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Solving Boundary Value Problems by Sinc Method and Geometric Sinc Method

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Abstract: This paper introduces an efficient numerical method for approximating solutions to geometric boundary value problems. We propose the multiplicative sinc–Galerkin method, tailored specifically for solving multiplicative differential equations. The method utilizes the geometric Whittaker cardinal function to approximate functions and their geometric derivatives. By reducing the geometric differential equation to a system of algebraic equations, we achieve computational efficiency. The method not only proves to be computationally efficient but also showcases a valuable symmetric property, aligning with inherent patterns in geometric structures. This symmetry enhances the method's compatibility with the often-present symmetries in geometric boundary value problems, offering both computational advantages and a deeper understanding of geometric calculus. To demonstrate the reliability and efficiency of the proposed method, we present several examples with both homogeneous and non-homogeneous boundary conditions. These examples serve to validate the method's performance in practice.

Keywords: geometric sinc method; non-Newtonian calculus; geometric vector space; geometric orthonormal set

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Citation: Darweesh, A.; Al-Khaled, K.; Algamara, M. Solving Boundary Value Problems by Sinc Method and Geometric Sinc Method. *Symmetry* **2024**, *16*, 411. https://doi.org/10.3390/sym16040411

Academic Editors: Sergei D. Odintsov, Pasquale Candito and Junesang Choi

Received: 24 February 2024
Revised: 11 March 2024
Accepted: 30 March 2024
Published: 1 April 2024

1. Introduction

Ordinary differential equations describing physical phenomena in science and engineering, such as electromagnetism and mechanical vibrations, often involve boundary value problems. These problems are tricky for numerical analysis because they are usually non-linear, making it challenging to find analytical solutions. In many cases, we cannot easily find these solutions, so we turn to numerical methods to obtain an approximate answer.

Various numerical methods and adjustments help us calculate an approximate solution for two-point boundary value problems. These methods include the shooting method [1], finite-difference method [2], finite-element method [3], least squares method [4], collection method [5], the Rayleigh–Ritz method [6], and the sinc–Galerkin method [7].

Over the last few decades, the sinc function and the sinc–Galerkin method have become very popular. The sinc method was introduced by Frank Stenger in [7] and was developed by him in [8]. A thorough study of sinc methods for two-point boundary value problems can be found in [9–12]. Compared to other numerical methods, this method differs in two primary ways: it decreases the error exponentially and provides reliable results at singular points. The sinc function, also known as the cardinal sine function, is a symmetric, wavelike function represented by $\mathrm{sinc}(x)$ that oscillates similarly to its close relative $\sin(x)$.

The sinc function is the foundation of signal analysis. It is critical for understanding many concepts involved in signal processing, making it the foundation of electronics. Its even symmetry, combined with its oscillating property, is extremely useful in a variety of modulation schemes. Symmetry principles are used in a variety of signal and data processing applications. For example, symmetrical Siamese neural networks are extremely efficient in image recognition tasks. Also, sinc functions are frequently used in applications requiring Fourier analysis. They are, for example, used to study Fourier analysis theory, which has theoretical applications because of its ability to maintain the original signal. This is met when the sampling frequency is at least twice the highest frequency in the signal. In general, the sinc function is widely used in communication systems for transmission and analysis because of its symmetry property, which aids in the frequency domain representation of various time domain signals. These applications demonstrate the need for appropriate methods to solve differential equations using sinc function methodology.

In 1972, Michael Grossman and Robert Katz [13] introduced a different kind of math known as multiplicative calculus, altering the traditional calculus created by Isaac Newton and Gottfried Wilhelm Leibnitz in the 17th century. It is sometimes referred to as non-Newtonian calculus. The operations in multiplicative calculus are called the multiplicative derivative and integral. Since then, numerous studies [14–18] have delved into this area.

Stanley [14], Bashirov et al. [15], and Riza et al. [17] discussed a popular form of non-Newtonian calculus. In contrast to Newton's and Leibnitz's, multiplicative calculus has a more limited application, focusing solely on positive functions. Misirli and Gurefe introduced multiplicative Adams–Bashforth–Moulton methods for solving differential equations [18].

Our geometric sinc method for solving multiplicative differential equations distinguishes itself from other existing approaches through its unique geometric framework. In contrast to the work in "Numerical methods for the multiplicative partial differential equations", which relies on explicit and implicit Euler as well as Crank–Nicolson algorithms [19], our method employs sinc functions and geometric principles.

The paper "Generalized Runge-Kutta Method with respect to the Non-Newtonian Calculus" explores non-Newtonian calculus applied to the Runge–Kutta method; see [20]. In comparison, our geometric sinc method utilizes sinc functions and geometric insights, offering a distinct perspective on solving multiplicative differential equations.

Furthermore, the paper "Multiplicative Finite Difference Methods" proposes finite difference schemes based on multiplicative calculus; see [17]. In contrast, our geometric sinc method integrates geometric concepts with sinc functions, potentially providing enhanced stability and accuracy in solving multiplicative differential equations.

Through systematic testing and analysis, the geometric sinc method aims to establish its efficacy and efficiency compared to these existing methods, highlighting the advantages of its geometric and sinc function-based approach in addressing the challenges posed by multiplicative differential equations.

The rest of this paper is organized as follows. In Section 2, we give some required definitions and consequences related to the sinc function, and we present its interpolation and quadrature rules. In Section 3, the basic definitions of geometric calculus are discussed, the geometric derivative and integral are presented, and some theorems are outlined. In Section 4, the geometric sinc function is defined, and its interpolation and quadrature rules are presented. In Section 5, the methodology of the geometric sinc–Galerkin method for solving geometric boundary value problems is introduced. In Section 6, we treat the non-homogeneous boundary conditions. Finally, in Section 7, we illustrate two examples for different cases, and we support these with numerical tables to show the significance of this method.

