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Block-Supersymmetric Polynomials on Spaces of Absolutely Convergent Series

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Abstract: In this paper, we consider a supersymmetric version of block-symmetric polynomials on a Banach space of two-sided absolutely summing series of vectors in $\mathbb{C}^s$ for some positive integer $s > 1$. We describe some sequences of generators of the algebra of block-supersymmetric polynomials and algebraic relations between the generators for the finite-dimensional case and construct algebraic bases of block-supersymmetric polynomials in the infinite-dimensional case. Furthermore, we propose some consequences for algebras of block-supersymmetric analytic functions of bounded type and their spectra. Finally, we consider some special derivatives in algebras of block-symmetric and block-supersymmetric analytic functions and find related Appell-type sequences of polynomials.

Keywords: symmetric analytic functions on Banach spaces; algebras of analytic functions; algebraic bases; block-symmetric polynomials

MSC: 46G20

1. Introduction

Symmetric functions on finite-dimensional vector spaces are standard objects in combinatorics and classical invariant theory (see, e.g., [1,2]). For infinite-dimensional spaces, investigations of symmetric polynomials were started by Nemirovski and Semenov in [3]. In particular, in [3], the authors constructed algebraic bases of algebras of symmetric real-valued polynomials on real Banach spaces ℓ_p and $L_p[0,1]$ for $1 \le p < \infty$. In [4], these results were generalized to separable sequence real Banach spaces with symmetric bases and to separable rearrangement invariant real Banach spaces, respectively. The cases of ℓ_∞ and L_∞ were considered in [5,6]. Since then, symmetric structures and mappings in infinite-dimensional Banach spaces have been studied by many authors (see, e.g., [7] and references therein).

In [8], Jawad and Zagorodnyuk considered supersymmetric polynomials and analytic functions on the space $\ell_1(\mathbb{Z}_0)$ of absolutely summable sequences (x_n), $n \in \mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. Supersymmetric polynomial generalizations for more general sequence spaces were considered in [9]. Applications of algebraic bases of supersymmetric polynomials to models of ideal gases in quantum physics were proposed in [10]. Supersymmetric polynomials over finite fields and their applications in cryptography were considered in [11]. Supersymmetric polynomials on finite-dimensional vector spaces were studied in [12,13].

Block-symmetric or MacMahon polynomials are natural generalizations of symmetric polynomials and can be considered symmetric polynomials on linear spaces of vector sequences.

Combinatorial properties of such polynomials are described in [14]. An algebraic basis of the algebra of all symmetric continuous complex-valued polynomials on the Cartesian power of the complex Banach space ℓ_p for some fixed $1 \le p < \infty$ was constructed in [15] (see also [16] for more details on the real case). Algebras of symmetric continuous polynomials on Cartesian products $\ell_{p_1} \times \cdots \times \ell_{p_s}$ for different $p_1, \ldots, p_s$ were considered

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in [17]. Some generalizations of the Newton formulas for algebraic bases of block-symmetric polynomials were obtained in [18]. Spectra of algebras of block-symmetric polynomials and holomorphic functions and algebraic structures on the spectra were considered in [19,20]. Zeros of block-symmetric polynomials were investigated in [21].

In this paper, we consider algebras of polynomials that are both block-symmetric and supersymmetric on infinite-dimensional Banach spaces of absolutely summable sequences of vectors in $\mathbb{C}^s$. The main goal of this research is to construct algebraic bases in these algebras and find some Newton-type relations between different bases. Furthermore, we propose some consequences for algebras of block-symmetric and supersymmetric analytic functions that are bounded on all bounded subsets and for derivatives on these algebras.

In Section 2, we provide a basic review of known preliminary results on supersymmetric and block-symmetric polynomials in ℓ_p-spaces. In Section 3, we introduce block-supersymmetric polynomials in corresponding Banach spaces and find analogs of classical algebraic bases of such polynomials and Newton-type relations between these bases. In addition, we discuss the finite-dimensional case and show some algebraic dependencies between generating elements. In Section 4, we consider algebras of block-supersymmetric analytic functions of bounded type and apply the obtained results on block-supersymmetric polynomials to their spectra. In Section 5, we consider some derivatives on the algebras of polynomials and analytic functions and construct related Appell-type polynomials.

2. Symmetric, Supersymmetric and Block-Symmetric Polynomials

2.1. Symmetric and Supersymmetric Polynomials

Let X be a complex Banach space with a symmetric basis (e_n). Let us recall that a Schauder basis (e_n) is *symmetric* if for every permutation (one-to-one map) $\sigma \in S_{\mathbb{N}}$, and the basis $(e_{\sigma(n)})$ is equivalent to (e_n), where $S_{\mathbb{N}}$ is the semigroup of all permutations on the set of all natural numbers $\mathbb{N}$. Therefore, we can uniquely represent every $x \in X$ as

$$x = (x_1, x_2, \ldots) = \sum_{n=1}^{\infty} x_n e_n.$$

A mapping F on X is said to be *symmetric* if

$$F(x_1, x_2, \ldots) = F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$$

for each $\sigma \in S_{\mathbb{N}}$. A function $P\colon X \to \mathbb{C}$ is a polynomial of degree m if the restriction of P to any finite-dimensional subspace of X is a polynomial of several variables of degree $\leq m$ and there is a finite-dimensional subspace V of X such that the restriction of P to V is a polynomial of degree m. We denote by $\mathcal{P}_s(X)$ the algebra of all continuous symmetric polynomials on X.

In the case $X = \ell_1$, polynomials

$$F_k(x) = \sum_{n=1}^{\infty} x_n^k, \quad k \in \mathbb{N},$$

form an algebraic basis of $\mathcal{P}_s(\ell_1)$ [4]. That is, for any polynomial $P \in \mathcal{P}_s(\ell_1)$, there is a unique polynomial of several complex variables $Q(t_1, \ldots, t_m)$, such that $P(x) = Q(F_1(x), \ldots, F_m(x))$. The algebraic basis is not unique, of course, and we will also use bases

$$G_n(x) = \sum_{i_1 < \ldots < i_n} x_{i_1} \cdots x_{i_n}$$

and

$$B_n(x) = \sum_{i_1 \leq \ldots \leq i_n} x_{i_1} \cdots x_{i_n}.$$

These bases are connected by known Newton formulas, which remains true for the infinite-dimensional case:

$$mG_m = \sum_{k=1}^{m} (-1)^{k-1} G_{m-k} F_k, \quad m \in \mathbb{N}, \tag{1}$$

$$mB_m = \sum_{k=1}^{m} B_{m-k} F_k, \quad m \in \mathbb{N}. \tag{2}$$

For these basis polynomials, there are the following generating functions:

$$\mathcal{F}(x)(t) = \sum_{n=1}^{\infty} t^{n-1} F_n(x),$$

$$\mathcal{G}(x)(t) = \sum_{n=0}^{\infty} t^n G_n(x), \quad G_0 = 1,$$

$$\mathcal{B}(x)(t) = \sum_{n=0}^{\infty} t^n B_n(x), \quad B_0 = 1.$$

It is well-known in combinatorics (see e.g., [22]) that

$$\mathcal{G}(x)(t) = \frac{1}{\mathcal{B}(-x)(t)}$$

and

$$\mathcal{G}(x)(t) = \exp\left(-\sum_{n=1}^{\infty} t^n \frac{F_n(-x)}{n} \right). \tag{3}$$

Let $\mathbb{Z}_0 = \mathbb{Z} \setminus 0$ and $\ell_1(\mathbb{Z}_0)$ be the Banach space of all sequences of complex numbers that are absolutely summing and indexed by numbers in $\mathbb{Z}_0$. Any element z in $\ell_1(\mathbb{Z}_0)$ has the representation

$$z = (\ldots, z_{-n}, \ldots, z_{-1}, z_1, \ldots, z_n, \ldots)$$

that can be written as

$$z = (y|x) = (\ldots, y_n, \ldots, y_1 | x_1, \ldots, x_n, \ldots)$$

with

$$||z|| = \sum_{i=-\infty}^{\infty} |z_i| = \|x\| + \|y\|,$$

where $z_n = x_n$, $z_{-n} = y_n$, and elements $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots)$, $n \in \mathbb{N}$ are in ℓ_1. Note that $x \mapsto (0|x)$ and $y \mapsto (y|0)$ are isometric embeddings of the copies of ℓ_1 into $\ell_1(\mathbb{Z}_0)$. A polynomial P on $\ell_1(\mathbb{Z}_0)$ is said to be *supersymmetric* if it is a finite algebraic combination of polynomials T_k, $k \in \mathbb{N}$,

$$T_k(y|x) = F_k(x) - F_k(y) = \sum_{i=1}^{\infty} x_i^k - \sum_{i=1}^{\infty} y_i^k.$$

In [8], it was shown that the following polynomials on $\ell_1(\mathbb{Z}_0)$,

$$W_n(y|x) = \sum_{k=1}^{n} G_k(x) B_{n-k}(-y), \quad n \in \mathbb{N},$$

form another basis in the algebra of supersymmetric polynomials. Moreover, in [10], it was observed that

$$\widetilde{W}_n(y|x) = \sum_{k=1}^{n} B_k(x) G_{n-k}(-y), \quad n \in \mathbb{N}$$

also forms an algebraic basis in the algebra of supersymmetric polynomials on $\ell_1(\mathbb{Z}_0)$ and produes some relations between $W_n(y|x)$ and $\widetilde{W}_n(y|x)$.

