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On Normed Algebras and the Generalized Maligranda–Orlicz Lemma

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Abstract: In this paper, we discuss some extensions of the Maligranda–Orlicz lemma. It deals with the problem of constructing a norm in a subspace of the space of bounded functions, for which it becomes a normed algebra so that the norm introduced is equivalent to the initial norm of the subspace. This is done by satisfying some inequality between these norms. We show in this paper how this inequality is relevant to the study of operator equations in Banach algebras. In fact, we study how to equip a subspace of the space of bounded functions with a norm equivalent to a given one so that it is a normed algebra. We give a general condition for the construction of such norms, which allows us to easily check whether a space with a given norm is an algebra with a pointwise product and the consequences of such a choice for measures of noncompactness in such spaces. We also study quasi-normed spaces. We introduce a general property of measures of noncompactness that allows the study of quadratic operator equations, prove a fixed-point theorem suitable for such problems, and complete the whole with examples and applications.

Keywords: Banach algebra; quasi-normed space; measure of noncompactness; quadratic operator equation

MSC: 46H10; 46J10; 46E25; 47H08; 47H09

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1. Introduction

In this paper we study real normed algebras from the point of view of their properties, including norms and measures of noncompactness. The main aim is to give a general construction of norms on subspaces X of the algebra of bounded functions $B([0,1])$ with the sup-norm such that they are normed algebras (or quasi-algebras), but at the same time, in these norms one can easily study properties of operators on such spaces: in particular, whether they are contractions with respect to some measures of noncompactness and applications in fixed-point theory. We give a general construction of such norms. Importantly, we show that the starting point for such studies is our generalized version of the Maligranda–Orlicz lemma ([1], Lemma 4.1), and what is new is that this is also the basis for the study of measures of noncompactness in normed algebras, where certain assumptions about the properties of measures (e.g., the (m)-property [2]) have been studied independently of the introduced norms.

We begin our motivation by pointing out an application of our results. Quadratic differential and integral problems are an interesting class from both a mathematical and a practical point of view ([3], for example). If we write the problem in operator form:

$$x = G(x) \cdot F(x), \quad x(t_0) = x_0, \tag{1}$$

where at least one of these operators is in an integral one, then the main question is to determine a function space X of solutions suitably chosen to satisfy some expected properties of the solutions, and in such a way as to control that these operators are well defined. Moreover, the (pointwise) product should be in X again, which leads us to Banach algebras. In the first part of this paper, we study an extension of the Maligranda–Orlicz lemma [1]. The proposed parametrized version of the inequality allows us to study many different norms on Banach algebras, making the considered spaces normalized Banach algebras. Using our lemma, we study analogous properties of measures of noncompactness on these spaces. Such a choice of space and norms allows fixed-point theorems to be used as the main tool to study these problems, with assumptions optimized for a particular problem. Up to now, the Maligranda–Orlicz lemma has generally been used as a tool to test whether a given concrete space is an algebra by pointwise multiplication [4]. However, the previous form of the lemma required a condition that was quite difficult to evaluate: hence, the restriction to the space of functions with bounded variation—although considered in a very different sense (cf. [4,5]). For example, in a recent paper [6], this approach is used to study sequences of generalized bounded variation. It may also be worth recalling that such variations have been studied in the context of Fourier series. Thus, for the study of norms (and measures of noncompactness, as will also be clarified in the paper) in Banach algebras and their further applications, this lemma can serve as a basic tool. As a last but not least result, let us turn our attention to the Darbo-type fixed-point theorem obtained by the previous results. Such theorems, of course, have many applications: let us recall perhaps only the study of the approximate controllability of hybrid Hilfer fractional differential inclusions with non-instantaneous impulses [7] or fractional quadratic integral equations [8].

Having a suitable tool at our disposal, we are able to study quadratic integral equations on various Banach algebras. Surprisingly, in many papers, the only case considered is the algebra of continuous functions (although some authors suggest a generalization of this case, cf. [9]). To the best of our knowledge, only a few such cases have been considered so far. Studies of quadratic problems in other Banach algebras are also in progress (e.g., [10] in Hölder spaces, but for compact operators, or [11] in spaces of regulated functions, but with supremum norm). In this paper, we present a general approach and prove some particularly interesting cases.

The key issue is the choice of the appropriate tool. It is known that for quadratic problems, due to either insufficient properties or really strong assumptions providing good properties of the product of operators, neither Banach nor Schauder fixed-point theorems are useful and we need a variant of the Darbo fixed-point theorem (e.g., [12]). Then, our general existence theorem for Banach algebras consist of checking the assumptions of this theorem, and then, we show that they are easy to check in concrete Banach algebras. An additional motivation is the construction of suitable Banach algebras that allow the study of quadratic problems for operators of fractional order (see, e.g., [3,8] for basic motivations), for which previous results (with some attempts, as in [8]) are limited to the space of continuous functions.

The question was posed in the work of [1], who studied many problems concerning the space of functions with generalized bounded variation; ([4,13]) concerns the construction of a norm to ensure continuity of multiplication in this norm and its application: for example, in quadratic operator equations. It is also important in the study of multiplier spaces (for example, [5]). Given a subspace $(X, \|\cdot\|_X)$ of the space of bounded functions, we have a question: how to find a norm $\|\cdot\|$ that is equivalent to $\|\cdot\|_X$ and for which $(X, \|\cdot\|)$ is a normalized normed algebra, i.e.,:

$$\|x \cdot y\| \leq \|x\| \cdot \|y\|.$$

We provide a universal method for constructing such norms. The starting point will always be a seminorm (or a norm) that defines belonging to the space X and some of its properties. And why do we study X space so much? It is because many problems either have natural solution spaces (e.g., operators in equations are self-invariant on such spaces) or it allows us to focus on different properties of solutions, such as those associated with certain seminorms. And we will do all this using a unified method and a single fixed-point theorem.

We will also discuss the case of arbitrary normed (and quasi-normed) algebras and the selection of norms in its subspaces so that they are also algebras with the action taken from the initial algebra. Without loss of generality, we will assume that these are real algebras. Importantly, we will show how to apply these results to the study of measures of noncompactness on Banach algebras and apply this to fixed-point theory. The relaxation of the triangle inequality in quasi-normed spaces makes them more general than normed spaces. Recall that it is precisely in the class of quasi-normed spaces that multiplication is jointly continuous. In particular, a natural case of quasi-normed spaces are L^p spaces ($0 < p < 1$). However, normed spaces have a more rigid structure that is often easier to work with in certain contexts. The choice between using a normed space or a quasi-normed space depends on the specific mathematical requirements and properties needed for the analysis or application at hand.

We have two main aims in this paper. First, we point out methods for creating norms on subspaces so that they are still normed algebras with the same operation, but we focus on the compactness arguments used in proofs of Darbo-type fixed-point theorems, i.e., we investigate measures of noncompactness of these spaces. We also study some properties of such measures related to quadratic problems and prove that they are related to properties of norms on normed algebras (Maligranda–Orlicz lemma). We unify and extend several properties related to this topic (such as the (m)-property [2,14,15], (WC)–Banach algebras [16,17], and the Maligranda–Orlicz lemma [1,4]).

Second, we will prove a variant of Darbo's fixed-point theorem for the product of operators and show how useful this version of the theorem can be. We propose to present a unified algorithm for proving theorems about the existence of solutions of quadratic equations in different Banach algebras, thus avoiding repetition and focusing on the technical elements of the proof and allowing us to be able to focus on the essential assumptions depending on the chosen solution space. The results obtained thus improve a number of known theorems, and we obtain a number of new results.

The paper is complemented by a number of examples of the relevance of the theorems considered.

2. Maligranda–Orlicz Lemma and Compactness Conditions

When studying differential and integral equations through their operator form, the fixed-point theorem becomes a key tool. For quadratic equations considered in normed (Banach) algebras, the operators must be contractions with respect to certain measures of noncompactness. Fundamental concepts are introduced, and the properties of these measures needed to study equations in Banach algebras are discussed.

