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On Ulam Stabilities of Delay Hammerstein Integral Equation

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Abstract: In this paper, we consider a Hammerstein integral equation (Hammerstein IE) in two variables with two variables of time delays. The aim of this paper is to investigate the Hyers–Ulam (HU) stability and Hyers–Ulam–Rassias (HUR) stability of the considered IE via Banach's fixed point theorem (Banach's FPT) and the Bielecki metric. The proofs of the new outcomes of this paper are based on these two basic tools. As the new contributions of the present study, here, for the first time, we develop the outcomes that can be found in the earlier literature on the Hammerstein IE, including variable time delays. The present study also includes complementary outcomes for the symmetry of Hammerstein IEs. Finally, a concrete example is given at the end of this study for illustrations.

Keywords: Hammerstein IE; HU stability; HUR stability; Banach FPT; Bielecki metric

MSC: 45D05; 45G10; 45J05; 45M10; 47H10

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1. Introduction

Extensive efforts have been spent on the study of HU stability, HUR stability, etc. There are several kinds of equations that include derivative(s), delay(s) or do not include them (see, e.g., Abbas and Benchohra [1], Akkouchi [2], Banaś and Rzepka [3,4], Banaś and Rzepka [5], Banaś and Chlebowicz [6], Banaś et al. [7], Castro and Guerra [8], Castro and Ramos ([9,10]), Castro and Simões ([11,12]), Ciplea et al. [13], Janfada and Sadeghi [14], Jung ([15,16]), Ngoc et al. [17], Öğrekçi et al. [18], Tunç and Tunç [19] and the references of these sources). The concepts of the HU, HUR, etc., for any equation kind under investigation, arise when we replace the equation under study with an inequality which acts as a perturbation of the equation. Hence, the stability question, in the sense of the HU or HUR, etc., for the equations under study, is how the solutions of the inequality will vary from that of the mathematical models taken under the study.

Accordingly, the Hammerstein IE is stable in the HU sense if, for every function satisfying the Hammerstein IE approximately, there exists a solution of the equation that is close to it. This means that the stability case of the Hammerstein IEs is how the solutions of the inequality vary from those of the considered Hammerstein IE. Despite the existence of extensive works for several types of equations, there are limited numbers on the study of these kinds of stabilities for IEs; in particular, there are only a few for the Hammerstein IEs. Presently, according to the documents of the current literature, we will report a few results related to the content of this study in the following lines.

As a start point in the relevant literature, the first Ulam stability work was achieved with regard to the functional equations by S.M. Ulam; see [20].

In 2007, using the fixed point method (FPM), Jung [15] studied the uniqueness of the solution and HUR stability of the Volterra IE:

$$y(x) = \int_c^x f(\tau, y(\tau)) d\tau.$$

Indeed, the work of Jung [15] has been a start point in the literature for the coming works with regard to Ulam stabilities of IEs.

The book of Jung [16] provides an overview of the theory of the stability of FEs and includes various results on the stability of the various kinds of FEs.

In 2009 and 2010, Castro and Ramos ([9,10]) dealt with the HU stability and HUR stabilities in the following nonlinear Volterra IEs without delay and with delay, respectively:

$$y(x) = \int_a^x f(x, \tau, y(\tau))d\tau$$

and

$$y(x) = \int_a^x f(x, \tau, y(\tau), y(\alpha(\tau)))d\tau.$$

In 2011, using the FPM, Akkouchi [2] discussed HU and HUR stabilities of the Volterra IE in Banach spaces:

$$f(x) = h(x) + \lambda \int_a^x G(x, y, f(y))dy.$$

In 2013, using the Banach FPT and the Bielecki metric, Castro and Guerra [8] studied the HUR stability of the nonlinear Volterra IE with a variable delay:

$$y(x) = g(x) + \Psi\left(\int_a^x k(x, t, y(t), y(\alpha(t)))d\tau\right).$$

In 2013 and 2023, benefiting from the FPM, Janfada and Sadeghi [14] and Öğrekçi et al. [18] focused on HU and HUR stabilities of the Volterra IE:

