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On Equation Manifolds, the Vinogradov Spectral Sequence, and Related Diffeological Structures

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Abstract: We consider basic diffeological structures that can be highlighted naturally within the theory of the Vinogradov spectral sequence and equation manifolds. These interrelated features are presented in a rigorous and accurate way, that complements some heuristic formulations appearing in very recent literature. We also propose a refined definition of the Vinogradov spectral sequence using diffeologies.

Keywords: diffeology; Frölicher space; Vinogradov spectral sequence; equation manifold

MSC: 58B10; 58A20; 37K05

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1. Introduction

Let us agree, for the purpose of this introduction, that a *full equation manifold* is defined as the set $\mathcal{E}M^{(\infty)}$ of all sequences of real numbers

$$(x_1, \dots, x_n, u^1, \dots, u^m, u^1_{x_1}, u^1_{x_2} \dots, \dots, u^i_{x_{i_1} x_{i_2} \dots x_{i_k}}, \dots)$$

satisfying a given differential equation and all its differential consequences. In standard cases, we would say that $\mathcal{E}M^{(\infty)}$ is an "in finite-dimensional manifold". For instance, we could imagine that if we are considering the heat equation, $\mathcal{E}M^{(\infty)}$ would be an in finite-dimensional hyperplane. The de Rham complex on a full equation manifold $\mathcal{E}M^{(\infty)}$ can be bigraded, exploiting the fact that this space can be equipped with a (flat) Ehresmann connection. This bigrading gives rise to the variational bicomplex on $\mathcal{E}M^{(\infty)}$. The spectral sequence associated with this bicomplex, obtained from filtration by vertical degree, is the Vinogradov spectral sequence. The importance of this structure for the study of geometric aspects of differential equations is highlighted in [1]. In addition, we cite the very relevant works [2–4] by Vinogradov, the textbook [5] edited by Krasil'shchik and Vinogradov, and we direct the reader to [1,5] for precise references to the work of some the "pioneers" such as Olver, Tulczyjew, Tsujishita, and Takens. We very much regret that the second reference listed in the bibliography of [1] has never appeared in print, as it would certainly be the standard treatise on the subject.

Now, the geometric study of equation manifolds is generally performed by restricting the domain of geometric objects to a smooth locus, that is, to a smooth submanifold of infinite jet space that is contained in the full equation manifold. The fact that this viewpoint is interesting is due to the (generally) non-triviality of the associated variational bicomplex; see for instance [6,7]. We question this procedure in the present work in view of [8,9], in which the notion of diffeology is highlighted as a necessary tool for the study of the geometry of jet spaces and related objects, and following our own papers [10,11]: in [10], we propose a Frölicher structure for diffieties (a *diffiety* is a structure more general than a full equation manifold. It appears prominently in the works of A. Vinogradov; see for instance [4,5]), and we highlight a sufficient condition that allows us to build the $\mathcal{C}$-spectral sequence (in this work, call we call this sequence the "Vinogradov spectral sequence" following [2,3], because of the use of the notation $\mathcal{C}$ in the context of Frölicher spaces); in [11], we introduce a hierarchy of differential equations posed on equation manifolds.

We chose to use diffeological spaces (and Frölicher spaces) in the current context primarily because of technical considerations. We wished to avoid introducing a new framework unnecessarily, but it appears to us that, given the current state of knowledge, diffeologies indeed provide a satisfactory structure for (infinite-dimensional) differential geometry, especially in the presence of complexities such as singularities (in the case of equation manifolds) or the absence of global atlases.

Our exposition is organized as follows: We give a short presentation of full equation manifolds of partial differential equations, as commonly defined in the existing literature in Section 2, and we present a short introduction to diffeologies and Frölicher spaces in Section 3. Notably, in this section, we establish a new result on the comparison of two natural diffeologies on a space of functions (see Theorem 1). Subsequently, in Sections 4 and 5, we explore the interplay between equation manifolds and diffeologies, we describe the diffeological content of an equation manifold, and we propose a diffeological Vinogradov spectral sequence. The spectral sequence presented herein is more general than the one appearing in [10]. We present our conclusions and outlook in Section 6.

2. On Equation Manifolds and Partial Differential Equations

Let us first focus on jet spaces. We consider a finite-dimensional fiber bundle E with typical fiber F (with $\dim(F) = n$) over a smooth manifold M of dimension m, which can be non-compact. We also consider a generic local trivialization $\varphi : U \times F \to E$ of E over an open subset U of M. The manifold M is the *space of independent variables*, and the typical fiber is the *space of dependent variables*.

Let us consider local coordinates x_i, $1 \leq i \leq n$ on M and u^α, $1 \leq \alpha \leq m$ on F. Assume that $s_1(x_i) = (x_i, s_1^\alpha(x_i))$ and $s_2(x_i) = (x_i, s_2^\alpha(x_i))$ are two local sections of the bundle $E \to M$ defined about a point $p = (x_i)$ in M. We say that s_1 and s_2 *agree to order k at $p \in M$* if s_1, s_2 and *all* the partial derivatives of the sections s_1 and s_2, up to order k, agree at p. This notion determines a coordinate-independent equivalence relation on local sections of E, see for instance [12] or [13]. We let $j^k(s)(p)$ represent the equivalence class of the section s at p; we call this equivalence class *the k-jet of s at p*.

Definition 1. *The* k-order jet bundle of E *is the space*

$$J^k(E) = \bigcup_{p \in M} J^k(p)\ ,$$

in which $J^k(p)$ denotes the set of all the k-jets $j^k(s)(p)$ of local sections s at p.