2. Sinc Function Preliminaries

In this section, we summarize the important concepts about the sinc method; for more details, see [10]. The function is defined for all x on the real line by

$$\operatorname{sinc}(x) \equiv \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is called the sinc function.

The translated sinc function is $\mathrm{sinc}[(x-kh)/h]$, for $h>0$ and $k=0,\pm1,\pm2,\ldots$, and the series $\sum_{k=-\infty}^{\infty} f(kh)\,\mathrm{sinc}[(z-kh)/h]$ is called the cardinal series of f.

The Paley–Wiener class of functions $B(h)$ is the family of entire functions f that, on the real line $x \to f(x,y)$ is in the space $\mathbf{L^2}(\mathbb{R})$, for all $y \in \mathbb{R}$, and in the complex plane f, satisfy

$$|f(z)| \le K \exp\left(\frac{\pi|z|}{h}\right)$$

for some $K>0$.

Theorem 1 ([10])**.** *Let $f \in B(h)$. Then, for all $z \in \mathbf{C}$, we have*

$$f(z) = \sum_{k=-\infty}^{\infty} f(kh)\,\mathrm{sinc}[(z-kh)/h]. \tag{1}$$

Moreover, if $f \in \mathbf{L^1}(\mathbf{R})$, then

$$\int_{-\infty}^{\infty} f(x)dx = h \sum_{k=-\infty}^{\infty} f(kh).$$

The sinc rules for a particular class of functions $B(\mathcal{E})$ have been created. The following definition describes the properties of such functions in $B(\mathcal{E})$.

Definition 1. *Let $\mathcal{E}$ be a simply connected domain in the complex plane and $\partial\mathcal{E}$ denote its boundary. Let a,b $(a \ne b)$ be points in $\mathcal{E}$, and let H be a conformal map from $\mathcal{E}$ to the infinite strip $\mathcal{S}$ of the width 2θ*

$$\mathcal{S} = \left\{ w = t + is : |s| < \theta \le \frac{\pi}{2} \right\},$$

where $H(a) = -\infty$ and $H(b) = \infty$. If the inverse map of H is denoted by J, define

$$\Lambda = \{ J(t) \in \mathcal{E} : -\infty < t < \infty \} = J(\mathbb{R})$$

and let

$$z_k = J(kh), \quad k = 0, \pm1, \pm2, \ldots$$

Definition 2. *Define $B(\mathcal{E})$ as the set of functions, F, that are analytic in $\mathcal{E}$ and satisfy*

$$\int_{J(t+L)} |F(z)|dz \to 0 \quad as \quad t \to \pm\infty,$$

where $L = \{ is : |s| < \theta \le \frac{\pi}{2} \}$, and, on the boundary of $\mathcal{E}$ (denoted by $\partial\mathcal{E}$), satisfies

$$N(F) = \int_{\partial\mathcal{E}} |F(z)dz| < \infty.$$

Theorem 2 ([10])**.** *Let $H'F \in B(\mathcal{E})$ and $h>0$. Let H be a conformal map of $\mathcal{E}$ onto the infinite strip $\mathcal{S}$. Further assume that there exist positive constants ζ, λ, and C such that*

$$|F(z)| \le C \begin{cases} \exp(-\zeta|H(z)|), & z \in \Lambda_a \\ \exp(-\lambda|H(z)|), & z \in \Lambda_b \end{cases}$$

where

$$\Lambda_a \equiv \{ z \in \Lambda : H(z) = x \in (-\infty, 0) \}$$

and

$$\Lambda_b \equiv \{ z \in \Lambda : H(z) = x \in [0, \infty) \}.$$

Make the selections

$$N = \left[\left|\frac{\zeta}{\lambda} M + 1\right|\right]$$

and

$$h = \sqrt{\frac{\pi\theta}{\zeta M}}.$$

Then, for all $x \in \Lambda$, the error

$$\epsilon_{M,N}(H'F) \equiv F(x) - \sum_{k=-M}^{N} F(z_k)\,\mathrm{sinc}\left(\frac{H(x) - kh}{h}\right)$$

is bounded by

$$\|\epsilon_{M,N}(H'F)\|_\infty \leq K\sqrt{M}\exp\left(-\sqrt{\pi\theta\zeta M}\right)$$

where K is a constant that depends solely on F, θ, and H.

Theorem 3 ([10])**.** *Assume $F \in B(\mathcal{E})$ and $h > 0$. Let H be a conformal map of $\mathcal{E}$ to the infinite strip $\mathcal{S}$. Consider $J = H^{-1}$, $z_k = J(kh)$, and $\Lambda = J(\mathbb{R})$. Assume that there exist positive constants ζ, λ, and C such that*

$$\left|\frac{F(x)}{H'(x)}\right| \leq C\begin{cases} \exp(-\zeta|H(x)|), & x \in \Lambda_a \\ \exp(-\lambda|H(x)|), & x \in \Lambda_b \end{cases}$$

where

$$\Lambda_a \equiv \{z \in \Lambda : H(z) = x \in (-\infty, 0)\}$$

and

$$\Lambda_b \equiv \{z \in \Lambda : H(z) = x \in [0, \infty)\}.$$

Make the selections

$$N = \left[\left|\frac{\zeta}{\lambda} M + 1\right|\right]$$

and

$$h = \sqrt{\frac{\pi\theta}{\zeta M}}.$$

So, the sinc trapezoidal quadrature rule is

$$I_{M,N}(F) \equiv \int_\Lambda F(z)dz - h\sum_{k=-M}^{N} \frac{F(z_k)}{H'(z_k)}$$

which is bounded by

$$|I_{M,N}(F)| \leq K\exp\left(-\sqrt{\pi\theta\zeta M}\right)$$

where K is a constant that depends solely on F, θ, and H.