$$x(t) = g(t, x(t)) + \int_0^t K(t, s, x(s))ds$$

In 2015, using the Schauder FPT, Abbas and Benchohra [1] obtained new results with regard to the HUR stability of the IE:

$$u(t) = f(t, u(t)) \int_0^t k(t, s)g(s, u(s))ds.$$

In 2018, using the FPM and Bielecki metric, Castro and Simões [12] obtained new outcomes concerning HUR, σ-semi HU and HU stabilities of the Fredholm and Volterra IEs:

$$y(x) = f\left(x, y(x), y(\alpha(x)), \int_a^b k(x, \tau, y(\tau), y(\beta(\tau)))d\tau\right).$$

and

$$y(x) = f\left(x, y(x), y(\alpha(x)), \int_a^x k(x, \tau, y(\tau), y(\beta(\tau)))d\tau\right).$$

In 2022, using Picard operators, Ciplea et al. [13] and Ngoc et al. [17] studied the HUR stability for Volterra–Hammerstein functional IEs in three variables.

Recently, Tunç and Tunç [19] obtained two results on HU and generalized HUR stabilities of an iterative functional IDE using the Banach FPT.

As the leading results for the motivation of this study, in 2017, Castro and Simões [11] considered the Hammerstein IE of the form

$$y(x) = p(x) + f(x, y(x)) \int_{a}^{x} g(x, \tau) h(\tau, y(\tau)) d\tau. \tag{1}$$

Using the FPM and the Bielecki metric, Castro and Simões [11] derived three new results with regard to HUR and HU stabilities of the Hammerstein IE (1) in the bounded interval case and the HUR stability in the infinite interval case.

On the other hand, in 2003, Banaś and Rzepka [3,4] focused on the following non-linear Volterra IE:

$$x(t) = f(t, x(t)) \int_{0}^{t} u(t, s, x(s)) ds.$$

Banaś and Rzepka [3,4] discussed the existence, the asymptotic stability and boundedness of solutions of this Volterra IE using a fixed point theorem and measures of noncompactness in fixed point theory.

Next, Banaś and Rzepka [5] studied the existence of solutions of a nonlinear quadratic Volterra IE by applying the same technique of [3,4] in conjunction with the classical Schauder FPT.

Banaś and Chlebowicz [6] considered the countably infinite systems of nonlinear IEs of the Volterra–Hammerstein type. They established the existence of solutions in the Banach space, where bounded and continuous functions are included, etc., using a suitable measure of noncompactness and a FPT of the Darbo type.

Banaś et al. [7] proved the existence of continuous and bounded solutions for certain nonlinear quadratic IEs of Volterra–Hammerstein types employing the technique of [3,4] in the Banach space.

Das et al. [21] investigated the existence of a solution for the fractional integral equation involving the Riemann–Liouville fractional integral via a new generalization of the Dorbo-type fixed point theorem.

Mohiuddine et al. [22] studied the existence of solutions of the system of nonlinear integral equations via a Dorbo-type theorem in tempered sequence spaces and so on. The authors [22] also constructed an illustrative example by taking an integral equation to validate their result.

Furthermore, several interesting results in the sense of Ulam stabilities, Lyapunov stabilities, some other related results with regard to qualitative behaviors of solutions for disparate classes of the IDEs, IEs and so on have been discussed by Chauhan et al. [23], Deep et al. [24], Gãvruţa [25], Graef and Tunç [26], Hammami and Hnia [27], Ngoc et al. [17], Petruşel et al. [28], Radu [29], Rassias [30], Shah et al. [31], Tunç and Tunç ([32–34]) and in the books or the articles cited in these sources together that are presented in the above ones.