It has been found that in order to achieve this, certain norms must be used to create a Banach algebra while at the same time introducing some measures of noncompactness. It is worth noting that the previously studied norms and measures can also be incorporated, leading to a substantial advancement of previous findings through a generalization of the Maligranda–Orlicz lemma ([1], Lemma 4.1).

In the first part of the paper, we restrict our attention to the case of function spaces closed under pointwise multiplication, which is a basis for the original result of Maligranda and Orlicz. In particular, we can study typical function spaces with finite variation (of various kinds), which become normed Banach algebras with pointwise multiplication of functions under the appropriate choice of norms, but we are not restricted to such a class of spaces.

Norms on Subalgebras

We begin with an extension of the Maligranda–Orlicz lemma. In this section, we focus on function spaces closed by pointwise multiplication. We start our considerations with a large space with this property, i.e., the space $(B([a,b], \|\cdot\|_\infty)$ of bounded functions on the interval $[a, b]$, and we normalize its subspaces so that these are also normed algebras with respect to pointwise multiplication and have the properties we are interested in.

We limit our attention to bounded functions. If the space of continuous functions $C(T)$ (on some topological space T) contains an unbounded function, then there is no submultiplicative norm on it ([18], Theorem 1 or [19]), and so it cannot be a normed algebra. It is also worth noting that in the space of continuous functions $C(K)$ on the compact topological space K, every norm for which $(C(K), \|\cdot\|)$ is a norm algebra has the property $\|x\| \geq \|x\|_\infty$ for every $x \in C(K)$ (Kaplansky [20]). In this case, it provides a partial solution to the norm equivalence problem also investigated in our paper.

However, it is worth noting that the main assumption we consider in the generalized Maligranda–Orlicz lemma is formulated in such a way as to cover both the original (parametrized) estimation and the case of submultiplicative seminorms. Thus, this condition can be tested by a direct estimation method but also by studying conditions for the submultiplicative seminorms (e.g., [21]). Note that we are not restricted to the space of continuous functions and their subspaces (see Section 5 for some examples). The general case of products in algebras other than the pointwise product will be considered later.

Consider the linear subspace $X \subset B([0,1])$, which is closed under pointwise multiplication. Again, we start with a large normed algebra, so the product is determined by this choice. First, we will investigate how to define a norm on X in such a way that we have a normed algebra again. Usually, the elements of X are defined as the finiteness of some certain seminorm, so let us assume that such a seminorm S is given. The choice of such a norm depends on the occurrence of certain conditions, which we capture in a single assumption so that it is easier to test and, moreover, as we will show in the examples, the parameters introduced allow a substantial extension of the class of useful norms. We are ready to prove the following parametrized extension of the Maligranda–Orlicz lemma:

Theorem 1. *Let $(X, \|\cdot\|_X)$, $X \subset B([0,1])$ be closed under pointwise multiplication and equip this space with the norm*

$$\|x\|_X = |x(0)| + S(x),$$

where S is a given seminorm on X.

Assume that there exist constants $k \geq 0$ and $l \geq 1$ such that for any $x, y \in X$, we have

$$S(x \cdot y) \leq l \cdot (\|x\|_\infty \cdot S(y) + S(x) \cdot \|y\|_\infty) + k \cdot S(x) \cdot S(y) \,. \tag{2}$$

Then:

(a) If $k > 0$, then the space X equipped with the norm

$$\|x\|_a = l \cdot \|x\|_\infty + k \cdot S(x) \quad \text{for } x \in X$$

is a normalized Banach algebra, i.e.,

$$\|x \cdot y\|_a \leq \|x\|_a \cdot \|y\|_a.$$

Moreover, the two norms $\|\cdot\|_X$ and $\|\cdot\|_a$ are equivalent.

(b) If $k = 0$, then the space X equipped with

$$\|x\|_b = \|x\|_\infty + \|x\|_X \quad (\text{or } \|x\|_b = \|x\|_\infty + S(x))$$

is also a Banach algebra.

If, in addition, there is a constant $M > 0$ such that $\|x\|_\infty \leq M \cdot \|x\|_X$ for all $x \in X$, then the norms $\|x\|_b$ and $\|x\|_X$ are equivalent.

Proof. (a) Let $k > 0$. It follows from our assumptions that for any choice of $l \geq 1$ and $k \geq 0$, $\|\cdot\|_a$ is a norm on X. We have the following estimate:

$$\begin{aligned}
\|x \cdot y\|_a &= l \cdot \|xy\|_\infty + k \cdot S(xy) \\
&\leq l \cdot \|x\|_\infty \cdot \|y\|_\infty + k \cdot l \cdot (\|x\|_\infty \cdot S(y) + S(x) \cdot \|y\|_\infty) + k^2 \cdot S(x) \cdot S(y) \\
&\leq l^2 \cdot \|x\|_\infty \cdot \|y\|_\infty + k \cdot l \cdot \|x\|_\infty \cdot S(y) + k \cdot l \cdot S(x) \cdot \|y\|_\infty + k^2 \cdot S(x)S(y) \\
&\leq (l \cdot \|x\|_\infty + k \cdot S(x)) \cdot (l \cdot \|y\|_\infty + k \cdot S(y)) = \|x\|_a \cdot \|y\|_a.
\end{aligned}$$

Since $X \subset (B([0,1]), \|\cdot\|_\infty)$, there exists a constant $N > 0$ such that $\|x\|_\infty \leq N \cdot \|x\|_X$ for all $x \in X$. Finally,

$$\|x\|_X = |x(0)| + S(x) \leq \|x\|_\infty + S(x) \leq \max\left\{l, \frac{1}{k}\right\} \cdot (l \cdot \|x\|_\infty + k \cdot S(x)) = \max\left\{l, \frac{1}{k}\right\} \cdot \|x\|_a$$

and

$$\|x\|_a = l \cdot \|x\|_\infty + k \cdot S(x) \leq l \cdot N \cdot \|x\|_X + k \cdot \|x\|_X = (l \cdot N + k) \cdot \|x\|_X.$$

The norms $\|\cdot\|_a$ and $\|\cdot\|_X$ are equivalent. Since $B([0,1]), \|\cdot\|_\infty$ is a Banach space, $(X, \|\cdot\|_s)$ with $s = a, b, c$ is complete too, so it is a Banach algebra.

(b) Now, assume that $k = 0$. Here, we proceed as in the original paper by Maligranda and Orlicz [1]. We have

$$\begin{aligned}
\|x \cdot y\|_b &= \|x \cdot y\|_\infty + |(x \cdot y)(0)| + S(xy) \\
&= 2 \cdot \|x\|_\infty \cdot \|y\|_\infty + l \cdot (\|x\|_\infty \cdot S(y) + S(x) \cdot \|y\|_\infty) \\
&\leq \|y\|_\infty \cdot (2 \cdot \|x\|_\infty + l \cdot S(x)) + \|x\|_\infty \cdot (2 \cdot \|y\|_\infty + l \cdot S(y)) \\
&\leq \max\{2, l\} \cdot \|y\|_\infty \cdot (\|x\|_\infty + S(x)) + \max\{2, l\} \cdot \|x\|_\infty \cdot (\|y\|_\infty + S(y)) \\
&\leq \max\{2, l\} \cdot \|x\|_b \cdot \|y\|_b.
\end{aligned}$$

Clearly, the constant $\max\{2, l\}$ can be replaced by l when we consider the case $\|x\|_b = \|x\|_\infty + S(x)$.

(c) Let $M > 0$ be a constant such that $\|x\|_\infty \leq M \cdot \|x\|_X$ for all $x \in X$. Then,

$$\|x\|_X \leq \|x\|_b = \|x\|_\infty + \|x\|_X \leq M \cdot \|x\|_X + \|x\|_X = (M + 1) \cdot \|x\|_X.$$

□

Clearly, the case $l = 1$ and $k = 0$ is consistent with the classical Maligranda–Orlicz lemma. Is the consideration of other constants relevant? Yes, we cannot restrict our attention to the original case, because in such a case, the estimate of $S(x \cdot y)$ need not be true, and we need larger constants (if they exist). Note that estimation of the above type with $l > 1$ is a typical property of seminorms in some spaces of bounded variation (cf. [4,13]).