3. Geometric Calculus

The concept of multiplicative calculus has developed over the last decade. It provides differentiation and integration tools that rely on multiplication rather than addition. Furthermore, multiplicative calculus has numerous applications, such as modeling finance, engineering, economics, biological image analysis, and literary works.

A complete ordered field is a field $(\mathcal{Y}, \oplus, \otimes)$ with an ordering relation, $\prec_Y$, that establishes a meaningful order among the elements in $\mathcal{Y}$. A crucial condition for a complete ordered field is "completeness", which means that any non-empty subset of $\mathcal{Y}$ that is bounded above must have a least upper bound (or supremum) within $\mathcal{Y}$. This completeness property ensures that the field has no "gaps" or missing elements, making it robust for mathematical analysis. The content in this section is summarized from references [13,14].

Consider any generator β with a range denoted by $\mathcal{B}$. In the realm of β-arithmetic, we introduce the operations and ordering relations as follows:

For β-Addition ($x \oplus y$), we determine it as follows

$$x \oplus y = \beta(\beta^{-1}(x) \oplus \beta^{-1}(y))$$

Similarly, for β-Subtraction ($x \ominus y$), β-Multiplication ($x \ominus y$), and β-Division ($x \ominus y$), the definitions are analogous:

$$x \ominus y = \beta(\beta^{-1}(x) \ominus \beta^{-1}(y))$$

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The β-Order ($x \prec_Y y$) is defined by the relation

$$x \prec_Y y \iff \beta^{-1}(x) \ominus \beta^{-1}(y) \dot{<} 0.$$

For the specific β-generator, $\beta(x) = e^x$ for x in the real numbers ($x \in \mathbb{R}$), and the inverse of β is given by $\beta^{-1}(x) = \ln(x)$. In this context, the β-arithmetic takes on a special significance known as geometric arithmetic.

For this choice, the geometric arithmetic operations become:

$$x \oplus y = \beta(\beta^{-1}(x) + \beta^{-1}(y)) = \exp((\ln(x) + \ln(y))) = xy.$$

$$x \ominus y = \beta(\beta^{-1}(x) - \beta^{-1}(y)) = \exp((\ln(x) - \ln(y))) = \frac{x}{y}$$

$$x \odot y = \beta(\beta^{-1}(x) \times \beta^{-1}(y)) = \exp((\ln(x) \times \ln(y))) = x^{\ln(y)}$$

$$x \oslash y = \beta(\beta^{-1}(x) / \beta^{-1}(y)) = \exp((\ln(x) / \ln(y))) = x^{\frac{1}{\ln(y)}} \quad \text{where } y \neq 1.$$

Now, we introduce the definition of a geometric derivative.

Definition 3. *Let $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a positive function. The *derivative of the function g is given by*

$$g^*(x) = \lim_{h \to 0} \left(\frac{g(x+h)}{g(x)} \right)^{\frac{1}{h}}.$$

If f is differentiable, then we have

$$f^*(x) = \lim_{h \to 0} \left(\frac{f(x+h)}{f(x)}) \right)^{\frac{1}{h}} = \exp\left(\frac{f'(x)}{f(x)} \right) = \exp\big((\ln \circ f)'(x)\big)$$

where $(\ln \circ f)(x) = \ln(f(x))$. In fact, the authors in [21] proved that for any β generator, $f : \mathbb{R} \to \mathcal{I}$ is differentiable if and only if it is β-differentiable.

If we repeat this procedure n times, we can conclude that if $f(x)$ is a positive function and $f^{(n)}(x)$ exists, then

$$f^{*(n)}(x) = \exp\Big((\ln \circ f)^{(n)}(x) \Big).$$

The following theorem summarizes some rules of *differentiation:

Theorem 4 ([13])**.** *Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions. Then, the following rules apply:*

(a) *Scaling: $(cf)'(x) = f'(x)$;*

(b) *Product:* $(fg)'(x) = f'(x)g'(x)$;

(c) *Quotient:* $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g'(x)}$;

(d) *Power:* $(f^h)'(x) = f'(x)^{h(x)} \cdot f(x)^{h'(x)}$;

(e) *Composition:* $(f \circ h)'(x) = f'(h(x))^{h'(x)}$.

In the following definition, we introduce the geometric Riemann integral.

Definition 4. *Let f be a positive bounded function on the closed interval $[a,b]$. Consider the partition $\mathscr{P} = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$. Consider the partition $\mathscr{P}$ and its associated numbers $\xi_1, \xi_2, \ldots, \xi_n$. The function f is said to be *integrable if there exists a number P having the property that for every $\epsilon > 0$, there exists a partition $\mathscr{P}_\epsilon$ of $[a,b]$ such that $|P(f,\mathscr{P}) - P| < \epsilon$ for every refinement $\mathscr{P}$ of $\mathscr{P}_\epsilon$ independently on the selection of the numbers associated*

$$|P(f,\mathscr{P}) - P| = \prod_{i=1}^{n} f(\xi_i)^{(x_i - x_{i-1})}.$$

*The symbol $\int_a^b f(x)dx^G$ denotes the *integral of f on $[a,b]$.*

Note that if f is positive and Riemann integrable on $[a,b]$, then it is *integrable on $[a,b]$ and

$$\int_a^b f(x)dx^G = \exp\left(\int_a^b (\ln \circ f)(x)dx\right).$$

In the following result, we provide some rules for the *integral.

Theorem 5 ([13])**.** *Let $f, g \in \mathbb{R}^+$ be *integrable on [a,b]. Then, the following holds*

(a) $\displaystyle\int_a^b (f(x))^k dx^G = \left(\int_a^b f(x)dx^G\right)^k$, *where* $k \in \mathbb{R}$.

(b) $\displaystyle\int_a^b (f(x)g(x))dx^G = \int_a^b f(x)dx^G \int_a^b g(x)dx^G$.