In the present paper, inspired by the study of Castro and Simões [11] and the studies presented above, we will deal with the Hammerstein IE with two variable delays:

$$y(x) = f(x, y(x), y(\tau(x))) \int_{a}^{x} g(x, \tau) h(\tau, y(\tau), y(\vartheta(\tau))) d\tau + p(x), \tag{2}$$

where $x,\ \tau \in [a,\ b]$, $a,\ b \in \mathbb{R}$, $a < b$, $p \in C([a,\ b], \mathbb{C})$, $\mathbb{C}$ is a closed subset of $\mathbb{R}$, $C([a,\ b], \mathbb{C})$ is the space of the continuous functions from $[a,\ b]$ to $\mathbb{C}$, $f,\ h \in C([a,\ b] \times \mathbb{C} \times \mathbb{C}, \mathbb{C})$ and $g \in C([a,\ b] \times [a,\ b], \mathbb{C})$ and g is the kernel of the Hammerstein IE.

From what we know, from the given literature review and that available from the international data bases, there is comprehensive research comprising books, papers, etc., with regard to the Ulam-type stabilities of numerous forms of the IDEs with and without time delays. When we check the same databases for Ulam-type stabilities of IEs with and without time delays, there is limited research on Ulam-type stabilities of IEs compared with the literature regarding IDEs. This case likely arises because of the difficulty of the topic for IEs. Meanwhile, Hammerstein IEs are known as particular kinds and classes of IEs.

In our information, we found only papers on Ulam-type stabilities of the Hammerstein IEs; see the papers of Castro and Simões [11], Ciplea et al. [13] and Ngoc et al. [17]. As it was noted above, the papers of Ciplea et al. [13] and Ngoc et al. [17] are related to the HUR stability for Volterra–Hammerstein functional IEs in several variables. Therefore, in a sense, we can say that Volterra–Hammerstein are functional IEs in several variables, and the results of Ciplea et al. [13] and Ngoc et al. [17] are not very related to the results of that obtained in this study. As for Ulam-type stabilities of Hammerstein IEs in two variables, to the best of our information, we only found the Hammerstein IE of Castro and Simões [11]. Truly, the IE of Benchohra [1], IE of Jung [15], IEs of Castro and Ramos ([9,10]) and Hammerstein IE of Castro and Simões [11] are particular cases of our Hammerstein IE (2).

Next, the IE of Benchohra [1], IE of Jung [15], IE of Castro and Ramos [9] and Hammerstein IE (1) of Castro and Simões [7] do not include any time delay. However, our IE, i.e., the Hammerstein IE (2), have two variable time delays and is more general than the IEs of Benchohra [1], Jung [15], Castro and Ramos [9] and Castro and Simões [11]. In addition, to our knowledge, there is no study concerning Ulam-type stabilities of Hammerstein IE including multiple time delays. Hence, the Ulam-type stabilities of Hammerstein IE deserve exploring. Therefore, our results improve or complement the results of Benchohra [1], Jung [15], Castro and Ramos [9] and Castro and Simões [11]. Additionally, the results of this study have new additives to the topic and content of this study.

The remainder of this study is regulated as noted below. The basic demarcations of HUR and HU stabilities of the Hammerstein IE (2), demarcation of the generalized metric and Banach FPT are arranged in Section 2. In Sections 3–5, the new contributions of this study as HUR and HU stabilities of the Hammerstein IE (2) in the finite interval case and the HUR stability of the Hammerstein IE (2) in the infinite intervals case are given by three new theorems, respectively. In Section 6, we clarify the new outcomes and improvements of this study via four items. As the final section, Section 7 consists of the conclusion of this study.

2. Fundamental Information

We now start with some fundamental information that is needed in the proofs of Ulam-type stabilities, i.e., the following definitions and theorem are useful to confirm HUR and HU stabilities of the delay in Hammerstein IE (2).

Let

$$Q = f(x, y(x), y(\tau(x))) \int_a^x g(x,\tau)h(\tau, y(\tau), y(\vartheta(\tau)))d\tau + p(x). \tag{3}$$

Definition 1. *If for each function y satisfying*

$$|y(x) - Q| \le \sigma(x), x \in [a,\ b],$$

where Q is same as Equation (3) and σ is a nonnegative function, there is a solution y_0 of the Hammerstein IE (2) and a $C \in \mathbb{R}, C > 0$, which is not dependent on y and y_0, such that

$$|y(x) - y_0(x)| \le C\sigma(x),$$

for all $x \in [a,\ b]$, then we state that the Hammerstein IE (2) admits the HUR stability.