Remark 1. *For practical reasons, in case (c), we assumed that there exists a constant $N > 0$ such that $\|x\|_\infty \leq N \cdot \|x\|_X$ for all $x \in X$. This assumption can be relaxed to the form:*

$$\|x_n\|_X \to 0 \quad \Rightarrow \quad \|x_n\|_\infty \to 0 \quad \text{for } n \to \infty.$$

Under this assumption, by the definition of $\|\cdot\|_b$, the norms $\|\cdot\|_X$ and $\|\cdot\|_b$ are equivalent.

This is actually a weaker condition, as the simplest example shows: $\|x\| = \|x\|_2$ on $C([0,1])$ satisfies the above condition, but $\|x\|_\infty$ cannot be estimated from above with a common constant N (cf. the sequence $x_n(t) = \sqrt{n} \cdot t^n$, $n \in \mathbb{N}$).

Remark 2. *A few more comments on the inequality (2): The case $l = 0$ is related to the problem of the submultiplicative property of seminorms, which has been studied intensively (see [21,22]). The case $k = 0$, $l = 1$ forms a basis for the original Maligranda–Orlicz lemma and is sufficient* *for most applications. Regarding the aspect of testing the equivalence of norms in our result, it is worth recalling that in some algebras there may be inequivalent multiplicative complete norms ([23], Corollary 6.3).*

Since our goal is to construct as many norms as possible for which the space is a normed algebra and to obtain analogous estimates for measures of noncompactness (and, thus, fixed-point theorems), we consider this in one general case of the form (2).

Seminorms that satisfy the condition (2) and are useful for defining multiple normed spaces are considered in the last part of the paper.

Example 1 (cf. [5], Example 12)**.** *Consider the space $RBV_p([0,1])$ of functions with Riesz finite p-variation (for some $p \geq 1$), i.e., with finite seminorms:*

$$Var_R^p(x) = \sup_{(t_i)} \sum_i \frac{|x(t_i) - x(t_{i-1})|^p}{|t_i - t_{i-1}|^p}$$

for subdivisions (t_i) of $[0,1]$.

Let $x(t) = y(t) = e^t \in RBV_p([0,1])$. As the functions are monotone, we can easily calculate their seminorms $Var_R^p(x) = Var_R^p(y) = \frac{e^p-1}{p}$.

Now consider the pointwise product xy and compute the integral:

$$Var_R^p(xy) = \int_0^1 |e^{2t}|^p \, dt = \frac{2^{p-1}(e^{2p}-1)}{p} = \frac{2^{p-1}(e^p+1)(e^p-1)}{p}.$$

Suppose, that this seminorm satisfies

$$Var_R^p(xy) \leq l \cdot (\|x\|_\infty \cdot Var_R^p(y) + Var_R^p(x) \cdot \|y\|_\infty) + k \cdot Var_R^p(x) \cdot Var_R^p(y)$$

for some constants $l \geq 1$, $k \geq 0$. Clearly, $\|x\|_\infty = \|y\|_\infty = e$. Thus,

$$\frac{2^{p-1}(e^p+1)(e^p-1)}{p} = Var_R^p(xy) \leq l \cdot \left(e \cdot \frac{e^p-1}{p} + \frac{e^p-1}{p} \cdot e \right) + k \cdot \frac{e^p-1}{p} \cdot \frac{e^p-1}{p}$$

and then

$$2^{p-1}(e^p+1) \leq 2 \cdot l \cdot e + k \cdot \frac{e^p-1}{p}.$$

Let us note that even in the classical case $k = 0$, we get

$$l \geq \frac{2^{p-1}(e^p+1)}{2 \cdot e}$$

and for large enough p ($\approx$1.351733), we get $l > 1$. The classical Maligranda–Orlicz lemma cannot be applied to this seminorm, but the proposed version may be useful. Our Theorem 1 thus gives an (alternative) proof that $(RBV_p([0,1]), \|\cdot\|_\infty + Var_R^p(\cdot))$ is a normed algebra.

Example 2. *Let $X = C^1([0,1])$ with $S(x) = \|x'\|_\infty$. Obviously, for $x_n(t) = t^n$, we get $S(x_n) = n$, and for $n > 1$, we have $S(x_n) \geq \|x\|_\infty$; then, the seminorm S does not have the estimate in Point (c) of Theorem 1. However, the inequality (2) still holds true, and Points (a) and (b) can be applied. Indeed,*

$$S(x \cdot y) = \|(x \cdot y)'\|_\infty = \|x' \cdot y + x \cdot y'\|_\infty \leq \|y\|_\infty \cdot S(x) + \|x\|_\infty \cdot S(y).$$

This space equipped with the norm $\|x\|_{cl} = \|x(0)\| + \|x'\|_\infty$ is a Banach algebra.

Thus, we have obtained an excellent tool for the study of norms for which the spaces are normed (Banach) algebras and it is not necessary to prove this property each time.

However, this is only the first aim of the paper, and in the next section, we will deal with the symmetry property of norms of spaces, measures of noncompactness in these spaces, and the consequences for the study of fixed-point products of operators in these spaces.

We proved a generalization of the Maligranda–Orlicz lemma in the space of bounded functions. Two questions arise: Does this apply only to normed spaces? Does it apply only to pointwise multiplication operations? We will answer these before giving examples of applications of the results.

In the last part of the paper, we will also show the consequences of the results obtained for measures of noncompactness and thus obtain new fixed-point theorems.

3. Towards a Theory of Quasi-Normed Algebras

A possible choice of constants different than the one in (2) leads us to another starting point.

Definition 1. *A quasi-seminorm on a vector space X is a real-valued map p on X that satisfies the following conditions:*

1. $p \geq 0$;
2. $p(sx) = |s|p(x)$ *for all* $x \in X$ *and all scalars* s;
3. *There exists a real* $k \geq 1$ *such that* $p(x+y) \leq k[p(x) + p(y)]$ *for all* $x, y \in X$.

As $p(x+x) = p(2x) = 2p(x) = p(x) + p(x)$, *the smallest possible* k *satisfies* $k \geq 1$ *and is called the quasi-triangle constant (or the modulus of concavity of the quasi-seminorm).*

A quasi-seminorm is a quasi-norm that also satisfies:

4. $x \in X$ *satisfies* $p(x) = 0$, *then* $x = 0$.

Any quasi-norm induces a metric topology on X, and if the quasi-normed space is complete with this metric, it is called a quasi-Banach space. With this definition. we can define a quasi-normed algebra:

Definition 2. *A quasi-normed space $(A, \|\cdot\|)$ is called a quasi-normed algebra if the vector space A is an algebra and there is a constant $K > 0$ such that*

$$\|x \cdot y\| \leq K \cdot \|x\| \cdot \|y\|$$

for all $x, y \in A$.

Example 3. *Let us consider the space $C_\alpha([0,1])$ consisting of continuous real-valued functions equipped with the quasi-norm*

$$\|x\|_\alpha = \sqrt[3]{\sup_{t \in [0,1]} e^{-\alpha t}|x(t)|^3}.$$

Indeed, we have

$$\sup_{t \in [0,1]} e^{-\alpha t}|x(t) + y(t)|^3 \leq 4 \cdot \left(\sup_{t \in [0,1]} e^{-\alpha t}|x(t)|^3 + \sup_{t \in [0,1]} e^{-\alpha t}|y(t)|^3 \right)$$

and, consequently,

$$\|x + y\|_\alpha \leq \sqrt[3]{4} \cdot (\|x\|_\alpha + \|y\|_\alpha).$$

This is a quasi-Banach space with the quasi-triangle constant $K = \sqrt[3]{4}$.

