quad + 0.4 \langle [0.35, 0.40], [0.35, 0.50]; [0.50, 0.60], [0.45, 0.60] \rangle \\
&= \langle [0.5006, 0.6009]; [0.3460, 0.5144] \rangle
\end{aligned}$$

Let us suppose that \tilde{P} , \tilde{Q} , and \tilde{R} are three sets of IIHFEs. Furthermore, $\tilde{q}_{ap} \in \tilde{P}$, $\tilde{q}_{bp} \in \tilde{Q}$, $\tilde{q}_p \in \tilde{R}$ for $(p = 1, 2, \dots, m)$. It can be easily verified that the proposed aggregation operator satisfies the following properties:

- (i) **Monotonicity:** If $\tilde{q}_{ap} \leq_{IIHF} \tilde{q}_{bp}$, for $p = 1, 2, \dots, m$, then $\sum_{p=1}^m \omega_p \tilde{q}_{ap} \leq_{IIHF} \sum_{p=1}^m \omega_p \tilde{q}_{bp}$.
- (ii) **Idempotency:** $\sum_{p=1}^m \omega_p \tilde{q}_{ap} =_{IIHF} \tilde{q}_p$, if $\tilde{q}_{ap} =_{IIHF} \tilde{q}_p$ for $p = 1, 2, \dots, m$.
- (iii) **Boundedness:** $\min_{1 \leq p \leq m} \tilde{q}_p \leq_{IIHF} \sum_{p=1}^m \omega_p \tilde{q}_p \leq_{IIHF} \max_{1 \leq p \leq m} \tilde{q}_p$.

In light of Definition 7, we may define a new score function and an accuracy function of the aggregated IIHFEs as follows.

Definition 12. The score function and accuracy function based on the min-max enclosure operator of the aggregated IIHFEs are defined as

$$Sc_{IIHFWMEAO} \left(\sum_{p=1}^m \omega_p \tilde{q}_p \right) = \frac{1}{2} \left[\left(2 - \prod_{p=1}^m (1 - \underline{q}_{p\mu}^l)^{\omega_p} + \prod_{p=1}^m (1 - \bar{q}_{p\mu}^l)^{\omega_p} \right) - \left(\prod_{p=1}^m (\underline{q}_{p\nu}^l) + \prod_{p=1}^m (\bar{q}_{p\nu}^u) \right) \right] \quad (2)$$

$$Ac_{IIHFWMEAO} \left(\sum_{p=1}^m \omega_p \tilde{q}_p \right) = \frac{1}{2} \left[\left(2 - \prod_{p=1}^m (1 - \underline{q}_{p\mu}^l)^{\omega_p} + \prod_{p=1}^m (1 - \bar{q}_{p\mu}^l)^{\omega_p} \right) + \left(\prod_{p=1}^m (\underline{q}_{p\nu}^l) + \prod_{p=1}^m (\bar{q}_{p\nu}^u) \right) \right]. \quad (3)$$

4. Matrix Games with IIHFE Payoffs

Suppose two players, viz., X_A and X_B , are involved in a matrix game. Let us assume that the pure strategies for Player X_A are denoted by $S_I = \{\alpha_p | p = 1, 2, \dots, m\}$.

$S_{II} = \{\beta_q | q = 1, 2, \dots, n\}$ represents the same for Player X_B . Furthermore, assume two sets Y and Z with the representations

$$Y = \{\mathbf{y} = (y_1, y_2, \dots, y_m)^T \in \mathbb{R}^{m+} | \sum_{p=1}^m y_p = 1\}$$

and $Z = \{\mathbf{z} = (z_1, z_2, \dots, z_n)^T \in \mathbb{R}^{n+} | \sum_{q=1}^n z_q = 1\}.$

Here, Y and Z are considered as the set of mixed strategies for Players X_A and X_B . \mathbb{R}^{m+} represents the m -dimensional Euclidean space, while \mathbb{R}^{n+} denotes n -dimensional Euclidean space. It is customary to assume that Player X_A is the maximizing player and Player X_B is the minimizing player. In this game, if Player X_A chooses pure strategy $\alpha_p (\in S_I)$ to maximize their gain and simultaneously, Player X_B takes $\beta_q (\in S_{II})$ as their pure strategy to minimize their loss, then the outcome of the game will be $\langle \alpha_p, \beta_q \rangle = \widetilde{q}_{pq}$, the pq^{th} entry of the payoff matrix $\tilde{\Gamma}$. The payoff matrix $\tilde{\Gamma}$ is considered as the outcome of Player X_A . \widetilde{q}_{pq} gathers asymmetric information in a possible range of players' hesitancy in terms of their acceptance and nonacceptance. Hence, \widetilde{q}_{pq} is in the form of an IIHFE, where $\widetilde{q}_{pq} = \langle \cup_{\Delta} [q_{pq\mu}^l, q_{pq\mu}^u]; \cup_{\Delta} [q_{pq\nu}^l, q_{pq\nu}^u] \rangle$. Consequently, the payoff matrix can be symbolized as

$$\tilde{\Gamma} = (\widetilde{q}_{pq})_{m \times n} = \left(\langle \cup_{\Delta} [q_{pq\mu}^l, q_{pq\mu}^u]; \cup_{\Delta} [q_{pq\nu}^l, q_{pq\nu}^u] \rangle \right)_{m \times n}.$$

Thus, we can express the game with IIHFE payoffs as $\tilde{H} = (Y, Z, \tilde{\Gamma})$.

If Player X_A takes \mathbf{y} and Player X_B takes \mathbf{z} as their mixed strategies, then the expected payoff of the maximizing player will be

$$\tilde{E}(\mathbf{y}, \mathbf{z}) = \mathbf{y}^T \tilde{\Gamma} \mathbf{z} = \sum_{p=1}^m \sum_{q=1}^n y_p \left(\langle \cup_{\Delta} [q_{pq\mu}^l, q_{pq\mu}^u]; \cup_{\Delta} [q_{pq\nu}^l, q_{pq\nu}^u] \rangle \right) z_q.$$

Using Theorem 1, the aggregated expected payoff can be obtained as

$$\begin{aligned} \tilde{E}(\mathbf{y}, \mathbf{z}) = & \left\langle \left[1 - \prod_{p=1}^m \prod_{q=1}^n (1 - q_{pq\mu}^l)^{y_p z_q}, 1 - \prod_{p=1}^m \prod_{q=1}^n (1 - \bar{q}_{pq\mu}^u)^{y_p z_q} \right]; \right. \\ & \left. \left[\prod_{p=1}^m \prod_{q=1}^n (q_{pq\nu}^l)^{y_p z_q}, \prod_{p=1}^m \prod_{q=1}^n (\bar{q}_{pq\nu}^u)^{y_p z_q} \right] \right\rangle. \end{aligned} \quad (4)$$

Motivated by the work of Li [6], we may conceptualize the solution of the matrix game with payoffs of IIHFEs as follows.

Definition 13. Suppose $\hat{\eta}$ and $\hat{\xi}$ are two IIHFEs. If there are some $\hat{\mathbf{y}} \in Y$ and $\hat{\mathbf{z}} \in Z$ such that for any other $\mathbf{y} \in Y$ and $\mathbf{z} \in Z$ they satisfy the requirements $\hat{\mathbf{y}}^T \tilde{\Gamma} \hat{\mathbf{z}} \geq_{IIHF} \hat{\eta}$ and $\mathbf{y}^T \tilde{\Gamma} \hat{\mathbf{z}} \geq_{IIHF} \hat{\xi}$, then $(\hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\eta}, \hat{\xi})$ is said to be the reasonable solution of the matrix game \tilde{H} with payoffs of IIHFEs.

Here, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are termed as the reasonable strategies for Player X_A and Player X_B , respectively, whereas $\hat{\eta}$ and $\hat{\xi}$ are said to be the reasonable values of the players.

Definition 14. Let us consider two sets, V_A and V_B , as the collection of all reasonable solutions of Player X_A and Player X_B , respectively. Suppose $\hat{\eta}^*$ and $\hat{\xi}^*$ are two IIHFEs such that $\hat{\eta}^* \in V_A$ and $\hat{\xi}^* \in V_B$. If there exist no such $\hat{\eta}$ and $\hat{\xi}$ satisfying $\hat{\eta} \geq_{IIHF} \hat{\eta}^*$ and $\hat{\xi} \leq_{IIHF} \hat{\xi}^*$, then the set $(\mathbf{y}^*, \mathbf{z}^*, \hat{\eta}^*, \hat{\xi}^*)$ is called the optimal solution of the matrix game \tilde{H} .

The strategy \mathbf{y}^* is called the maximin strategy for Player X_A , and similarly, \mathbf{z}^* is termed as the minimax strategy for Player X_B . Furthermore, these $\tilde{\eta}^*$ and $\tilde{\zeta}^*$ are called the values of the game for Players X_A and X_B , respectively.

Let Player X_B choose a pure strategy β_q and Player X_A pick up the mixed strategy $\mathbf{y} \in Y$, then Player X_A 's expected payoff reduces to

$$\tilde{E}_q = \left\langle \left[1 - \prod_{p=1}^m (1 - \underline{\rho}_{pq\mu}^l)^{y_p}, 1 - \prod_{p=1}^m (1 - \bar{\rho}_{pq\mu}^u)^{y_p} \right]; \left[\prod_{p=1}^m (\underline{\rho}_{pqv}^l)^{y_p}, \prod_{p=1}^m (\bar{\rho}_{pqv}^u)^{y_p} \right] \right\rangle.$$

The minimum of \tilde{E}_q is calculated as

$$\begin{aligned} \tilde{\Phi} &= \left\langle [\phi_\mu^{l-}, \phi_\mu^{u+}]; [\phi_\nu^{l-}, \phi_\nu^{u+}] \right\rangle \\ &= \left\langle \min_q \left\{ \left[1 - \prod_{p=1}^m (1 - \underline{\rho}_{pq\mu}^l)^{y_p}, 1 - \prod_{p=1}^m (1 - \bar{\rho}_{pq\mu}^u)^{y_p} \right] \right\}; \max_q \left\{ \left[\prod_{p=1}^m (\underline{\rho}_{pqv}^l)^{y_p}, \prod_{p=1}^m (\bar{\rho}_{pqv}^u)^{y_p} \right] \right\} \right\rangle. \end{aligned}$$

From the aforementioned expression, it is seen that $\tilde{\Phi}$ is a function of \mathbf{y} . Now Player X_A should opt $\mathbf{y}^* \in Y$ to maximize this $\tilde{\Phi}$, i.e., to obtain

$$\begin{aligned} \tilde{\Phi}^* &= \tilde{\Phi}(\mathbf{y}^*) = \left\langle [\phi_\mu^{*l-}, \phi_\mu^{*u+}]; [\phi_\nu^{*l-}, \phi_\nu^{*u+}] \right\rangle \\ &= \left\langle \max_{\mathbf{y} \in Y} \min_q \left\{ \left[1 - \prod_{p=1}^m (1 - \underline{\rho}_{pq\mu}^l)^{y_p}, 1 - \prod_{p=1}^m (1 - \bar{\rho}_{pq\mu}^u)^{y_p} \right] \right\}; \right. \\ &\quad \left. \min_{\mathbf{y} \in Y} \max_q \left\{ \left[\prod_{p=1}^m (\underline{\rho}_{pqv}^l)^{y_p}, \prod_{p=1}^m (\bar{\rho}_{pqv}^u)^{y_p} \right] \right\} \right\rangle. \end{aligned} \quad (5)$$

By the knowledge of the definition of IIHFE (4) and its mathematical operations (5), we can say that $\tilde{\Phi}^*$ is an IIHFE. Such \mathbf{y}^* is termed as the maximin strategy for Player X_A and $\tilde{\Phi}^*$ is said to be the gain floor for Player X_A .