The jet bundle $J^k(E)$ possesses a natural manifold structure; it fibers over $J^l(E)$, $l < k$, and also over M. Local coordinates on $J^k(E)$ are

$$(x_i, u_0^\alpha, u_{i_1}^\alpha, u_{i_1 i_2}^\alpha, \ldots, u_{i_1 \ldots i_k}^\alpha)\ ,$$

in which

$$u_0^\alpha(j^k(s)(p)) = s^\alpha(p)\ , \quad u_i^\alpha(j^k(s)(p)) = \frac{\partial s^\alpha}{\partial x_i}(p)\ , \quad u_{i_1 i_2}^\alpha(j^k(s)(p)) = \frac{\partial^2 s^\alpha}{\partial x_{i_1} \partial x_{i_2}}(p)\ ,$$

and so forth, where $j^k(s)(p) \in J^k E$ and $(x^i) \mapsto (x^i, s^\alpha(x^i))$ is any local section in the equivalence class $j^k(s)(p)$. Simply put, we think of the variables $x_1, \ldots, x_n$, $u^1, \ldots, u^m$, and partial derivatives $u^{\alpha}_{i_1,\ldots i_j}$ (up to the kth order) as adapted coordinates on $J^k(E)$. In these coordinates, the projection map $\pi^k_M : J^k(E) \to M$ (the "source map") given by $(x_i, u^{\alpha}_0, u^{\alpha}_{i_1}, \ldots, u^{\alpha}_{i_1\ldots i_k}) \mapsto (x_i)$ factors through the projection map $\pi^k : J^k(E) \to E$; on a local generic trivialization, $\varphi : U \times F \to E$, $\pi^k = \pi^k_M \times \beta^k$ where β^k is the local target map. The projections $\pi^k_l : J^k(E) \to J^l(E)$, $l < k$ are defined in obvious ways.

We can be a little more precise: for $e \in E$, $(\pi^k)^{-1}(e)$ is a vector space modeled on the direct sum

$$\bigoplus_{i=1}^{k} L^i(\mathbb{R}^m, \mathbb{R}^n) \ ,$$

where $L^i(\mathbb{R}^m, \mathbb{R}^n)$ is the space of i-linear symmetric mappings on $\mathbb{R}^m$ with values in $\mathbb{R}^n$. Therefore, $J^k(E)$ can be understood as:

- A finite-dimensional vector bundle over E with typical fiber $\bigoplus_{i=1}^{k} L^i(\mathbb{R}^m, \mathbb{R}^n)$.
- A finite-dimensional fiber bundle over M with typical fiber

$$F \times \left(\bigoplus_{i=1}^{k} L^i(\mathbb{R}^m, \mathbb{R}^n) \right) .$$

An extended exposition of the properties of k-jets can be found in [12–14].

The *infinite jet bundle* $\pi^{\infty}_M : J^{\infty}(E) \to M$ is the inverse limit of the sequence of jet bundles

$$\ldots \to J^{k+1}(E) \to J^k(E) \to \ldots \to J^0(E) = E \to M \tag{1}$$

under the standard projections $\pi^k_l : J^k(E) \to J^l(E)$, $k > l$ and $\pi^k_M : J^k(E) \to M$. The section of π^{∞}_M corresponding to the family of compatible sections $\{j^k(s)\}$ is denoted by

$$j^{\infty}(s) : M \to J^{\infty}(E) \ .$$

A kth-order partial differential equation (hereafter understood as either a scalar equation or a system) $\Delta = 0$, in which Δ depends on independent variables $x_1, \ldots, x_n$, dependent variables $u^1, \ldots, u^m$, and a finite number of partial derivatives of u^{α} with respect to the variables x_j, clearly determines a subset of an appropriate jet bundle $J^k(E)$, in which $E \to M$ is an $(n+m)$-dimensional fiber bundle over an n-dimensional manifold M. In this paper, *we define the equation manifold of* $\Delta = 0$ *as* $\mathcal{EM}^k = \Delta^{-1}(0) \subset J^k(E)$ (or $\mathcal{EM}^k_{\Delta}$, if there is risk of confusion). In spite of its name, the set $\mathcal{EM}^k_{\Delta}$ may not be a bona fide manifold, that is, it may carry singularities or hidden constraints (indeed, a subtler definition of equation manifolds would take into account the symmetries of the theory (for example, the diffeomorphism invariance of Einstein equations), but we will not consider this case in this article). In order to circumvent this problem, almost all authors restricted their investigation to a smooth submanifold (in the classical sense) of $J^k(E)$ contained in $\mathcal{EM}_{\Delta}$ and, in this section, we will follow their lead. Such a smooth submanifold of $J^k(E)$, set-theoreticaly included in $\mathcal{EM}_{\Delta}$, will be called in this paper a *smooth locus*. The restriction to a smooth locus is actually the standard way to deal with differential geometric structures on $\mathcal{EM}_{\Delta}$. It is very interesting to note that this viewpoint was already well established by the end of the XIXth century, as Goursat's book on second order partial differential equations (see, e.g., [15] [Chp. II]) testifies.

Intrinsic definitions of jet bundles and equation manifolds appear for instance in [5]; see also our paper [11] for a brief review. As we just stated, we assume here that $\mathcal{EM}^{(k)} \subset J^k(E)$ is a submanifold of $J^k(E)$ such that the map

$$\pi^k \mid_{\mathcal{EM}^{(k)}} : \mathcal{EM}^{(k)} \to M$$

is also a bundle. Then, a local section s of π is a *solution of the system of PDEs* $\Delta = 0$ if and only if the graph of the section $j^k(s)$ is contained in $\mathcal{EM}^{(k)}$. The set

$$\mathcal{EM}^{(k+n)} = \left\{ j^{k+n}(s)(p) \mid \text{the graph of } j^k(s) \text{ is tangent to } \mathcal{EM}^{(k)} \text{ with order} \geq n \text{ at } j^k(s)(p) \in \mathcal{EM}^{(k)} \right\}$$

is the *n-th prolongation of $\mathcal{EM}^{(k)}$*.

Let us go back to the infinite jet bundle $J^\infty(E)$. We say that a function $f : J^\infty(E) \to \mathbb{R}$ is *smooth* if it factors through a finite-order jet bundle, that is, if $f = f_k \circ \pi_k^\infty$ for some smooth function $f_k : J^k(E) \to \mathbb{R}$, in which $\pi_k^\infty : J^\infty(E) \to J^k(E)$ denotes the canonical projection from $J^\infty E$ onto $J^k(E)$. A natural class of differential forms on $J^\infty(E)$ is defined in an analogous way. With respect to tangent vectors, we note that the sequence of finite-dimensional bundle projections (1) allows us to form the sequence of tangent spaces

$$\ldots \to TJ^{k+1}(E) \to TJ^k(E) \to \ldots \to TE \to TM\,. \tag{2}$$

The inverse limit of this sequence is the tangent bundle $TJ^\infty(E)$. *There is a unique distribution $\mathcal{C}$ on $J^\infty(E)$, such that, for any point $m \in M$ and any local section s of π, we have, on a neighbourhood of m,*

$$\mathcal{C}_{j^\infty(s)(m)} = j^\infty(s)_*(T_m M)\,. \tag{3}$$

This distribution has rank equal to $\dim(M)$ and is called *the Cartan distribution on $J^\infty(E)$*.