Remark 1. *We can consider the space ℓ_p as the space of functions $x_n = x(n)$ on the set of positive integers $\mathbb{N}$ with values in $\mathbb{C}$. Hence, ℓ_p is a short notation for $\ell_p(\mathbb{N}, \mathbb{C})$, and $\ell_1(\mathbb{Z}_0)$ is a short notation for $\ell_1(\mathbb{Z}_0, \mathbb{C})$. In the general case, if Y is a normed space and $\mathfrak{A}$ is a set of indexes, the notation $\ell_p(\mathfrak{A}, Y)$, $1 \le p < \infty$ means the normed space of Y-valued functions $x_n = x(n)$, $n \in \mathfrak{A}$ such that*

$$\|x\| = \left(\sum_{n\in\mathfrak{A}} \|x_n\|^p \right)^{1/p}.$$

It is easy to see that if a polynomial $P(y|x)$ on $\ell_1(\mathbb{Z}_0)$ is supersymmetric, then it is invariant with respect to a semigroup of mappings on $\ell_1(\mathbb{Z}_0)$ consisting of all permutations of coordinates separately for x and for y, and affine operators of the form

$$(y|x) = (\ldots, y_n, \ldots, y_1|x_1, \ldots, x_n, \ldots) \mapsto (\ldots, y_n, \ldots, y_1, a|a, x_1, \ldots, x_n, \ldots), \quad a \in \mathbb{C}. \tag{4}$$

The following examples show each condition itself does not imply supersymmetry.

Example 1. *Let*

$$P(y|x) = (F_k(x))^2 - (F_k(y))^2 = \Big(\sum_{n=1}^{\infty} x_n^k \Big)^2 - \Big(\sum_{n=1}^{\infty} y_n^k \Big)^2$$

for some $k \in \mathbb{N}$. Then, $P(y|x)$ is invariant with respect to all permutations of coordinates separately for x and for y, but for $a = 1$,

$$\begin{aligned} P(y|x) &= (F_k(x) - F_k(y))(F_k(x) + F_k(y)) \neq (F_k(x) - F_k(y))(F_k(x) + F_k(y) + 2) \\ &= P(\ldots, y_n, \ldots, y_1, 1|1, x_1, \ldots, x_n, \ldots). \end{aligned}$$

Thus, P is not supersymmetric.

Example 2. *Let*

$$P(y|x) = \sum_{n=1}^{\infty} (x_n - y_n)^k$$

for some natural $k > 1$. Then, $P(y|x)$ is invariant with respect to the action of (4) but not invariant with respect to all permutations of coordinates separately for x and for y. Therefore, it is not supersymmetric. Note that $P(y|x)$ is invariant with respect to all simultaneous permutations of coordinates of x and y but it is not enough.

2.2. Block-Symmetric Polynomials

We denote by $\ell_p(\mathbb{C}^s) = \ell_p(\mathbb{N}, \mathbb{C}^s)$, $1 \le p < \infty$ the linear space of all sequences

$$x = (x_1, x_2, \ldots, x_j, \ldots), \tag{5}$$

such that $x_j = (x_j^{(1)}, \ldots, x_j^{(s)}) \in \mathbb{C}^s$ for $j \in \mathbb{N}$, and the series $\sum\limits_{j=1}^{\infty} \sum\limits_{r=1}^{s} \left| x_j^{(r)} \right|^p$ converges. We also use the representation

$$x = \left(\begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \\ \vdots \\ x_1^{(s)} \end{pmatrix}, \begin{pmatrix} x_2^{(1)} \\ x_2^{(2)} \\ \vdots \\ x_2^{(s)} \end{pmatrix}, \ldots, \begin{pmatrix} x_j^{(1)} \\ x_j^{(2)} \\ \vdots \\ x_j^{(s)} \end{pmatrix}, \ldots \right)$$

for x. Vectors x_j in (5) are called vector coordinates of x. The linear space $\ell_p(\mathbb{C}^s)$ is endowed with the norm

$$\|x\| = \left(\sum_{j=1}^{\infty}\sum_{r=1}^{s}\left|x_j^{(r)}\right|^p\right)^{1/p}$$

which is a Banach space. A polynomial P on the space $\ell_p(\mathbb{C}^s)$ is called *block-symmetric (or vector-symmetric)* if:

$$P(x_1, x_2, \ldots, x_m, \ldots) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}, \ldots)$$

for every permutation $\sigma \in S_{\mathbb{N}}$ and $x_m \in \mathbb{C}^s$. Let us denote by $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ the algebra of all block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$.

More information about algebra $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ can be found in [15,17] and references therein. In the paper, we concentrate on the case $p = 1$ because in this case, we have explicit representations of different algebraic bases. Furthermore, the case $p = 1$ has a physical meaning in applications for modeling quantum ideal gases [10]. Note that in combinatorics, block-symmetric polynomials on finite-dimension spaces are called *MacMahon symmetric polynomials* (see [14]).

Throughout this paper, we consider multi-indexes $\mathbf{k} = (k_1, k_2, \ldots, k_s)$ with non-negative integer entries $k_1, k_2, \ldots, k_s$, and we will use the standard notations $|\mathbf{k}| = k_1 + k_2 + \cdots + k_s$, and $\mathbf{k}! = k_1!k_2!\cdots k_s!$.

According to [15], polynomials

$$H^{\mathbf{k}}(x) = H^{k_1,k_2,\ldots,k_s}(x) = \sum_{j=1}^{\infty}\prod_{\substack{r=1\\|k|\geq\lceil p\rceil}}^{s}(x_j^{(r)})^{k_r}$$

form an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$, $1 \le p < \infty$, where $x = (x_1, \ldots, x_m, \ldots)$ are in $\ell_p(\mathbb{C}^s)$, and $x_j = (x_j^{(1)}, \ldots, x_j^{(s)}) \in \mathbb{C}^s$. According to [14], these polynomials are called the power sum MacMahon symmetric functions. For example, $H^{n,0,\ldots,0}(x) = \left(x_1^{(1)}\right)^n + \left(x_2^{(1)}\right)^n + \cdots$.

In the case of the space $\ell_1(\mathbb{C}^s)$, there is another important algebraic basis (see [1,14,20]):

$$R^{\mathbf{k}}(x) = R^{k_1,k_2,\ldots,k_s}(x) = \sum_{\substack{i_1^j<\cdots<i_{k_j}^j\\1\le j\le s}}^{\infty}\prod_{j=1}^{s}x_{i_1^j}^{(j)}\ldots x_{i_{k_j}^j}^{(j)}. \tag{6}$$

In [14], these polynomials are called the elementary MacMahon symmetric functions. For example,

$$R^{1,2,0\ldots,0}(x) = \sum_{i,j_1<j_2} x_i^{(1)}x_{j_1}^{(2)}x_{j_2}^{(2)}.$$

Let $\mathcal{H}(x)(t)$ and $\mathcal{R}(x)(t)$ be a formal series

$$\mathcal{H}(x)(t) = \sum_{|k|=1}^{\infty}\prod_{i=1}^{s}t_i^{k_i}H^{\mathbf{k}}(x),$$

$$\mathcal{R}(x)(t) = \sum_{|k|=0}^{\infty}\prod_{i=1}^{s}t_i^{k_i}R^{\mathbf{k}}(x), \quad R^{\mathbf{0}} = 1,$$

which are also called generating functions (see [14]). From [14,20], we know that

$$\mathcal{R}(x)(t) = \sum_{|k|=0}^{\infty}\prod_{i=1}^{s}t_i^{k_i}R^{\mathbf{k}}(x) = \prod_{i=1}^{\infty}(1 + x_i^{(1)}t_1 + \cdots + x_i^{(s)}t_s). \tag{7}$$

From [14], it follows that there is one more algebraic basis of homogeneous polynomials $E^{\mathbf{k}}(x)$, which can be defined from the generating function

$$\mathcal{E}(x)(t) = \sum_{|k|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} E^{\mathbf{k}}(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i^{(1)} t_1 - \cdots - x_i^{(s)} t_s}, \quad E^{\mathbf{0}} = 1. \tag{8}$$

These polynomials are called the complete homogeneous MacMahon symmetric functions.