Similarly, if Player X_A takes pure strategy α_p and Player X_B chooses $\mathbf{z} \in Z$ as a mixed strategy, then Player X_B 's expected payoff is transformed into

$$\tilde{E}_p = \left\langle \left[1 - \prod_{q=1}^n (1 - \underline{\rho}_{pq\mu}^l)^{z_q}, 1 - \prod_{q=1}^n (1 - \bar{\rho}_{pq\mu}^u)^{z_q} \right]; \left[\prod_{q=1}^n (\underline{\rho}_{pqv}^l)^{z_q}, \prod_{q=1}^n (\bar{\rho}_{pqv}^u)^{z_q} \right] \right\rangle. \quad (6)$$

The maximum of \tilde{E}_p is calculated as

$$\begin{aligned} \tilde{\Psi} &= \left\langle [\psi_\mu^{l-}, \psi_\mu^{u+}]; [\psi_\nu^{l-}, \psi_\nu^{u+}] \right\rangle \\ &= \left\langle \max_p \left\{ \left[1 - \prod_{q=1}^n (1 - \underline{\rho}_{pq\mu}^l)^{z_q}, 1 - \prod_{q=1}^n (1 - \bar{\rho}_{pq\mu}^u)^{z_q} \right] \right\}; \min_p \left\{ \left[\prod_{q=1}^n (\underline{\rho}_{pqv}^l)^{z_q}, \prod_{q=1}^n (\bar{\rho}_{pqv}^u)^{z_q} \right] \right\} \right\rangle. \end{aligned}$$

Obviously, the function $\tilde{\Psi}$ depends only on \mathbf{z} . In this scenario, Player X_B should choose $\mathbf{z}^* \in Z$ to minimize this $\tilde{\Psi}$, i.e., to obtain

$$\begin{aligned} \tilde{\Psi}^* &= \tilde{\Psi}(\mathbf{z}^*) = \left\langle [\psi_\mu^{*l-}, \psi_\mu^{*u+}]; [\psi_\nu^{*l-}, \psi_\nu^{*u+}] \right\rangle \\ &= \left\langle \min_{\mathbf{z} \in Z} \max_p \left\{ \left[1 - \prod_{q=1}^n (1 - \underline{\rho}_{pq\mu}^l)^{z_q}, 1 - \prod_{q=1}^n (1 - \bar{\rho}_{pq\mu}^u)^{z_q} \right] \right\}; \right. \\ &\quad \left. \max_{\mathbf{z} \in Z} \min_p \left\{ \left[\prod_{q=1}^n (\underline{\rho}_{pqv}^l)^{z_q}, \prod_{q=1}^n (\bar{\rho}_{pqv}^u)^{z_q} \right] \right\} \right\rangle. \end{aligned} \quad (7)$$

Clearly, $\tilde{\Psi}^*$ is an IIHFE. This \mathbf{z}^* is termed as the minimax strategy for Player X_B , and $\tilde{\Psi}^*$ is said to be the loss ceiling for Player X_B .

Theorem 2. If $\tilde{\Phi}^*$ is the gain of Player X_A and $\tilde{\Psi}^*$ is the loss of Player X_B , then $\tilde{\Phi}^* \leq_{IIHF} \tilde{\Psi}^*$.

Proof. It is obvious that $\max_q \left\{ \sum_{p=1}^m y_p \ln(1 - \varrho_{pq\mu}^l) \right\} \geq \sum_{p=1}^m y_p \ln(1 - \varrho_{pq\mu}^l)$.

This implies

$$\begin{aligned} \max_q \left\{ \sum_{p=1}^m y_p \ln(1 - \varrho_{pq\mu}^l) \right\} &\geq \sum_{q=1}^n \left(\sum_{p=1}^m y_p \ln(1 - \varrho_{pq\mu}^l) \right) z_q \quad [as, z_q \geq 0 \forall q, \sum_{q=1}^n z_q = 1] \\ \text{i.e.,} \quad \max_q \left\{ \sum_{p=1}^m y_p \ln(1 - \varrho_{pq\mu}^l) \right\} &\geq \sum_{p=1}^m \sum_{q=1}^n y_p z_q \ln(1 - \varrho_{pq\mu}^l). \end{aligned} \quad (8)$$

Furthermore, we have $\sum_{q=1}^n z_q \ln(1 - \varrho_{pq\mu}^l) \geq \min_p \left\{ \sum_{q=1}^n z_q \ln(1 - \varrho_{pq\mu}^l) \right\}$.

This implies

$$\begin{aligned} \sum_{p=1}^m \left(\sum_{q=1}^n z_q \ln(1 - \varrho_{pq\mu}^l) \right) y_p &\geq \min_p \left\{ \sum_{q=1}^n z_q \ln(1 - \varrho_{pq\mu}^l) \right\} \quad [as, y_p \geq 0 \forall p, \sum_{p=1}^m y_p = 1] \\ \text{i.e.,} \quad \sum_{p=1}^m \sum_{q=1}^n y_p z_q \ln(1 - \varrho_{pq\mu}^l) &\geq \min_p \left\{ \sum_{q=1}^n z_q \ln(1 - \varrho_{pq\mu}^l) \right\}. \end{aligned} \quad (9)$$

Combining Equations (8) and (9), we have

$$\begin{aligned} \max_q \left\{ \sum_{p=1}^m y_p \ln(1 - \varrho_{pq\mu}^l) \right\} &\geq \sum_{p=1}^m \sum_{q=1}^n y_p z_q \ln(1 - \varrho_{pq\mu}^l) \geq \min_p \left\{ \sum_{q=1}^n z_q \ln(1 - \varrho_{pq\mu}^l) \right\} \\ \text{i.e.,} \quad \max_q \left\{ \sum_{p=1}^m y_p \ln(1 - \varrho_{pq\mu}^l) \right\} &\geq \min_p \left\{ \sum_{q=1}^n z_q \ln(1 - \varrho_{pq\mu}^l) \right\} \\ \text{i.e.,} \quad \max_q \left\{ \prod_{p=1}^m (1 - \varrho_{pq\mu}^l)^{y_p} \right\} &\geq \min_p \left\{ \prod_{q=1}^n (1 - \varrho_{pq\mu}^l)^{z_q} \right\} \\ \text{Hence,} \quad \min_{y \in Y} \max_q \left\{ \prod_{p=1}^m (1 - \varrho_{pq\mu}^l)^{y_p} \right\} &\geq \max_{z \in Z} \min_p \left\{ \prod_{q=1}^n (1 - \varrho_{pq\mu}^l)^{z_q} \right\}. \\ \text{Thus,} \quad \max_{y \in Y} \min_q \left\{ 1 - \prod_{p=1}^m (1 - \varrho_{pq\mu}^l)^{y_p} \right\} &\geq \min_{z \in Z} \max_p \left\{ 1 - \prod_{q=1}^n (1 - \varrho_{pq\mu}^l)^{z_q} \right\}. \end{aligned} \quad (10)$$

Similarly, we can show that

$$\max_{y \in Y} \min_q \left\{ 1 - \prod_{p=1}^m (1 - \bar{\varrho}_{pq\mu}^u)^{y_p} \right\} \geq \min_{z \in Z} \max_p \left\{ 1 - \prod_{q=1}^n (1 - \bar{\varrho}_{pq\mu}^u)^{z_q} \right\}. \quad (11)$$

$$\text{Again,} \quad \max_q \left\{ \sum_{p=1}^m y_p \ln(\varrho_{pqv}^l) \right\} \geq \sum_{p=1}^m y_p \ln(\varrho_{pqv}^l)$$

implies

$$\begin{aligned} \max_q \left\{ \sum_{p=1}^m y_p \ln(\varrho_{pqv}^l) \right\} &\geq \sum_{q=1}^n \left(\sum_{p=1}^m y_p \ln(\varrho_{pqv}^l) \right) z_q \quad [as, z_q \geq 0 \forall q, \sum_{q=1}^n z_q = 1] \\ \text{i.e.,} \quad \max_q \left\{ \sum_{p=1}^m y_p \ln(\varrho_{pqv}^l) \right\} &\geq \sum_{p=1}^m \sum_{q=1}^n y_p z_q \ln(\varrho_{pqv}^l). \\ \text{Furthermore,} \quad \sum_{q=1}^n z_q \ln(\varrho_{pqv}^l) &\geq \min_p \left\{ \sum_{q=1}^n z_q \ln(\varrho_{pqv}^l) \right\} \end{aligned} \quad (12)$$

implies

$$\sum_{p=1}^m \left(\sum_{q=1}^n z_q \ln \left(\underline{\varrho}_{pqv}^l \right) \right) y_p \geq \min_p \left\{ \sum_{q=1}^n z_q \ln \left(\underline{\varrho}_{pqv}^l \right) \right\} \quad [as, y_p \geq 0 \forall p, \sum_{p=1}^m y_p = 1]$$

i.e.,
$$\sum_{p=1}^m \sum_{q=1}^n y_p z_q \ln \left(\underline{\varrho}_{pqv}^l \right) \geq \min_p \left\{ \sum_{q=1}^n z_q \ln \left(\underline{\varrho}_{pqv}^l \right) \right\}. \quad (13)$$

Combining Equations (12) and (13), we have

$$\max_q \left\{ \sum_{p=1}^m y_p \ln \left(\underline{\varrho}_{pqv}^l \right) \right\} \geq \sum_{p=1}^m \sum_{q=1}^n y_p z_q \ln \left(\underline{\varrho}_{pqv}^l \right) \geq \min_p \left\{ \sum_{q=1}^n z_q \ln \left(\underline{\varrho}_{pqv}^l \right) \right\}$$

i.e.,
$$\max_q \left\{ \sum_{p=1}^m y_p \ln \left(\underline{\varrho}_{pqv}^l \right) \right\} \geq \min_p \left\{ \sum_{q=1}^n z_q \ln \left(\underline{\varrho}_{pqv}^l \right) \right\}$$

i.e.,
$$\max_q \left\{ \prod_{p=1}^m \left(\underline{\varrho}_{pqv}^l \right)^{y_p} \right\} \geq \min_p \left\{ \prod_{q=1}^n \left(\underline{\varrho}_{pqv}^l \right)^{z_q} \right\}.$$

Hence,
$$\min_{z \in Z} \max_q \left\{ \prod_{p=1}^m \left(\underline{\varrho}_{pqv}^l \right)^{y_p} \right\} \geq \max_{z \in Z} \min_p \left\{ \prod_{q=1}^n \left(\underline{\varrho}_{pqv}^l \right)^{z_q} \right\}. \quad (14)$$

By an similar argument, we show the following:

$$\min_{z \in Z} \max_q \left\{ \prod_{p=1}^m \left(\bar{\varrho}_{pqv}^u \right)^{y_p} \right\} \geq \max_{z \in Z} \min_p \left\{ \prod_{q=1}^n \left(\bar{\varrho}_{pqv}^u \right)^{z_q} \right\} \quad (15)$$

Thus, from Equations (10), (11), (14), and (15), we may write

$$\begin{aligned} & \left\{ \frac{1}{2} \left[\max_{y \in Y} \min_q \left\{ 1 - \prod_{p=1}^m \left(1 - \underline{\varrho}_{pq\mu}^l \right)^{y_p} \right\} + \max_{y \in Y} \min_q \left\{ 1 - \prod_{p=1}^m \left(1 - \bar{\varrho}_{pq\mu}^u \right)^{y_p} \right\} \right] - \right. \\ & \left. \frac{1}{2} \left[\min_{y \in Y} \max_q \left\{ \prod_{p=1}^m \left(\underline{\varrho}_{pqv}^l \right)^{y_p} \right\} + \min_{y \in Y} \max_q \left\{ \prod_{p=1}^m \left(\bar{\varrho}_{pqv}^u \right)^{y_p} \right\} \right] \right\} \leq \\ & \left\{ \frac{1}{2} \left[\min_{z \in Z} \max_p \left\{ 1 - \prod_{q=1}^n \left(1 - \underline{\varrho}_{pq\mu}^l \right)^{z_q} \right\} + \min_{z \in Z} \max_p \left\{ 1 - \prod_{q=1}^n \left(1 - \bar{\varrho}_{pq\mu}^u \right)^{z_q} \right\} \right] - \right. \\ & \left. \frac{1}{2} \left[\max_{z \in Z} \min_p \left\{ \prod_{q=1}^n \left(\underline{\varrho}_{pqv}^l \right)^{z_q} \right\} + \max_{z \in Z} \min_p \left\{ \prod_{q=1}^n \left(\bar{\varrho}_{pqv}^u \right)^{z_q} \right\} \right] \right\}. \end{aligned}$$

This relation implies $Sc_{IIHFWMEAO}(\tilde{\Phi}^*) \leq Sc_{IIHFWMEAO}(\tilde{\Psi}^*)$ (by Definition 12). Thus, by the ranking property of IIHFEs, we have $\tilde{\Phi}^* \leq_{IIHF} \tilde{\Psi}^*$, which completes the theorem. \square

Now, according to Equation (5) and Definitions 13 and 14, to obtain the maximin strategy y^* and the gain floor $\tilde{\Phi}^*$, we have to solve the following bi-objective interval-valued intuitionistic hesitant fuzzy programming model (BOIIHFPM):

$$\begin{aligned}
& \max \left\{ \left[\phi_{\mu}^{l-}, \phi_{\mu}^{u+} \right] \right\}, \min \left\{ \left[\phi_{\nu}^{l-}, \phi_{\nu}^{u+} \right] \right\} \\
\text{subject to } & \left[1 - \prod_{p=1}^m \left(1 - \underline{\rho}_{pq\mu}^l \right)^{y_p}, 1 - \prod_{p=1}^m \left(1 - \bar{\rho}_{pq\mu}^u \right)^{y_p} \right] \geq_{IIHF} \left[\phi_{\mu}^{l-}, \phi_{\mu}^{u+} \right] \quad (q = 1, 2, \dots, n) \\
& \left[\prod_{p=1}^m \left(\underline{\rho}_{pq\nu}^l \right)^{y_p}, \prod_{p=1}^m \left(\bar{\rho}_{pq\nu}^u \right)^{y_p} \right] \leq_{IIHF} \left[\phi_{\nu}^{l-}, \phi_{\nu}^{u+} \right] \quad (q = 1, 2, \dots, n) \\
& 0 \leq \phi_{\mu}^{l-}, \phi_{\mu}^{u+}, \phi_{\nu}^{l-}, \phi_{\nu}^{u+} \leq 1 \\
& \sum_{p=1}^m y_p = 1, \quad y_p \geq 0 \quad (p = 1, 2, \dots, m).
\end{aligned} \tag{16}$$