In the case of quasi-normed spaces, the assumption of a constant l in the inequality (2), which is not assumed to be equal to 1, is particularly relevant. In many studies of functions with finite variation, the resulting spaces are only quasi-algebras. Interesting results, including estimates of type (2) and constants, can be found in [24], Theorems 1 and 2. It is also worth noting that the mentioned paper considers the full seminorm estimates of the type (2), as proposed in this paper, with the constants $l, k \neq 0$ (cf. [24], Section 4).

It is worth recalling that in $X = BV^p([0,1])$ for $S(x) = Var^p(x)$ with

$$Var^p(x) = \sup_{(t_i)} \sum_i |x(t_i) - x(t_{i-1})|^p$$

for subdivisions (t_i) of $[0,1]$, we have that

$$Var^p(x \cdot y) \leq Var^p(x) \cdot Var^p(y)$$

(i.e., with $l = 1$) is satisfied if and only if $p \leq 1$ (see [1], Theorem 1). Recall that a counterexample for the constant $l = 1$ and for $p > 1$ can also be found in [1]. The same holds true in $RBV^p([0,1])$ for Riesz p-finite variation ($0 < 0 \leq 1$):

Proposition 1. *For any* $0 < p \leq 1$,

$$Var^p_R(x \cdot y) \leq Var^p_R(x) \cdot Var^p_R(y),$$

where $x, y \in RBV_p([0,1])$.

The simple proof is a consequence of the characterization of functions from $RBV^p_R([0,1])$: $x \in RBV^p_R([0,1])$ if and only if x is absolutely continuous on $[0,1]$ and its derivative $x' \in L_p([0,1])$. In this case, we have $Var^p_R(x) = \int_0^1 |x'(t)|^p \, dt$. By calculating the derivative of the product and using the simple estimate $(a+b)^p \leq a^p + b^p$ for $0 < p < 1$ and $a, b > 0$, we obtain the expected inequality.

In contrast to the classical version of the Maligranda–Orlicz lemma, the existence of a constant K in the definition of a quasi-norm does not significantly affect the proof, i.e., analogous to Theorem 1, we obtain the following:

Proposition 2. *Assume* $(X, \|\cdot\|_X)$ *is a quasi-normed algebra consisting of bounded functions with the quasi-norm* $\|x\|_X = |x(0)| + p(x)$. *Here,* p *is a given quasi-seminorm on* X *with the following property: there exist constants* $k > 0$ *and* $l \geq 1$ *for any* $x, y \in X$ *for which we have*

$$p(x \cdot y) \leq l \cdot (\|x\|_X \cdot p(y) + p(x) \cdot \|y\|_X) + k \cdot p(x) \cdot p(y). \tag{3}$$

Then, the space X *equipped with the norm*

$$\|x\|_a = l \cdot \|x\|_X + k \cdot p(x) \quad \text{for } x \in X$$

is a quasi-normed algebra, so

$$\|x \cdot y\|_a \leq \|x\|_a \cdot \|y\|_a.$$

The proof is analogous to that of Theorem 1. Note that the constant M does not affect the proof here, unlike other properties (e.g., measures of noncompactness examined later) based on quasi-norms.

Example 4. *Let us return again to Example 3. Note that the inequality (3) holds for* $\|\cdot\|_\alpha$:

$$\|x \cdot y\|_\alpha = \sqrt[3]{\sup_{t \in [0,1]} e^{-\alpha t} |x(t)y(t)|^3} \leq \|x\|_\infty \cdot \|y\|_\alpha + \|y\|_\infty \cdot \|x\|_\alpha \,,$$

and hence, the space $(C_\alpha([0,1]), \|\cdot\|_\infty + \|\cdot\|_\alpha)$ *is a quasi-Banach algebra.*

Quasi-normed spaces play an important role in a wide range of investigations (see, for example, [25]), and the properties of the norm that determine whether they are quasi-normed algebras (or quasi-Banach algebras) are as important as those in the norm case.

Here, we prepare the ground for a unified treatment of the topic by studying norms and quasi-norms simultaneously. However, the problems in these spaces are not the same and need to be studied separately. However, there is not complete symmetry between the two cases.

In this paper, in keeping with its aim and scope, we study only quasi-norms, and the consequences of this for measures of noncompactness or fixed-point theorems require extensive studies and will be presented later.

4. Measures of Noncompactness

We now turn to the properties of measures of noncompactness, which seem to be essential in fixed-point studies for the product of operators or in so-called hybrid fixed-point theorems.

However, we should emphasize that measures of non-convexity are a topological tool for studies in metric spaces, so there is no obstacle to applying our results to quasi-normed spaces. It is worth noting, however, that this has interesting applications beyond the intended scope of our paper and is worth exploring in the future. For example, refs. [26,27] investigate measures of noncompactness in interpolation theory between quasi-Banach spaces.

In previous papers (e.g., [14,15,28]), it was usually assumed that only measures of noncompactness with special additional properties of the measure of noncompactness of the product of sets could be useful in proofs of fixed-point theorems and, consequently, for quadratic equations or hybrid operators, and it was shown that such measures exist. In this paper, we will show why these properties of these measures are useful and not others, and we will prove the full symmetry of the properties of a certain class of measures of noncompactness with the properties of the norms in the algebras studied and demonstrate the usefulness of the result obtained earlier. We will conclude this section with a new fixed-point theorem.

If X is a subset of a quasi-Banach space E, then $\overline{X}$ and $convX$ denote the closure and convex closure of X, respectively. By B_r, we denote a ball centered at θ with a radius r. The standard algebraic operations on sets are denoted by the symbols $k \cdot X$ and $X + Y$. Moreover, by $\mathcal{M}_E$, we denote the family of all nonempty and bounded subsets of E, and by $\mathcal{N}_E$, we denote its subfamily consisting of all relatively compact subsets of E. Thanks to its universality and generality, we will apply an axiomatic approach to the concept of the measure of noncompactness.

Definition 3 ([12])**.** *A mapping $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

(i) $\mu(X) = 0 \Leftrightarrow X \in \mathcal{N}_E$;
(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$;
(iii) $\mu(\overline{X}) = \mu(convX) = \mu(X)$;
(iv) $\mu(\lambda X) = |\lambda|\, \mu(X)$, *for* $\lambda \in \mathbb{R}$;
(v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$;
(vi) $\mu(X \bigcup Y) = \max\{\mu(X), \mu(Y)\}$;
(vii) *If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \cdots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.*

Let us recall two classical examples. The Hausdorff measure of noncompactness $\beta_H(A)$ (cf. [12]) is defined as follows:

$$\beta_H(A) = inf\{r > 0 : \text{there exists a finite subset Y of A such that } x \subset Y + B_r\},$$

where A is an arbitrary nonempty and bounded subset of E. Furthermore, let $\alpha(A)$ denotes the Kuratowski measure of noncompactness of the (bounded) set $A \subset E$, i.e., the infimum of the number d such that A admits a finite covering by sets with diameters smaller than d.

It is sometimes convenient to replace (i) with the axiom $\mu(X) = 0 \Rightarrow X \in \mathcal{N}_E$, which allows, for example, analogous treatment of measures of noncompactness, e.g., the diameter of a set or its norm (in a generalized sense, i.e., with different sets of axioms).

We also recall that measures of noncompactness can be defined in metric spaces, countably normed spaces, and even locally convex spaces. In such cases, we consider different sets of axioms, e.g., for the case of quasi-normed spaces, which is of interest in this paper, weaker requirements should be adopted, e.g.,

$$\mu_X(A+B) \le c_X \cdot (\mu_X(A) + \mu_X(B)),$$

instead of the algebraic semi-additivity (cf. [25,29]), where c_X is the quasi-triangle constant in X. Considering quasi-normed spaces instead of normed ones in the paper does not change essential properties of the measures, such as their mutual equivalence (cf. [25]).

Nevertheless, it turns out that among all classical realizations of measures of noncompactness, the Hausdorff measure seems to be the most convenient for and useful in applications. It should be noted, however, that this measure depends on the norm in the space under consideration, and we will exploit this fact.