From (7) and (8), we have

$$\mathcal{R}(x)(t) = \frac{1}{\mathcal{E}(-x)(t)}. \tag{9}$$

Proposition 1. *For the generating function $\mathcal{R}(x)(t)$, we have the following relation:*

$$\mathcal{R}(x)(t) = \exp\left(- \sum_{|\mathbf{k}|=1}^{\infty} \prod_{i=1}^{\infty} t_i^{k_i} \frac{(|\mathbf{k}|-1)! H^{\mathbf{k}}(-x)}{\mathbf{k}!} \right), \tag{10}$$

where $x = (x^{(1)}, \ldots, x^{(s)}) \in \ell_1(\mathbb{C}^s)$ and $t = (t_1, \ldots, t_s)$.

Proof. For polynomials $F_k(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})$, $G_k(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})$, by Formula (3), we have

$$\mathcal{G}(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})(\lambda) = \exp\left(- \sum_{n=1}^{\infty} \lambda^n \frac{F_n(-(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)}))}{n} \right), \tag{11}$$

where $\lambda \in \mathbb{C}$.

The generating functions $\mathcal{G}(t_1 x^{(1)} + t_2 x^{(2)} + \ldots + t_s x^{(s)})(\lambda)$ in the case $\lambda = 1$ can be rewritten as

$$\mathcal{G}(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})(1) = \sum_{n=0}^{\infty} G_n(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)}). \tag{12}$$

On the other hand, each of polynomials $F_m(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})$ and $G_m(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})$ have a representation as a linear combination of block-symmetric polynomials $H^{\mathbf{k}}(x)$ and $R^{\mathbf{k}}(x)$, respectively. Indeed, from direct calculations,

$$G_n(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)}) = \sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} R^{\mathbf{k}}(x) \tag{13}$$

and

$$F_n(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)}) = \sum_{|\mathbf{k}|=n} \frac{|\mathbf{k}|!}{\mathbf{k}!} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} H^{\mathbf{k}}(x), \tag{14}$$

where $x = (x^{(1)}, \ldots, x^{(s)})$. From (12) and (13), we have that

$$\begin{aligned} \mathcal{G}(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})(1) &= \sum_{n=0}^{\infty} G_n(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)}) \\ &= \sum_{n=0}^{\infty} \sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} R^{\mathbf{k}}(x) \\ &= \sum_{|\mathbf{k}|=0}^{\infty} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} R^{\mathbf{k}}(x) = \mathcal{R}(x)(t). \end{aligned} \tag{15}$$

In the case $\lambda = 1$, from (11), (14) and (15) we obtain

$$\begin{aligned}
\mathcal{R}(x)(t) &= \mathcal{G}(t_1x^{(1)} + t_2x^{(2)} + \cdots + t_sx^{(s)})(1) \\
&= \exp\left(-\sum_{n=1}^{\infty} 1^n \frac{F_n(-(t_1x^{(1)} + t_2x^{(2)} + \cdots + t_sx^{(s)}))}{n} \right) \\
&= \exp\left(-\sum_{n=1}^{\infty} \frac{\sum_{|\mathbf{k}|=n} \frac{|\mathbf{k}|!}{\mathbf{k}!} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} H^{\mathbf{k}}(-x)}{n} \right) \\
&= \exp\left(-\sum_{|\mathbf{k}|=1}^{\infty} \prod_{i=1}^{s} t_i^{k_i} \frac{(|\mathbf{k}|-1)! H^{\mathbf{k}}(-x)}{\mathbf{k}!} \right).
\end{aligned}$$

□

Let us consider more examples of block-symmetric polynomials.

Example 3. *Let P be a symmetric polynomial on ℓ_1. Then $P(a_1x^{(1)} + \cdots + a_sx^{(s)})$ is a block symmetric polynomial on $\ell_1(\mathbb{C}^s)$, where $x^{(j)} \in \ell_1$ and a_j are some nonzero numbers. Formulas (13) and (14) show that the block-symmetric polynomials of such type are important. In particular, polynomial in Example 2 is block-symmetric for $s = 2$, $x^{(1)} = x$, and $x^{(2)} = y$.*

Example 4. *Let M_r be the linear space of $r \times r$ matrices for some integer $r > 1$. Denote by $\ell_1(M_r)$ the space of absolutely summing sequences of matrices $u = (u_1, u_2, \ldots)$, $u_n \in M_r$. Clearly, $\ell_1(M_r)$ is isomorphic to $\ell_1(\mathbb{C}^s)$ for $s = r^2$. Then,*

$$P(u) = \sum_{n=1}^{\infty} \det(u_n)$$

is a block-symmetric polynomial on $\ell_1(M_r)$.

2.3. Newton-Type Formulas for Block-Symmetric Polynomials

Let ω be the isomorphism of $\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$ to itself, defined so that $\omega(H^{\mathbf{k}}) = -H^{\mathbf{k}}$ for every multi-index $\mathbf{k}$. In other words, if $P \in \mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$ is of the form

$$P(x) = Q(H^{\mathbf{k}}, H^{\mathbf{m}}, \ldots, H^{\mathbf{r}})$$

for some polynomial Q of several variables, then

$$\omega(P)(x) = Q\big(\omega(H^{\mathbf{k}}), \omega(H^{\mathbf{m}}), \ldots, \omega(H^{\mathbf{r}})\big).$$

It is clear that $\omega(H^{\mathbf{k}})(x) = (-1)^{|\mathbf{k}|+1}(H^{\mathbf{k}})(-x)$ for every multi-index $\mathbf{k}$, and $\omega^2(P) = P$ for every $P \in \mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$. The following result generalizes a well-known fact in the case of symmetric polynomials (see p. 4, [22]).

Proposition 2. *For every multi-index $\mathbf{k}$,*

$$\omega(R^{\mathbf{k}}) = E^{\mathbf{k}} \quad \text{and} \quad \omega(E^{\mathbf{k}}) = R^{\mathbf{k}}.$$

Proof. According to relations (9) and (10), we have

$$\begin{aligned}\mathcal{R}(x)(t) &= \exp\left(-\sum_{|\mathbf{k}|=1}^{\infty}\prod_{i=1}^{\infty} t_i^{k_i}\frac{(|\mathbf{k}|-1)!H^{\mathbf{k}}(-x)}{\mathbf{k}!}\right)\\ &= \exp\left(\sum_{|\mathbf{k}|=1}^{\infty}\prod_{i=1}^{\infty} t_i^{k_i}\frac{(|\mathbf{k}|-1)!(-1)^{|\mathbf{k}|+1}H^{\mathbf{k}}(x)}{\mathbf{k}!}\right)\\ &= \exp\left(\sum_{|\mathbf{k}|=1}^{\infty}\prod_{i=1}^{\infty} t_i^{k_i}\frac{(|\mathbf{k}|-1)!\omega(H^{\mathbf{k}})(x)}{\mathbf{k}!}\right)\\ &= \omega(\mathcal{E})(x)(t).\end{aligned}$$

By the definition of $E^{\mathbf{k}}$, (8) we can see that $\omega(E^{\mathbf{k}}) = R^{\mathbf{k}}$. Thus, $E^{\mathbf{k}} = \omega^2(E^{\mathbf{k}}) = \omega(R^{\mathbf{k}})$. □

For given multi-indexes $\mathbf{k}$ and $\mathbf{q}$, we denote by $\mathbf{k}-\mathbf{q} = (k_1 - q_1, k_2 - q_2, \ldots, k_s - q_s)$. In addition, we write $\mathbf{k} \geq \mathbf{q}$ whenever $k_1 \geq q_1$, $k_2 \geq q_2$, ..., $k_s \geq q_s$. In [18], the following generalization of Newton's formula is proved (1).

$$nR^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|}(-1)^{j-1}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}R^{\mathbf{k}-\mathbf{q}}. \tag{16}$$

Using Proposition 2 and Equation (16), we can prove an analog of Newton's Formula (2).