Again, according to Equation (7) and Definitions 13 and 14, to calculate the minimax strategy \mathbf{z}^* and the loss ceiling $\bar{\Psi}^*$ for Player X_B , we have to solve the following BOIIHFPM:

$$\begin{aligned}
& \min \left\{ \left[\psi_{\mu}^{l-}, \psi_{\mu}^{u+} \right] \right\}, \max \left\{ \left[\psi_{\nu}^{l-}, \psi_{\nu}^{u+} \right] \right\} \\
\text{subject to } & \left[1 - \prod_{q=1}^n \left(1 - \underline{\rho}_{pq\mu}^l \right)^{z_q}, 1 - \prod_{q=1}^n \left(1 - \bar{\rho}_{pq\mu}^u \right)^{z_q} \right] \leq_{IIHF} \left[\psi_{\mu}^{l-}, \psi_{\mu}^{u+} \right] \quad (p = 1, 2, \dots, m) \\
& \left[\prod_{q=1}^n \left(\underline{\rho}_{pq\nu}^l \right)^{z_q}, \prod_{q=1}^n \left(\bar{\rho}_{pq\nu}^u \right)^{z_q} \right] \geq_{IIHF} \left[\psi_{\nu}^{l-}, \psi_{\nu}^{u+} \right] \quad (p = 1, 2, \dots, m) \\
& 0 \leq \psi_{\mu}^{l-}, \psi_{\mu}^{u+}, \psi_{\nu}^{l-}, \psi_{\nu}^{u+} \leq 1 \\
& \sum_{q=1}^n z_q = 1, \quad z_q \geq 0 \quad (q = 1, 2, \dots, n).
\end{aligned} \tag{17}$$

It is not a facile task to evaluate the optimal solutions of the BOIIHFPMs, as depicted in Problems (16) and (17). Thus, we have to find a way out so that it becomes easier to calculate the expected result of the problem. Keeping all these in mind, we extend the methodology proposed by Li [6] for intuitionistic payoffs in hesitant fuzzy scenarios.

For $0 \leq \phi_{\mu}^{l-}, \phi_{\mu}^{u+} \leq 1$, it is clear that $(\max\{\phi_{\mu}^{l-}, \phi_{\mu}^{u+}\})$ is equivalent to $(\min\{[1 - \phi_{\mu}^{u+}, 1 - \phi_{\mu}^{l-}]\})$. Thus, by the weighted average method, the objective function of Problem (16) transforms into $(\min\{\tau[1 - \phi_{\mu}^{u+}, 1 - \phi_{\mu}^{l-}] + (1 - \tau)[\phi_{\nu}^{l-}, \phi_{\nu}^{u+}]\})$. As each interval assigned here is some IIHFE, this addition must follow the rule of addition of IIHFEs (as defined by Chen and Xu [41]). Hence, the objective function will be

$$\min \left\{ \left[1 - \left(\phi_{\mu}^{u+} \right)^{\tau} \left(\phi_{\nu}^{l-} \right)^{1-\tau}, 1 - \left(\phi_{\mu}^{l-} \right)^{\tau} \left(\phi_{\nu}^{u+} \right)^{1-\tau} \right] \right\} \tag{18}$$

where τ is a weight, which takes values from $[0, 1]$. This τ stands for the perception parameters of the player's choice, which is completely assigned by the players.

Again, $[1 - \prod_{p=1}^m (1 - \underline{\rho}_{pq\mu}^l)^{y_p}, 1 - \prod_{p=1}^m (1 - \bar{\rho}_{pq\mu}^u)^{y_p}] \geq_{IIHF} [\phi_{\mu}^{l-}, \phi_{\mu}^{u+}]$ implies $[\prod_{p=1}^m (1 - \bar{\rho}_{pq\mu}^u)^{y_p}, \prod_{p=1}^m (1 - \underline{\rho}_{pq\mu}^l)^{y_p}] \leq_{IIHF} [1 - \phi_{\mu}^{u+}, 1 - \phi_{\mu}^{l-}]$ for $q = 1, 2, \dots, n$, as $0 \leq 1 - \prod_{p=1}^m (1 - \underline{\rho}_{pq\mu}^l)^{y_p}, 1 - \prod_{p=1}^m (1 - \bar{\rho}_{pq\mu}^u)^{y_p}, \phi_{\mu}^{l-}, \phi_{\mu}^{u+} \leq 1$. Furthermore, we have $[\prod_{p=1}^m (\underline{\rho}_{pq\nu}^l)^{y_p}, \prod_{p=1}^m (\bar{\rho}_{pq\nu}^u)^{y_p}] \leq_{IIHF} [\phi_{\nu}^{l-}, \phi_{\nu}^{u+}]$ for $q = 1, 2, \dots, n$. Thus, utilizing the weights, we can aggregate the constraints of Problem (16) as

$$\begin{aligned}
& \tau \left[\prod_{p=1}^m (1 - \bar{\rho}_{pq\mu}^u)^{y_p}, \prod_{p=1}^m (1 - \underline{\rho}_{pq\mu}^l)^{y_p} \right] + (1 - \tau) \left[\prod_{p=1}^m (\underline{\rho}_{pq\nu}^l)^{y_p}, \prod_{p=1}^m (\bar{\rho}_{pq\nu}^u)^{y_p} \right] \leq_{IIHF} \\
& \tau [1 - \phi_{\mu}^{u+}, 1 - \phi_{\mu}^{l-}] + (1 - \tau) [\phi_{\nu}^{l-}, \phi_{\nu}^{u+}], \quad q = 1, 2, \dots, n.
\end{aligned}$$

Upon calculation, this gives

$$\left[1 - \left\{ \prod_{p=1}^m (1 - \bar{\varrho}_{pq\mu}^u)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m (\varrho_{pqv}^l)^{y_p} \right\}^{1-\tau}, 1 - \left\{ \prod_{p=1}^m (1 - \varrho_{pq\mu}^l)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m (\bar{\varrho}_{pqv}^u)^{y_p} \right\}^{1-\tau} \right] \leq \\ \left[1 - (\phi_\mu^{u+})^\tau (\phi_v^{l-})^{1-\tau}, 1 - (\phi_\mu^{l-})^\tau (\phi_v^{u+})^{1-\tau} \right], \quad q = 1, 2, \dots, n.$$

From this, we can say that

$$\begin{aligned} 1 - \left\{ \prod_{p=1}^m (1 - \bar{\varrho}_{pq\mu}^u)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m (\varrho_{pqv}^l)^{y_p} \right\}^{1-\tau} &\leq 1 - (\phi_\mu^{u+})^\tau (\phi_v^{l-})^{1-\tau} \quad (q = 1, 2, \dots, n) \\ 1 - \left\{ \prod_{p=1}^m (1 - \varrho_{pq\mu}^l)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m (\bar{\varrho}_{pqv}^u)^{y_p} \right\}^{1-\tau} &\leq 1 - (\phi_\mu^{l-})^\tau (\phi_v^{u+})^{1-\tau} \quad (q = 1, 2, \dots, n) \\ 0 &\leq \phi_\mu^{l-}, \phi_\mu^{u+}, \phi_v^{l-}, \phi_v^{u+} \leq 1 \\ \sum_{p=1}^m y_p &= 1, \quad y_p \geq 0 \quad (p = 1, 2, \dots, m). \end{aligned} \quad (19)$$

Combining the objective function (18) with the constraint (19), Problem (16) may be transformed into the following interval-valued mathematical programming problem:

$$\begin{aligned} \min & \left\{ \left[1 - (\phi_\mu^{u+})^\tau (\phi_v^{l-})^{1-\tau}, 1 - (\phi_\mu^{l-})^\tau (\phi_v^{u+})^{1-\tau} \right] \right\} \\ \text{subject to} & \quad 1 - \left\{ \prod_{p=1}^m (1 - \bar{\varrho}_{pq\mu}^u)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m (\varrho_{pqv}^l)^{y_p} \right\}^{1-\tau} \leq 1 - (\phi_\mu^{u+})^\tau (\phi_v^{l-})^{1-\tau} \quad (q = 1, 2, \dots, n) \\ & \quad 1 - \left\{ \prod_{p=1}^m (1 - \varrho_{pq\mu}^l)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m (\bar{\varrho}_{pqv}^u)^{y_p} \right\}^{1-\tau} \leq 1 - (\phi_\mu^{l-})^\tau (\phi_v^{u+})^{1-\tau} \quad (q = 1, 2, \dots, n) \\ & \quad 0 \leq \phi_\mu^{l-}, \phi_\mu^{u+}, \phi_v^{l-}, \phi_v^{u+} \leq 1 \\ & \quad \sum_{p=1}^m y_p = 1, \quad y_p \geq 0 \quad (p = 1, 2, \dots, m). \end{aligned} \quad (20)$$

Let us take $1 - (\phi_\mu^{u+})^\tau (\phi_v^{l-})^{1-\tau} = \phi^l$ and $1 - (\phi_\mu^{l-})^\tau (\phi_v^{u+})^{1-\tau} = \phi^u$. Clearly, $\phi^l, \phi^u \geq 0$. Then, rewriting Problem (20), we have

$$\begin{aligned} \min & \left\{ \left[\phi^l, \phi^u \right] \right\} \\ \text{subject to} & \quad 1 - \left\{ \prod_{p=1}^m (1 - \bar{\varrho}_{pq\mu}^u)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m (\varrho_{pqv}^l)^{y_p} \right\}^{1-\tau} \leq \phi^l \quad (q = 1, 2, \dots, n) \\ & \quad 1 - \left\{ \prod_{p=1}^m (1 - \varrho_{pq\mu}^l)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m (\bar{\varrho}_{pqv}^u)^{y_p} \right\}^{1-\tau} \leq \phi^u \quad (q = 1, 2, \dots, n) \\ & \quad 0 \leq \phi^l, \phi^u \leq 1 \\ & \quad \sum_{p=1}^m y_p = 1, \quad y_p \geq 0 \quad (p = 1, 2, \dots, m). \end{aligned} \quad (21)$$

Clearly, Problem (21) is a nonlinear programming problem having an interval objective function. According to Ishibuchi and Tanaka [43], the objective function of the aforementioned problem can be resolved as

$$\min \left\{ \phi^u \right\}, \min \left\{ \frac{\phi^l + \phi^u}{2} \right\}. \quad (22)$$

For the sake of the computation of Problem (22), we make use of the following lemma.

Lemma 1 ([6]). Assume that D is the set of constraints, and $d_1, d_2 \in D$. Then the following equalities are valid:

$$\min_{d_1+d_2 \in D} \{d_1 + d_2\} = \min_{d_1 \in D} \{d_1\} + \min_{d_2 \in D} \{d_2\} \quad (23)$$

$$\text{and, } \max_{d_1+d_2 \in D} \{d_1 + d_2\} = \max_{d_1 \in D} \{d_1\} + \max_{d_2 \in D} \{d_2\}. \quad (24)$$

Using Equation (23) of Lemma 1, we may say that the objective function (22) is equivalent to the following:

$$\min \left\{ \phi^u \right\}, \min \left\{ \frac{\phi^l}{2} \right\} + \min \left\{ \frac{\phi^u}{2} \right\} \quad (25)$$

which is equivalent to

$$\min \left\{ \phi^l \right\}, \min \left\{ \phi^u \right\}. \quad (26)$$

Thus, utilizing the objective function (26), Problem (21) can be rewritten as

$$\begin{aligned} & \min \left\{ \phi^l \right\}, \min \left\{ \phi^u \right\} \\ \text{subject to } & 1 - \left\{ \prod_{p=1}^m \left(1 - \bar{e}_{pq\mu}^u \right)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m \left(\underline{e}_{pqv}^l \right)^{y_p} \right\}^{1-\tau} \leq \phi^l \quad (q = 1, 2, \dots, n) \\ & 1 - \left\{ \prod_{p=1}^m \left(1 - \underline{e}_{pq\mu}^l \right)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m \left(\bar{e}_{pqv}^u \right)^{y_p} \right\}^{1-\tau} \leq \phi^u \quad (q = 1, 2, \dots, n) \\ & 0 \leq \phi^l, \phi^u \leq 1 \\ & \sum_{p=1}^m y_p = 1, \quad y_p \geq 0 \quad (p = 1, 2, \dots, m). \end{aligned} \quad (27)$$

Motivated by the work of Li [6], we can aggregate this problem as follows:

$$\begin{aligned} & \min \left\{ \frac{\phi^l + \phi^u}{2} \right\} \\ \text{subject to } & \frac{1}{2} \left[1 - \left\{ \prod_{p=1}^m \left(1 - \bar{e}_{pq\mu}^u \right)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m \left(\underline{e}_{pqv}^l \right)^{y_p} \right\}^{1-\tau} + \right. \\ & \left. 1 - \left\{ \prod_{p=1}^m \left(1 - \underline{e}_{pq\mu}^l \right)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m \left(\bar{e}_{pqv}^u \right)^{y_p} \right\}^{1-\tau} \right] \leq \frac{\phi^l + \phi^u}{2} \quad (q = 1, 2, \dots, n) \\ & 0 \leq \phi^l, \phi^u \leq 1 \\ & \sum_{p=1}^m y_p = 1, \quad y_p \geq 0 \quad (p = 1, 2, \dots, m). \end{aligned} \quad (28)$$

Substituting ϕ in place of $\frac{\phi^l + \phi^u}{2}$ in Problem (28), we obtain

$$\begin{aligned}
& \min \{ \phi \} \\
\text{subject to } & \frac{1}{2} \left[2 - \left\{ \prod_{p=1}^m \left(1 - \bar{\varrho}_{pq\mu}^u \right)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m \left(\varrho_{pqv}^l \right)^{y_p} \right\}^{1-\tau} - \right. \\
& \left. \left\{ \prod_{p=1}^m \left(1 - \varrho_{pq\mu}^l \right)^{\tau y_p} \right\} \left\{ 1 - \prod_{p=1}^m \left(\bar{\varrho}_{pqv}^u \right)^{y_p} \right\}^{1-\tau} \right] \leq \phi \quad (q = 1, 2, \dots, n) \\
& 0 \leq \phi \leq 1 \\
& \sum_{p=1}^m y_p = 1, \quad y_p \geq 0 \quad (p = 1, 2, \dots, m).
\end{aligned} \tag{29}$$

Now, by solving the NLPP (29), we find out the maximin strategy \mathbf{y}^* and the crisp equivalent ϕ^* of $\bar{\Phi}^*$ for the maximizing Player X_A .