(c) $\displaystyle\int_a^b \left(\frac{f(x)}{g(x)}\right)dx^G = \frac{\int_a^b f(x)dx^G}{\int_a^b g(x)dx^G}$.

(d) $\displaystyle\int_a^b f(x)dx^G = \int_a^c f(x)dx^G \int_c^b f(x)dx^G$, *where* $a \le c \le b$.

Theorem 6 ([13])**.** *(Multiplicative Integration by Parts). Let f: $[a,b] \longrightarrow \mathbb{R}^+$ be *differentiable and g: $[a,b] \longrightarrow \mathbb{R}$ be differentiable so that f^g is *integrable. Then*

$$\int_a^b f^*(x)^{g(x)} dx^G = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \frac{1}{\int_a^b f(x)^{g'(x)} dx^G}.$$

In order to present our suggested method, we need to introduce the concept of geometric vector space; for more details, see [22].

Definition 5. *A geometric vector space over the geometric field $\mathbb{R}^+$ is a set $\boldsymbol{V}$ on which two operations, denoted by $+$ and $\cdot$, satisfy the following for all $u, v, w \in \boldsymbol{V}$, and $\lambda, \mu \in \mathbb{R}^+$:*

1. $u + v = v + u$.
2. $(u + v) + w = u + (v + w)$.
3. *There exist* $0 \in \boldsymbol{V}$ *such that for all* $u \in \boldsymbol{V}$, $0 + u = u$.
4. *For all* $u \in \boldsymbol{V}$, *there exist* $(-u) \in \boldsymbol{V}$, *such that* $u + (-u) = (-u) + u = 0$.
5. *There exist* $1_G \in \mathbb{R}^+$ *such that for all* $u \in \boldsymbol{V}$, $1_G \cdot u = u$.

6. $\lambda \cdot (\mu \cdot u) = (\lambda^{\ln \mu)}) \cdot u.$
7. $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v.$
8. $(\lambda\mu) \cdot u = (\lambda \cdot u) + (\mu \cdot u).$

4. Geometric Sinc Function

We start this section by introducing the geometric sinc function.

Definition 6. *The function defined for all $x \in \mathbb{R}$ by*

$$\operatorname{sinc}_G(x) := \exp(\operatorname{sinc}(x))$$

is called the geometric sinc function.

Now, we introduce the geometric cardinal infinite product.

Definition 7. *Let f be a positive function and let $h > 0$. Define the infinite product*

$$C_G(f, h)(x) = \prod_{k=-\infty}^{\infty} f(kh) \odot \exp\left(\operatorname{sinc}\left(\frac{x - kh}{h}\right)\right).$$

Whenever this series converges it is called the geometric cardinal function of f.

One of the most important tools is the inner product.

Definition 8. *A geometric real inner product space is a real geometric vector space $\boldsymbol{V}$ where a real geometric inner product is defined. A geometric inner product space is a mapping. $\langle ., . \rangle_* : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}^+$ such that the following hold for each $f, g, h \in \boldsymbol{V}$, and $\lambda \in \mathbb{R}^+$:*

1. $\langle f, f \rangle_* \geq 1.$
2. $\langle f, f \rangle_* = 1$ *iff* $f = 1.$
3. $\langle f \oplus g, h \rangle_* = \langle f, h \rangle_* \oplus \langle g, h \rangle_*$
4. $\langle f, g \rangle_* = \langle g, f \rangle_*$
5. $\langle \lambda \odot f, g \rangle_* = \lambda \odot \langle f, g \rangle_*.$

The crucial and fundamental concept in the present method is the orthonormal set.

Definition 9. *Let $p \geq 1$, and I be any interval of real numbers. We define the geometric L_p space by $L_G^p(I) := \{f : \ln(f) \in L^p(I)\}$.*

Definition 10. *A set $\{\xi_k\}_{k=-\infty}^{\infty}$ contained in a geometric inner space is called geometric orthonormal if*

$$\langle \xi_k, \xi_n \rangle_* = \begin{cases} 1, & n \neq k \\ e, & n = k \end{cases}$$

Theorem 7. *The set $\left\{ \left(\operatorname{sinc}_G\left(\frac{x-kh}{h}\right)\right)^{\frac{1}{\sqrt{h}}} \right\}_{k=-\infty}^{\infty}$ is a geometric orthonormal set.*

Proof.

$$\begin{aligned}
&\int_{-\infty}^{\infty} \operatorname{sinc}_G\left(\frac{x-kh}{h}\right) \odot \operatorname{sinc}_G\left(\frac{x-jh}{h}\right) dx^G \\
&\qquad = \int_{-\infty}^{\infty} \left(\exp\left(\operatorname{sinc}\left(\frac{x-kh}{h}\right)\right)\right)^{\ln\left(\exp\left(\operatorname{sinc}\left(\frac{x-jh}{h}\right)\right)\right)} dx^G \\
&\qquad = \int_{-\infty}^{\infty} \exp\left(\operatorname{sinc}\left(\frac{x-kh}{h}\right) \operatorname{sinc}\left(\frac{x-jh}{h}\right)\right) dx^G \\
&\qquad = \exp\left(\int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{x-kh}{h}\right) \operatorname{sinc}\left(\frac{x-jh}{h}\right) dx\right) \\
&\qquad = \exp\left(h\delta_{kj}^{(0)}\right) \\
&\qquad = \begin{cases} 1, & k \neq j \\ \exp(h), & k = j. \end{cases}
\end{aligned}$$

Hence, the set $\left\{\left(\operatorname{sinc}_G\left(\frac{x-kh}{h}\right)\right)^{\frac{1}{\sqrt{h}}}\right\}_{k=-\infty}^{\infty}$ is a geometric orthonormal set. □

Now, we prove the geometric sinc interpolation and quadrature formula.