Theorem 1. *For every multi-index* $\mathbf{k}$,

$$nE^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}E^{\mathbf{k}-\mathbf{q}}. \tag{17}$$

Proof. Applying the isomorphism ω to Equation (16), we have

$$\begin{aligned}nE^{\mathbf{k}} = n\omega(R^{\mathbf{k}}) &= \sum_{j=1}^{|\mathbf{k}|}(-1)^{j-1}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}\omega(H^{\mathbf{q}}R^{\mathbf{k}-\mathbf{q}})\\ &= \sum_{j=1}^{|\mathbf{k}|}(-1)^{j-1}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}(-1)^{|\mathbf{q}|+1}H^{\mathbf{q}}E^{\mathbf{k}-\mathbf{q}}\\ &= \sum_{j=1}^{|\mathbf{k}|}(-1)^{2j}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}E^{\mathbf{k}-\mathbf{q}}\\ &= \sum_{j=1}^{|\mathbf{k}|}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}E^{\mathbf{k}-\mathbf{q}}.\end{aligned}$$

□

2.4. The Finite-Dimensional Case

Let us consider a more detailed case $\ell_1^m(\mathbb{C}^s) = \ell_1(\mathbb{N}_m, \mathbb{C}^s)$, where $\mathbb{N}_m = \{1, \ldots, m\}$. In other words, $\ell_1^m(\mathbb{C}^s)$ is an sm-dimensional complex space consisting of sequences of length m of vectors in $\mathbb{C}^s$.

Note that every function on $\ell_1^m(\mathbb{C}^s)$ depends on sm independent variables. We say that a function on $\ell_1^m(\mathbb{C}^s)$ is *totally symmetric* if it is invariant with respect to all possible permutations of these variables. Clearly, every totally symmetric function is block-symmetric. There are exactly sm totally symmetric algebraically independent polynomials $\ell_1^m(\mathbb{C}^s)$. If we restrict the basis (6) to $\ell_1^m(\mathbb{C}^s)$, we obtain

$$\sum_{l=1}^{m} \frac{(l+1)(l+2)\cdots(l+s-1)}{(s-1)!}$$

generators of $\mathcal{P}_{vs}(\ell_1^m(\mathbb{C}^s))$. From classic results of invariant theory (see Lemma 5 of [1]), there are at least N algebraic dependencies between these generators, where

$$N = \sum_{l=1}^{m} \frac{(l+1)(l+2)\cdots(l+s-1)}{(s-1)!} - sm.$$

Thus, in the finite-dimensional case, generating elements of the algebra of block-symmetric polynomials on $\ell_1^m(\mathbb{C}^s)$ are always algebraically dependent if $s > 1$.

We say that a system of generators $\tau_{vs}(\ell_1^m(\mathbb{C}^s))$ of $\mathcal{P}_{vs}(\ell_1^m(\mathbb{C}^s))$ is *reasonable* if it contains sm totally symmetric algebraically independent polynomials. In [23], it is shown how to find algebraic dependencies in a reasonable system of the generators for polynomials that are not totally symmetric.

Example 5. *Let* $\left(\begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix}, \begin{pmatrix} x_2^{(1)} \\ x_2^{(2)} \end{pmatrix}\right) \in \ell_1^2(\mathbb{C}^2)$. *For the generating elements $H^{\mathbf{k}}(x)$, the following identity holds (see [23]):*

$$\eta_5^2 - \eta_1\eta_2\eta_5 + \frac{1}{2}\eta_3\eta_2^2 + \frac{1}{2}\eta_4\eta_1^2 - \eta_3\eta_4 \equiv 0,$$

where

$$\begin{aligned}
\eta_1 &= H^{1,0} = x_1^{(1)} + x_2^{(1)}, \\
\eta_2 &= H^{0,1} = x_1^{(2)} + x_2^{(2)}, \\
\eta_3 &= H^{2,0} = \left(x_1^{(1)}\right)^2 + \left(x_2^{(1)}\right)^2, \\
\eta_4 &= H^{0,2} = \left(x_1^{(2)}\right)^2 + \left(x_2^{(2)}\right)^2, \\
\eta_5 &= H^{1,1} = x_1^{(1)}x_1^{(2)} + x_2^{(1)}x_2^{(2)}.
\end{aligned}$$

Let

$$\begin{aligned}
\nu_1 &= E^{1,0} = x_1^{(1)} + x_2^{(1)}, \\
\nu_2 &= E^{0,1} = x_1^{(2)} + x_2^{(2)}, \\
\nu_3 &= E^{2,0} = x_1^{(1)}x_2^{(1)} + \left(x_1^{(1)}\right)^2 + \left(x_2^{(1)}\right)^2, \\
\nu_4 &= E^{0,2} = x_1^{(2)}x_2^{(2)} + \left(x_1^{(2)}\right)^2 + \left(x_2^{(2)}\right)^2, \\
\nu_5 &= E^{1,1} = x_1^{(1)}x_2^{(2)} + x_2^{(1)}x_1^{(2)} + 2x_1^{(1)}x_1^{(2)} + 2x_2^{(1)}x_2^{(2)}.
\end{aligned}$$

Then, the following identity holds:

$$\nu_5^2 - 3\nu_1\nu_2\nu_5 + 3\nu_2^2\nu_3 + 3\nu_1^2\nu_4 - 4\nu_3\nu_4 \equiv 0.$$

3. Block-Supersymmetric Polynomials

3.1. Bases of Block-Supersymmetric Polynomials

We will use the following short notation $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ for the Banach space $\ell_1(\mathbb{Z}_0, \mathbb{C}^s)$. In other words, $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ is the space of sequences

$$z = (\ldots, z_{-n}, \ldots, z_{-1}, z_1, \ldots, z_n, \ldots)$$

$$= (y|x) = (\ldots, y_n, \ldots, y_1|x_1, \ldots, x_n, \ldots)$$

with

$$||z|| = \sum_{i=-\infty}^{\infty} ||z_i|| = \sum_{i=-\infty}^{\infty}\sum_{j=1}^{s} |z_i^{(j)}|,$$

where both $x = (x_1, \ldots, x_n, \ldots)$ and $y = (y_1, \ldots, y_n, \ldots)$ are in $\ell_1(\mathbb{C}^s)$, and $x_i = (x_i^{(1)}, \ldots, x_i^{(s)})$ and $y_i = (y_i^{(1)}, \ldots, y_i^{(s)})$ are in $\mathbb{C}^s$ with $z_n = x_n, z_{-n} = y_n$ for $n \in \mathbb{N}$.

Let us consider the next polynomials on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$:

$$T^{\mathbf{k}}(z) = H^{\mathbf{k}}(x) - H^{\mathbf{k}}(y) = \sum_{j=1}^{\infty} \prod_{\substack{r=1 \\ |k|\geq 1}}^{s} (x_j^{(r)})^{k_r} - \sum_{j=1}^{\infty} \prod_{\substack{r=1 \\ |k|\geq 1}}^{s} (y_j^{(r)})^{k_r}, \mathbf{k} = (k_1, \ldots, k_s). \tag{18}$$

Definition 1. *A polynomial P on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ is called block-supersymmetric if it is an algebraic combination of polynomials $\{T^{\mathbf{k}}\}_{|\mathbf{k}|=1}^{\infty}$. That is, P can be written as a finite sum of finite products of polynomials in $\{T^{\mathbf{k}}\}_{|\mathbf{k}|=1}^{\infty}$ and constants. We denote by $\mathcal{P}_{vsup}$ the algebra of all block-supersymmetric polynomials on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$.*

Proposition 3. *Polynomials $T^{\mathbf{k}}$ are algebraically independent and so $\{T^{\mathbf{k}}\}_{|\mathbf{k}|=1}^{\infty}$ is an algebraic basis in $\mathcal{P}_{vsup}$.*

Proof. Let us suppose for a contradiction that there is a non-trivial polynomial Q of several variables such that

$$Q(T^{\mathbf{k}}(y|x), T^{\mathbf{m}}(y|x), \ldots, T^{\mathbf{r}}(y|x)) \equiv 0$$

for some finite sequence of multi-indexes $(\mathbf{k}, \mathbf{m}, \ldots, \mathbf{r})$. This equality will still be true if we restrict it to elements $(0|x)$, $x \in \ell_1(\mathbb{C}^s)$. But $T^{\mathbf{k}}(0|x) = H^{\mathbf{k}}(x)$ for every multi-index $\mathbf{k}$. Thus,

$$Q(H^{\mathbf{k}}(x), H^{\mathbf{m}}(x), \ldots, H^{\mathbf{r}}(x)) \equiv 0$$

which contradicts the algebraic independence of polynomials $(H^{\mathbf{k}})$. □

Let us consider the following relation of equivalence: $z \sim w$, for z, w in $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ if and only if $T^{\mathbf{k}}(z) = T^{\mathbf{k}}(w)$ for every $|\mathbf{k}| \geq 1$. The quotient set $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})/\sim$ will be denoted by $\mathcal{M}$. Let $[z] \in \mathcal{M}$ be the class of equivalence containing $z \in \ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$. Clearly, any block-supersymmetric polynomial P is well-defined on $\mathcal{M}$ by $P([z]) = P(z)$.