Taking the n-th prolongation of $\mathcal{EM}^{(k)}$ and restricting the projections π^{k+n}_{k+n-1} appropriately, we obtain the sequence of maps

$$\ldots \to \mathcal{EM}^{(k+n)} \xrightarrow{\pi^{k+n}_{k+n-1}} \mathcal{EM}^{(k+n-1)} \to \ldots \to EM^{(k)} \tag{4}$$

which (under natural conditions of regularity, assuming that all prolongations $S^{(k+n)}$ are submanifolds of $J^{k+n}(E)$, which implies that all projections in (4) are bundle mappings; see [14]), we obtain the *infinite prolongation $\mathcal{EM}^{(\infty)}$ of the given PDE system*. We call $\mathcal{EM}^{(\infty)}$ the *full equation manifold* of the system.

The sequence (4) allows us to form a corresponding sequence of tangent spaces as in (2). Its inverse limit defines $T\mathcal{EM}^{(\infty)}$, and we define differential forms on $\mathcal{EM}^{(\infty)}$ via pull-back by the canonical inclusion $\iota : \mathcal{EM}^{(\infty)} \hookrightarrow J^\infty E$.

The restriction of the Cartan distribution $\mathcal{C}$ to $\mathcal{EM}^{(\infty)}$ will be denoted by $\mathcal{C}_{\mathcal{EM}^{(\infty)}}$ and it will be called *the Cartan distribution on $\mathcal{EM}^{(\infty)}$*. A crucial remark is that it satisfies the integrability condition

$$\left[\mathcal{C}_{\mathcal{EM}^{(\infty)}}, \mathcal{C}_{\mathcal{EM}^{(\infty)}}\right] \subset \mathcal{C}_{\mathcal{EM}^{(\infty)}}\,. \tag{5}$$

3. A Crash Introduction to Diffeological and Frölicher Spaces

Geometry on spaces extending beyond conventional manifolds requires specialized treatment, as noted, for instance, in [16] [Preface], and as sketched out in the preceding section. We believe that, for this purpose, diffeological spaces (or, the more restricted framework of Frölicher spaces used in [17] in a mathematical physics context; see also [18–20]) can serve as a natural setting, particularly for the geometric study of differential equations.

We define diffeological and Frölicher spaces following [16,21–23]. To be succinct, our presentation follows, roughly, the recent review [24]. Other relevant references for our specific needs are [8,9,20,25–32].

Definition 2. *Let X be a non-empty set. A p-parametrization on X is a map from an open subset of $\mathbb{R}^p$ to X. A diffeology on X is a set $\mathcal{P}$ of parametrizations on X such that*

- *For all $p \in \mathbb{N}$, any constant map $\mathbb{R}^p \to X$ is in $\mathcal{P}$.*

- *Let $\{f_i : O_i \to X\}_{i \in I}$ be a family of compatible maps that extend to a map $f : \bigcup_{i \in I} O_i \to X$. If $\{f_i : O_i \to X\}_{i \in I} \subset \mathcal{P}$, then $f \in \mathcal{P}$.*
- *Let $f \in \mathcal{P}$ be a p-parametrization with domain O, $O' \subset \mathbb{R}^q$ an open set, and $g : O' \to O$ a smooth map. Then, $f \circ g \in \mathcal{P}$.*

If $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ are diffeological spaces, a map $f : X \to X'$ is smooth if and only if $f \circ \mathcal{P} \subset \mathcal{P}'$.

Definition 3. *A **Frölicher** space is a triple $(X, \mathcal{F}, \mathcal{C})$ such that $\mathcal{C}$ is a set of paths $\mathbb{R} \to X$, $\mathcal{F}$ is a set of functions from X to $\mathbb{R}$, and these sets satisfy:*

- *a function $f : X \to \mathbb{R}$ is in $\mathcal{F}$ if and only if, for any $c \in \mathcal{C}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$;*
- *a path $c : \mathbb{R} \to X$ is in $\mathcal{C}$ if and only if, for any $f \in \mathcal{F}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.*

If $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ are Frölicher spaces, a map $f : X \to X'$ is smooth if and only if $\mathcal{F}' \circ f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})$.

For example, if X is a finite-dimensional smooth manifold, we can take $\mathcal{F}$ as the set of all smooth maps from X to $\mathbb{R}$, and $\mathcal{C}$ the set of all smooth paths from $\mathbb{R}$ to X. If X is paracompact, it can be proven that indeed $(X, C^\infty(X, \mathbb{R}), C^\infty(\mathbb{R}, X))$ is a Frölicher structure on X, and that the smooth (in the standard sense) maps between two such manifolds are precisely the smooth maps in the Frölicher sense. Another interesting class of examples is provided by Fréchet spaces (see [19]). A general construction of Frölicher structures is the following:
Any family of maps $\mathcal{F}_g$ from X to $\mathbb{R}$ generates a Frölicher structure $(X, \mathcal{F}, \mathcal{C})$ by setting

- $\mathcal{C} = \{c : \mathbb{R} \to X \text{ such that } \mathcal{F}_g \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})\}$;
- $\mathcal{F} = \{f : X \to \mathbb{R} \text{ such that } f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})\}$.

A Frölicher space $(X, \mathcal{F}, \mathcal{C})$ carries a natural topology, namely, the pull-back topology of $\mathbb{R}$ via $\mathcal{F}$. It is known (see [22]) that in the finite-dimensional manifold example just discussed, the underlying topology of the Frölicher structure is the same as the manifold topology.