Remark 1. *Letting $\sigma(x) = \theta$, $\theta \ge 0$, $\theta \in \mathbb{R}$ in Definition 1, we say that the Hammerstein IE (2) admits the HU stability.*

In truth, the FPM and the generalized Bielecki metric are well-known and very effective tools to study Ulam-type stabilities. The FPM involves certain fixed point results with regard to the usual metric and the generalized metric. Hence, we will now introduce the demarcation of the generalized metric on a set $X, X \neq \varnothing$.

Definition 2 (Castro and Simões [11], Diaz and Margolis [35]). *Let $X \neq \varnothing$. A function $d : X \times X \to [0, +\infty]$ is called a generalized metric on the set X if the function d satisfies the following three properties:*
(M1) $d(x, y) = 0 \Leftrightarrow x = y$;
(M2) $d(x, y) = d(y, x), \forall x,\ y \in X$;
(M3) $d(x, z) \leq d(x, y) + d(y, z), \forall x,\ y,\ z \in X$.

According to Definition 2, it is clear that the range of the generalized metric includes the infinity, "∞".

In the just-presented setting of the generalized metric, the following benefits from the well-known Banach FPT. This theorem is one of the main ideas upon which the desired results can be obtained.

Theorem 1 (Miahiet al. [36], Diaz and Margolis [35]). *Let (X, d) be a complete generalized metric space and $\Theta : X \to X$ be a strictly contractive mapping with the Lipschitz constant $L < 1$. Then, for each given element $x \in X$, either $d(\Theta^n x, \Theta^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0, such that*
(1^0) *$d(\Theta^k x, \Theta^{k+1} x) < \infty$ for all $n \geq n_0$;*
(2^0) *the sequence $\{\Theta^n x\}_{n \in \mathbb{N}}$ converges to a fixed point $y*$ of Θ;*
(3^0) *$y*$ is the unique fixed point of Θ in the set $\Delta = \{y \in X : d(\Theta^{n_0} x, y) < \infty\}$;*
(4^0) *$d(y, y*) \leq \frac{1}{1-L} d(y, \Theta y)$ for all $y \in \Delta$.*

It is better for the reader to state the following variant of the Banach FPT.

Theorem 2 (Baker [37]). *Suppose (Y, ρ) is a complete metric space and $T : Y \to Y$ is a contraction (for some $\lambda \in [0, 1)$, $\rho(T(x), T(y)) \leq \lambda \rho(x, y)$ for all $x, y \in Y$). Also suppose that $u \in Y$, $\delta > 0$, and*

$$\rho(u, T(u)) \leq \delta.$$

Then, there exists a unique $p \in Y$ such that $p = T(p)$. Moreover,

$$\rho(u, p) \leq \frac{\delta}{1 - \lambda}.$$

In this article, we consider the space continuous functions, which are represented by $C([a, b])$ on $[a, b]$, and endowed with a generalization of the Bielecki metric described by

$$d(u, v) = \sup_{x \in [a,b]} \frac{|u(x) - v(x)|}{\sigma(x)}, \tag{4}$$

where σ is a nondecreasing continuous function from $[a, b]$ to $(0, \infty)$. We also remember that $(C([a, b]), d)$ is a complete metric space (see, Castro and Simões [11]).

Let $C([.])$ and $C([.], [.])$ denote $C([a, b], [a, b])$ and $C([a, b])$ throughout the study, respectively.

3. The HUR Stability of the Hammerstein IE

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation, as well as the experimental conclusions that can be drawn.

We will now deal with the new hypotheses for the HUR stability of the Hammerstein IE (2) on a finite interval. The first HUR stability result of this paper with regard to the Hammerstein IE (2) is presented in the following Theorem 3.

Theorem 3. *Assume that the following conditions are satisfied:*

(As1) $\tau, \vartheta \in C([.],[.])$ *such that* $\tau(x) \le x$ *and* $\vartheta(x) \le x$, $\forall x \in [a,b]$, $\mu_1,\ \mu_2 \in C([.],\mathbb{R}^+)$, $\mathbb{R}^+ = [0,\infty)$.