More examples of measures of noncompactness can be found in [12]. To distinguish between measures of noncompactness μ in different spaces (if necessary), we will indicate an appropriate space as an index, i.e., μ_E, μ_X, etc.

4.1. A Property of Measures of Noncompactness

It is known that the fact that the pointwise product of functions in a function space is in the same space again does not imply the submultiplicative property of any norm on that space. However, if this initial norm satisfies some natural inequality, we can show that the norm is submultiplicative. We will study a similar property of measures of noncompactness in such spaces.

Since measures of noncompactness are a well-known tool for solving differential and integral problems; we need a special property of such measures for quadratic problems. Such a property was first formulated when studying quadratic equations on the space $C([a,b])$ ((m)-property in [2]). In a more general form, it was defined and used in [30], which includes a proof of this more general form for the Hausdorff measure of noncompactness ([30], Lemma 3.2).

However, we will show here that this condition is not very restrictive, and, perhaps surprisingly, the idea is based on the property of norms on Banach algebras. So far, this property has been studied independently of the norm property that the space is a Banach algebra. In the study of quadratic problems, it has been important to find a suitable measure with the desired property. We will show how to find such a measure and show that it depends on the assumed norm of the space.

This kind of condition was first formulated by Maligranda and Orlicz [1] for a special measure of noncompactness: namely, for the norm of a set. Indeed, we are motivated to do so by the Maligranda–Orlicz lemma ([1]) formulated in terms of norms (which is also a special case of measures of noncompactness: satisfying condition (i) only for a set $\{0\}$).

We prove a parameterized version of this lemma that can be used directly for many norms. In this paper, we will show how this inequality is relevant to the study of operator equations in Banach algebras (cf. [1]).

Let us emphasize the condition (2). Since the norm of a set is a measure of noncompactness in the sense of Definition 3, it is sufficient when we study norm contractions. However, this lemma can be extended to a wider class of measures of noncompactness. In the case we consider, we need the following:

Condition (H): Fix a triple of Banach spaces X, Y, and Z with the chosen measures of noncompactness in each of them: μ_X, μ_Y, and μ_Z, respectively. We say that the bilinear operator $H : X \times Y \to Z$ satisfies condition (H) if for all bounded subsets $A \subset X$, $B \subset Y$ there exist constants $b, c > 0$ such that

$$\mu_Z(H(A,B)) \le b \cdot \mu_X(A) + c \cdot \mu_Y(B). \tag{4}$$

The above condition also holds when we consider Banach algebras with H other than the pointwise multiplication. However, in this paper, we consider the property on Banach algebras in the last case, i.e., for $H(t,s) = s \cdot t$. The general case will be considered elsewhere. In the special case $X = Y = Z = C([0,1])$, our condition coincides with the property (m) discussed in [2] (with $b = \|B\|_\infty$ and $c = \|A\|_\infty$). In [2], this property was proved for some special measures of noncompactness ([2], Theorems 2.3 and 2.4) on $C([0,1])$. For reasons concerning the general form of condition (H), see [30,31].

In the considered case of quadratic problems under consideration, this condition has a special form for a fixed measure of noncompactness on the Banach algebra. Let $A \cdot B$ denote the (Minkowski) product of the sets A and B, i.e., the set $\{a \cdot b : a \in A, b \in B\}$. Note that for bounded set A, B, the product $A \cdot B$ is bounded.

Condition (HA): for a given measure of noncompactness μ_X on the (quasi-)Banach algebra X for all bounded subsets $A, B \subset X$, there exist constants $b, c > 0$ such that

$$\mu_X(A \cdot B) \le b \cdot \mu_X(A) + c \cdot \mu_X(B). \tag{5}$$

This condition is important when we need to prove the contraction property of the product of operators. For a fixed bounded set E in a Banach algebra X, we examine the set $E_1 = G(E) \cdot F(E)$, and to check the contraction property of the product of operators, we need to estimate a measure of noncompactness of this set. We have a simple lemma showing the relation between this condition and the contraction condition for the product:

Lemma 1. *Let $(X, \|\cdot\|_X)$ be a quasi-Banach algebra of bounded functions with the product $(x \cdot y)(t) = x(t) \cdot y(t)$, and let μ_X be a measure of noncompactness in X satisfying condition (HA). Then for arbitrary bounded subsets $A, B \subset X$, there exists a constant k such that*

$$\mu_X(A \cdot B) \le k \cdot \max\{\mu_X(A), \mu_X(B)\}.$$

Proof. Let $A, B \subset X$ be bounded sets. As for the measure of noncompactness μ_X, condition (HA) holds:

$$\mu_X(A \cdot B) \le b \cdot \mu_X(A) + c \cdot \mu_X(B) \le \max\{b, c\} \cdot \max\{\mu_X(A), \mu_X(B)\}.$$

□

When studying quadratic problems, we need to find a measure of noncompactness μ_X on the putative Banach algebra that satisfies this condition. Although this hypothesis (HA) is not very restrictive (in view of Theorem 1) and the problem of equivalence of measures of noncompactness remains unchanged from the case of normed spaces, the following lemma can also be useful by simplifying many calculations in quasi-Banach algebras:

Lemma 2. *Suppose, that μ_X and ν_X are equivalent measures of noncompactness on a quasi-Banach algebra X and that μ_X satisfies condition (HA). Then μ_X also satisfies this condition.*

Proof. Since μ_X and ν_X are equivalent measures of noncompactness, we can find some constant $m, M > 0$ such that for any bounded set $A, B \subset X$, we have

$$m \cdot \nu_X(A) \le \mu_X(A) \le M \cdot \nu_X(A),$$

and, as μ_X satisfies condition (HA), there exist constants $b, c > 0$ such that

$$\mu_X(A \cdot B) \le b \cdot \mu_X(A) + c \cdot \mu_X(B).$$

Then,

$$m \cdot \nu_X(A \cdot B) \le \mu_X(A \cdot B) \le b \cdot \mu_X(A) + c \cdot \mu_X(B) \le b \cdot M \cdot \mu_X(A) + c \cdot M\mu_X(B),$$

and finally,

$$\nu_X(A \cdot B) \le \frac{b \cdot M}{m}\nu_X(A) + \frac{c \cdot M}{m}\nu_X(B),$$

i.e., condition (HA) is satisfied. □

Now, we are ready to show how the Maligranda–Orlicz lemma implies the existence of such a measure of noncompactness on any Banach algebra, and so by Lemma 2, we obtain some interesting measures with this property. By applying property (HA), we will improve Lemma 1, but most importantly, we will obtain a generalization of Theorem 1 to the Hausdorff measure of noncompactness (see also [16], Lemma 2.4).

Lemma 3. *Let $(X, \|\cdot\|_X)$ be a Banach algebra for which the elements are bounded functions. Let $x, y \in X$, and assume that*

$$\|x \cdot y\|_X \le l \cdot (\|x\|_\infty \cdot \|y\|_X + \|x\|_X \cdot \|y\|_\infty) + k \cdot \|x\|_X \cdot \|y\|_X$$

for some constants $k \ge 0$ and $l \ge 1$. Then, the Hausdorff measure of noncompactness β in the space X equipped with the norm

$$\|x\| = l \cdot \|x\|_\infty + k \cdot \|x\|_X$$

has the following property: for arbitrary bounded sets $A, B \subset X$

$$\beta(A \cdot B) \le \|B\|_\infty \cdot \beta(A) + \|A\|_\infty \cdot \beta(B) + \beta(A) \cdot \beta(B). \tag{6}$$

Moreover, there exist constants $b, c > 0$ such that for arbitrary bounded sets $A, B \subset X$,

$$\beta(A \cdot B) \le b \cdot \beta(A) + c \cdot \beta(B). \tag{7}$$

Proof. Let $A, B \subset X$ be bounded. From the definition of β, for arbitrary $\varepsilon > 0$, there exists a finite set $K_1 \subset X$ such that

$$A \subset K_1 + B(0, r_1 + \varepsilon),$$

where $r_1 = \beta(A)$, and there exists a finite set $K_2 \subset X$ such that

$$B \subset K_2 + B(0, r_2 + \varepsilon),$$

where $r_2 = \beta(B)$. Then,

$$\begin{aligned} A \cdot B &\subset (K_1 + B(0, r_1 + \varepsilon)) \cdot (K_2 + B(0, r_2 + \varepsilon)) \\ &\subset K_1 \cdot K_2 + K_1 \cdot B(0, r_2 + \varepsilon) + K_2 \cdot B(0, r_1 + \varepsilon) + B(0, r_1 + \varepsilon) \cdot B(0, r_2 + \varepsilon). \end{aligned}$$

Clearly, $K_1 \cdot K_2$ is finite, and $B(0, r_1 + \varepsilon) \cdot B(0, r_2 + \varepsilon) \subset B(0, (r_1 + \varepsilon) \cdot (r_2 + \varepsilon))$. Even though the sets K_1 and K_2 need not be contained in A and B, respectively, without loss of generality, we can assume that $\|K_1\|_\infty \le \|A\|_\infty$ and $\|K_2\|_\infty \le \|B\|_\infty$.