The category of Frölicher spaces, along with smooth maps between them, exhibits closure under various crucial set-theoretical operations. Notably, these operations include products, quotients, inverse/direct limits, and subsets, as detailed in [16]. Furthermore, it is worth noting that the category of smooth manifolds and smooth maps constitutes a full subcategory of the category of Frölicher spaces. In turn, Frölicher spaces themselves are a subcategory of the category of diffeological spaces.

Let us discuss a little more the relation between Frölicher spaces and diffeologies. If $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we define a natural diffeology on X following [18,20]; we consider the family of maps

$$\mathcal{P}_\infty(\mathcal{F}) = \coprod_{p \in \mathbb{N}} \{ f : D(f) \subset \mathbb{R}^p \to X;\ D(f) \text{ open and } \mathcal{F} \circ f \in C^\infty(D(f), \mathbb{R}) \} , \qquad (6)$$

in which $C^\infty(D(f), \mathbb{R})$ indicates the set of functions from $D(f)$ to $\mathbb{R}$ that are smooth in the usual sense. If X is a differentiable manifold, this diffeology is Souriau's *nebula diffeology* (see [23]).

The following important proposition, taken from [21,23], shows one way in which the category of diffeological spaces is more "convenient" that the category of finite-dimensional smooth manifolds.

Proposition 1. *Let $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ be two diffeological spaces. There exists a diffeology $\mathcal{P} \times \mathcal{P}'$ on $X \times X'$ made of plots $g : O \to X \times X'$ that decompose as $g = f \times f'$, where $f : O \to X \in \mathcal{P}$ and $f' : O \to X' \in \mathcal{P}'$. We call it the **product diffeology**. This construction extends to infinite (maybe not countable) products.*

This proposition applied to Frölicher spaces yields the following result (compare with [22]):

Proposition 2. *Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces equipped with their natural diffeologies $\mathcal{P}$ and $\mathcal{P}'$. There is a natural Frölicher space structure on $X \times X'$ whose contours $\mathcal{C} \times \mathcal{C}'$ are precisely the 1-plots of $\mathcal{P} \times \mathcal{P}'$.*

We can also state the above result for infinite products; we simply take Cartesian products of the plots, or of the contours.

Now, we let $(X, \mathcal{P})$ be a diffeological space, X' be a set, and $f : X \to X'$ be a map. The **push-forward diffeology** $f_*(\mathcal{P})$, is the coarsest diffeology (i.e., minimal for inclusion) among all the diffeologies on X' which contains $f \circ \mathcal{P}$, that is, for which f is smooth. A smooth map $f : (X, \mathcal{P}) \to (X', \mathcal{P}')$ between two diffeological spaces is called a **subduction** if it is surjective and $f_*(\mathcal{P}) = \mathcal{P}'$. Conversely, if $(X', \mathcal{P}')$ is a diffeological space, X is a set, and $f : X \to X'$, the **pull-back diffeology** $f^*(\mathcal{P}')$ is the diffeology on X which is maximal for inclusion, for which f is smooth. These definitions allow us to consider quotients following [23] and [21] [p. 27].

Proposition 3. *Let $(X, \mathcal{P})$ be a diffeological space and $\mathcal{R}$ an equivalence relation on X. There is a natural diffeology on $X/\mathcal{R}$, denoted by $\mathcal{P}/\mathcal{R}$, defined as the push-forward diffeology on $X/\mathcal{R}$ that is induced by the quotient projection $X \to X/\mathcal{R}$.*

Given a subset $X_0 \subset X$, where X is a Frölicher space or a diffeological space, we can equip X_0 with structures induced by X:

1. If X is equipped with a diffeology $\mathcal{P}$, we define a diffeology $\mathcal{P}_0$ on X_0 (the **subset** or **trace diffeology**; see [21,23]) by setting
$$\mathcal{P}_0 = \{ p \in \mathcal{P} : \text{The image of } p \text{ is a subset of } X_0 \} .$$
2. If $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we take as a generating set of maps $\mathcal{F}_g$ on X_0 the restrictions of the maps $f \in \mathcal{F}$. In this case, the contours (respectively, the induced diffeology) on X_0 are the contours (respectively, the plots) on X whose images are subsets of X_0.

Our last general construction is the functional diffeology. Its existence implies the crucial fact that the category of diffeological spaces is Cartesian closed. Our discussion follows [21].

Let $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ be diffeological spaces and let $S \subset C^\infty(X, X')$ be a set of smooth maps. The **functional diffeology** on S is the diffeology $\mathcal{P}_S$ made of plots
$$\rho : D(\rho) \subset \mathbb{R}^k \to S$$
such that, for each $p \in \mathcal{P}$, the maps $\Phi_{\rho,p} : (x, y) \in D(p) \times D(\rho) \mapsto \rho(y)(x) \in X'$ are plots of $\mathcal{P}'$. We have (see [21] [Chapter 1]):

Proposition 4.

1. *Let X, Y, Z be diffeological spaces. Then,*
$$C^\infty(X \times Y, Z) = C^\infty(X, C^\infty(Y, Z)) = C^\infty(Y, C^\infty(X, Z))$$
as diffeological spaces equipped with functional diffeologies.
2. *The composition of smooth maps is smooth for functional diffeologies, and the functional diffeology on $C^\infty(X, Y)$ is the largest diffeology for which the evaluation map*
$$ev : X \times C^\infty(X, Y) \to Y$$

is smooth.

Now, we discuss "fiber bundles" within the broader context of diffeologies. To begin, we note that, for a given algebraic structure, we can define a corresponding compatible diffeological (or Frölicher) structure. For instance, as outlined in [21] [pp. 66–68], if $\mathbb{R}$ is equipped with its canonical diffeology (or Frölicher structure), a diffeological (respectively, Frölicher) $\mathbb{R}$-vector space is an $\mathbb{R}$-vector space for which addition and scalar multiplication are smooth. This observation is further discussed in [19].

In the realm of finite-dimensional smooth manifolds, classical fiber bundles are defined by the following elements:

- a smooth manifold E called total space.
- a smooth manifold X called base space.
- a smooth submersion $\pi : E \to X$ called the fiber bundle projection.
- a smooth manifold F called the typical fiber. F satisfies the condition: $\forall x \in X, \pi^{-1}(x)$ is a smooth submanifold of E diffeomorphic to F.
- a smooth atlas on X with domains $U \subset X$, such that $\pi^{-1}(U)$ is an open submanifold of E diffeomorphic to $U \times F$. These domains U constitute a system of local trivializations for the fiber bundle.