As in [8,20], we can introduce an algebraic operation "$\bullet$" of the "symmetric addition" on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ by

$$z \bullet w = (y \bullet v|x \bullet u) = (\ldots, v_n, y_n, \ldots, v_1, y_1|x_1, u_1, \ldots, x_n, u_n, \ldots),$$

where $z = (y|x), w = (v|u)$. We denote by $z^- = (y|x)^- = (x, y)$ the "symmetric inverse" element to z. It is easy to see that $(z^-)^- = z$ and $z \bullet z^- \sim (0|0)$. These operations can be extended to $\mathcal{M}$ by

$$[z] \bullet [w] = [z \bullet w] \quad \text{and} \quad [z]^- = [z^-], \quad z, w \in \ell_1(\mathbb{C}^s_{\mathbb{Z}_0})). \tag{19}$$

Similarly to in [8], some obvious basic properties of "$\bullet$" can be formulated in the following theorem.

Theorem 2. *The following statements hold:*

1. *$T^{\mathbf{k}}(z \bullet w) = T^{\mathbf{k}}(z) + T^{\mathbf{k}}(w)$ for every $\mathbf{k} = (k_1, \ldots, k_s)$.*
2. *The algebraic operations on $\mathcal{M}$, defined in (19), do not depend on the choice of representatives.*

3. *The set $\mathcal{M}$, together with operations in (19), form a commutative group, where $0 = (0|0)$ is the zero element.*
4. *$z \sim 0$ if and only if we can write $z = (d|s)$ for some $d, s \in \ell_1(\mathbb{C}^s)$ and $H^{\mathbf{k}}(d) = H^{\mathbf{k}}(s)$ for all $\mathbf{k} = (k_1, \ldots, k_s)$. In other words, the set of nonzero vector coordinates of d coincides with the set of nonzero vector coordinates of s.*

Other algebraic and topological structures on $\mathcal{M}$ in more general situations were considered in [9].

Let us denote by Λ an algebraic isomorphism from $\mathcal{P}_{vs} = \mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$ to $\mathcal{P}_{vsup}$ such that

$$\Lambda\colon H^{\mathbf{k}} \longmapsto T^{\mathbf{k}}, \quad \mathbf{k} = (k_1, \ldots, k_s).$$

Since Λ is an algebraic isomorphism, we have the following proposition.

Proposition 4. *If $\{P_n\}_{n=1}^{\infty}$ is an algebraic basis in $\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$, then $\{\Lambda(P_n)\}_{n=1}^{\infty}$ is an algebraic basis in $\mathcal{P}_{vsup}$.*

For a given $y \in \ell_1(\mathbb{C}^s)$, we denote by $\Lambda_y(P)(x) = \Lambda(P)(y|x)$, $P \in \mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$. It is easy to check that $\Lambda_y(x \bullet y) = \Lambda(P)(y|x \bullet y) = \Lambda(P)(0|x) = P(x)$.

Theorem 3. *Let $\Lambda(R^{\mathbf{n}}) = W^{\mathbf{n}}$. Then*

$$W^{\mathbf{n}}(y|x) = \sum_{\mathbf{k} \le \mathbf{n}} R^{\mathbf{k}}(x) E^{\mathbf{n}-\mathbf{k}}(-y), \quad \mathbf{n} = (n_1, \ldots, n_s), \tag{20}$$

and

$$\mathcal{W}(y|x)(t) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} W^{\mathbf{k}}(y|x) = \frac{\mathcal{R}(x)(t)}{\mathcal{R}(y)(t)}, \tag{21}$$

where the equality holds for all vectors $t \in \mathbb{C}^s$ in the common domains of convergence.

Proof. Due to [20], we know that

$$\mathcal{R}(x \bullet y)(t) = \mathcal{R}(x)(t)\mathcal{R}(y)(t), \quad x, y \in \ell_1(\mathbb{C}^s). \tag{22}$$

Hence, for any fixed $y \in \ell_1(\mathbb{C}^s)$,

$$\begin{aligned}\Lambda_y(\mathcal{R}(x \bullet y)(t)) &= \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} \Lambda(R^{\mathbf{k}})(y|x) \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(y) \\ &= \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} W^{\mathbf{k}}(y|x) \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(y).\end{aligned}$$

On the other hand,

$$\Lambda_y(\mathcal{R}(x \bullet y)(t)) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(x).$$

Therefore,

$$\sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} W^{\mathbf{k}}(y|x) \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(y) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(x). \tag{23}$$

From (23), we have

$$\mathcal{W}(y|x)(t)\mathcal{R}(y)(t) = \mathcal{R}(x)(t)$$

and so (21) holds. Using Formula (9), we obtain

$$\sum_{|\mathbf{k}|=0}^{\infty}\prod_{i=1}^{s}t_i^{k_i}W^{\mathbf{k}}(y|x)=\sum_{|\mathbf{k}|=0}^{\infty}\prod_{i=1}^{s}t_i^{k_i}R^{\mathbf{k}}(x)\sum_{|\mathbf{k}|=0}^{\infty}\prod_{i=1}^{s}t_i^{k_i}E^{\mathbf{k}}(-y)$$

$$=\sum_{|\mathbf{k}|=0}^{\infty}\prod_{i=1}^{s}t_i^{k_i}\sum_{\mathbf{j}\le\mathbf{k}}R^{\mathbf{j}}(x)E^{\mathbf{k}-\mathbf{j}}(-y).$$

From here, (20) follows. □

Corollary 1. *For the generating function $\mathcal{W}((y|x)\bullet(d|b))(t)$, the following identity holds*

$$\mathcal{W}((y|x)\bullet(d|b))(t)=\mathcal{W}(y|x)(t)\mathcal{W}(d|b)(t),\quad x,y,d,b\in\ell_1(\mathbb{C}^s).$$

Proof. The equality follows from (22) and (21). □

Corollary 2. *For every multi-index $\mathbf{n}=(n_1,\dots,n_s)$ and $x,y,b,d\in\ell_1(\mathbb{C}^s)$ we have*

$$W^{\mathbf{n}}((y|x)\bullet(d|b))=\sum_{\mathbf{k}\le\mathbf{n}}W^{\mathbf{k}}(y|x)W^{\mathbf{n}-\mathbf{k}}(d|b),$$

$$R^{\mathbf{n}}(x\bullet b)=\sum_{\mathbf{k}\le\mathbf{n}}R^{\mathbf{k}}(x)R^{\mathbf{n}-\mathbf{k}}(b),$$

and

$$E^{\mathbf{n}}(y\bullet d)=\sum_{\mathbf{k}\le\mathbf{n}}E^{\mathbf{k}}(y)E^{\mathbf{n}-\mathbf{k}}(d).$$

Proof. From Corollary 1, we have

$$\sum_{|\mathbf{n}|=0}^{\infty}\prod_{i=1}^{s}t_i^{n_i}W^{\mathbf{n}}((y|x)\bullet(d|b))=\sum_{|\mathbf{k}|=0}^{\infty}\prod_{i=1}^{s}t_i^{k_i}W^{\mathbf{k}}(y|x)\sum_{|\mathbf{l}|=0}^{\infty}\prod_{i=1}^{s}t_i^{l_i}W^{\mathbf{l}}(d|b).$$

Equating coefficients at the same monomials $\prod_{i=1}^{s}t_i^{n_i}$, we have verified the first equality. The second equality follows from the first one for the case $W^{\mathbf{n}}((0|x)\bullet(0|b))$, and the third one for the case $W^{\mathbf{n}}((y|0)\bullet(d|0))$. □

In [20], it is observed that

$$\mathcal{R}(x)(t)=\prod_{i=1}^{\infty}(1+x_i^{(1)}t_1+\cdots+x_i^{(s)}t_s)\tag{24}$$

and the product absolutely converges to an entire function on $\mathbb{C}^s$. From (21) and (24) it follows that for every fixed $u=(y|x)$, $\mathcal{W}(u)(t)$ is a meromorphic function on $\mathbb{C}^s$ of the form

$$\mathcal{W}(u)(t)=\frac{\prod_{i=1}^{\infty}(1+x_i^{(1)}t_1+\cdots+x_i^{(s)}t_s)}{\prod_{i=1}^{\infty}(1+y_i^{(1)}t_1+\cdots+y_i^{(s)}t_s)}.\tag{25}$$

Proposition 5. *For each $u=[(y|x)]\in\mathcal{M}$*

$$\mathcal{W}(u)(t)=\exp\left(-\sum_{|\mathbf{k}|=1}^{\infty}\prod_{i=1}^{\infty}t_i^{k_i}\frac{(|\mathbf{k}|-1)!T^{\mathbf{k}}(-u)}{\mathbf{k}!}\right),\tag{26}$$

where $-u=[(-y|-x)]$ and the equality holds on the common domain of convergence.