Let $z \in K_1$ and $y \in B(0, r_2 + \varepsilon)$ be arbitrary. Then by Theorem 1 $\|z \cdot y\| \le \|z\| \cdot \|y\| \le \|K_1\| \cdot (r_2 + \varepsilon)$. Consequently, $K_1 \cdot B(0, r_2 + \varepsilon)$ can be covered by a finite number of balls with radius $\|K_1\|_\infty \cdot (r_2 + \varepsilon)$. Similar reasoning leads to the conclusion that $K_2 \cdot B(0, r_1 + \varepsilon)$ can be covered by a finite number of balls with radius $\|K_2\|_\infty \cdot (r_1 + \varepsilon)$.

From the properties of the Hausdorff measure of noncompactness, we obtain

$$\begin{aligned} \beta(A \cdot B) &\le \beta(K_1 \cdot B(0, r_2 + \varepsilon)) + \beta(K_2 \cdot B(0, r_1 + \varepsilon)) + \beta(B(0, r_1 + \varepsilon) \cdot B(0, r_2 + \varepsilon)) \\ &\le \|A\|_\infty \cdot (r_2 + \varepsilon) + \|B\|_\infty \cdot (r_1 + \varepsilon) + \beta(B(0, (r_1 + \varepsilon)) \cdot (r_2 + \varepsilon)) \\ &= \|A\|_\infty \cdot (\beta(B) + \varepsilon) + \|B\|_\infty \cdot (\beta(A) + \varepsilon) + (\beta(A) + \varepsilon) \cdot (\beta(B) + \varepsilon). \end{aligned}$$

By passing to the limit with $\varepsilon \to 0^+$, we get the first expected inequality. The second follows from the estimation $\beta(A) \leq \|A\|_\infty, \beta(B) \leq \|B\|_\infty$ and then either $b = 2\|B\|_\infty$ and $c = \|A\|_\infty$ or $b = \|B\|_\infty$ and $c = 2\|A\|_\infty$. □

In this paper, we are concerned with the choice of norms or quasi-norms that allow a space to have the properties of an algebra. However, the previous results about measures of noncompactness apply to normed spaces. Of course, it is also possible to consider measures of noncompactness on quasi-normed spaces and study their analogous properties. Note, however, that these will not be fully analogous results since the balls in such spaces need not be convex and the measures themselves will not be, e.g., convex-invariant (condition (iii) of Definition 3 will not be satisfied) and will not have all the other properties of Definition 3.

It is worth adding that the above lemma can be easily extended to quasi-Banach algebras since it only requires the estimation of the quasi-norm of the sum of the elements and the multiplicativity property along with the definition of a measure of noncompactness based on properties of balls (convex or non-convex). This immediately leads to the following conclusion regarding the satisfaction of condition (HA) in quasi-normed spaces:

Corollary 1. *Let $(X, \|\cdot\|_X)$ be a quasi-Banach algebra with the quasi-triangle constant c for which the elements are bounded functions. Let $x, y \in X$ and assume that*

$$\|x \cdot y\|_X \leq l \cdot (\|x\|_\infty \cdot \|y\|_X + \|x\|_X \cdot \|y\|_\infty) + k \cdot \|x\|_X \cdot \|y\|_X$$

for some constants $k \geq 0$ and $l \geq 1$. Then, the Hausdorff measure of noncompactness β in the quasi-normed space X equipped with the quasi-norm

$$\|x\| = l \cdot \|x\|_\infty + k \cdot \|x\|_X$$

has the following property: for arbitrary bounded sets $A, B \subset X$,

$$\beta(A \cdot B) \leq c \cdot (\|B\|_\infty \cdot \beta(A) + \|A\|_\infty \cdot \beta(B) + \beta(A) \cdot \beta(B)). \tag{8}$$

Moreover, there exist constants $p, q > 0$ such that for arbitrary bounded sets $A, B \subset X$,

$$\beta(A \cdot B) \leq p \cdot \beta(A) + q \cdot \beta(B). \tag{9}$$

A recommended survey of the properties that distinguish the case of measures of noncompactness in normed and quasi-normed spaces is in [25,29]. Let us give one example of a significant difference between the two cases.

In particular, the Hausdorff measure of noncompactness is investigated here. In [25] Lemma 11.6, a variant of Mönch's characterization of this measure of noncompactness is proved, i.e., for finite-dimensional subspaces $X_1 \subseteq X_2 \subseteq \ldots$, and for any countable bounded subset $A = \{x_1, x_2, \ldots\} \subset X$, we obtain

$$\beta(A) \leq c \lim_{k\to\infty} \limsup_{n\to\infty} dist(x_n, X_k), \tag{10}$$

where c is just the quasi-triangle constant for X. Recall that, unlike in this case, for (separable) normed linear spaces, this estimate is equal and has constant $c = 1$ ([25], Lemma 11.7). In the case of the Kuratowski measure of noncompactness, it is worth recommending a comparison of the measures of noncompactness in the two cases cited: for example, in [25], Theorem 13.9, Proposition 13.11, and all other results in the book that use such measures. The full study is extensive and beyond the scope of this paper and will be presented in subsequent papers. We will refine the remark after the presentation of the Darbo-type fixed-point theorem.

Note that the above proof is based on the properties of the norm and the balls in that norm, so it is worth noting that the above estimation of the product of sets is also true for the DeBlasi measure of weak noncompactness (e.g., [32]) in the so-called (WC)-

algebras considered: for example, in [16,28] when the Banach algebra is equipped with a norm of type $\|x\| = \|x\|_\infty + \|x\|_X$. Note, however, that in normed algebras, the inequality under consideration implies that the product in the algebra is norm-continuous, while (WC)-algebras require weak–weak (sequential) continuity of this action.

Remark 3. *The considerations so far are part of the proof methods for integral (or operator) quadratic equations, i.e., the study of the equations $x = F(x) \cdot G(x)$ in a certain Banach algebra E (cf. [31,33]). Nevertheless, in certain questions, not only are products of two operators studied but a larger, though finite, number of operators is studied. It is worth noting that the results obtained can easily be extended and applied to problems of the form $x = \Pi_{i=1}^N F_i(x)$.*

To facilitate future applications, let us state:

Corollary 2. *Let $(X, \|\cdot\|_X)$ be the Banach algebra of bounded functions with the product $(x \cdot y)(t) = x(t) \cdot y(t)$, and let μ_X be a measure of noncompactness in X satisfying the following condition: for arbitrary bounded sets $A_i \subset X$, $i = 1, 2, ..., N$, there exist constants $b_i > 0$, $i = 1, 2, ..., N$, such that*

$$\mu_X(A \cdot B) \leq \sum_{i=1}^{N} b_i \cdot \mu_X(A_i). \tag{11}$$

Then, for arbitrary bounded subsets $A_i \subset X$, $i = 1, 2, ..., N$, there exists a constant k such that

$$\mu_X(\Pi_{i=1}^N A_i) \leq \sum_{i=1}^{N} b_i \cdot \mu_X(A_i) \leq k \cdot \max\{\mu_X(A_i) : i = 1, ..., N\}.$$

To prove this Corollary, it is enough to use the method of induction and rely on Lemma 1.