Theorem 8. *Let f(x) be a positive function and* $\ln f \in B(h)$*. Then*

$$f(x) = \prod_{k=-\infty}^{\infty} f(kh) \odot \exp\left(\operatorname{sinc}\left(\frac{x-kh}{h}\right)\right), for\ all\ x \in \mathbb{R}.$$

Furthermore, if $f \in L_G^1(\mathbb{R})$,

$$\int_{-\infty}^{\infty} f(x) dx^G = \prod_{k=-\infty}^{\infty} (f(kh))^h.$$

Proof. Let $f(x)$ be a positive function with $\ln f \in B(h)$. Using Equation (1), we obtain

$$\ln(f(x)) = \sum_{k=-\infty}^{\infty} \ln(f(kh)) \operatorname{sinc}\left(\frac{x-kh}{h}\right).$$

Thus,

$$\begin{aligned}
f(x) &= \exp\left(\left(\sum_{k=-\infty}^{\infty} \ln(f(kh)) \operatorname{sinc}\left(\frac{x-kh}{h}\right)\right)\right) \\
&= \prod_{k=-\infty}^{\infty} \exp\left(\ln(f(kh)) \operatorname{sinc}\left(\frac{x-kh}{h}\right)\right) \\
&= \prod_{k=-\infty}^{\infty} f(kh)^{\operatorname{sinc}\left(\frac{x-kh}{h}\right)} \\
&= \prod_{k=-\infty}^{\infty} f(kh) \odot \exp\left(\operatorname{sinc}\left(\frac{x-kh}{h}\right)\right).
\end{aligned}$$

Now, assume that $f \in L^1_G(\mathbb{R})$. By applying the sinc quadrature rule in Equation (1), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx^G &= \int_{-\infty}^{\infty} \ln f(x)dx \\ &= \exp\left(\left(h\sum_{k=-\infty}^{\infty} \ln f(kh)\right)\right) \\ &= \prod_{k=-\infty}^{\infty} (f(kh))^h,\end{aligned}$$

which completes the proof. □

Theorem 9. *Let $H' \ln F \in B(\mathcal{E})$ with $h > 0$. Let H be a conformal map from $\mathcal{E}$ to the infinite strip $\mathcal{S}$. Assume $J = H^{-1}$, $z_k = J(kh)$, and $\Lambda = J(\mathbb{R})$. Assume that there are positive constants ζ, λ, and C such that*

$$\big|\ln F(x)\big| \leq C\big(\exp(-\zeta|H(x)|)1_{\Lambda_a} + \exp(-\lambda|H(x)|)1_{\Lambda_b}\big)$$

where 1_{Λ_a} and 1_{Λ_b} denote, respectively, the characteristic of

$$\Lambda_a \equiv \{z \in \Lambda : H(z) = x < 0\}$$

and

$$\Lambda_b \equiv \{z \in \Lambda : H(z) = x > 0\}.$$

Based on the selections

$$N = \left[\left|\frac{\zeta}{\lambda}M + 1\right|\right]$$

and

$$h = \sqrt{\frac{\pi\theta}{\zeta M}}.$$

Then, for all $x \in \Lambda$, the error

$$\epsilon^G_{M,N}(H'F) \equiv F(x) / \prod_{k=-M}^{N} (F(z_k)) \odot \exp\left(\operatorname{sinc}\left(\frac{H(x) - kh}{h}\right)\right)$$

is bounded by

$$\|\epsilon^G_{M,N}(H'F)\|_\infty \leq K \ \exp\left(M^{1/2} \exp\left(-\sqrt{\pi\theta\zeta M}\right)\right),$$

K is a constant that depends solely on F, θ, and H..

Proof. Suppose that $H' \ln F \in B(\mathcal{E})$. By using Theorem 2, we obtain

$$\ln F(x) = \sum_{k=-M}^{N} \ln(F(z_k)) \operatorname{sinc}\left(\frac{H(x) - kh}{h}\right).$$

Thus,

$$
\begin{aligned}
F(x) &= \exp(\ln F(x)) \\
&= \exp\left(\left(\sum_{k=-M}^{N} \ln(F(z_k))\,\mathrm{sinc}\left(\frac{H(x)-kh}{h}\right)\right)\right) \\
&= \prod_{k=-M}^{N} \exp\left(\ln(F(z_k))\,\mathrm{sinc}\left(\frac{H(x)-kh}{h}\right)\right) \\
&= \prod_{k=-M}^{N} F(z_k)^{\mathrm{sinc}\left(\frac{H(x)-kh}{h}\right)} \\
&= \prod_{k=-M}^{N} F(z_k) \odot \exp\left(\mathrm{sinc}\left(\frac{H(x)-kh}{h}\right)\right).
\end{aligned}
$$

Hence, the error

$$
\epsilon^{G}_{M,N}(H'F) \equiv F(x) / \prod_{k=-M}^{N} F(z_k)^{\mathrm{sinc}\left(\frac{H(x)-kh}{h}\right)}
$$

is bounded by

$$
\begin{aligned}
\|\epsilon^{G}_{M,N}(H'F)\|_\infty &= \exp(\|\epsilon_{M,N}(H'F)\|_\infty) \\
&\leq \exp\left(K_1 M^{1/2} \exp\left(-\sqrt{\pi\theta\zeta M}\right)\right) \\
&= K \exp\left(M^{1/2} \exp\left(-\sqrt{\pi\theta\zeta M}\right)\right)
\end{aligned}
$$