(As2) $p \in C([a,\ b],\mathbb{C})$, $f \in C([a,\ b] \times \mathbb{C} \times \mathbb{C},\mathbb{C})$, $g \in C([a,\ b] \times [a,\ b],\mathbb{C})$, $M_0 = \max_{x\in[a,b]} |f(x,y(x),y(\tau(x)))|, y \in \mathbb{C}$.

Moreover, we assume that there are constants $\beta_1,\ \beta_2 \in [0,1)$, $\beta_1 + \beta_2 < 1$, *and functions* $\mu_1,\ \mu_2$ *from (As1) such that*

$$\int_a^x |g(x,\tau)|\mu_1(\tau)\sigma(\tau)d\tau \le \beta_1\sigma(x),$$

$$\int_a^x |g(x,\tau)|\mu_2(\tau)\sigma(\tau)d\tau \le \beta_2\sigma(x),$$

and $h \in C([a,\ b] \times \mathbb{C} \times \mathbb{C},\mathbb{C})$,

$$\begin{aligned}&|h(x,u(x),u(\vartheta(x))) - h(x,v(x),v(\vartheta(x)))|\\&\quad\le \mu_1(x)|u(x) - v(x)| + \mu_2(x)|u(\vartheta(x)) - v(\vartheta(x))|, \forall x \in [a,\ b], \forall u,v \in C([.]).\end{aligned}$$

If $y \in C([.])$ *is such that*

$$|y(x) - Q| \le \sigma(x), x \in [a,\ b],$$

where Q *is defined by* (3), σ *is a nonnegative function and*

$$M_0(\beta_1 + \beta_2) < 1,$$

then there exists a unique function $y_0 \in C([.])$ *so that*

$$y_0(x) = p(x) + f(x,y_0(x),y_0(\tau(x)))\int_a^x g(x,\tau)h(\tau,y_0(\tau),y_0(\vartheta(\tau)))d\tau$$

and

$$|y(x) - y_0(x)| \le \frac{\sigma(x)}{1 - M_0(\beta_1 + \beta_2)}, \forall x \in [a,\ b].$$

Hence, according the conditions above, we infer that the Hammerstein IE (2) *admits the HUR stability.*

Proof. We now keep in view the operator

$$T := C([.]) \to C([.]),$$

which is depicted by

$$\begin{aligned}(Tu)(x) =&f(x,u(x),u(\tau(x)))\int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau + p(x),\\&\forall t \in [a,b] \text{ and } \forall u \in C([.]).\end{aligned}$$

Utilizing the conditions of Theorem 3, we will now verify that the T is strictly contractive with regard to the metric (4). Assuming that $u,\ v\in C([.])$, we obtain

$$\begin{aligned}
|(Tu)(x)-(Tu)(x_0)| \le{}& |p(x)-p(x_0)| \\
&+|f(x,u(x),u(\tau(x)))\int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
&-f(x_0,u(x_0),u(\tau(x_0)))\int_a^{x_0} g(x_0,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau| \\
\le{}&|p(x)-p(x_0)| \\
&+M_0\left|\int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau-\int_a^{x_0} g(x_0,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau\right| \\
={}&|p(x)-p(x_0)| \\
&+M_0|\int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau-\int_a^x g(x_0,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
&+\int_a^x g(x_0,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau-\int_a^{x_0} g(x_0,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau| \\
\le{}&|p(x)-p(x_0)| \\
&+M_0\left(\int_a^x |g(x,\tau)-g(x_0,\tau)|\ |h(\tau,u(\tau),u(\vartheta(\tau)))|d\tau\right) \\
&+M_0\left|\int_{x_0}^x g(x_0,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau\right|\to 0
\end{aligned}$$

when $x\to x_0$.