Consequently, a smooth fiber bundle is determined by the quadruple (E, X, F, π); the definitions of π and X allow us to identify systems of local trivializations. For brevity, a fiber bundle is often denoted by the projection map $\pi : E \to X$.

Now, there exist diffeological spaces which carry no atlas, so if we wish to generalize the notion of a fiber bundle, the condition of having an atlas, as in the last item above, is not a priori necessary. We mention two structures that have appeared in the literature: the notion of a quantum structure (a generalization of a *principal* fiber bundle) (see [23]), and the notion of a vector pseudo-bundle (see [30]). The common trend (see [23,30–32]) is to describe fibered objects by means of a total diffeological space E, a diffeological space X, and a canonical smooth projection $\pi : E \to X$ such that $\pi^{-1}(x)$, $x \in X$, is endowed with a (smooth) algebraic structure, without assuming the existence of local trivializations.

Definition 4. *Let E and X be two diffeological spaces and let $\pi : E \to X$ be a smooth surjective map. Then, (E, π, X) is a **diffeological fiber pseudo-bundle** if and only if π is a subduction.*

We do not assume that there exists a typical fiber, in agreement with Pervova's diffeological vector pseudo-bundles.

Definition 5. *Let $\pi : E \to X$ be a diffeological fiber pseudo-bundle and let $\mathbb{K}$ be a diffeological field. We set*

$$E^{(2)} = \coprod_{x \in X} \{(u, v) \in E^2 \mid (u, v) \in \pi^{-1}(x)\} ,$$

and we equip it with the pull-back diffeology of the canonical map $E^{(2)} \to E^2$. Then, $\pi : E \to X$ is a diffeological $\mathbb{K}$-vector pseudo-bundle if there exist the following :

- *a smooth fiberwise map $\cdot : \mathbb{K} \times E \to E$;*
- *a smooth fiberwise map $+ : E^{(2)} \to E$;*

such that $\forall x \in X$, $(\pi^{-1}(x), +, \cdot)$ is a diffeological $\mathbb{K}$-vector space.

Now we state a new property which will be of importance in the sequel. We need to recall the notion of *weak topology* on the space $\Gamma^\infty(X, E)$ of smooth sections of a vector bundle E over a finite-dimensional manifold X. We assume that E is locally trivializable with a typical fiber being a Fréchet space, and that X is equipped with a Riemannian metric. The weak topology is defined via a countable covering of X by a family (K_i) of compact subsets of X, which are assumed to be contained in the domains of local trivializations of

E. It depends neither on the choice of family (K_i) nor on the underlying trivialization. We define seminorms

$$p(s) = \sup_{x \in K} \sup_{(v_\alpha) \in B_\alpha(T_x X)} p'(D^\alpha s(x)(v_\alpha)) \,,$$

where p' is a continuous semi-norm on E_x, α is a multi-index, $v_\alpha \in (T_x X)^{|\alpha|}$, and $B_\alpha(T_x X) = \left\{ v_\alpha \in (T_x X)^{|\alpha|} \,:\, ||v_\alpha|| \leq 1 \right\}$. Hereafter, when we mention "differentiability with respect to the weak topology", we mean Fréchet differentiability measured with the help of these semi-norms p. This Fréchet topology is, roughly speaking, the topology of uniform convergence on compact subsets K of X of derivatives at any order.

Theorem 1. *Let E be a vector bundle over a finite-dimensional (maybe non-compact) manifold X. Assume that E is locally trivializable and with a typical fiber being a Fréchet vector space V. Let $\Gamma^\infty(X,E) \subset C^\infty(X,E)$ be the set of smooth sections of E. Then, the nebulae diffeology on $\Gamma^\infty(X,E)$ coincides with the functional diffeology.*

The functional diffeology on $\Gamma^\infty(X,E)$ is precisely the subset diffeology on $\Gamma^\infty(X,E)$ induced by pull-back through the set-theoretic inclusion $\Gamma^\infty(X,E) \subset C^\infty(X,E)$ of the functional diffeology of $C^\infty(X,E)$ (see [24]); the nebulae diffeology that we consider here is $\mathcal{P}_\infty(\Gamma^\infty(X,E))$, in which $\Gamma^\infty(X,E)$ is considered as a Fréchet diffeological space (in fact, a Fréchet manifold).

Proof. Let us fix $n = dim X$. For the sake of clarity, let us note by $\mathcal{P}_F$ the functional diffeology on $\Gamma^\infty(X,E)$ and by $\mathcal{P}_W$ the nebulae diffeology $\mathcal{P}_\infty(\Gamma^\infty(X,E))$ on $\Gamma^\infty(X,E)$. $\mathcal{P}_W$ makes the evaluation map

$$ev : X \times \Gamma^\infty(X,E) \to E$$

smooth and, therefore,we have that $\mathcal{P}_W \subset \mathcal{P}_F$. Let us prove that $\mathcal{P}_F \subset \mathcal{P}_W$. For this, according to the remarks on smoothness for one-dimensional and nebulae diffeologies, in, e.g., [19,33], it is sufficient to prove that any smooth path γ from $\mathbb{R}$ to $(\Gamma^\infty(X,E), \mathcal{P}_F)$ is also smooth for the weak topology on $\Gamma^\infty(X,E)$. Let K be a compact subset of X that we assume in the domain of a local trivialization of X without loss of generality, and let p be a semi-norm on V. Let α be a multi-index. $\frac{d^2}{dt^2} D^\alpha \gamma \in C^\infty(\mathbb{R} \times K, V)$; hence, there exists $m > 0$ such that

$$\sup_{(t,x \in [-1;1] \times K)} p\left(\frac{d^2}{dt^2} D^\alpha \gamma\right) \leq m$$

which implies that

$$\sup_{x \in K} p\left(\frac{d}{dt} D^\alpha \gamma(t,x) - \frac{d}{dt} D^\alpha \gamma(0,x)\right) \leq mt$$

and, therefore, there exists $m' > 0$ such that

$$\sup_{(x \in K} p\left(\frac{D^\alpha \gamma(t,x) - D^\alpha \gamma(0,x)}{t} - \frac{d}{dt} D^\alpha \gamma(0,x)\right) \leq m' t.$$

These estimates are valid for any compact subspace K of X, for any multi-index α, and for any semi-norm p on V. Therefore, γ is differentiable at $t = 0$ for the weak topology. The same holds for any parameter t. Therefore, any smooth path with respect to p_F is smooth for $\mathcal{P}_W$. Let $p : D(p) \to \Gamma^\infty(X,E)$ be a plot in $\mathcal{P}_F$. Then, it is smooth on each affine segment of $D(p)$ and, evaluating smoothness on any straight segment on $D(p)$, we get that p is also Gâteaux smooth and, hence, smooth for the weak topology, which proves that $\mathcal{P}_F \subset \mathcal{P}_W$. $\square$

We finish this section with a short discussion on differential forms.