□

Theorem 10. *Consider* $\ln F \in B(\mathcal{E})$ *and* $h > 0$*. Let* H *be a conformal map from* $\mathcal{E}$ *to the infinite strip* $\mathcal{S}$*. Let* $J = H^{-1}$*,* $x_k = J(kh)$*, and* $\Lambda = J(\mathbb{R})$*. Assume that there are positive constants* ζ*,* λ*, and* C *such that*

$$
\left|\frac{\ln F(x)}{H'(x)}\right| \leq C\big(\exp(-\zeta|H(x)|)1_{\Lambda_a} + \exp(-\lambda|H(x)|)1_{\Lambda_b}\big)
$$

where 1_{Λ_a} *and* 1_{Λ_b} *denote, respectively, the characteristic of*

$$
\Lambda_a \equiv \{z \in \Lambda : H(z) = x < 0\}
$$

and

$$
\Lambda_b \equiv \{z \in \Lambda : H(z) = x > 0\}.
$$

Make the selections

$$
N = \left[\left|\frac{\zeta}{\lambda} M + 1\right|\right]
$$

and

$$
h = \sqrt{\frac{\pi\theta}{\zeta M}}
$$

Then, the geometric sinc trapezoidal quadrature rule is

$$
I^{G}_{M,N}(F) \equiv \int_\Lambda F(x) dx^G / \prod_{k=-M}^{N} (F(x_k))^{\frac{h}{H'(x_k)}}
$$

and it holds that

$$
|I^{G}_{M,N}(F)| \leq \exp\left(K\ \exp\left(-\sqrt{\pi\theta\zeta M}\right)\right),
$$

where K is a constant depending only on F, θ, and H.

Proof. Suppose that $\ln F \in B(\mathcal{E})$. By using Theorem 3, we obtain

$$\int_{\Lambda} \ln F(x) dx = h \sum_{k=-M}^{N} \frac{\ln F(x_k)}{H'(x_k)}.$$

Thus,

$$\exp\left(\int_{\Lambda} \ln F(x) dx\right) = \exp\left(h \sum_{k=-M}^{N} \frac{\ln F(x_k)}{H'(x_k)}\right),$$

which yields

$$\begin{aligned} \int_{\Lambda} F(x) dx^G &= \prod_{k=-M}^{N} \exp\left(h \frac{\ln F(x_k)}{H'(x_k)}\right) \\ &= \prod_{k=-M}^{N} F(x_k)^{\frac{h}{H'(x_k)}}. \end{aligned}$$

Hence, the error

$$I_{M,N}^G(F) \equiv \int_{\Lambda} F(x) dx^G / \prod_{k=-M}^{N} (F(x_k))^{\frac{h}{H'(x_k)}}$$

is bounded by

$$\begin{aligned} |I_{M,N}^G(F)| &= \exp(|I_{M,N}(F)|) \\ &\leq \exp\left(K_1 \exp\left(-\sqrt{\pi\theta\zeta M}\right)\right) \\ &= K \exp\left(\exp\left(-\sqrt{\pi\theta\zeta M}\right)\right) \end{aligned}$$

□

5. Geometric Sinc–Galerkin Method

Consider the geometric boundary value problem

$$L_* y(x) \equiv y^{**}(x).(y^*(x))^{a_1(x)}.(y(x))^{a_0(x)} = f(x), \quad a < x < b, \tag{2}$$

subject to the boundary conditions

$$y(a) = y(b) = 1.$$

Denote the geometric sinc function by

$$S_G(j,h) = \mathrm{sinc}_G\left(\frac{x - jh}{h}\right)$$

Let

$$\chi_j = S_G(j,h) \circ H$$

where $H(x)$ is a conformal mapping determined by the interval (a,b). Hence, we obtain

$$\left(y^{**}(x).(y^*(x))^{a_1(x)}.(y(x))^{a_0(x)}\right)^{\chi_j.w} = (f(x))^{\chi_j.w}, \tag{3}$$

where $w(x)$ is a second continuously differentiable weight function chosen such that $w(a) = w(b) = 0$. For the nodes $x_k = H^{-1}(kh)$, let $y_k = y(x_k)$. Then, consider

$$y_m(x) = \prod_{k=-M}^{N} y_k \odot S_G(k,h) \circ H(x), \ \ m = M + N + 1$$

Now, if we integrate geometrically both sides of (3) and use Theorem 5, we have

$$\int_a^b (y^{**}(x))^{\chi_k(x).w(x)} dx^G . \int_a^b \Big((y^*(x))^{a_1(x)} \Big)^{\chi_k(x).w(x)} dx^G . \int_a^b \Big((y(x))^{a_0(x)} \Big)^{\chi_k.w} dx^G \\ = \int_a^b ((f(x))^{\chi_j.w})^{\chi_k(x).w(x)}. \tag{4}$$

The first integral in (4) can be treated via the geometric integration of parts with the use of $w(b) = w(a) = 0$ and $y(a) = y(b) = 1$, to obtain

$$\int_a^b (y^{**}(x))^{\chi_k.w(x)} dx^G = \\ \left[\prod_{i=0}^{1} \Big(y^{*(1-i)}(x) \Big)^{\left(w(x)\, \mathrm{sinc}\left(\frac{H(x)-kh}{h}\right) \right)^{*(i)}} \right]_a^b \int_a^b (y(x))^{(\mathrm{sinc}\left(\frac{H(x)-kh}{h}\right) w(x))''} dx^G \\ = \int_a^b (y(x))^{(\mathrm{sinc}\left(\frac{H(x)-kh}{h}\right) w(x))''} dx^G.$$

By using the quadrature rule in Theorem 10, we obtain

$$\int_a^b (y^{**}(x))^{\chi_k.w(x)} dx^G = \int_a^b (y(x))^{\sum_{i=0}^{2} S_G(H(x),h)^{(i)}(x) g_{2,i}(x)} dx^G \tag{5}$$