Next, we consider the BOIHFPM (17). As $0 \leq \psi_\mu^{l-}, \psi_\mu^{u+} \leq 1$, so we may say that $(\min\{\psi_\mu^{l-}, \psi_\mu^{u+}\})$ is equivalent to $(\max\{[1 - \psi_\mu^{u+}, 1 - \psi_\mu^{l-}]\})$. Hence by the weighted average method, the objective function of the fuzzy programming problem (17) can be transformed into

$$\max \left\{ \left[1 - \left(\psi_\mu^{u+} \right)^\tau \left(\psi_\nu^{l-} \right)^{1-\tau}, 1 - \left(\psi_\mu^{l-} \right)^\tau \left(\psi_\nu^{u+} \right)^{1-\tau} \right] \right\} \tag{30}$$

Again, $[1 - \prod_{q=1}^n (1 - \varrho_{pq\mu}^l)^{z_q}, 1 - \prod_{q=1}^n (1 - \bar{\varrho}_{pq\mu}^u)^{z_q}] \leq_{IHF} [\psi_\mu^{l-}, \psi_\mu^{u+}]$ implies $[\prod_{q=1}^n (1 - \bar{\varrho}_{pq\mu}^u)^{z_q}, \prod_{q=1}^n (1 - \varrho_{pq\mu}^l)^{z_q}] \geq_{IHF} [1 - \psi_\mu^{u+}, 1 - \psi_\mu^{l-}]$ for $p = 1, 2, \dots, m$, as $0 \leq 1 - \prod_{q=1}^n (1 - \varrho_{pq\mu}^l)^{z_q}, 1 - \prod_{q=1}^n (1 - \bar{\varrho}_{pq\mu}^u)^{z_q}, \psi_\mu^{l-}, \psi_\mu^{u+} \leq 1$. Furthermore, here we have $[\prod_{q=1}^n (\varrho_{pqv}^l)^{z_q}, \prod_{q=1}^n (\bar{\varrho}_{pqv}^u)^{z_q}] \geq_{IHF} [\psi_\nu^{l-}, \psi_\nu^{u+}]$ for $p = 1, 2, \dots, m$. Thus, utilizing the weights, we can aggregate the constraints of Problem (17) as

$$\begin{aligned}
& \tau \left[\prod_{q=1}^n (1 - \bar{\varrho}_{pq\mu}^u)^{z_q}, \prod_{q=1}^n (1 - \varrho_{pq\mu}^l)^{z_q} \right] + (1 - \tau) \left[\prod_{q=1}^n (\varrho_{pqv}^l)^{z_q}, \prod_{q=1}^n (\bar{\varrho}_{pqv}^u)^{z_q} \right] \geq_{IHF} \\
& \tau [1 - \psi_\mu^{u+}, 1 - \psi_\mu^{l-}] + (1 - \tau) [\psi_\nu^{l-}, \psi_\nu^{u+}], \quad p = 1, 2, \dots, m.
\end{aligned}$$

After simplification, we have

$$\begin{aligned}
& \left[1 - \left\{ \prod_{q=1}^n (1 - \bar{\varrho}_{pq\mu}^u)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n (\varrho_{pqv}^l)^{z_q} \right\}^{1-\tau}, 1 - \left\{ \prod_{q=1}^n (1 - \varrho_{pq\mu}^l)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n (\bar{\varrho}_{pqv}^u)^{z_q} \right\}^{1-\tau} \right] \geq \\
& \left[1 - \left(\psi_\mu^{u+} \right)^\tau \left(\psi_\nu^{l-} \right)^{1-\tau}, 1 - \left(\psi_\mu^{l-} \right)^\tau \left(\psi_\nu^{u+} \right)^{1-\tau} \right], \quad p = 1, 2, \dots, m.
\end{aligned}$$

From this, we can say

$$\begin{aligned}
& 1 - \left\{ \prod_{q=1}^n (1 - \bar{\varrho}_{pq\mu}^u)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n (\varrho_{pqv}^l)^{z_q} \right\}^{1-\tau} \geq 1 - \left(\psi_\mu^{u+} \right)^\tau \left(\psi_\nu^{l-} \right)^{1-\tau} \quad (p = 1, 2, \dots, m) \\
& 1 - \left\{ \prod_{q=1}^n (1 - \varrho_{pq\mu}^l)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n (\bar{\varrho}_{pqv}^u)^{z_q} \right\}^{1-\tau} \geq 1 - \left(\psi_\mu^{l-} \right)^\tau \left(\psi_\nu^{u+} \right)^{1-\tau} \quad (p = 1, 2, \dots, m) \\
& 0 \leq \psi_\mu^{l-}, \psi_\mu^{u+}, \psi_\nu^{l-}, \psi_\nu^{u+} \leq 1 \\
& \sum_{q=1}^n z_q = 1, \quad z_q \geq 0 \quad (q = 1, 2, \dots, n).
\end{aligned} \tag{31}$$

Thus, combining the objective function, depicted in Equation (30), with the constraints given in (31), Problem (17) can be transformed into Problem (32) as

$$\begin{aligned}
& \max \left\{ \left[1 - \left(\psi_{\mu}^{u+} \right)^{\tau} \left(\psi_{\nu}^{l-} \right)^{1-\tau}, 1 - \left(\psi_{\mu}^{l-} \right)^{\tau} \left(\psi_{\nu}^{u+} \right)^{1-\tau} \right] \right\} \\
\text{subject to } & 1 - \left\{ \prod_{q=1}^n \left(1 - \bar{e}_{pq\mu}^u \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(e_{pq\nu}^l \right)^{z_q} \right\}^{1-\tau} \geq 1 - \left(\psi_{\mu}^{u+} \right)^{\tau} \left(\psi_{\nu}^{l-} \right)^{1-\tau} \quad (p = 1, 2, \dots, m) \\
& 1 - \left\{ \prod_{q=1}^n \left(1 - e_{pq\mu}^l \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\bar{e}_{pq\nu}^u \right)^{z_q} \right\}^{1-\tau} \geq 1 - \left(\psi_{\mu}^{l-} \right)^{\tau} \left(\psi_{\nu}^{u+} \right)^{1-\tau} \quad (p = 1, 2, \dots, m) \\
& 0 \leq \psi_{\mu}^{l-}, \psi_{\mu}^{u+}, \psi_{\nu}^{l-}, \psi_{\nu}^{u+} \leq 1 \\
& \sum_{q=1}^n z_q = 1, \quad z_q \geq 0 \quad (q = 1, 2, \dots, n).
\end{aligned} \tag{32}$$

Let us denote $1 - (\psi_{\mu}^{u+})^{\tau} (\psi_{\nu}^{l-})^{1-\tau}$ as ψ^l and $1 - (\psi_{\mu}^{l-})^{\tau} (\psi_{\nu}^{u+})^{1-\tau}$ as ψ^u , respectively. It is clear that $0 \leq \psi^l, \psi^u \leq 1$. Thus, rewriting Problem (32), we have

$$\begin{aligned}
& \max \left\{ \left[\psi^l, \psi^u \right] \right\} \\
\text{subject to } & 1 - \left\{ \prod_{q=1}^n \left(1 - \bar{e}_{pq\mu}^u \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(e_{pq\nu}^l \right)^{z_q} \right\}^{1-\tau} \geq \psi^l \quad (p = 1, 2, \dots, m) \\
& 1 - \left\{ \prod_{q=1}^n \left(1 - e_{pq\mu}^l \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\bar{e}_{pq\nu}^u \right)^{z_q} \right\}^{1-\tau} \geq \psi^u \quad (p = 1, 2, \dots, m) \\
& 0 \leq \psi^l, \psi^u \leq 1 \\
& \sum_{q=1}^n z_q = 1, \quad z_q \geq 0 \quad (q = 1, 2, \dots, n).
\end{aligned} \tag{33}$$

Problem (33) is a nonlinear programming problem having an interval-valued objective function. By a similar argument of the maximization Problem (21), the objective function of (33) can be written as

$$\max \left\{ \psi^l \right\}, \max \left\{ \frac{\psi^l + \psi^u}{2} \right\}. \tag{34}$$

Utilizing Equation (24) of Lemma 1, the objective function (34) can be rewritten as

$$\max \left\{ \psi^l \right\}, \max \left\{ \frac{\psi^l}{2} \right\} + \max \left\{ \frac{\psi^u}{2} \right\} \tag{35}$$

which is also equivalent to

$$\max \left\{ \psi^l \right\}, \max \left\{ \psi^u \right\}. \tag{36}$$

Thus, utilizing the objective function (36), Problem (33) can be rewritten as

$$\begin{aligned}
& \max \left\{ \psi^l \right\}, \max \left\{ \psi^u \right\} \\
\text{subject to} \quad & 1 - \left\{ \prod_{q=1}^n \left(1 - \bar{\varrho}_{pq\mu}^u \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\varrho_{pqv}^l \right)^{z_q} \right\}^{1-\tau} \geq \psi^l \quad (p = 1, 2, \dots, m) \\
& 1 - \left\{ \prod_{q=1}^n \left(1 - \varrho_{pq\mu}^l \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\bar{\varrho}_{pqv}^u \right)^{z_q} \right\}^{1-\tau} \geq \psi^u \quad (p = 1, 2, \dots, m) \\
& 0 \leq \psi^l, \psi^u \leq 1 \\
& \sum_{q=1}^n z_q = 1, \quad z_q \geq 0 \quad (q = 1, 2, \dots, n).
\end{aligned} \tag{37}$$

This is also a bi-objective programming problem, which can be aggregated as

$$\begin{aligned}
& \max \left\{ \frac{\psi^l + \psi^u}{2} \right\} \\
\text{subject to} \quad & \frac{1}{2} \left[1 - \left\{ \prod_{q=1}^n \left(1 - \bar{\varrho}_{pq\mu}^u \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\varrho_{pqv}^l \right)^{z_q} \right\}^{1-\tau} + \right. \\
& \left. 1 - \left\{ \prod_{q=1}^n \left(1 - \varrho_{pq\mu}^l \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\bar{\varrho}_{pqv}^u \right)^{z_q} \right\}^{1-\tau} \right] \geq \frac{\psi^l + \psi^u}{2} \quad (p = 1, 2, \dots, m) \\
& 0 \leq \psi^l, \psi^u \leq 1 \\
& \sum_{q=1}^n z_q = 1, \quad z_q \geq 0 \quad (q = 1, 2, \dots, n).
\end{aligned} \tag{38}$$

Taking ψ in place of $\frac{\psi^l + \psi^u}{2}$ in Problem (38), we obtain

$$\begin{aligned}
& \max \left\{ \psi \right\} \\
\text{subject to} \quad & \frac{1}{2} \left[2 - \left\{ \prod_{q=1}^n \left(1 - \bar{\varrho}_{pq\mu}^u \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\varrho_{pqv}^l \right)^{z_q} \right\}^{1-\tau} - \right. \\
& \left. \left\{ \prod_{q=1}^n \left(1 - \varrho_{pq\mu}^l \right)^{\tau z_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\bar{\varrho}_{pqv}^u \right)^{z_q} \right\}^{1-\tau} \right] \geq \psi \quad (p = 1, 2, \dots, m) \\
& 0 \leq \psi \leq 1 \\
& \sum_{q=1}^n z_q = 1, \quad z_q \geq 0 \quad (q = 1, 2, \dots, n).
\end{aligned} \tag{39}$$

Solving Problem (39), we can derive the minimax strategy \mathbf{z}^* and the crisp equivalent ψ^* of $\tilde{\Psi}^*$ for the minimizing Player X_B .