Proof. In [8], it was proved that for the case $s = 1$,

$$\mathcal{W}(y|x)(t) = \exp\left(-\sum_{n=1}^{\infty} t^n \frac{T_n(-u)}{n}\right). \tag{27}$$

From the straightforward computations, we have

$$\begin{aligned}
&T_n(-y^{(1)}t_1 - \cdots - y^{(s)}t_s| - x^{(1)}t_1 - \cdots - x^{(s)}t_s) = \\
&= F_n(-x^{(1)}t_1 - \cdots - x^{(s)}t_s) - F_n(-y^{(1)}t_1 - \cdots - y^{(s)}t_s) = \\
&= \sum_{|\mathbf{k}|=n} \frac{|\mathbf{k}|!}{\mathbf{k}!} t_1^{k_1} \cdots t_s^{k_s} (H^{\mathbf{k}}(-x) - H^{\mathbf{k}}(-y)) = \\
&= \sum_{|\mathbf{k}|=n} \frac{|\mathbf{k}|!}{\mathbf{k}!} t_1^{k_1} \cdots t_s^{k_s} T^{\mathbf{k}}(-u).
\end{aligned} \tag{28}$$

On the other hand,

$$\begin{aligned}
&\mathcal{W}(y^{(1)}t_1 + \cdots + y^{(s)}t_s|x^{(1)}t_1 + \cdots + x^{(s)}t_s)(1) = \frac{\mathcal{G}(x^{(1)}t_1 + \cdots + x^{(s)}t_s)(1)}{\mathcal{G}(y^{(1)}t_1 + \cdots + y^{(s)}t_s)(1)} \\
&= \frac{\mathcal{R}(x)(t)}{\mathcal{R}(y)(t)} = \mathcal{W}(u)(t).
\end{aligned} \tag{29}$$

Substituting (28) into (27), and taking into account that $|\mathbf{k}| = n$ and (29), we obtain (26). □

Theorem 4. *Let $u = (y|x) \neq (0|0)$. The equations $T^{\mathbf{k}}(v) = \lambda T^{\mathbf{k}}(u)$ for all multi-indexes $\mathbf{k}$ and a number $\lambda \in \mathbb{C}$ has a solution $v \in \mathcal{M}$ if and only if λ is an integer number.*

Proof. Let $\lambda = n \in \mathbb{Z}$. If $n = 0$, then $v = 0$. If $n > 0$, then $v = \underbrace{u \bullet \ldots \bullet u}_{n}$. If $n < 0$, then $v = \underbrace{u^- \bullet \ldots \bullet u^-}_{n}$.

Now, let $\lambda \notin \mathbb{Z}$. According to (26)

$$\mathcal{W}(v)(t) = (\mathcal{W}(u)(t))^{\lambda}.$$

But $(\mathcal{W}(u)(t))^{\lambda}$ is not a meromorphic function if $\lambda \notin \mathbb{Z}$. Thus, we have a contradiction the representation (25) for $\mathcal{W}(v)(t)$. □

3.2. Newton-Type Formulas for Block-Supersymmetric Polynomials

To obtain some Newton-type formulas for block-supersymmetric polynomials, we have to apply the isomorphism Λ to corresponding Newton-type formulas for block-symmetric polynomials. Applying Λ to (16), we obtain

$$nW^{\mathbf{k}} = \Lambda(nR^{\mathbf{k}}) = \sum_{j=1}^{|\mathbf{k}|} (-1)^{j-1} \sum_{\substack{|\mathbf{q}|=j \\ \mathbf{k} \geq \mathbf{q}}} \frac{|\mathbf{q}|!}{\mathbf{q}!} \Lambda(H^{\mathbf{q}}) \Lambda(R^{\mathbf{k}-\mathbf{q}}) = \sum_{j=1}^{|\mathbf{k}|} (-1)^{j-1} \sum_{\substack{|\mathbf{q}|=j \\ \mathbf{k} \geq \mathbf{q}}} \frac{|\mathbf{q}|!}{\mathbf{q}!} T^{\mathbf{q}} W^{\mathbf{k}-\mathbf{q}}. \tag{30}$$

We denote $\widetilde{W}^{\mathbf{k}} = \Lambda(E^{\mathbf{k}})$. Then, applying Λ to (17), we have

$$n\widetilde{W}^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|} \sum_{\substack{|\mathbf{q}|=j \\ \mathbf{k} \geq \mathbf{q}}} \frac{|\mathbf{q}|!}{\mathbf{q}!} T^{\mathbf{q}} \widetilde{W}^{\mathbf{k}-\mathbf{q}}. \tag{31}$$

Thus, we have proven the following theorem.

Theorem 5. *The algebraic bases $T^{\mathbf{k}}$, $W^{\mathbf{k}}$ and $\widetilde{W}^{\mathbf{k}}$ of $\mathcal{P}_{vsup}$ are connected by Newton-type relations* (30) *and* (31).

Note that the isomorphism ω defined in Section 2.3 can be extended to an isomorphism Ω of $\mathcal{P}_{vsup}$ setting $\Omega = \Lambda \circ \omega \circ \Lambda^{-1}$. In other words, $\Omega(T^{\mathbf{k}}) = -T^{\mathbf{k}}$ for every multi-index $\mathbf{k}$. Furthermore, it is easy to check that $\Omega(W^{\mathbf{k}}) = \widetilde{W}^{\mathbf{k}}$, Ω^2 is the identity operator, and $\Omega(P)(y|x) = P(x|y)$ for every $P \in \mathcal{P}_{vsup}$. Thus,

$$\widetilde{W}^{\mathbf{k}} = \Omega(W^{\mathbf{k}}) = \Omega\left(\sum_{\mathbf{q}\leq\mathbf{k}} R^{\mathbf{q}}(x)E^{\mathbf{k}-\mathbf{q}}(-y)\right) = \sum_{\mathbf{q}\leq\mathbf{k}} E^{\mathbf{q}}(x)R^{\mathbf{k}-\mathbf{q}}(-y).$$

3.3. The Finite Dimensional Case for Block-Supersymmetric Polynomials

Let us denote by $\left(\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})\right)_{p,q}$ the finite-dimensional Banach space of all sequences

$$z = (z_{-p}, \ldots, z_{-1}|z_1, \ldots, z_q) = (y|x) = (y_p, \ldots, y_1|x_1, \ldots, x_q).$$

Clearly, $\left(\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})\right)_{p,q}$ is a subspace of $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$. There are

$$\sum_{l=1}^{p+q} \frac{(l+1)(l+2)\ldots(l+s-1)}{(s-1)!}$$

homogeneous polynomials $T^{\mathbf{k}}$ for $|\mathbf{k}| \leq p+q$ and $s(p+q)$ independent variables. Thus, the system of generators consisting of the restrictions of $T^{\mathbf{k}}$ to $\left(\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})\right)_{p,q}$ must have at least

$$N = \sum_{l=1}^{p+q} \frac{(l+1)(l+2)\ldots(l+s-1)}{(s-1)!} - s(p+q)$$

algebraic dependencies. The same is true if we will take another algebraic basis instead of $T^{\mathbf{k}}$.