4.2. Fixed-Point Theorems

It should be noted that the next theorem is a step towards greatly simplifying the study of quadratic equations (and similarly equations on n-tuples). On the one hand, we will now present the proof algorithm, and on the other hand, we will show how to perform the crucial step of this algorithm. When studying quadratic solutions in Banach algebras, one should simply separately check the continuity and contraction conditions of the operators under study in a given norm (and their invariant sets). Then, using condition (2), make sure both that it is a Banach algebra (Theorem 1) and that the measure of noncompactness allows the construction of the contraction condition for the product of the operators (Lemma 3).

This is a very important step because it allows real research on equations on Banach algebras. The vast majority of current papers is practically on the space of continuous functions and the supremum norm. This should be improved, e.g., fractional-order integral operators are invariant on special spaces of Hölder type (cf. [8]), and this is worth studying, as in the proposed algorithm. Such a study of the properties of operators in special spaces is given, for example, in [34,35]. In particular, it is useful to study such equations in algebras larger than continuous functions (e.g., the space of regulated functions, cf. [11,30]).

If we are interested in quadratic-type problems, instead of technical proofs, we propose an algorithm in a few unified steps:

1. Analyze the problem to determine a type of interesting operators;
2. Choose an appropriate Banach algebra of (expected) solutions;
3. Verify (separately) acting, boundedness, and continuity conditions for considered operators;
4. Find an invariant set T;
5. Determine the measure of noncompactness μ_X on X satisfying the condition (HA);
6. Check the contraction property for F;
7. Verify the contraction condition;
8. Finally, apply the proposed fixed-point theorem.

We propose to follow this idea by proving the existence theorem for solutions of quadratic equations in Banach algebra and for some interesting operators F and G. Due to the proposed algorithm and the fact that in this kind of application contraction constants seem to be important (since they should be less than 1), we will prove a version of the Darbo fixed-point theorem in a form adapted to quadratic problems.

As a consequence of Lemma 3 and applying the contraction property (7), we immediately obtain the following Darbo fixed-point theorem for the product of operators in quasi-Banach algebras (cf. [14,25]). Note that this version emphasizes the separate study of each operator and gives a method for selecting the appropriate norm in the Banach algebra.

Theorem 2. *Let $(X, \|\cdot\|_X)$ be a Banach normed algebra for which the elements are bounded functions. Let $x, y \in X$, and assume that*

$$\|x \cdot y\|_X \leq l \cdot (\|x\|_\infty \cdot \|y\|_X + \|x\|_X \cdot \|y\|_\infty) + k \cdot \|x\|_X \cdot \|y\|_X$$

for some constants $k \geq 0$ and $l \geq 1$.

Assume that T is a nonempty, bounded, closed, and convex subset of X; and the operators $A : T \to X$ and $B : T \to X$ are continuous, with $A(T), B(T)$ being bounded in X. Moreover, assume that $H(x) = A(x) \cdot B(x) \in T$ whenever $x \in T$. If A and B are β_X contractions with constants k_1 and k_2, respectively, and $k = \|A(T)\|_X \cdot k_2 + \|B(T)\|_X \cdot k_1 < \frac{1}{2}$, then there exists at least one fixed point for the operator H in the set T.

Proof. The proof will be the same as in the proposed algorithm. In fact, we will show that the operator H is a contraction with respect to some measure of noncompactness and that it satisfies the assumptions of the Darbo fixed-point theorem.

As for any bounded subset U of T, in view of Lemma 3, we have

$$\begin{aligned}
\beta_X(A(U) \cdot B(U)) &\leq \|B\|_\infty \cdot \beta_X(A(U)) + \|A\|_\infty \cdot \beta_X(B(U)) + \beta_X(A(U)) \cdot \beta_X(B) \\
&\leq 2\|A\|_\infty \cdot \beta_X(A(U)) + 2\|B\|_\infty \cdot \beta_X(B(U)) \\
&\leq 2k_1\beta_X(U) + 2k_2\beta_X(U) = (2k_1 + 2k_2) \cdot \beta_X(U) = k \cdot \beta_X(U).
\end{aligned}$$

Since $k < 1$, we are done. Recall that the acting, boundedness, and continuity conditions are strictly dependent on the operators considered, and, importantly, the chosen norm $\|\cdot\|_X$ plays a key role in this.

Nevertheless, most of these assumptions have already been studied for classical operators and can be used directly in the proof. This gives a general outline of how to proceed with the proof. □

A key role in the application of other measures of noncompactness is played by Lemma 2 and condition (HA).

Corollary 3. *Given the result of Lemma 2, the Hausdorff measure of noncompactness can be replaced by any measure of noncompactness equivalent to it.*

A known case is the following:

Corollary 4 ([14]). *Let X be a Banach algebra. Assume that T is a nonempty, bounded, closed, and convex subset of X, and the operators $A : T \to X$ and $B : T \to X$ are continuous, with $A(T), B(T)$ being bounded in X. Moreover, assume that $H = A \cdot B$ transforms T into itself. If*

1. *There exists a constant $k_1 > 0$ such that A satisfies an inequality $\mu_X(A(U)) \leq k_1 \cdot \mu_X(U)$ for arbitrary bounded subset U of T;*
2. *There exists a constant $k_2 > 0$ such that B satisfies an inequality $\mu_X(B(U)) \leq k_2 \cdot \mu_X(U)$ for arbitrary bounded subset U of T;*
3. *μ_X satisfies the condition (m);*
4. *$\|A(T)\|_X \cdot k_2 + \|B(T)\|_X \cdot k_1 < 1$;*

then there exists at least one fixed point for the operator H in the set T.

This corollary was proved by Banaś in the special case of Banach algebras $X = C(I, \mathbb{R})$.

Remark 4. *When investigating the properties of measures of noncompactness, we have so far done so in the case of normed algebras. However, it is possible to obtain similar estimates in quasi-normed spaces, as will be shown in a forthcoming paper. This is a more comprehensive problem, as we must also note the need to study a non-convex set as the domain of the operator H. Such results for the standard operators (i.e., with $H = A \cdot Id = A$) have been obtained in quasi-normed spaces (cf. [25,29]), but ensuring invariance for the domain of the quadratic operator requires research from scratch from the foundations of quasi-Banach space geometry and would therefore be too extensive to present here.*

Due to the need to keep the results of the paper consistent, we will limit ourselves to giving an example of how the new fixed-point theorem helps in the study of quadratic integral equations. Let us give a simple example by referring to an existing result obtained in the Banach algebra $C(I)$ for a compact space I. In the paper [9], a quadratic problem is solved on $C([0, a])$. It can easily be generalized to the case of the Banach algebra of regulated functions (as this space is taken with the supremum norm, there is no essential difference in the proof: some weaker conditions apply). In the case of non-quadratic equations, we usually expect different properties of the solutions, i.e., different spaces X in which we define the operator F. For example, we study the boundedness and continuity of the operator on such a space, e.g., Hölder (see [8,10,34,36]). Since this is the case, if we introduce a norm according to our result, and it is equivalent to the original norm on X, then the properties of F are the same in this new norm. We can obtain "for free" an existence of solutions for quadratic problems of the form $x = F(x) \cdot F(x)$ or, more generally, $x = F(x) \cdot G(x)$, where both operators have been studied on X.

Open problem: In general, it is an open problem of how to characterize a set T in quasi-normed algebras so that its pointwise product of fixed operators has the following property: $F(T) \cdot G(T) \subset T$.