These solutions (\mathbf{y}^*, ϕ^*) and (\mathbf{z}^*, ψ^*) are called the optimal solutions of the NLPPs (29) and (39), respectively.

In order to make the solution of the matrix games with payoffs of IIHFes, we converted the BOIIHFPMs into two NLPPs, utilizing the mathematical operations of IIHFes. Here an inevitable question arises as to the acceptability of the optimal values, obtained by solving these two NLPPs for the proposed problem. The following theorem is stated to answer such a query.

Theorem 3. $(\mathbf{y}^*, \tilde{\Phi}^*)$ and $(\mathbf{z}^*, \tilde{\Psi}^*)$ are the noninferior solutions of the interval-valued intuitionistic hesitant fuzzy programming problems (16) and (17), respectively, for $0 \leq \tau \leq 1$, if (\mathbf{y}^*, ϕ^*) and (\mathbf{z}^*, ψ^*) are the optimal solutions of the NLPPs (29) and (39), respectively, where $\tilde{\Phi}^* =$

$\langle [\phi_\mu^{*l-}, \phi_\mu^{*u+}]; [\phi_\nu^{*l-}, \phi_\nu^{*u+}] \rangle$ and $\tilde{\Psi}^* = \langle [\psi_\mu^{*l-}, \psi_\mu^{*u+}]; [\psi_\nu^{*l-}, \psi_\nu^{*u+}] \rangle$, satisfying the following relations:

$$\begin{aligned} \phi^* &= \frac{1}{2} [2 - (\phi_\mu^{*u+})^\tau (\phi_\nu^{*l-})^{1-\tau} - (\phi_\mu^{*l-})^\tau (\phi_\nu^{*u+})^{1-\tau}], \\ \text{and} \quad \psi^* &= \frac{1}{2} [2 - (\psi_\mu^{*u+})^\tau (\psi_\nu^{*l-})^{1-\tau} - (\psi_\mu^{*l-})^\tau (\psi_\nu^{*u+})^{1-\tau}]. \end{aligned}$$

Proof. Let us assume that $(\mathbf{y}^*, \tilde{\Phi}^*)$ (16). Then there must exist a feasible solution $(\hat{\mathbf{y}}, \hat{\Phi})$, where $\hat{\mathbf{y}} \in Y$ and $\hat{\Phi} = \langle [\hat{\phi}_\mu^{l-}, \hat{\phi}_\mu^{u+}]; [\hat{\phi}_\nu^{l-}, \hat{\phi}_\nu^{u+}] \rangle$, such that

$$\begin{aligned} 1 - \left\{ \prod_{p=1}^m (1 - \underline{e}_{pq\mu}^l)^{\hat{y}_p} \right\} &\geq \hat{\phi}_\mu^{l-}; \quad 1 - \left\{ \prod_{p=1}^m (1 - \bar{e}_{pq\mu}^u)^{\hat{y}_p} \right\} \geq \hat{\phi}_\mu^{u+}, \quad (q = 1, 2, \dots, n) \\ \prod_{p=1}^m (\underline{e}_{pq\nu}^l)^{\hat{y}_p} &\leq \hat{\phi}_\nu^{l-}; \quad \prod_{p=1}^m (\bar{e}_{pq\nu}^u)^{\hat{y}_p} \leq \hat{\phi}_\nu^{u+}, \quad (q = 1, 2, \dots, n) \\ 0 &\leq \hat{\phi}_\mu^{l-}, \hat{\phi}_\mu^{u+}, \hat{\phi}_\nu^{l-}, \hat{\phi}_\nu^{u+} \leq 1 \\ \sum_{p=1}^m \hat{y}_p &= 1, \quad \hat{y}_p \geq 0 \quad (p = 1, 2, \dots, m). \end{aligned} \quad (40)$$

Furthermore,

$$\hat{\phi}_\mu^{l-} \geq \phi_\mu^{*l-}, \quad \hat{\phi}_\mu^{u+} \geq \phi_\mu^{*u+}, \quad \hat{\phi}_\nu^{l-} \leq \phi_\nu^{*l-}, \quad \hat{\phi}_\nu^{u+} \leq \phi_\nu^{*u+}. \quad (41)$$

As $0 \leq \tau \leq 1$, so from Problem (40), it is derived that

$$\begin{aligned} \frac{1}{2} \left[2 - \left\{ \prod_{p=1}^m (1 - \bar{e}_{pq\mu}^u)^{\tau \hat{y}_p} \right\} \left\{ 1 - \prod_{p=1}^m (\underline{e}_{pq\nu}^l)^{\hat{y}_p} \right\}^{1-\tau} - \left\{ \prod_{p=1}^m (1 - \underline{e}_{pq\mu}^l)^{\tau \hat{y}_p} \right\} \left\{ 1 - \prod_{p=1}^m (\bar{e}_{pq\nu}^u)^{\hat{y}_p} \right\}^{1-\tau} \right] &\leq \\ \frac{1}{2} \left[2 - (\hat{\phi}_\mu^{u+})^\tau (\hat{\phi}_\nu^{l-})^{1-\tau} - (\hat{\phi}_\mu^{l-})^\tau (\hat{\phi}_\nu^{u+})^{1-\tau} \right], \quad (q = 1, 2, \dots, n) \\ 0 &\leq \hat{\phi}_\mu^{l-}, \hat{\phi}_\mu^{u+}, \hat{\phi}_\nu^{l-}, \hat{\phi}_\nu^{u+} \leq 1 \\ \sum_{p=1}^m \hat{y}_p &= 1, \quad \hat{y}_p \geq 0 \quad (p = 1, 2, \dots, m). \end{aligned} \quad (42)$$

and

$$\frac{1}{2} \left[2 - (\phi_\mu^{*u+})^\tau (\phi_\nu^{*l-})^{1-\tau} - (\phi_\mu^{*l-})^\tau (\phi_\nu^{*u+})^{1-\tau} \right] < \frac{1}{2} \left[2 - (\hat{\phi}_\mu^{u+})^\tau (\hat{\phi}_\nu^{l-})^{1-\tau} - (\hat{\phi}_\mu^{l-})^\tau (\hat{\phi}_\nu^{u+})^{1-\tau} \right]. \quad (43)$$

Let us take $\hat{\phi} = \frac{1}{2} [2 - (\hat{\phi}_\mu^{u+})^\tau (\hat{\phi}_\nu^{l-})^{1-\tau} - (\hat{\phi}_\mu^{l-})^\tau (\hat{\phi}_\nu^{u+})^{1-\tau}]$. Thus, Problem (42) can be written as

$$\begin{aligned} \frac{1}{2} \left[2 - \left\{ \prod_{p=1}^m (1 - \bar{e}_{pq\mu}^u)^{\tau \hat{y}_p} \right\} \left\{ 1 - \prod_{p=1}^m (\underline{e}_{pq\nu}^l)^{\hat{y}_p} \right\}^{1-\tau} - \left\{ \prod_{p=1}^m (1 - \underline{e}_{pq\mu}^l)^{\tau \hat{y}_p} \right\} \left\{ 1 - \prod_{p=1}^m (\bar{e}_{pq\nu}^u)^{\hat{y}_p} \right\}^{1-\tau} \right] &\leq \hat{\phi}, \\ (q = 1, 2, \dots, n) \\ 0 &\leq \hat{\phi}_\mu^{l-}, \hat{\phi}_\mu^{u+}, \hat{\phi}_\nu^{l-}, \hat{\phi}_\nu^{u+} \leq 1 \\ \sum_{p=1}^m \hat{y}_p &= 1, \quad \hat{y}_p \geq 0 \quad (p = 1, 2, \dots, m). \end{aligned} \quad (44)$$

Now, Problem (40) shows that $(\hat{\mathbf{y}}, \hat{\phi})$ is a feasible solution of Problem (29). Furthermore, from Equation (41), we have $\hat{\phi} < \phi^*$, which makes a contradiction to the fact that (\mathbf{y}^*, ϕ^*)

is the optimal solution of (29). Thus, our assumption was wrong and thereafter, we may conclude that if (\mathbf{y}^*, ϕ^*) is the optimal solution of the NLPP (29), then $(\mathbf{y}^*, \tilde{\Phi}^*)$ is the noninferior solution of the interval-valued hesitant fuzzy programming problem (16) for $0 \leq \tau \leq 1$.

Similarly, assume that $(\mathbf{z}^*, \tilde{\Psi}^*)$ is not a noninferior solution of the BOIHFPM (17). Then there exists a feasible solution $(\hat{\mathbf{z}}, \hat{\Psi})$, where $\hat{\mathbf{z}} \in Z$ and $\hat{\Psi} = \langle [\hat{\psi}_\mu^{l-}, \hat{\psi}_\mu^{u+}]; [\hat{\psi}_\nu^{l-}, \hat{\psi}_\nu^{u+}] \rangle$, such that

$$\begin{aligned} 1 - \left\{ \prod_{q=1}^n \left(1 - \underline{e}_{pq\mu}^l \right)^{\hat{z}_q} \right\} &\leq \hat{\psi}_\mu^{l-}; \quad 1 - \left\{ \prod_{q=1}^n \left(1 - \bar{e}_{pq\mu}^u \right)^{\hat{z}_q} \right\} \leq \hat{\psi}_\mu^{u+}, \quad (p = 1, 2, \dots, m) \\ \prod_{q=1}^n \left(\underline{e}_{pq\mu}^l \right)^{\hat{z}_q} &\geq \hat{\psi}_\mu^{l-}; \quad \prod_{q=1}^n \left(\bar{e}_{pq\mu}^u \right)^{\hat{z}_q} \geq \hat{\psi}_\mu^{u+}, \quad (p = 1, 2, \dots, m) \\ 0 &\leq \hat{\psi}_\mu^{l-}, \hat{\psi}_\mu^{u+}, \hat{\psi}_\nu^{l-}, \hat{\psi}_\nu^{u+} \leq 1 \\ \sum_{q=1}^n \hat{z}_q &= 1, \quad \hat{z}_q \geq 0 \quad (q = 1, 2, \dots, n). \end{aligned} \quad (45)$$

Furthermore,

$$\hat{\psi}_\mu^{l-} \leq \phi_\mu^{*l-}, \quad \hat{\psi}_\mu^{u+} \leq \phi_\mu^{*u+}, \quad \hat{\psi}_\nu^{l-} \geq \phi_\nu^{*l-}, \quad \hat{\psi}_\nu^{u+} \geq \phi_\nu^{*u+}. \quad (46)$$

Since $0 \leq \tau \leq 1$, from Problem (45), it is therefore derived that

$$\begin{aligned} \frac{1}{2} \left[2 - \left\{ \prod_{q=1}^n \left(1 - \bar{e}_{pq\mu}^u \right)^{\tau \hat{z}_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\underline{e}_{pq\mu}^l \right)^{\hat{z}_q} \right\}^{1-\tau} - \left\{ \prod_{q=1}^n \left(1 - \underline{e}_{pq\mu}^l \right)^{\tau \hat{z}_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\bar{e}_{pq\mu}^u \right)^{\hat{z}_q} \right\}^{1-\tau} \right] &\geq \\ \frac{1}{2} \left[2 - \left(\hat{\psi}_\mu^{u+} \right)^\tau \left(\hat{\psi}_\mu^{l-} \right)^{1-\tau} - \left(\hat{\psi}_\mu^{l-} \right)^\tau \left(\hat{\psi}_\mu^{u+} \right)^{1-\tau} \right], \quad (p = 1, 2, \dots, m) \\ 0 &\leq \hat{\psi}_\mu^{l-}, \hat{\psi}_\mu^{u+}, \hat{\psi}_\nu^{l-}, \hat{\psi}_\nu^{u+} \leq 1 \\ \sum_{q=1}^n \hat{z}_q &= 1, \quad \hat{z}_q \geq 0 \quad (q = 1, 2, \dots, n). \end{aligned} \quad (47)$$

and

$$\frac{1}{2} \left[2 - \left(\psi_\mu^{*u+} \right)^\tau \left(\psi_\nu^{*l-} \right)^{1-\tau} - \left(\psi_\mu^{*l-} \right)^\tau \left(\psi_\nu^{*u+} \right)^{1-\tau} \right] > \frac{1}{2} \left[2 - \left(\hat{\psi}_\mu^{u+} \right)^\tau \left(\hat{\psi}_\mu^{l-} \right)^{1-\tau} - \left(\hat{\psi}_\mu^{l-} \right)^\tau \left(\hat{\psi}_\mu^{u+} \right)^{1-\tau} \right]. \quad (48)$$