Definition 6 ([23])**.** *Let $(X, \mathcal{P})$ be a diffeological space and let V be a vector space equipped with a differentiable structure. A V-valued n-differential form α on X (noted $\alpha \in \Omega^n(X, V)$) is a map*

$$\alpha : \{p : O_p \to X\} \in \mathcal{P} \mapsto \alpha_p \in \Omega^n(O_p ; V)$$

such that the following conditions are true:

- *Let $x \in X$. $\forall p, p' \in \mathcal{P}$ such that $x \in Im(p) \cap Im(p')$; the forms α_p and $\alpha_{p'}$ are of the same order n.*
- *Moreover, let $y \in O_p$ and $y' \in O_{p'}$. If $(X_1, \ldots, X_n)$ are n germs of paths in $Im(p) \cap Im(p')$, and if there exist two systems of n-vectors $(Y_1, \ldots, Y_n) \in (T_y O_p)^n$ and $(Y'_1, \ldots, Y'_n) \in (T_{y'} O_{p'})^n$, if $p_*(Y_1, \ldots, Y_n) = p'_*(Y'_1, \ldots, Y'_n) = (X_1, \ldots, X_n)$,*

$$\alpha_p(Y_1, \ldots, Y_n) = \alpha_{p'}(Y'_1, \ldots, Y'_n).$$

We note by

$$\Omega(X; V) = \oplus_{n \in \mathbb{N}} \Omega^n(X, V)$$

the set of V-valued differential forms.

We note that if there do not exist n linearly independent vectors $(Y_1, \ldots, Y_n)$ as in the last point of the definition, then $\alpha_p = 0$ at y. Furthermore, if $(\alpha, p, p') \in \Omega(X, V) \times \mathcal{P}^2$ and there exists $g \in C^\infty(D(p), D(p'))$ (in the usual sense) such that $p' \circ g = p$, then $\alpha_p = g^* \alpha_{p'}$.

Proposition 5. *The set $\mathcal{P}(\Omega^n(X, V))$ made of maps $q : x \mapsto \alpha(x)$ from an open subset O_q of a finite-dimensional vector space to $\Omega^n(X, V)$ such that, for each $p \in \mathcal{P}$,*

$$\{x \mapsto \alpha_p(x)\} \in C^\infty(O_q, \Omega^n(O_p, V))$$

is a diffeology on $\Omega^n(X, V)$.

We can define the wedge product and the exterior differential of differential forms working on plots of a diffeology. These operations have the same properties as the wedge product and the differential of standard differential forms.

4. Jets, Equation Manifolds, Diffeologies, and Frölicher Spaces

In this section, we specify some structures existing on mapping spaces and jet spaces. First, let us propose an example that shows that the point of view considered in most of the literature, of restricting to a smooth locus of a given equation manifold, has some limitations.

Example 1. *If*

$$\Delta(x, y, y') = x \cos(y) \sin(4y') \; ,$$

the equation manifold is the subset $\mathcal{EM} = \{\Delta = 0\}$ in $J^1(\mathbb{R}^2)$. It has infinite singular points. Let us assume that we restrict $\mathcal{EM}$ to a neighbourhood U of a regular point. The solutions $x \mapsto (x, y(x))$ we would find are simply constant solutions or affine solutions, and we may lose information depending on our choice of U. Now, these solutions extend to piecewise affine (and hence only piecewise smooth) solutions on $\mathcal{EM}$. If, instead of looking for a manifold structure, we consider an adapted diffeology, and we consider only stationary parametrizations at singular points, these piecewise affine solutions would be smooth, in a diffeological sense. Thus, globally speaking, talking about $\mathcal{EM}$ as a "subset equipped with an adapted diffeology", is less ambiguous than thinking of it as a "submanifold", and it relies only on the chosen parametrizations.

First, let us go back to jet spaces. The bundle $J^\infty(E)$ is a vector bundle over E where the typical fiber is constituted by the **formal series**

$$\sum_{i=1}^{+\infty} L^i(\mathbb{R}^m, \mathbb{R}^n)$$

and projection map $\pi_E^\infty : J^\infty(E) \to E$, and it is also a vector bundle over $J^k(E)$ with fiber

$$\sum_{i=k+1}^{+\infty} L^i(\mathbb{R}^m, \mathbb{R}^n)$$

and projection map $\pi_k^\infty : J^\infty(E) \to J^k(E)$.

The inverse limit $J^\infty(E)$ is a topological space: a basis for the topology on $J^\infty(E)$ is the collection of all sets of the form $(\pi_k^\infty)^{-1}(W)$, in which W is a relatively compact open subset of $J^k(E)$, $k \geq 0$. Moreover, it is easy to prove that the typical fiber $\sum_{i=1}^{+\infty} L^i(\mathbb{R}^m, \mathbb{R}^n)$ of $J^\infty(E)$ is a Fréchet space equipped with the semi-norms

$$||.||_i = ||.||_{L^i(\mathbb{R}^m,\mathbb{R}^n)} \circ \pi_i$$

where $\pi_i : \sum_{i=1}^{+\infty} L^i(\mathbb{R}^m, \mathbb{R}^n) \to L^i(\mathbb{R}^m, \mathbb{R}^n)$ is the canonical projection, and that $J^\infty(E)$ is a Fréchet manifold modeled over $\mathbb{R}^{m+n} \oplus \sum_{i=1}^{+\infty} L^i(\mathbb{R}^m, \mathbb{R}^n)$.