Example 6. *Now let $\left(\ell_1(\mathbb{C}^2_{\mathbb{Z}_0})\right)_{1,1}$ be the space of all vectors*

$$(y|x) = (y_1|x_1) = \left(\begin{pmatrix} y_1^{(1)} \\ y_1^{(2)} \end{pmatrix} \middle| \begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix}\right).$$

Then, using rutin computations, we can obtain the following identity for the generating elements (18) *restricted to $\left(\ell_1(\mathbb{C}^2_{\mathbb{Z}_0})\right)_{1,1}$*

$$\xi_5\xi_1\xi_2 - \frac{1}{2}\xi_1^2\xi_4 - \frac{1}{2}\xi_2^2\xi_3 \equiv 0,$$

where

$$T^{(1,0)} = x_1^{(1)} - y_1^{(1)} = \xi_1,$$
$$T^{(0,1)} = x_1^{(2)} - y_1^{(2)} = \xi_2,$$
$$T^{(2,0)} = (x_1^{(1)})^2 - (y_1^{(1)})^2 = \xi_3,$$
$$T^{(0,2)} = (x_1^{(2)})^2 - (y_1^{(2)})^2 = \xi_4,$$
$$T^{(1,1)} = x_1^{(1)}x_1^{(2)} - y_1^{(1)}y_1^{(2)} = \xi_5.$$

This identity can be checked by direct substitution of the generating elements.

For the generating elements of the form (20) *in* $\mathcal{P}_{vsup}\left(\left(\ell_1(\mathbb{C}^2_{\mathbb{Z}_0})\right)_{1,1}\right)$, *we obtain the following identity:*

$$\omega_5\omega_1\omega_2 - \omega_1^2\omega_4 - \omega_2^2\omega_3 \equiv 0,$$

where

$$W^{(1,0)} = x_1^{(1)} - y_1^{(1)} = \omega_1,$$
$$W^{(0,1)} = x_1^{(2)} - y_1^{(2)} = \omega_2,$$
$$W^{(2,0)} = -x_1^{(1)}y_1^{(1)} + (y_1^{(1)})^2 = \omega_3,$$
$$W^{(0,2)} = -x_1^{(2)}y_1^{(2)} + (y_1^{(2)})^2 = \omega_4,$$
$$W^{(1,1)} = 2y_1^{(1)}y_1^{(2)} - x_1^{(1)}y_1^{(2)} - x_1^{(2)}y_1^{(1)} = \omega_5.$$

4. Applications for Algebras of Block-Supersymmetric Analytic Functions

Let $\mathcal{H}_b^{vsup}$ be the completion of $\mathcal{P}_{vsup}$ with respect to the topology of uniform convergence on a bounded subset. This is a locally convex metrizable topology, generated by the following countable family of norms

$$\|P\|_r = \sup_{\|u\|\le r} |P(u)|, \quad P \in \mathcal{P}_{vsup},$$

where r belongs to the set of positive rational numbers. Elements of $\mathcal{H}_b^{vsup}$ will be called block-supersymmetric analytic (or entire) functions of bounded type on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$.

Let us denote by M_b^{vsup} the spectrum of $\mathcal{H}_b^{vsup}$, that is, the set of all continuous nonzero complex homomorphisms (characters) of $\mathcal{H}_b^{vsup}$. Clearly, for every point $u = [(y|x)] \in \mathcal{M}$ there is character $\delta_u \in M_b^{vsup}$ such that $\delta_u(f) = f(y|x), f \in \mathcal{H}_b^{vsup}$. Conversely, if $u \neq v$ as elements of $\mathcal{M}$, then $\delta_u \neq \delta_v$. Thus, we can consider $\mathcal{M}$ as a subset of M_b^{vsup}. Various algebras of entire analytic functions on Banach spaces and their spectra have been investigated by many authors. Investigations of spectra of algebras $\mathcal{H}_b(X)$ of all entire functions of bounded type on Banach spaces X were started by Aron, Cole, and Gamelin in [24], where the authors observed that the spectrum of $\mathcal{H}_b(X)$ may have a complicated structure; in particular, it contains extended point-evaluation functionals associated with points of the second dual space X^{**} (see also [25,26]). Subalgebras of $\mathcal{H}_b(X)$ of symmetric analytic functions with respect to permutations of basis vectors of $X = \ell_p$ and their spectra were studied in [27,28] and others (see [29] and references therein), with respect to continual permutations in symmetric structures of $X = L_p$ in [6,30,31] and others, and with respect to abstract groups of operators in [7,32]. There are two important questions about the spectrum of a subalgebra $\mathcal{H}_0$ of $\mathcal{H}_b(X)$. The first one is related to the structure of point evaluation functionals. It is clear that for two different points $x, y \in X$ point evaluation functionals δ_x and δ_y are equal on $\mathcal{H}_0$ if and only if $f(x) = f(y)$ for every $f \in \mathcal{H}_0$ (in this case, we say that $x \sim y$). Thus, algebraic and topological structures of the set of point evaluation functionals can be defined as the corresponding structures of the quotient set $X/\sim$. The second question is about the existence and some possible description of characters that are not point evaluation functionals.

By the definition, $\mathcal{H}_b^{vsup}$ is the minimal closed subalgebra of $\mathcal{H}_b(\ell_1(\mathbb{C}^s_{\mathbb{Z}_0}))$, which contains $\mathcal{P}_{vsup}$. The set of point evaluation functionals of $\mathcal{H}_b^{vsup}$ can be associated with $\mathcal{M}$. Algebraic and topological structures on such quotient sets for more general cases were considered in [9]. On the other hand, it is well-known that if the subalgebra of polynomials of a given algebra of entire functions of bounded type has an algebraic basis, then every character is completely defined by its values on the basis polynomials (for details on countably generated algebras, see, e.g., [33]). In particular, for the algebraic basis $\{W^{\mathbf{n}}\}$

in $\mathcal{H}_b^{vsup}$, every character $\varphi \in M_b^{vsup}$ can be represented by the countable set $\{\varphi(W^{\mathbf{n}})\}$ or, equivalently, by the function

$$\varphi(\mathcal{W}(t)) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} \varphi(W^{\mathbf{k}}).$$

If $\varphi = \delta_u$ is a point evaluation functional, that is, $\delta_u(f) = f(u)$, $f \in H_b^{vsup}$ for some fixed $u \in \mathcal{M}$, then $\varphi(\mathcal{W}(t)) = \mathcal{W}(u)(t)$ is a meromorphic function of the form (25). Therefore, we have a description of point evaluation functionals of $\mathcal{H}_b^{vsup}$ in terms of meromorphic functions. Let us show that $\mathcal{H}_b^{vsup}$ supports characters that are not point evaluation functionals. Similar results for different algebras were obtained in [8,20,27,29,34].

Let $\mu = (\mu_1, \ldots, \mu_s)$ and $\lambda = (\lambda_1, \ldots, \lambda_s)$ be nonzero vectors in $\mathbb{C}^s$. Consider

$$u_n = \left(\begin{pmatrix} 0 \\ \cdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \cdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \frac{\mu_1}{n} \\ \cdots \\ \frac{\mu_s}{n} \end{pmatrix}, \ldots, \begin{pmatrix} \frac{\mu_1}{n} \\ \cdots \\ \frac{\mu_s}{n} \end{pmatrix} \right|$$

$$\left. \begin{pmatrix} \frac{\lambda_1}{n} \\ \cdots \\ \frac{\lambda_s}{n} \end{pmatrix}, \ldots, \begin{pmatrix} \frac{\lambda_1}{n} \\ \cdots \\ \frac{\lambda_s}{n} \end{pmatrix}, \begin{pmatrix} 0 \\ \cdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \cdots \\ 0 \end{pmatrix} \right) =$$

$$= \left(0, \ldots, 0, \frac{\mu}{n}, \ldots, \frac{\mu}{n} \middle| \frac{\lambda}{n}, \ldots, \frac{\lambda}{n}, 0, \ldots, 0 \right).$$

From the compactness reasons, it follows that the sequence of characters $\{\delta_{v_n}\}$ must have a cluster point $\psi_{\lambda,\mu} \in M_b^{vsup}$. Thus,

$$\psi_{\lambda,\mu}(\mathcal{W}(t)) = \lim_{n\to\infty} \frac{\sum\limits_{|\mathbf{k}|=0}^{\infty} \prod\limits_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(\lambda/n, \ldots, \lambda/n, 0, \ldots, 0)}{\sum\limits_{|\mathbf{k}|=0}^{\infty} \prod\limits_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(\mu/n, \ldots, \mu/n, 0, \ldots, 0)}.$$

From [20], we know that

$$\lim_{n\to\infty} \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(\lambda/n, \ldots, \lambda/n, 0, \ldots, 0) = \exp\left(\sum_{i=1}^{s} \lambda_i t_i \right).$$

Hence,

$$\psi_{\lambda,\mu}(\mathcal{W}(t)) = \exp\left(\sum_{i=1}^{s} (\lambda_i - \mu_i) t_i \right).$$

Comparing this formula with (25), we can deduce that $\psi_{\lambda,\mu}$ is not a point evaluation functional if $\lambda \neq \mu$. Thus, we have proven the following result.