Let us take $\hat{\psi} = \frac{1}{2} [2 - (\hat{\psi}_\mu^{u+})^\tau (\hat{\psi}_\nu^{l-})^{1-\tau} - (\hat{\psi}_\mu^{l-})^\tau (\hat{\psi}_\nu^{u+})^{1-\tau}]$. Thus, Problem (47) can be written as

$$\begin{aligned} \frac{1}{2} \left[2 - \left\{ \prod_{q=1}^n \left(1 - \bar{e}_{pq\mu}^u \right)^{\tau \hat{z}_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\underline{e}_{pq\mu}^l \right)^{\hat{z}_q} \right\}^{1-\tau} - \left\{ \prod_{q=1}^n \left(1 - \underline{e}_{pq\mu}^l \right)^{\tau \hat{z}_q} \right\} \left\{ 1 - \prod_{q=1}^n \left(\bar{e}_{pq\mu}^u \right)^{\hat{z}_q} \right\}^{1-\tau} \right] &\geq \hat{\psi}, \\ (p = 1, 2, \dots, m) \\ 0 &\leq \hat{\psi}_\mu^{l-}, \hat{\psi}_\mu^{u+}, \hat{\psi}_\nu^{l-}, \hat{\psi}_\nu^{u+} \leq 1 \\ \sum_{q=1}^n \hat{z}_q &= 1, \quad \hat{z}_q \geq 0 \quad (q = 1, 2, \dots, n). \end{aligned} \quad (49)$$

Hence, we can infer that $(\hat{\mathbf{z}}, \hat{\psi})$ is a feasible solution of Problem (39). Furthermore, from Equation (46), we have $\hat{\psi} < \psi^*$, which contradicts the fact that (\mathbf{z}^*, ψ^*) is the optimal solution of Problem (39). Thus, our assumption is wrong, and hence, we may conclude that if (\mathbf{z}^*, ψ^*) is the optimal solution of the NLPP (39), then $(\mathbf{z}^*, \tilde{\Psi}^*)$ is the noninferior solution of the interval-valued hesitant fuzzy programming problem (17) for $0 \leq \tau \leq 1$. \square

Now, we can summarize the rendered methodology for solving matrix games with payoffs of IIHFEs in the form of a solution algorithm in the subsequent subsection.

Algorithm

- Step 1: Consider a matrix game $\tilde{\Gamma}$ whose payoffs are taken as IIHFEs, say, \tilde{q}_{pq} , where $p = 1, 2, \dots, m$ and $q = 1, 2, \dots, n$.
- Step 2: To solve the game, we have to derive two BOIIHFPMs, as depicted in Problems (16) and (17).
- Step 3: To handle the situation, the fuzzy programming problems are converted into two nonlinear programming Problems (29) and (39), respectively, by using the mathematical operations of IIHFEs.
- Step 4: To obtain the optimal strategies y^* for Player X_A and z^* for Player X_B , the NLPPs (29) and (39) are solved using WOLFRAM MATHEMATICA 9.0 software.
- Step 5: Utilizing the optimal strategies y^* and z^* in Equation (4), we can find the aggregated expected payoff of Player X_A .

To justify the proposed methodology we present the subsequent section.

5. Numerical Application

In reality, decision makers (DMs) have to rely on different sources of information. In most cases, DMs collect secondary data. So, DMs have to face several ambiguities about the accuracy of the information. In addition, occasionally DMs are provided with various asymmetric information in terms of some possible intervals. An IIHFE is the appropriate tool to assess these types of data sets. To validate the proposed methodology, this section discusses a real-life problem.

Application to Preventing Cybercrime

Nowadays, fraudulent email, theft of financial and corporate data, cyber-extortion, etc., are familiar words to everyone. These all are included in cybercrime. Cybercrime represents criminal activities that always target a networked device to hack information. In most cases, it is carried out by some well-managed organizations. They use highly advanced technologies to damage the network. According to the report of the NCRB [44], in the first half of the year 2020 only, almost 4.83 million cyber-attacks were reported globally. That means the number stands at 18 per minute. In India alone, the report of cybercrime in 2020 is nine times of the total from 2013. So, it has become a heightened threat to the safety of the citizen's life. Figure 1 depicts the state-wise reported number of cybercrimes in the last three years in India.

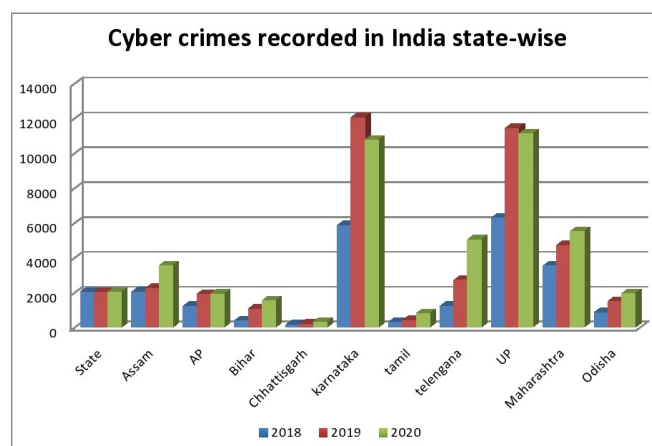


Figure 1. State-wise reported number of cybercrimes in last three years.

Here, both the hackers and the defenders act intelligently and anticipate every move of the opponent. We can portray such a relationship of the hackers and the defenders in the context of the theory of matrix games. Suppose the defenders are chosen as Player X_A and the hackers are chosen as Player X_B . The aim of the defenders, i.e., Player X_A , is to maximize the protection of all kinds of networked devices. For that they take two strategies:

α_1 : Installing a powerful firewall.

α_2 : Using updated software.

Meanwhile, the hackers, or Player X_B , use the following strategies:

β_1 : Breaking the security password.

β_2 : Sending suspicious links.

Suppose the crime control branch makes an annual report on cybercrime. For that, they recruit three persons (DMs) to gather asymmetric information. Studying the recent cases, the DMs assess their decisions (here, the payoff matrices) against the activities of the players (here, the defenders and the hackers). Since all of the activities are controlled by the human mind, the decisions are imprecise. So, capturing the hesitancy of the hackers' and the defenders' minds in terms of the degree of acceptance and rejection of a decision, we portray the matrix game under an IIHF environment. The payoff matrices assessed by three DMs are listed in Tables 1–3.

Table 1. Payoff matrix assigned by DM 1.

Strategies	β_1	β_2
α_1	$\widetilde{d}_{11}^1 = \langle [0.70, 0.80], [0.75, 0.90], [0.70, 0.85]; [0.50, 0.60], [0.50, 0.65] \rangle$	$\widetilde{d}_{12}^1 = \langle [0.65, 0.80], [0.65, 0.85]; [0.70, 0.85] \rangle$
α_2	$\widetilde{d}_{21}^1 = \langle [0.35, 0.40], [0.35, 0.50], [0.30, 0.40]; [0.50, 0.60], [0.45, 0.60] \rangle$	$\widetilde{d}_{22}^1 = \langle [0.65, 0.70]; [0.35, 0.45], [0.35, 0.50] \rangle$

Table 2. Payoff matrix assigned by DM 2.

Strategies	β_1	β_2
α_1	$\widetilde{d}_{11}^2 = \langle [0.70, 0.80], [0.75, 0.95]; [0.50, 0.60], [0.50, 0.70] \rangle$	$\widetilde{d}_{12}^2 = \langle [0.65, 0.70], [0.60, 0.75]; [0.60, 0.65], [0.50, 0.65] \rangle$
α_2	$\widetilde{d}_{21}^2 = \langle [0.30, 0.50]; [0.40, 0.55] \rangle$	$\widetilde{d}_{22}^2 = \langle [0.60, 0.70], [0.65, 0.75]; [0.35, 0.40] \rangle$

Table 3. Payoff matrix assigned by DM 3.

Strategies	β_1	β_2
α_1	$\widetilde{d}_{11}^3 = \langle [0.65, 0.75]; [0.55, 0.70] \rangle$	$\widetilde{d}_{12}^3 = \langle [0.52, 0.60], [0.50, 0.60]; [0.35, 0.40], [0.30, 0.45] \rangle$
α_2	$\widetilde{d}_{21}^3 = \langle [0.35, 0.45]; [0.50, 0.65], [0.50, 0.70] \rangle$	$\widetilde{d}_{22}^3 = \langle [0.60, 0.75], [0.60, 0.70], [0.65, 0.75]; [0.30, 0.50] \rangle$

We can signify the decisions as follows. Suppose Player X_A chooses strategy α_1 and Player X_B chooses strategy β_1 . Then according to the second DM, the initiative to prevent cybercrime is effective with a belief of 70–80% and with a disbelief of 50–60%. Here, the whole process is controlled by the human mind. So some circumstantial conditions may bring changes in the effectiveness of the strategies. Depending upon the situation, the second DM states that the strategies might be effective with a belief of 75–95% and

disbelief of 50–70%. Therefore, the decision of the second DM can be assessed in the form of an IIHFE as $\widetilde{d}_{11}^2 = \langle [0.70, 0.80], [0.75, 0.95]; [0.50, 0.60], [0.50, 0.70] \rangle$. Actually, \widetilde{d}_{11}^2 measures the damage prevention in terms of uncertainty. We can describe the other payoffs in a similar manner.

Suppose, the weights of the DMs are assigned as 0.3, 0.4, and 0.3, respectively. Thus, the aggregated payoff matrix $\widetilde{\mathbf{M}}$ with IIHFE entries can be written as

$$\widetilde{\mathbf{M}} = \begin{matrix} & \beta_1 & \beta_2 \\ \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} & \begin{pmatrix} \widetilde{q}_{11} & \widetilde{q}_{12} \\ \widetilde{q}_{21} & \widetilde{q}_{22} \end{pmatrix} \end{matrix}.$$

where \widetilde{q}_{pq} is aggregated as $\widetilde{q}_{pq} = 0.3 \widetilde{d}_{pq}^1 + 0.4 \widetilde{d}_{pq}^2 + 0.3 \widetilde{d}_{pq}^3$, $p = 1, 2; q = 1, 2$.

Applying the aggregation rule as defined in Definition 11, we can obtain each \widetilde{q}_{pq} as an IIHFE. The expressions of \widetilde{q}_{pq} ($p = 1, 2; q = 1, 2$) are given in Table 4.

Table 4. Aggregated value of the decisions.

\widetilde{q}_{pq} ($p = 1, 2; q = 1, 2$)	Aggregated IIHFES
\widetilde{q}_{11}	$\langle [0.6858, 0.7861], [0.7079, 0.8771], [0.7025, 0.8263], [0.7234, 0.9002], [0.6858, 0.8038], [0.7079, 0.8873], [0.5145, 0.6284], [0.5145, 0.6684], [0.5145, 0.6437], [0.5145, 0.6846] \rangle$
\widetilde{q}_{12}	$\langle [0.4608, 0.7358], [0.6104, 0.7358], [0.5941, 0.7544], [0.5891, 0.7544], [0.4608, 0.7576], [0.6104, 0.7576], [0.5941, 0.7747], [0.5941, 0.7747], [0.5365, 0.6089], [0.5123, 0.7309], [0.4969, 0.6089], [0.4745, 0.7309] \rangle$
\widetilde{q}_{21}	$\langle [0.3304, 0.4565], [0.3304, 0.4854], [0.3154, 0.4565], [0.4573, 0.5936], [0.4573, 0.6069], [0.4431, 0.5936], [0.4431, 0.6069] \rangle$
\widetilde{q}_{22}	$\langle [0.6157, 0.7160], [0.6157, 0.7000], [0.6309, 0.7160], [0.6356, 0.7359], [0.6356, 0.7211], [0.6500, 0.7359], [0.3342, 0.4437], [0.3342, 0.4573] \rangle$

Now, our objective is to find the optimal strategies for both players, as well as the expected aggregated payoff of Player X_A .