The space $J^\infty(E)$ is also a Frölicher space, equipped with a Frölicher structure derived from the inverse limit of the Frölicher structures

$$\left(J^k(E), C^\infty(J^k(E), \mathbb{R}), C^\infty(\mathbb{R}, J^k(E)) \right),$$

$k \geq 0$, and differential geometric arguments can be carried out on $J^\infty(E)$ without explicitly relying on its (local) Fréchet manifold characterization. Instead, we can utilize its Frölicher and diffeology structures. In fact, we have found that this is precisely the approach taken in works such as [1,5,6].

If P is some finite-dimensional manifold, the class of smooth functions $f : J^\infty(E) \to P$ used in [1,6] is determined by the following condition: there exist $k \geq 0$ and a smooth function $f_k : J^k(E) \to P$ for some $k \geq 0$, such that $f = f_k \circ \pi_k^\infty$ (in Section 2, we used the $P = \mathbb{R}$ case of this definition), and a function $f : P \to J^\infty(E)$ is smooth if, for any finite-dimensional manifold Q and any smooth map $g : J^\infty(E) \to Q$, the composition $g \circ f$ is smooth.

This concept of smoothness is included within the broader notion of smoothness for Frölicher spaces. The "smooth functions" defined in the previous paragraph form a space $\mathcal{F}_0$ that generates a Frölicher structure, and the corresponding diffeology is the set of maps $\mathcal{P}$ from any open subset $O \subset \mathbb{R}^k$, $k \geq 1$ to $J^\infty(E)$, such that $\mathcal{F}_0 \circ \mathcal{P} \subset \bigcup_O C^\infty(O, \mathbb{R})$, where $C^\infty(O, \mathbb{R})$ is the usual set of smooth functions from O to $\mathbb{R}$.

We also note that, when considering sections in $\Gamma^\infty(X, J^\infty(E)) \subset C^\infty(X, J^\infty(E))$, we can describe a Fréchet structure if X is compact, and we can produce a nice topological vector space structure using, e.g., the Whitney topology; if X is not compact, see [22]; however, functional diffeology seems to be necessary in applications. This is the one that is used in [8]. Because of Theorem 1, this choice is the same as choosing the diffeological structure (that is, the nebulae diffeology) induced by the weak topology of $\Gamma^\infty(X, J^\infty(E))$, defined by the uniform convergence of derivatives of any order on any compact subset of X.

In order to define vector fields and the tangent bundle $TJ^\infty(E)$, we use inverse limits again: $TJ^\infty(E)$ is the inverse limit of the sequence

$$\ldots \to TJ^k(E) \to TJ^{k-1}(E) \to \ldots \to TE\,,$$

and, therefore, it is also equipped with a natural Frölicher structure. Vector fields are smooth functions $V : J^\infty(E) \to TJ^\infty(E)$ satisfying $T\pi \circ V = id_{J^\infty(E)}$, in which $T\pi : TJ^\infty(E) \to J^\infty(E)$ is the canonical projection; locally, they correspond precisely to formal series derivations (see, for example, [11]). This is interesting, because on general Frölicher spaces, derivations may not coincide with vector fields, as explained in [22] (see also [19] and references therein). *This definition of $TJ^\infty(E)$ coincides with the standard definition of the tangent bundle of $J^\infty(E)$* (see, for instance, [13] [Section 7.2]).

Now, we consider a fixed partial differential equation (or system) of order k denoted as $\Delta = 0$; we assume that $\Delta \in C^\infty(J^k(E), \mathbb{R}^m)$ for some $m \geq 1$. Let us consider $\mathcal{EM}^{(k)} = \Delta^{-1}(0)$. Example 1 suggests that it might be overly optimistic to expect $\mathcal{EM}^{(k)}$ to possess the structure of a smooth manifold. As noted in Section 2, researchers traditionally address this challenge by restricting $\mathcal{EM}^{(k)}$ to a smooth locus within $J^k(E)$. In explicit examples, this restriction can be defined by a unique embedding of an open subset of Euclidean space into $J^k(E)$. Then, subject to some technical assumptions (see [14]), the equation manifold $\mathcal{EM}^{(\infty)}$ is constructed via prolongations. This approach allows us to apply standard geometrical arguments, but it comes at the cost of losing global information, leading to the somewhat unsettling notion that we are engaging in "formal geometry".

Diffeologies and Frölicher spaces offer a solution. Let us *define* $\mathcal{EM}^{(\infty)}$ as the subset of $J^\infty(E)$ formed by all points in $J^\infty(E)$ that satisfy $\Delta = 0$ and *all* its differential consequences. Given that $J^\infty(E)$ is a Frölicher space with the nebulae diffeology $\mathcal{P}_\infty$, we can introduce the **subset diffeology** on $\mathcal{EM}^{(\infty)} \subset J^\infty(E)$. This is achieved by considering the maps $p \in \mathcal{P}_\infty$ whose range lies within $\mathcal{EM}^{(\infty)}$. Thus, we obtain $\mathcal{EM}^{(\infty)}$ as a diffeological (and Frölicher) space, bypassing the need for finite-dimensional prolongations and, thus, eliminating the obstacles presented in [14] and the assumptions of Section 2.

In this context, we define tangent vectors and differential forms on $\mathcal{EM}^{(\infty)}$ in a natural manner (it is worth noting that the definition of vectors in general diffeological spaces involves some subtleties addressed in [24,34,35] and also discussed in [9,19]): $T\mathcal{EM}^{(\infty)}$ is defined by pull-back of $TJ^\infty(E)$ via the inclusion map, or equivalently, it consists of all germs of paths in $TJ^\infty(E)$ with values in $\mathcal{EM}^{(\infty)}$. Differential forms are constructed from local forms that are "compatible on parametrizations". As in the case of smooth functions, the differential forms appearing in standard literature on infinite-dimensional geometry of differential equations are included in the differential forms just defined in terms of Frölicher/diffeological structures. In fact, when $p \in \mathcal{P}_\infty$ is a particular smooth embedding into $J^k(E) \subset J^\infty(E)$, we recognize exactly the definition of differential forms as outlined in [1,6,7] (see also [13]). We stress that an advantage of the approach via diffeologies is that it also carries natural compatibility conditions for a global definition of differential forms on $\mathcal{EM}^{(\infty)}$. Exterior differential calculus and the de Rham differential can also be defined globally in this framework. We refer to [21] for details.