5. Examples and Applications

It remains to be shown how the discussed properties of norms and measures of noncompactness allow application to quadratic problems. We should emphasize that this approach is most effective when we are able to find analytic formulas for the measures of noncompactness in Banach algebras, since it is then easiest to check whether the operators in question are contractions with respect to such measures. This is also the reason for the problems with applying such a procedure in quasi-Banach spaces (cf. (10)). It is interesting to investigate which spaces are a natural choice when studying quadratic differential equations and with which norms these are Banach algebras, but our results also give more information about the properties of the solutions than just their continuity.

Example 5. *Consider the Banach algebra consisting of bounded but not necessarily continuous functions. It is a Banach algebra with pointwise multiplication that contains the space $C([0, 1])$. It is not sufficiently studied and it is not a subspace of $C([0, 1])$; then, we pay more attention to it by recalling some interesting results. It is not widely known that $G([0, 1])$ is not a subspace of $C([0, 1])$, but it is still a Banach function algebra since it is isometric to the algebra $C(K)$ of continuous functions on a compact non-metrizable Hausdorff space K:*

$$\mathbb{K} = \{(t, 0) : 0 < t \le 1\} \cup \{(t, 1) : 0 \le t \le 1\} \cup \{(t, 2) : 0 \le t < 1\}$$

which is called the Alexandroff (or: the Alexandroff–Urysohn) arrow space. When this set is equipped with the order topology given by the lexicographic order (i.e., $(s, i) \prec (t, j)$ if either $s < t$ or $s = t$ and $i < j$) (cf. [11]), this space contains functions having finite one-side limits at every point, so *the functions are possibly discontinuous but are bounded functions, and again, when equipped with the sup-norm, the space* $(G([0,1]), \|\cdot\|_\infty)$ *is a normalized Banach algebra:*

$$\|x \cdot y\|_\infty \leq \|x\|_\infty \cdot \|y\|_\infty.$$

If we consider a seminorm (a modulus of equi-regularity) ω_G *([11]), then it satisfies (2), and then* $(G([0,1]), \|\cdot\|_\omega)$ *equipped with the norm*

$$\|x\|_\omega = \|x\|_\infty + \omega_G(x)$$

is also a normed (Banach) algebra.

Example 6. *We consider another space consisting of possibly discontinuous bounded functions: the space* $BV([0,1])$ *when equipped with the supremum norm is a normalized algebra but not a Banach algebra (with the supremum norm, it is not a complete space). In this space, we can consider some interesting norms. First of all, let us note that the variation function is a classical seminorm with properties that ensure that the assumptions of Theorem 1 are satisfied: namely,* $S(x) = Var(x,[0,1])$*. This motivates us to use the following norm:*

$$\|x\| = \|x\|_\infty + Var(x,[0,1])$$

instead of the classical one: $\|x\|_{BV} = |x(0)| + Var(x,[0,1])$*. An excellent discussion of the topic of norms, their properties, and the equivalence problem can be found in the book [13].*

We started with examples of algebras that do not only contain continuous functions. It is remarkable that these examples show that the same seminorms (such as $\int_0^1 |x'(t)|\, dt$) can be used on different spaces, but also that in a given space, the choice of the norm guaranteeing the continuity of the product is not unique. Let us look at some spaces covered by our approach.

Example 7. *The space of Lipschitz continuous functions, i.e., satisfying, for some* $L \geq 0$*, the condition*

$$|x(t) - x(s)| \leq L \cdot |t - s|.$$

This space equipped with the norm $\|x\|_{Lip} = |x(0)| + Lip(x)$*, where*

$$Lip(x) = \sup_{s \neq t} \frac{|x(t) - x(s)|}{|t - s|}$$

is a Banach algebra and it could be a classic example of the applicability of Theorem 1. This is because $S(x) = Lip(x)$ *satisfies the assumption of Theorem 1, and then, this space equipped with the norm*

$$\|x\|_{Lip} = \|x\|_\infty + Lip(x)$$

is a normalized Banach algebra. Again, this norm seems to be the best choice for carrying out proofs for many integral operators in this space.

Example 8. *The space of Hölder continuous functions with exponent* $p < 1$*, i.e., satisfying the condition*

$$|x(t) - x(s)| \leq L \cdot |t - s|^p$$

for some $L \geq 0$*. It is known that* $(H_p, \|\cdot\|_C + lip_p(\cdot))$*, where*

$$S(x) = lip_p(x) = \sup_{s \neq t} \frac{|x(t) - x(s)|}{|t - s|^p}$$

is a normalized Banach algebra (see Theorem 1).

Example 9. *We can consider a slightly larger space than H_α (since there are functions belonging to $AC([0,1])$ that are not Hölder continuous for any $\alpha \in (0,1]$)—cf. [13], for example. It is clear that $AC([0,1]) \subset C([0,1]) \cap BV([0,1])$. As in the case of $C([0,1])$, we get the Banach algebra $(AC([0,1]), \|\cdot\|_\infty)$:*

$$\|x \cdot y\|_\infty \leq \|x\|_\infty \cdot \|y\|_\infty.$$

Another interesting norm is the following one:

$$||x|| = \int_a^b (|x(t)| + |x'(t)|)\, dx.$$

Sometimes, the norm $\|\cdot\|_{BV}$ is found to be used, which we understand to mean:

$$\|x\|_{AC} = |x(0)| + \int_0^1 |x'(t)|\, dt$$

, and a seminorm $S(x) = \int_0^1 |x'(t)|\, dt$ satisfies condition (2); by Theorem 1, we obtain that

$$\|x\|_1 = \|x\|_\infty + \int_0^1 |x'(t)|\, dt$$

is a submultiplicative norm on that space (recall that for $x \in AC([0,1])$, we have $Var(x) = \int_0^1 |x'(t)|\, dt$—see [13], Theorem 3.19). However, these norms are equivalent.

Example 10. *Let us also recall the space of classical solutions for differential problems of the first order. The space of continuously differentiable functions C^1 with the norm $\|x\|_{cl} = \|x(0)\| + \|x'\|_\infty$ or with*

$$\|x\|_{C^1} = \|x\|_\infty + \|x'\|_\infty$$

is a Banach algebra.

We are not able to collect here all interesting cases of algebras constructed according to our approach. Let us mention some other algebras: the algebra of functions with (p,α)-bounded variation, which is a generalization of the Riesz p-variation [37]; the algebra of functions with bounded total Φ-variation in the Schramm sense [38], Theorem 3.8; or a number of results concerning quasi-Banach algebras (see, for example, [39]).

This section could end with a theorem about the existence of solutions of the equation $x = F(x) \cdot G(x)$ in the chosen Banach algebra X. Note, however, that the presented algorithm is universal and involves the selection of a space corresponding to solutions with given properties (generally indicated by finite values of the seminorm, e.g., variation of a function). We further construct the appropriate norm in X (see Theorem 1) and study the operators on this space—importantly—completely independently of the purpose of considering this question. Such results for many spaces are the subject of independent research (cf. [25,34,36]).

Because of Theorem 2, we are interested in contractions with respect to some measures of noncompactness in X space, but we can also consider conditions such as Lipschitz continuity, complete continuity, or sums of such operators. Lemmas 1 or 3 and Theorem 2 are used to connect them within $T = F \cdot G$. There is no need to make extremely technical estimates of T in X space, and this is usually the longest and most difficult part of the proof. We take the liberty of recommending that people check how this can simplify known proofs (e.g., in the algebra of regulated functions [11] or in some Hölder-type algebras [8]), but above all, we encourage readers to apply it to new quadratic problems and to various Banach algebras, not just $C([0,1])$. Thanks to Corollary 2, this scheme can be used in cubic equations or on n-tuples. See [40], Lemma 2.3, for methods of creating norms on n-tuples.

6. Conclusions

The paper points out a general method for norming subspaces of the space of bounded functions so that it is a normed algebra, and the norm is convenient for the study of both operators on these subspaces and measures of noncompactness. This allows the study of operator quadratic equations by means of the obtained fixed-point theorem and in a unified way for all such spaces. We generalize and unify a number of previous results for normed and quasi-normed algebras.

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