According to Problems (29) and (39), we construct Problems (50) and (51) to find the optimal strategies.

$$\begin{aligned} & \min \{ \phi \} \\ \text{s.t.} \quad & \frac{1}{2} \left[2 - \left\{ 0.0998^{y_1} 0.5146^{y_2} \right\}^\tau \left\{ 1 - 0.5145^{y_1} 0.4431^{y_2} \right\}^{1-\tau} - \left\{ 0.3142^{y_1} 0.6846^{y_2} \right\}^\tau \left\{ 1 - 0.6846^{y_1} 0.6069^{y_2} \right\}^{1-\tau} \right] \leq \phi \\ & \frac{1}{2} \left[2 - \left\{ 0.2253^{y_1} 0.2641^{y_2} \right\}^\tau \left\{ 1 - 0.4745^{y_1} 0.3342^{y_2} \right\}^{1-\tau} - \left\{ 0.5392^{y_1} 0.3843^{y_2} \right\}^\tau \left\{ 1 - 0.7309^{y_1} 0.4573^{y_2} \right\}^{1-\tau} \right] \leq \phi \\ & 0 \leq \phi \leq 1 \\ & y_1 + y_2 = 1, \quad y_p \geq 0 \quad (p = 1, 2) \end{aligned} \quad (50)$$

and

$$\begin{aligned} & \max \{ \psi \} \\ \text{s.t.} \quad & \frac{1}{2} \left[2 - \left\{ 0.0998^{z_1} 0.2253^{z_2} \right\}^\tau \left\{ 1 - 0.5145^{z_1} 0.4745^{z_2} \right\}^{1-\tau} - \left\{ 0.3142^{z_1} 0.5392^{z_2} \right\}^\tau \left\{ 1 - 0.6846^{z_1} 0.7309^{z_2} \right\}^{1-\tau} \right] \geq \psi \\ & \frac{1}{2} \left[2 - \left\{ 0.5146^{z_1} 0.2641^{z_2} \right\}^\tau \left\{ 1 - 0.4431^{z_1} 0.3342^{z_2} \right\}^{1-\tau} - \left\{ 0.6846^{z_1} 0.3843^{z_2} \right\}^\tau \left\{ 1 - 0.6069^{z_1} 0.4573^{z_2} \right\}^{1-\tau} \right] \geq \psi \\ & 0 \leq \psi \leq 1 \\ & z_1 + z_2 = 1, \quad z_q \geq 0 \quad (q = 1, 2). \end{aligned} \quad (51)$$

Here, τ is considered as the perception parameter of the players' choice. We solve NLPPs (50) and (51) using MATHEMATICA 9.0 software and obtain optimal strategies

y^*, z^* , the crisp equivalents of ϕ^* , and ψ^* for different values of the perception parameters. Furthermore, substituting the values of y^*, z^*, ϕ^*, ψ^* in Equation (4), the aggregated expected payoff $\tilde{E}_s(y, z)$ for Player X_A is obtained.

The results obtained for different values of the perception parameter $\tau \in [0, 1]$ are listed in Table 5.

Table 5. Result table.

τ	y^*	ϕ^*	z^*	ψ^*	$\tilde{E}_s(y, z)$	$Sc(\tilde{E}_s)$
0	(0,1)	0.5063	(1,0)	0.5063	$\langle [0.3154, 0.4854]; [0.4431, 0.6069] \rangle$	0.2492
0.2	(0,1)	0.5250	(1,0)	0.5250	$\langle [0.3154, 0.4854]; [0.4431, 0.6069] \rangle$	0.2492
0.4	(0,1)	0.5337	(0,1)	0.5337	$\langle [0.6157, 0.7359]; [0.3342, 0.4573] \rangle$	0.5601
0.6	(0,1)	0.5883	(0,1)	0.5883	$\langle [0.6157, 0.7359]; [0.3342, 0.4573] \rangle$	0.5601
0.8	(0.4518, 0.5482)	0.6301	(0.0096, 0.9904)	0.6339	$\langle [0.5518, 0.7542]; [0.3920, 0.5659] \rangle$	0.3481
1	(0.4898, 0.5102)	0.6510	(0.1336, 0.8664)	0.6682	$\langle [0.4315, 0.5861]; [0.5314, 0.6536] \rangle$	0.1494

The changes of score values of the expected payoffs for different values of the perception parameter τ for Player X_A are depicted in Figure 2.

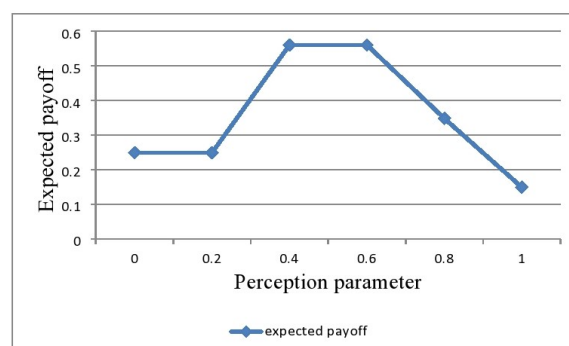


Figure 2. Change of expected payoffs with respect to the perception parameter τ .

From Table 5 and Figure 2, the following comments are listed:

- Table 5 shows that for $\tau = 0.8$, we obtain the optimal strategies for the players as $y^* = (0.4518, 0.5482)$ and $z^* = (0.0096, 0.9904)$. Furthermore, Player X_A 's aggregated expected payoff is obtained as $\langle [0.5518, 0.7542]; [0.3920, 0.5659] \rangle$. In other words, we may say that if Player X_A 's, i.e., the defenders', control strategies are α_1 with a probability of 0.4518 and α_2 with a probability of 0.5482 while Player X_B 's, i.e., the hackers', governing strategies are β_1 with a probability of 0.0096 and β_2 with a probability of 0.9904, and also, Player X_A is very much optimistic toward the information, then the DMs come to an inference that the damage will be reduced with a surety from 55.18% to 75.42% and with a disbelief from 39.20% to 56.59%.
- Table 5 shows that $\phi^* \leq \psi^*$ for all the values of the perception parameter τ . This symbolizes that the crisp equivalent of the amount of gain of Player X_A does not exceed the crisp equivalent amount of loss of Player X_B .
- Actually, the perception parameter captures the behavior of the players. From Table 5, we can see the crisp equivalent gain of Player X_A and the crisp equivalent loss of Player X_B both are gradually increased with the increment of the perception parameter chosen by the players. That indicates the gain value of Player X_A is maximum when he/she is optimistic enough.
- The score values of expected payoff for Player X_A are increased for $\tau \in [0, 0.5]$ and are decreased for $\tau \in [0.5, 1]$. This indicates that we can achieve the expected payoff for Player X_A at the maximum level when the players are neutral in nature. This shows the importance of the role of the perception parameter in our proposed model.

5. From Table 5, it is observed that the optimal strategies of the players change significantly with the changes in the perception parameter τ . Both of the players use pure strategies only when the perception parameter τ lies in the interval $[0, 0.6]$. Otherwise, the players use mixed strategies with some probabilities. For example, when $\tau = 0.8$, Player X_A uses strategies α_1 and α_2 with probabilities 0.4518 and 0.5482, respectively, while Player X_B uses strategies β_1 and β_2 with probabilities 0.0096 and 0.9904, respectively. Therefore, the players have the freedom to change their strategies with the changes in τ to optimize their expected payoffs.

6. Comparative Analysis and Discussion

Sections 6.1 and 6.2 justify the superiority of the rendered model in comparison with the presented methods of Bhaumik and Roy [39] and Xue et al. [45].

6.1. Comparison with Bhaumik and Roy [39]

In 2019, Bhaumik and Roy [39] introduced a special form of IIHFE in the literature along with a new aggregation operator. Aiming to consider the hesitancy of the players in terms of both their acceptance and rejection of the possible range of asymmetric information, they applied the notion of IIHFEs in matrix games. To make the solution to this problem, Bhaumik and Roy derived two interval-valued intuitionistic hesitant fuzzy programming models as follows:

$$\begin{aligned} & \max \left\langle \bigcup_{\Delta} [V_{\mu}^l, V_{\mu}^u]; \bigcup_{\Delta} [V_{\nu}^l, V_{\nu}^u] \right\rangle \\ \text{subject to } & \sum_{p=1}^m \left\langle \bigcup_{\Delta} [e_{pq\mu}^l, e_{pq\mu}^u]; \bigcup_{\Delta} [e_{pq\nu}^l, e_{pq\nu}^u] \right\rangle y_p \geq_{IIHF} \left\langle \bigcup_{\Delta} [V_{\mu}^l, V_{\mu}^u]; \bigcup_{\Delta} [V_{\nu}^l, V_{\nu}^u] \right\rangle \\ & \sum_{p=1}^m y_p = 1; \quad y_p \geq 0 \text{ for } p = 1, 2, \dots, m. \end{aligned} \quad (52)$$

and

$$\begin{aligned} & \min \left\langle \bigcup_{\Delta} [W_{\mu}^l, W_{\mu}^u]; \bigcup_{\Delta} [W_{\nu}^l, W_{\nu}^u] \right\rangle \\ \text{subject to } & \sum_{q=1}^n \left\langle \bigcup_{\Delta} [e_{pq\mu}^l, e_{pq\mu}^u]; \bigcup_{\Delta} [e_{pq\nu}^l, e_{pq\nu}^u] \right\rangle z_q \leq_{IIHF} \left\langle \bigcup_{\Delta} [W_{\mu}^l, W_{\mu}^u]; \bigcup_{\Delta} [W_{\nu}^l, W_{\nu}^u] \right\rangle \\ & \sum_{q=1}^n z_q = 1; \quad z_q \geq 0 \text{ for } q = 1, 2, \dots, n. \end{aligned} \quad (53)$$

where $\langle \bigcup_{\Delta} [V_{\mu}^l, V_{\mu}^u]; \bigcup_{\Delta} [V_{\nu}^l, V_{\nu}^u] \rangle = \tilde{V}$ and $\langle \bigcup_{\Delta} [W_{\mu}^l, W_{\mu}^u]; \bigcup_{\Delta} [W_{\nu}^l, W_{\nu}^u] \rangle = \tilde{W}$ are Player X_A 's gain floor and Player X_B 's loss ceiling, respectively.

Taking the defined ranking approach (assigned by Bhaumik and Roy [39]), these two problems are converted into the following linear programming models:

$$\begin{aligned} & \max \frac{1}{2} \left[\left(\underline{V}_{\mu}^l + \overline{V}_{\mu}^u \right) - \left(\underline{V}_{\nu}^l + \overline{V}_{\nu}^u \right) \right] \\ \text{subject to } & \sum_{p=1}^m \left[\underline{e}_{pq\mu}^l, \overline{e}_{pq\mu}^u \right] y_p \geq \left[\underline{V}_{\mu}^l, \overline{V}_{\mu}^u \right]; \quad \sum_{p=1}^m \left[\underline{e}_{pq\nu}^l, \overline{e}_{pq\nu}^u \right] y_p \leq \left[\underline{V}_{\nu}^l, \overline{V}_{\nu}^u \right] \\ & \underline{V}_{\mu}^l \leq \overline{V}_{\mu}^u; \quad \underline{V}_{\nu}^l \leq \overline{V}_{\nu}^u; \quad 0 \leq \overline{V}_{\mu}^u + \overline{V}_{\nu}^u \leq 1 \\ & \sum_{p=1}^m y_p = 1; \quad y_p \geq 0 \text{ for } p = 1, 2, \dots, m. \end{aligned} \quad (54)$$

and

$$\begin{aligned}
& \max \frac{1}{2} \left[\left(\underline{W}_\mu^l + \overline{W}_\mu^u \right) - \left(\underline{W}_v^l + \overline{W}_v^u \right) \right] \\
\text{subject to} \quad & \sum_{q=1}^n \left[\underline{\varrho}_{pq\mu}^l, \overline{\varrho}_{pq\mu}^u \right] z_q \leq \left[\underline{W}_\mu^l, \overline{W}_\mu^u \right]; \sum_{q=1}^n \left[\underline{\varrho}_{pqv}^l, \overline{\varrho}_{pqv}^u \right] z_q \geq \left[\underline{W}_v^l, \overline{W}_v^u \right] \\
& \underline{W}_\mu^l \leq \overline{W}_\mu^u; \underline{W}_v^l \leq \overline{W}_v^u; 0 \leq \overline{W}_\mu^u + \overline{W}_v^u \leq 1 \\
& \sum_{q=1}^n z_q = 1; \quad z_q \geq 0 \text{ for } q = 1, 2, \dots, n.
\end{aligned} \tag{55}$$

where $\langle [\underline{V}_\mu^l, \overline{V}_\mu^u]; [\underline{V}_v^l, \overline{V}_v^u] \rangle$, $\langle [\underline{W}_\mu^l, \overline{W}_\mu^u]; [\underline{W}_v^l, \overline{W}_v^u] \rangle$, and $\langle [\underline{\varrho}_{pq\mu}^l, \overline{\varrho}_{pq\mu}^u]; [\underline{\varrho}_{pqv}^l, \overline{\varrho}_{pqv}^u] \rangle$ are the aggregated form of the preassigned gain floor, loss ceiling, and payoffs, respectively.