5. Global Cartan Distributions and Vinogradov Spectral Sequence

There exist some problems in the definition of the tangent space of a diffeological space, as anticipated in the last paragraph of the previous section. Let us be more precise. The equivalent definitions of tangent space of a finite-dimensional manifold extend to definitions of tangent space of a Frölicher or diffeological space, but in non-equivalent ways; we refer to [24] for details. In particular, the tangent space defined by germs of smooth paths, the tangent space defined by infinitesimal action of smooth diffeomorphisms, and the tangent space defined by local derivations, give three different spaces on finite-dimensional examples with singularities (see [36] or [24]). Therefore, defining in addition a *bracket* of vector fields is a hard task (see [9]) and a definition via the Lie algebra of the group of diffeomorphisms along the lines of [10] is only partially satisfying, since it assumes additional properties of the group of diffeomorphisms that may appear artificial, or at least, may not be automatically fulfilled [9].

Therefore, we have to specify our definition of a tangent space before giving a global definition of the Cartan distribution along the lines of (3) and of the fundamental property (5). As we mentioned in Section 2, the Cartan distribution is defined on a smooth locus $\mathcal{L} \subset \mathcal{EM}$, and it determines a space of derivations on the space of smooth functions from $\mathcal{L}$ to $\mathbb{R}$. Thus, it seems natural to use the internal–external tangent cone, noted by $C^{ie}T\mathcal{EM}^{(\infty)}$, along the lines of [24]. For simplicity, we also consider the corresponding internal tangent space, but we omit it in the notations since it carries no ambiguity in our constructions.

Definition 7. *We note by $\mathcal{C}(\mathcal{L})$ the Cartan distribution associated with a smooth locus $\mathcal{L} \subset \mathcal{EM}$. The **global Cartan distribution** on $\mathcal{EM}$ is the set $\mathcal{C}$ defined by*

$$\mathcal{C} = \bigcup_{\mathcal{L} \text{ smooth locus in } \mathcal{EM}} \mathcal{C}(\mathcal{L}) \subset C^{ie}T\mathcal{EM}^{(\infty)}.$$

This definition is consistent because the Cartan distribution is unique for each fixed smooth locus $\mathcal{L}$. Therefore, $\mathcal{C}$ is only a fiber (pseudo-)bundle whose fibers are cones, and, therefore, the bracket of vector fields (and hence involutivity) on the Cartan distribution may be difficult to define without the help of an additional structure. We propose a new definition of involution the following way:

Definition 8. *Let $\mathcal{DIS} \subset C^{ie}T\mathcal{EM}^{(\infty)}$ be a sub pseudo-bundle of $C^{ie}T\mathcal{EM}^{(\infty)}$. Let*

$$\Omega^*_{\mathcal{DIS}} = \{\alpha \in \Omega^*(\mathcal{EM}) \mid \alpha|_{\mathcal{DIS}=0}\}.$$

*Then, $\mathcal{DIS}$ is **involutive** if*

$$d(\Omega^*_{\mathcal{DIS}}) \subset \Omega^*_{\mathcal{DIS}},$$

where d is the de Rham differential on differential forms.

Remark 1. *This definition remains valid when replacing $\mathcal{EM}^{(\infty)}$ by any diffeological space X, and it coincides with the classical notion of involutive distribution for a (classical) distribution $\mathcal{DIS} \subset TX$ over a finite-dimensional manifold X.*

Now we can build the **Vinogradov spectral sequence** along the lines of [2–4]: the algebra of differential forms

$$\Omega^*(\mathcal{EM}) = \sum_{k \in \mathbb{N}} \Omega^*(\mathcal{EM})$$

is a graded differential algebra for the de Rham differential operator. Let us define

$$C\Omega^k(\mathcal{EM}) = \left\{\alpha \in \Omega^k \mid \alpha|_{\mathcal{C}} = 0\right\},$$

$$\mathcal{C}\Omega(\mathcal{EM}) = \sum_{k \in \mathbb{N}} \mathcal{C}\Omega^k(\mathcal{EM}) = \Omega^*_{DIS} \text{ with } \mathcal{DIS} = \mathcal{C}$$

and

$$\mathcal{C}^l\Omega(\mathcal{EM}) = \mathcal{C}\Omega(\mathcal{EM})^{\wedge l}.$$

Since $\mathcal{C}$ is involutive, $\mathcal{C}$ is a differential ideal, and so is $\mathcal{C}^l\Omega(\mathcal{EM})$ for $l \in \mathbb{N}^*$. Therefore, we can define

$$E_0^{p,q} = \frac{\mathcal{C}^p\Omega^{p+q}(\mathcal{EM})}{\mathcal{C}^{p+1}\Omega^{p+q}(\mathcal{EM})},$$

$$E_{r+1}^{p,q} = H^*(E_r^{p,q}),$$

and

$$\mathcal{C}E(\mathcal{EM}) = \left\{E_r^{p,q}\right\},$$

equipped with the restriction/extension of the de Rham differential to the corresponding spaces. This is our version of Vinogradov's spectral sequence.

6. Conclusions and Outlook

We claim in this paper that the geometry of equation manifolds can be reinterpreted in the very general and flexible framework of diffeologies without apparent restriction. In this work, we have considered a diffeological version of one of the fundamental constructions in the area, the Vinogradov spectral sequence.

After dealing with the Vinogradov sequence, our aim is to propose a "diffeological way" to understand the variational bicomplex. However, we believe that we have to consider with caution some natural approaches that may be too straightforward in this context, because most notions in diffeologies, even if safely generalized, can be extended in non-equivalent ways. One striking example is, in our opinion, the five different tangent bundles of diffeological spaces actually developed in the existing literature, each one of them with its own motivations and applications (see, e.g., the review [24]).

We feel that the same difficulties may appear in the present context. Various constructions of the variational bicomplex may lead to non-equivalent global generalizations.

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