For the coming action, we will show that the running conditions of Theorem 3 allow that T is strictly contractive in accordance with the metric (4). Moreover, using (As1) and (As2) of Theorem 3, for all $u,v\in C([.])$, we deduce that

$$\begin{aligned}
d(Tu,Tv)={}&\sup_{x\in[a,b]}\frac{|(Tu)(x)-(Tv)(x)|}{\sigma(x)} \\
={}&\sup_{x\in[a,b]}\frac{1}{\sigma(x)}\Bigg|f(x,u(x),u(\tau(x)))\int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
&-f(x,v(x),v(\tau(x)))\int_a^x g(x,\tau)h(\tau,v(\tau),v(\vartheta(\tau)))d\tau\Bigg| \\
\le{}&M_0\sup_{x\in[a,b]}\frac{1}{\sigma(x)}\int_a^x |g(x,\tau)|\ |h(\tau,u(\tau),u(\vartheta(\tau)))-h(\tau,v(\tau),v(\vartheta(\tau)))|d\tau \\
\le{}&M_0\sup_{x\in[a,b]}\frac{1}{\sigma(x)}\int_a^x |g(x,\tau)|\mu_1(t)|u(\tau)-v(\tau)|d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ M_0 \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int\limits_a^x |g(x,\tau)|\mu_2(t)|u(\vartheta(\tau)) - v(\vartheta(\tau))|d\tau \\
=&M_0 \sup_{x\in[a,b]} \frac{\int\limits_a^x |g(x,\tau)|\mu_1(t)\sigma(\tau)\frac{|u(\tau)-v(\tau)|}{\sigma(\tau)}d\tau}{\sigma(x)} \\
&+ M_0 \sup_{x\in[a,b]} \frac{\int\limits_a^x |g(x,\tau)|\mu_1(t)\sigma(\vartheta(\tau))\frac{|u(\vartheta(\tau))-v(\vartheta(\tau))|}{\sigma(\vartheta(\tau))}d\tau}{\sigma(x)} \\
\leq&M_0 \sup_{\tau\in[a,b]} \frac{|u(\tau)-v(\tau)|}{\sigma(\tau)} \sup_{x\in[a,b]} \frac{\int\limits_a^x |g(x,\tau)|\mu_1(t)\sigma(\tau)d\tau}{\sigma(x)} \\
&+ M_0 \sup_{\tau\in[a,b]} \frac{|u(\vartheta(\tau))-v(\vartheta(\tau))|}{\sigma(\vartheta(\tau))} \sup_{x\in[a,b]} \frac{\int\limits_a^x |g(x,\tau)|\mu_1(t)\sigma(\vartheta(\tau))d\tau}{\sigma(x)} \\
\leq&\beta_1 M_0 \sup_{\tau\in[a,b]} \frac{|u(\tau)-v(\tau)|}{\sigma(\tau)} + \beta_2 M_0 \sup_{\tau\in[a,b]} \frac{|u(\tau)-v(\tau)|}{\sigma(\tau)} \\
\leq&(\beta_1+\beta_2)M_0 d(u,v).
\end{aligned}
$$

Next, the condition $(\beta_1+\beta_2)M_0 < 1$ implies that T is strictly contractive. Utilizing the Banach FPT, we have

$$d(y,y_0) \leq \frac{1}{1-M_0(\beta_1+\beta_2)} d(Ty,y),$$

which implies that the Hammerstein IE (2) has the HUR.

Next, we also have

$$\sup_{x\in[a,b]} \frac{|y(x)-y_0(x)|}{\sigma(x)} \leq \frac{1}{1-M_0(\beta_1+\beta_2)}.$$

Hence,

$$\frac{|y(x)-y_0(x)|}{\sigma(x)} \leq \frac{1}{1-M_0(\beta_1+\beta_2)}$$

and consequently the idea of Theorem 3 holds. This outcome completes the proof. □

4. The HU Stability of the Hammerstein IE

In this part, we will construct new conditions in relation to the HU stability of the Hammerstein IE (2). Here, as in Section 3, we will use the non-decreasing continuous function σ, which is defined from $[a,b]$ to $(0,\infty)$ and continues to benefit from the Bielecki metric of (4).