Using the sinc quadrature rule on the right-hand side of (5), we obtain

$$\int_a^b (y^{**}(x))^{\chi_k.w(x)} dx^G = \prod_{j=-M}^{N} (y(x_j))^{\frac{h}{H'(x_j)} \sum_{i=0}^{2} \frac{\delta_{jk}^{(i)} \ g_{2,i}(x_j)}{h^i}} \tag{6}$$

$$= \prod_{j=-M}^{N} \prod_{i=0}^{2} \left(\exp\left(\frac{\delta_{jk}^{(i)} \ g_{2,i}}{h^i} \right) \right)^{\frac{h}{H'(x_j)} \ln\left(y(x_j)\right)} \tag{7}$$

where

$$g_{2,2} = w(H')^2$$

$$g_{2,1} = wH'' + 2w'H',$$

and

$$g_{2,0} = w''.$$

Now, the second geometric integral in (4), which contains the first geometric derivative of y, takes the following form

$$\int_a^b \Big((y^*(x))^{a_1(x)} \Big)^{\chi_k(x).w(x)} dx^G = \int_a^b (y^*(x))^{a_1(x)} \odot \exp\left(\mathrm{sinc}\left(\frac{H(x)-kh}{h} \right) \right) \odot w_G(x) dx^G \\ = \int_a^b (y^*(x))^{a_1(x)\, \mathrm{sinc}\left(\frac{x-kh}{h}\right) w(x)} dx^G.$$

Using geometric integration by parts to remove the geometric derivative from the dependent variable y, we obtain

$$\int_a^b \Big((y^*(x))^{a_1(x)}\Big)^{\chi_k(x).w(x)} dx^G = B^*_{T,1}\left(\int_a^b (y(x))^{(\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)a_1(x)w(x))'} dx^G\right)^{-1}. \quad (8)$$

where the boundary term

$$B^*_{T,1} = (y(x))^{\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)a_1(x)w(x)}\Big|_a^b = 1.$$

Thus, (8) may be written as

$$\int_a^b (y^*(x))^{a_1(x)\operatorname{sinc}\left(\frac{x-kh}{h}\right)w(x)} dx^G = \left(\int_a^b (y(x))^{\sum_{i=0}^{1} S(H(x),h)^{(i)}(x) g_{1,i}(x)} dx^G\right)^{-1} \quad (9)$$

where

$$g_{1,1} = (a_1 w)H'(x)$$
$$g_{1,0} = (a_1 w)'.$$

Using the sinc quadrature rule on the right-hand side of (9), we obtain

$$\begin{aligned}\int_a^b (y^*(x))^{a_1(x)\operatorname{sinc}\left(\frac{x-kh}{h}\right)w(x)} dx^G &= \prod_{j=-M}^{N} (y(x_j))^{\frac{-h}{H'(x_j)}\sum_{i=0}^{1}\frac{\delta_{jk}^{(i)} g_{1,i}(x)}{h^i}} \\ &= \prod_{j=-M}^{N}\prod_{i=0}^{1}\left(\exp\left(\frac{\delta_{jk}^{(i)} g_{1,i}}{h^i}\right)\right)^{\frac{-h}{H'(x_j)}\ln(y(x_j))}.\end{aligned}$$

Next, we treat the third integral in (4) as follows:

$$\begin{aligned}&\int_a^b \Big((y(x))^{a_0(x)}\Big)^{\chi_k.w} dx^G \\ &\quad= \int_a^b (y(x))^{a_0(x)} \odot \exp\left(\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)\right) \odot w_G(x) dx^G \\ &\quad= \int_a^b (y(x))^{a_0(x)\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)w(x)} dx^G \\ &\quad= \exp\left(h\frac{a_0(x_k)w(x_k)}{H'(x_k)}\ln(y(x_k))\right).\end{aligned}$$

Finally, we evaluate the integral on the left-hand side of (4):

$$\begin{aligned}&\int_a^b ((f(x))^{\chi_j.w})^{\chi_k(x).w(x)} \\ &\quad= \int_a^b f(x) \odot \exp\left(\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)\right) \odot w_G(x) dx^G \\ &\quad= \int_a^b (f(x))^{\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)w(x)} dx^G \\ &\quad= \exp\left(h\frac{w(x_k)}{H'(x_k)}\ln(f(x_k))\right).\end{aligned}$$

Define

$$(\delta_G)^{(i)}_{jk} = \exp\left(h^i\right) \odot \frac{d^p}{dx^{pG}}[S_G(j,h)(x)]_{x=x_k}.$$

Then, we have

$$\begin{aligned}
(\delta_G)^{(2)}_{jk} &= \exp\left(h^2\right) \odot \frac{d^2}{dx^{2G}}[S_G(j,h)(x)]_{x=x_k} \\
&= \exp\left(h^2\right) \odot \frac{d^2}{dx^{2G}}\left[\exp\left(\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)\right]_{x=x_k}\right) \\
&= \exp\left(h^2\right)\frac{d^2}{dx^2}\left[\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)\right]_{x=x_k} \\
&= \exp\left(\delta^{(2)}_{jk}\right) \\
&= \begin{cases} \exp\left(\frac{-\pi^2}{3}\right), & j=k \\ \exp\left(\frac{-2(-1)^{k-j}}{(k-j)^2}\right), & j\neq k \end{cases}.
\end{aligned}$$

In a similar manner,

$$\begin{aligned}
(\delta_G)^{(1)}_{jk} &= \exp(h) \odot \frac{d}{dx^G}[S_G(j,h)(x)]_{x=x_k} \\
&= \exp(h) \odot \frac{d}{dx^G}\left[\exp\left(\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)\right)\right]_{x=x_k} \\
&= \exp\left(h\frac{d}{dx}\left[\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)\right]_{x=x_k}\right) \\
&= \exp\left(\delta^{(1)}_{jk}\right) \\
&= \begin{cases} 1, & j=k \\ \exp\left(\frac{(-1)^{k-j}}{(k-j)}\right), & j\neq k \end{cases}.
\end{aligned}$$