Theorem 6. *There is a family of characters $\psi_{\lambda,\mu}$, $\lambda, \mu \in \mathbb{C}^s$, $\lambda \neq \mu$ of $\mathcal{H}_b^{vsup}$ that are not point evaluation functionals, and $\psi_{\lambda,\mu}(\mathcal{W}(t))$ is defined by* (25)*.*

5. Derivatives and Appell-Type Polynomials

Let us recall that a sequence $P_n(t)$, $n = 0, 1, 2, \ldots$ of polynomials of a complex variable is an Appell sequence if $P_n'(t) = nP_{n-1}(t)$. There is a large number of studies on Appell-type polynomial families in the literature (see, e.g., [35–37]). Appell-type polynomials of several variables were considered in [38].

In [39], a specific derivative was introduced associated with the operation "•" on the algebra of symmetric polynomials. This derivative was extended to supersymmetric polynomials in [40]. We consider it in the cases of block-symmetric and block-supersymmetric polynomials and found corresponding Appell-type polynomials.

Let us denote by $\mathbb{I}_j$, $1 \le j \le s$ the following element in $\ell_1(\mathbb{C}^s)$,

$$\mathbb{I}_j = \left(\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots \right),$$

where 1 is on the jth place of the first vector coordinate, and the rest of the coordinates are zeros.

Definition 2. *Let $P \in \mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$. For every $1 \le j \le s$, we define*

$$D_j(P)(x) = \lim_{t_j \to 0} \frac{P(x \bullet t_j \mathbb{I}_j) - P(x)}{t_j}, \quad x \in \ell_1(\mathbb{C}^s), \quad t_j \in \mathbb{C}.$$

Using easy standard computations, we can check that the operator D_j is linear on $\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$ and for any polynomials P and Q in the domain of D_j,

$$D_j(PQ) = D_j(P)Q + PD_j(Q),$$

that is, D_j is a derivative on the algebra $\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$. Let $\mathbf{n} = (n_1, \ldots, n_s)$ be a multi-index. Taking into account that $H^{\mathbf{n}}(x \bullet z) = H^{\mathbf{n}}(x) + H^{\mathbf{n}}(z)$, and

$$H^{\mathbf{n}}(t_j \mathbb{I}_j) = t_j^{n_j},$$

we have

$$D_j(H^{\mathbf{n}})(x) = \lim_{t_j \to 0} \frac{t_j^{n_j}}{t_j} = \begin{cases} 1 & \text{if } n_j = 1 \\ 0 & \text{if } n_j \neq 1. \end{cases}$$

Since $\{H^{\mathbf{n}}\}$ is an algebraic basis, D_j is well-defined on the whole space of block symmetric polynomials $\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$. Moreover, the following theorem shows that it can be extended to the space of block-symmetric analytic functions of bounded type. Denote by $\mathcal{H}_b^{vs}(\ell_1(\mathbb{C}^s))$ the algebra of all block-symmetric analytic functions of bounded type, that is, $\mathcal{H}_b^{vs}(\ell_1(\mathbb{C}^s))$ is the closure of $\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$ in $\mathcal{H}_b(\ell_1(\mathbb{C}^s))$.

Theorem 7. *For every $1 \le j \le s$, the derivative D_j is continuous with respect to the topology of $\mathcal{H}_b^{vs}(\ell_1(\mathbb{C}^s))$, and so can be extended by continuity and linearity to $\mathcal{H}_b^{vs}(\ell_1(\mathbb{C}^s))$.*

Proof. Let Φ be the forward shift operator from $\ell_1(\mathbb{C}^s)$ to itself defined by

$$\Phi(x_1, \ldots, x_n, \ldots) = (0, x_1, \ldots, x_n, \ldots),$$

where $x_n \in \mathbb{C}^s$ and 0 is the zero-vector in $\mathbb{C}^s$. Φ is a continuous linear operator, and so, the composition operator $C_\Phi(f) = f \circ \Phi$ is continuous on $\mathcal{H}_b^{vs}(\ell_1(\mathbb{C}^s))$. The operator D_j can be represented as the composition of C_Φ with the Gâteaux derivative in direction $\mathbb{I}_j$. Since the Gâteaux derivative in any direction is continuous on the space of analytic functions of bounded type, the operator D_j is continuous. □

We can extend operator D_j to block-supersymmetric analytic functions of bounded type in two ways, setting

$$D_j^+(f)(y|x) = \lim_{t_j \to 0} \frac{f(y|x \bullet t_j \mathbb{I}_j) - f(x)}{t_j},$$

and

$$D_j^-(f)(y|x) = \lim_{t_j \to 0} \frac{f(y \bullet t_j \mathbb{I}_j|x) - f(x)}{t_j}.$$

The same argument as in Theorem 7 implies that both D_j^+ and D_j^- are well-defined and continuous on the algebra of block-supersymmetric analytic functions of bounded type. Let us compute the derivatives on different bases.

Example 7. *Suppose that* $\mathbf{n} = (n_1, \ldots, n_s)$ *is such that* $n_j \neq 0$. *Then,*

$$R^{\mathbf{n}}(x \bullet t^j \mathbb{I}_j) = \sum_{\mathbf{k} \le \mathbf{n}} R^{\mathbf{k}}(\mathbb{I}_j) R^{\mathbf{n}-\mathbf{k}}(x) = t_j R^{n_1, \ldots, n_j - 1, \ldots, n_s}(x)$$

because,

$$R^{\mathbf{k}}(\mathbb{I}_j) = \begin{cases} 1 & \text{if } \mathbf{n} = (\underbrace{0, \ldots, 1}_{j}, 0 \ldots) \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$D_j(R^{\mathbf{n}}) = \frac{\partial}{\partial t_j} t_j R^{n_1, \ldots, n_j - 1, \ldots, n_s} = R^{n_1, \ldots, n_j - 1, \ldots, n_s}.$$

If $n_j = 0$, *then* $D_j(R^{\mathbf{n}}) = 0$.

Proposition 6. *It* $n_j \neq 0$, *then* $D_j(E^{\mathbf{n}}) = E^{n_1, \ldots, n_j - 1, \ldots, n_s}$, *and if* $n_j = 0$, *then* $D_j(E^{\mathbf{n}}) = 0$.

Proof. The proof can be obtained from Example 7, the Newton-type Formula (17), and the simple induction with respect to n_j. □

Let us consider the case $s = 1$. Then $j = 1$, and we denote $D = D_1$. Furthermore, $G_n = R^{n,0,0,\ldots}$ and $B_n = E^{n,0,0,\ldots}$. Thus, $D(G_n) = G_{n-1}$ and $D(B_n) = B_{n-1}$. In addition, $D^+(W_n) = W_{n-1}$ and $D^-(W_n) = W_{n-1}$ (c.f. [40]). Using these equalities, we can construct Appell-type symmetric and supersymmetric polynomials.

Corollary 3. *For a given sequence of polynomials* P_n, *let* $Q_n = n!P_n$.

1. *If* P_n *are symmetric polynomials on* ℓ_1 *of the form* $P_n = G_n$ *or* $P_n = B_n$, *then* $D(Q_n) = nQ_{n-1}$.
2. *If* P_n *is a supersymmetric polynomial on* $\ell_1(\mathbb{Z}_0)$ *of the form* $P_n = W_n$, *then* $D^+(Q_n) = nQ_{n-1}$, *if* $P_n = (-1)^n W_n$, *then* $D^-(Q_n) = nQ_{n-1}$.

6. Conclusions

Algebras of block-supersymmetric polynomials on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ admit algebraic bases, and we constructed some of them and found Newton-type relations between different bases. We established some algebraic properties of block-supersymmetric polynomials and found applications to the description of spectra of algebras of block-supersymmetric analytic functions of bounded type. Furthermore, we considered a special derivative associated with the "symmetric shift" operator $x \mapsto x \bullet a$ on the algebra of block-symmetric polynomials and extended it to both algebras of block-symmetric and of block-supersymmetric analytic functions of bounded type. Some related sequences of Appell-type symmetric and supersymmetric polynomials are constructed.

These results are at the intersection of combinatorics and functional analysis. On the other hand, symmetric and supersymmetric polynomials are applicable in cryptography [11] and quantum physics [10,41]. Therefore, we can expect that the obtained relations will be useful for modeling quantum ideal gases and in the information theory.

Further investigations will consider analytic and algebraic structures on the spectrum of the algebra of block-supersymmetric analytic functions of bounded type, in particular the question about the existence of an analytic manifold structure on the spectrum. Furthermore, we will consider the case when the dimension s of blocks is infinite.

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