Theorem 4. *Assume that (As1) and (As2) hold. Moreover, if* $\in C([.])$ *is such that*

$$|y(x) - Q| \leq \theta, x \in [a,\ b],$$

where Q is defined by (3), $\theta \geq 0$ *and*

$$M_0(\beta_1+\beta_2) < 1,$$

then there is a unique function $y_0 \in C([.])$ *so that*

$$y_0(x) = p(x) + f(x, y_0(x), y_0(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_0(\tau), y_0(\vartheta(\tau)))d\tau$$

and

$$|y(x) - y_0(x)| \leq \frac{\theta}{1 - M_0(\beta_1 + \beta_2)}, \forall x \in [a,\ b].$$

Hence, according the conditions above, we infer that the Hammerstein IE (2) *admits the HU stability.*

Proof. We now start with the operator

$$T := C([.]) \to C([.]),$$

which is depicted by

$$(Tu)(x) = f(x, u(x), u(\tau(x))) \int_a^x g(x,\tau)h(\tau, u(\tau), u(\vartheta(\tau)))d\tau + p(x),$$

$$\forall t \in [a,b] \text{ and } \forall u \in C([.]).$$

Depending on the conditions of Theorem 4, we will now prove that the operator T is strictly contractive with regard to the metric (4). Letting $u,\ v \in C([.])$, we obtain

$$\begin{aligned}
d(Tu, Tv) = \sup_{x\in[a,b]} & \frac{|(Tu)(x) - (Tv)(x)|}{\sigma(x)} \\
\leq & M_0 \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x |g(x,\tau)|\ |h(\tau, u(\tau), u(\vartheta(\tau))) - h(\tau, v(\tau), v(\vartheta(\tau)))|d\tau \\
\leq & M_0 \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x |g(x,\tau)|\mu_1(t)|u(\tau) - v(\tau)|d\tau \\
& + M_0 \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x |g(x,\tau)|\mu_2(t)|u(\vartheta(\tau)) - v(\vartheta(\tau))|d\tau \\
= & M_0 \sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)|\mu_1(t)\sigma(\tau)\frac{|u(\tau)-v(\tau)|}{\sigma(\tau)}d\tau}{\sigma(x)} \\
& + M_0 \sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)|\mu_1(t)\sigma(\vartheta(\tau))\frac{|u(\vartheta(\tau))-v(\vartheta(\tau))|}{\sigma(\vartheta(\tau))}d\tau}{\sigma(x)} \\
\leq & M_0 \sup_{\tau\in[a,b]} \frac{|u(\tau) - v(\tau)|}{\sigma(\tau)} \sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)|\mu_1(t)\sigma(\tau)d\tau}{\sigma(x)}
\end{aligned}$$

$$\begin{aligned}&+M_0\sup_{\tau\in[a,b]}\frac{|u(\vartheta(\tau))-v(\vartheta(\tau))|}{\sigma(\vartheta(\tau))}\sup_{x\in[a,b]}\frac{\int_a^x|g(x,\tau)|\mu_1(t)\sigma(\vartheta(\tau))d\tau}{\sigma(x)}\\ \le&\beta_1M_0\sup_{\tau\in[a,b]}\frac{|u(\tau)-v(\tau)|}{\sigma(\tau)}+\beta_2M_0\sup_{\tau\in[a,b]}\frac{|u(\tau)-v(\tau)|}{\sigma(\tau)}\\ =&(\beta_1+\beta_2)M_0d(u,v),\end{aligned}$$

i.e., it is clear that

$$d(Tu,Tv)\le(\beta_1+\beta_2)M_0d(u,v).$$

Hence, in an analogous way to the proof of Theorem 3, we can apply the Banach FPT of this study, which ensures the HU stability for the Hammerstein IE. Indeed, due to the fact that $(\beta_1+\beta_2)M_0<1$, it follows from the results obtained above that T is strictly contractive and the idea of Theorem 4 holds. This is the final step of the proof. □