Now, for the matrix \tilde{M} , the LP problems according to Problems (54) and (55) are constructed as

$$\begin{aligned}
& \max \frac{1}{2} \left[\left(\underline{V}_\mu^l + \overline{V}_\mu^u \right) - \left(\underline{V}_v^l + \overline{V}_v^u \right) \right] \\
\text{subject to} \quad & 0.3142y_1 + 0.6846y_2 \geq \underline{V}_\mu^l; \quad 0.5392y_1 + 0.3843y_2 \geq \underline{V}_\mu^l \\
& 0.0998y_1 + 0.5146y_2 \geq \overline{V}_\mu^u; \quad 0.2253y_1 + 0.2641y_2 \geq \overline{V}_\mu^u \\
& 0.5145y_1 + 0.4431y_2 \leq \underline{V}_v^l; \quad 0.4745y_1 + 0.3342y_2 \leq \underline{V}_v^l \\
& 0.6846y_1 + 0.6069y_2 \leq \overline{V}_v^l; \quad 0.7309y_1 + 0.4573y_2 \leq \overline{V}_v^l \\
& \underline{V}_\mu^l \leq \overline{V}_\mu^u; \quad \underline{V}_v^l \leq \overline{V}_v^u; \quad 0 \leq \overline{V}_\mu^u + \overline{V}_v^u \leq 1 \\
& y_1 + y_2 = 1; \quad y_p \geq 0 \text{ for } p = 1, 2.
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
& \max \frac{1}{2} \left[\left(\underline{W}_\mu^l + \overline{W}_\mu^u \right) - \left(\underline{W}_v^l + \overline{W}_v^u \right) \right] \\
\text{subject to} \quad & 0.3142z_1 + 0.5392z_2 \leq \underline{W}_\mu^l; \quad 0.6846z_1 + 0.3843z_2 \leq \underline{W}_\mu^l \\
& 0.0998z_1 + 0.2253z_2 \leq \overline{W}_\mu^u; \quad 0.5146z_1 + 0.2641z_2 \leq \overline{W}_\mu^u \\
& 0.5145z_1 + 0.4745z_2 \geq \underline{W}_v^l; \quad 0.4431z_1 + 0.3342z_2 \geq \underline{W}_v^l \\
& 0.6846z_1 + 0.7309z_2 \geq \overline{W}_v^l; \quad 0.6069y_1 + 0.4573y_2 \geq \overline{W}_v^l \\
& \underline{W}_\mu^l \leq \overline{W}_\mu^u; \quad \underline{W}_v^l \leq \overline{W}_v^u; \quad 0 \leq \overline{W}_\mu^u + \overline{W}_v^u \leq 1 \\
& z_1 + z_2 = 1; \quad z_q \geq 0 \text{ for } q = 1, 2.
\end{aligned} \tag{57}$$

The computational results, obtained by solving Problems (56) and (57), are listed in Table 6.

Table 6. Results obtained by Bhaumik and Roy's [39] method.

y^*	\tilde{V}^*	$Sc(\tilde{V}^*)$	z^*	\tilde{W}^*	$Sc(\tilde{W}^*)$	$\tilde{E}_b(y, z)$	$Sc(\tilde{E}_b)$
(0,1)	$\langle [0.2641, 0.2850]; [0.4431, 0.6069] \rangle$	0.5009	(0.2949, 0.7051)	$\langle [0.4728, 0.4830]; [0.3663, 0.5014] \rangle$	0.0881	$\langle [0.2529, 0.3446]; [0.3706, 0.3751] \rangle$	0.1482

Analyzing the computed results portrayed in Tables 5 and 6, the following observations are made:

- (i) In our proposed method, the perception parameter plays a significant role. Table 5 shows that Player X_A 's gain-floor, Player X_B 's loss ceiling, and Player X_A 's expected payoff are changed with the changes in the perception parameter τ . It is observed that when the players are optimistic toward the information, i.e., when τ lies in $[0.5, 1]$, players have better gain values. Hence, the players have the option to

choose different parameter values to obtain better gains. However, in Bhaumik and Roy's [39] methodology, there is no such parameter that can improve the gain of the player.

- (ii) In classical game theory, there is a most celebrated result that the gain floor of the winning player never exceeds the loss ceiling of the defeated player. In Theorem 2, we showed that this endures also in the IIHF environment. However, if we solve the same problem by Bhaumik and Roy's [39] approach, we have $Sc(\tilde{V}^*) > Sc(\tilde{W}^*)$. Table 6 shows that the gain floor \tilde{V}^* is greater than the loss ceiling \tilde{W}^* , which contradicts the statement of Theorem 2.
- (iii) Comparing Tables 5 and 6, we have $Sc(\tilde{E}_b) < Sc(\tilde{E}_s)$. This implies $\tilde{E}_b <_{IIHF} \tilde{E}_s$ for all values of τ , the perception parameter. This implies that our proposed methodology gives better results as compared to Bhaumik and Roy's [39] approach.
- (iv) Furthermore, it is worth noting that we need to preserve the hesitant character of IIHFEs. However, to preserve the character, if we apply mathematical operations for IIHFEs in the interval-valued intuitionistic hesitant fuzzy programming models, it yields two NLPP models instead of LP models. This phenomenon raises an essential question on the validation of Bhaumik and Roy's [39] approach.

So, considering all these aspects, we can claim that our proposed model is superior to the existing model.

6.2. Comparison with Xue et al. [45]

This subsection provides a comparison with the celebrated methodology of matrix games discussed by Xue et al. [45]. Xue et al. [45] used hesitant fuzzy elements as the payoffs of a matrix game. To make a comparison, we promote the hesitant fuzzy payoffs assessed by Xue et al. [45] into suitable interval-valued hesitant fuzzy payoffs. The newly promoted payoff matrix \tilde{H}^* is assigned in Table 7.

Table 7. Payoff matrix \tilde{H}^* in tabular form.

Strategy	ρ_1	ρ_2	ρ_3	ρ_4
ρ_1	$\langle [0.17, 0.22], [0.28, 0.32] \rangle$	$\langle [0.56, 0.62], [0.68, 0.72], [0.75, 0.82] \rangle$	$\langle [0.48, 0.52], [0.56, 0.62] \rangle$	$\langle [0.35, 0.43], [0.56, 0.62] \rangle$
ρ_2	$\langle [0.45, 0.55], [0.65, 0.72] \rangle$	$\langle [0.17, 0.22], [0.28, 0.32] \rangle$	$\langle [0.36, 0.42], [0.56, 0.62] \rangle$	$\langle [0.38, 0.44], [0.47, 0.53] \rangle$
ρ_3	$\langle [0.36, 0.42], [0.48, 0.54] \rangle$	$\langle [0.67, 0.72], [0.75, 0.82] \rangle$	$\langle [0.05, 0.12], [0.15, 0.22], [0.27, 0.32] \rangle$	$\langle [0.35, 0.43], [0.54, 0.63] \rangle$
ρ_4	$\langle [0.48, 0.52], [0.56, 0.62], [0.75, 0.82] \rangle$	$\langle [0.78, 0.82], [0.85, 0.95] \rangle$	$\langle [0.77, 0.82], [0.85, 0.92] \rangle$	$\langle [0.28, 0.32], [0.35, 0.45] \rangle$

Now, to solve this matrix game, here, we use Problems (29) and (39). Problems (29) and (39) are two NLPPs that contain both membership and nonmembership values (i.e., the acceptance and nonacceptance of the players toward the information) of the aggregated payoffs. However, the payoffs described in Table 7 only contain the membership values. So, we switch off all nonmembership functions in Problems (29) and (39). Next, to reach the optimal values, we have to solve NLPPs (58) and (59).

$$\begin{aligned}
& \min \phi \\
\text{subject to} & \quad \frac{1}{2} \left[2 - 0.68^{y_1} 0.28^{y_2} 0.46^{y_3} 0.18^{y_4} - 0.83^{y_1} 0.55^{y_2} 0.64^{y_3} 0.52^{y_4} \right] \leq \phi \\
& \quad \frac{1}{2} \left[2 - 0.18^{y_1} 0.68^{y_2} 0.18^{y_3} 0.05^{y_4} - 0.44^{y_1} 0.83^{y_2} 0.33^{y_3} 0.22^{y_4} \right] \leq \phi \\
& \quad \frac{1}{2} \left[2 - 0.38^{y_1} 0.38^{y_2} 0.68^{y_3} 0.08^{y_4} - 0.52^{y_1} 0.64^{y_2} 0.95^{y_3} 0.23^{y_4} \right] \leq \phi \\
& \quad \frac{1}{2} \left[2 - 0.38^{y_1} 0.47^{y_2} 0.37^{y_3} 0.55^{y_4} - 0.65^{y_1} 0.62^{y_2} 0.65^{y_3} 0.72^{y_4} \right] \leq \phi \quad (58)
\end{aligned}$$

and

$$\begin{aligned}
& \max \psi \\
\text{subject to} & \quad \frac{1}{2} \left[2 - 0.68^{y_1} 0.18^{y_2} 0.38^{y_3} 0.38^{y_4} - 0.83^{y_1} 0.44^{y_2} 0.52^{y_3} 0.65^{y_4} \right] \leq \psi \\
& \quad \frac{1}{2} \left[2 - 0.28^{y_1} 0.68^{y_2} 0.38^{y_3} 0.47^{y_4} - 0.55^{y_1} 0.83^{y_2} 0.64^{y_3} 0.62^{y_4} \right] \leq \psi \\
& \quad \frac{1}{2} \left[2 - 0.46^{y_1} 0.18^{y_2} 0.68^{y_3} 0.37^{y_4} - 0.64^{y_1} 0.33^{y_2} 0.95^{y_3} 0.65^{y_4} \right] \leq \psi \\
& \quad \frac{1}{2} \left[2 - 0.18^{y_1} 0.05^{y_2} 0.08^{y_3} 0.55^{y_4} - 0.52^{y_1} 0.22^{y_2} 0.23^{y_3} 0.72^{y_4} \right] \leq \psi \quad (59)
\end{aligned}$$

By solving Problems (58) and (59), we obtain the optimal strategies for Players X_A and X_B along with their game values. In Table 8, we list both the results obtained by the proposed method and Xue et al.'s [45] method.

Table 8. Comparative results of proposed method and Xue et al.'s [45] method.

Outcomes	Proposed Method	Xue et al.'s [45] Method
\mathbf{y}^*	(0.3147, 0.5946, 0.0907, 0)	(0.5, 0.5, 0, 0)
\mathbf{z}^*	(0.4442, 0.2498, 0.3060, 0)	(0.1243, 0, 0.1014, 0.7743)
ϕ^*	0.4890	0.1455
ψ^*	0.9778	0.9354

The following observations are recorded:

- (i) Table 8 shows that we can obtain better optimal values for both players if we choose our rendered model. Thus, it is clear that the proposed model is superior to the model explored by Xue et al. [45].
- (ii) Using the optimal strategies \mathbf{y}^* and \mathbf{z}^* for Players X_A and X_B , respectively, if we calculate the expected payoff for Player X_A , we have $E^* = [0.3718, 0.6063]$. The score function of this expected payoff is $Sc(E^*) = 0.4891$, which gives a better value than the existing one [45].
- (iii) Moreover, Xue et al. [45] considered only the hesitancy of decision makers to portray the matrix game problem. However, occasionally in practical problems, DMs fail to judge the asymmetric information scenario properly. As a result, they cannot assign any precise value. Consequently, they have to choose some interval values to assign the payoffs. In that sense, delineating a matrix game problem in an intuitionistic interval-valued hesitant fuzzy environment is more realistic in the literature.

7. Conclusions

In reality, occasionally we fail to gather accurate information about an event, and we have to rely on a possible range of information. Consequently, decision makers have some hesitancy about the asymmetric information. To deal with such a situation, we actualize a matrix game problem with interval-valued intuitionistic hesitant fuzzy entries. To solve this, we derive two bi-objective interval-valued intuitionistic hesitant fuzzy programming

models. It is difficult to handle such programming models traditionally. To establish the model, we define a new aggregation operator, viz., the min-max enclosure aggregation operator, for IIHFEs. Based on this operator, the ranking order of the IIHFEs is assigned. To obtain the optimal values, two interval-valued intuitionistic hesitant fuzzy programming problems are derived, which are converted into two nonlinear programming models with the perception parameters of the players.

In classical game theory, there is a most celebrated result that the gain floor of the maximizing player is always less than or equal to the loss ceiling of the minimizing player. In our discussion, we showed that this result is also valid in the IIHF environment. Here, we have explained that the perception parameter of the players plays an important role in this model. Depending on this parameter, the value of the game and the expected payoff of the conquering player are changed. To solve this problem, we have to change the fuzzy programming problem to a nonlinear programming problem. However, we were able to show that the solution obtained from the BOIIHFPMs is a noninferior solution to the solution obtained from the derived NLPPs. With a valid comparison, we showed that our executed model is superior to the existing methodologies proposed by Bhaumik and Roy [39] and Xue et al. [45].

To check the cogency of the proposed methodology, we presented an imprecise and asymmetric information scenario of hackers and of the defenders attempting to prevent cybercrime. Here, it is observed that both the gain of the winning player and the loss of the defeated player increase with the increment of the perception parameter of the player.

Besides several advantages, the proposed methodology has some limitations. One of the major limitations is that we failed to calculate the gains and losses of the players in terms of IIHFEs. Another limitation is that we generated the solution procedure by presuming the existence of a solution for matrix games in the IIHF environment. However, we cannot establish any theorems about it. Therefore, further investigation is needed in the future, which can be considered a future study of the rendered methodology.

This methodology can be applied to various decision-making problems in an IIHF environment. Moreover, in various conflicting situations, portraying the situation in terms of a possible range of acceptance and nonacceptance by the players, along with the degree of hesitancy of the players towards the information, such as various management issues, war science, psychological issues, etc., can be resolved by our proposed methodology.

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