Finally,

$$\begin{aligned}
(\delta_G)^{(0)}_{jk} &= e \odot [S_G(j,h)(x)]_{x=x_k} \\
&= e \odot \left[\exp\left(\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)\right)\right]_{x=x_k} \\
&= \exp\left(\left[\operatorname{sinc}\left(\frac{H(x)-kh}{h}\right)\right]_{x=x_k}\right) \\
&= \exp\left(\delta^{(0)}_{jk}\right) \\
&= \begin{cases} e, & j=k \\ 1, & j\neq k \end{cases}.
\end{aligned}$$

6. Treatment of the Boundary Conditions

In the previous section, the geometric sinc–Galerkin technique for homogeneous boundary conditions was developed. This technique is practical because the geometric sinc functions, $S_G(j,h) \circ H(x)$, are one at the interval's endpoints. This paper's geometric sinc–Galerkin applies to Equation (2) with non-homogeneous boundary conditions. We simply change the problem to one with uniform boundary conditions. For example, consider Equation (2) subject to the boundary conditions

$$y(a) = a_0, \quad y(b) = b_0. \tag{10}$$

The non-homogeneous boundary conditions in (10) can be transformed into homogeneous boundary conditions by defining

$$v(x) = y(x) \ \exp(cx + d), \tag{11}$$

where

$$c = \frac{\ln(b_0) - \ln(a_0)}{a - b}, \quad d = \frac{\ln(a_0)\ b - \ln(b_0)\ a}{a - b}.$$

The transformation in (11) will convert (4), a geometric second-order boundary value problem of the same type, in terms of $v(x)$, with the new boundary conditions $v(a) = 1$, $v(b) = 1$. Therefore, our approximate solution using the geometric sinc–Galerkin method becomes

$$v_m(x) = \prod_{k=-M}^{N} v_k \odot S_G(k,h) \circ H(x)$$

and the solution $y(x)$ of (4) is $y(x) = v(x) \ \exp(-(cx + d))$.

7. Numerical Examples

For the examples in this section, we consider $M = N = 100$, $h = \frac{\pi}{10}$, and $\zeta = \lambda = \frac{1}{2}$. The included problems demonstrate the method's ease of implementation and assembly, parameter selection, and exponential convergence rate. Choosing examples with known solutions enables a more comprehensive error analysis. The examples presented here are intended to demonstrate the convergence and accuracy of the error using our approach.

Example 1. *Consider the second-order boundary value problem.*

$$y(x)y''(x) - (y'(x))^2 + 2y^2(x) = 0, \quad 0 < x < 1$$
$$y(0) = y(1) = 1.$$

The corresponding geometric second-order differential equation is

$$y^{**}(x) = e^{-2},$$

with the boundary conditions

$$y(0) = y(1) = 1$$

The exact solution is $y(x) = e^{x(1-x)}$. The solution y to the problem is approximated by

$$y_m(x) = \prod_{k=-M}^{N} y_j \odot \mathrm{sinc}_G\left(\frac{x - kh}{h}\right), \quad m = M + N + 1.$$

To determine unknown coefficients, use the discrete geometric sinc–Galerkin system to evaluate the unknown coefficient $\{y_j\}_{j=-M}^{N}$

$$\prod_{j=-M}^{N} \prod_{i=0}^{2} \left(\exp\left(\frac{\delta_{jk}^{(i)} g_{2,i}}{h^i} \right) \right)^{\frac{h}{H'(x_j)} y_j} = \exp\left(h \frac{w(x_k)}{H'(x_k)} f(x_k) \right), \quad k = -M, \ldots, N.$$

The error table for the obtained solution at the given mesh points is displayed in Table 1.

Table 1. Numerical values for Examples 1 and 2.

(a) Numerical Values for Examples 1.		(b) Numerical Values for Example 2.	
x_i	G-SGM	x_i	G-SGM
0.1	1.82299×10^{-13}	0.1	2.64233×10^{-14}
0.2	9.19265×10^{-14}	0.2	1.03251×10^{-14}
0.3	1.65423×10^{-13}	0.3	1.57652×10^{-14}
0.4	1.80078×10^{-13}	0.4	2.33147×10^{-15}
0.5	3.90799×10^{-13}	0.5	1.60982×10^{-14}
0.6	1.80078×10^{-13}	0.6	1.249×10^{-14}
0.7	1.64979×10^{-13}	0.7	7.54952×10^{-15}
0.8	9.23706×10^{-13}	0.8	8.54872×10^{-15}
0.9	1.82077×10^{-13}	0.9	1.77636×10^{-15}

Example 2. *Consider the geometric second-order boundary value problem*

$$y^{**}(x) = \exp(2y(x)), \quad 0 < x < 1$$

with the non-homogeneous boundary conditions

$$y(0) = 1, \ y(1) = \frac{1}{4}.$$

The exact solution is $y(x) = \frac{1}{(1+x)^2}$. *Let* $v(x) = 4^{-x}y(x)$. *The geometric boundary value problem becomes*

$$v^{**}(x) = \exp\big(2(4^{-x})v(x)\big), \quad 0 < x < 1$$

with the homogeneous boundary conditions

$$v(0) = 1, \ v(1) = 1.$$

The discrete geometric sinc–Galerkin system for the determination of the unknown coefficients $\{v_j\}_{j=M}^{N}$ *is given by*

$$\prod_{j=-M}^{N}\prod_{i=0}^{2}\left(\exp\left(\frac{\delta_{jk}^{(i)} g_{2,i}}{h^i}\right)\right)^{\frac{h}{H'(x_j)}v_j} = \exp\left(\frac{hw(x_k)}{H'(x_k)}(2)(4^{-x_k})v(x_k)\right), \quad k = -M, \dots, N.$$

The error table for the obtained solution at the given mesh points is displayed in Table 1.

Author Contributions: Formal analysis, investigation, and writing—original draft preparation A.D., K.A.-K. and M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: Authors would like to thank Jordan University of Science and Technology, especially the College of Graduate Students as a part of this work is taken from the third author's master thesis. As well as authors are grateful to the Editor for handling the full submission of the manuscript.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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