5. The HUR Stability of the Hammerstein IE in the Infinite Interval

In Section 5, we will study the HUR stability of the Hammerstein IE in infinite intervals. Therefore, we will now take into account corresponding intervals $[a,\infty)$, $(-\infty,b]$, for some fixed $a,b\in\mathbb{R}$, and $\mathbb{R}=(-\infty,\infty)$. Hence, we will consider the Hammerstein IE:

$$y(x)=f(x,y(x),y(\tau(x)))\int_a^x g(x,\tau)h(\tau,y(\tau),y(\vartheta(\tau)))d\tau+p(x),x\in[a,\ \infty),\tag{5}$$

where a is a fixed number, $\tau,\vartheta\in C([a,\ \infty),[a,\ \infty))$, $p\in C([a,\ \infty),\mathbb{C})$ and $f\in C([a,\infty)\times\mathbb{C}\times\mathbb{C},\mathbb{C})$ are bounded functions, $h\in C([a,\ \infty)\times\mathbb{C}\times\mathbb{C},\mathbb{C})$ and $g\in C([a,\ \infty)\times[a,\ \infty),\mathbb{C})$, and g is the kernel of (5).

Consider a fixed non-decreasing continuous function φ from $[a,\ \infty)$ to $(\varepsilon,\ \omega)$, for some $\varepsilon>0,\omega>0$. Let $C^b([a,\infty))$ be the space of bounded continuous functions endowed with the weight metric

$$d^b(u,v)=\sup_{t\in[a,\infty)}\frac{|u(x)-v(x)|}{\varphi(x)}.\tag{6}$$

The final new result of this study with regard to the HUR stability of the Hammerstein IE (5) according to the infinite intervals is introduced in the following theorem.

Theorem 5. *Suppose the following conditions are held:*

(As3) $\tau,\vartheta\in C([a,\ \infty),[a,\ \infty))$ *such that* $\tau(x)\le x$ *and* $\vartheta(x)\le x,\forall x\in[a,\ \infty)$, $\mu_1,\ \mu_2\in C([a,\infty),\mathbb{R}^+),\mathbb{R}^+=[0,\infty)$.

(As4) $p\in C([a,\ \infty),\mathbb{C})$ *and* p *is a bounded function,*

$$f\in C([a,\ \infty)\times\mathbb{C}\times\mathbb{C},\mathbb{C}),g\in C([a,\ \infty)\times[a,\ \infty),\mathbb{C}),$$

$$M_0=\max_{x\in[a,\ \infty)}|f(x,y(x),y(\tau(x)))|,y\in\mathbb{C},$$

and

$$\int_a^x g(x,\tau)h(\tau,z(\tau),z(\vartheta(\tau)))d\tau$$

is a bounded continuous function for any bounded continuous function z.

Furthermore, we assume that there are constants $\beta_1,\ \beta_2\in[0,1),\beta_1+\beta_2<1$, *and functions* $\mu_1,\ \mu_2$ *from (As3), such that*

$$\int_a^x|g(x,\tau)|\mu_1(\tau)\varphi(\tau)d\tau\le\beta_1\varphi(x),$$

$$\int_a^x |g(x,\tau)|\mu_2(\tau)\varphi(\tau)d\tau \le \beta_2\varphi(x),$$

and the function

$$h \in C([a,\ \infty) \times \mathbb{C} \times \mathbb{C}, \mathbb{C})$$

satisfies the inequality

$$\begin{aligned} &|h(x,u(x),u(\vartheta(x))) - h(x,v(x),v(\vartheta(x)))| \\ &\quad \le \mu_1(x)|u(x) - v(x)| + \mu_2(x)|u(\vartheta(x)) - v(\vartheta(x))|, \\ &\quad \forall x \in [a,\infty), \forall u,v \in C^b([a,\infty)). \end{aligned}$$

Let $y \in C^b([a,\ \infty))$ *with*

$$|y(x) - Q| \le \varphi(x), x \in [a,\ \infty),$$

where Q is defined by (3)*, the function* φ *is defined above and*

$$M_0(\beta_1 + \beta_2) < 1.$$

Then, there is a unique function $y_0 \in C_b([a,\ \infty))$*, such that*

$$y_0(x) = f(x,y_0(x),y_0(\tau(x))) \int_a^x g(x,\tau)h(\tau,y_0(\tau),y_0(\vartheta(\tau)))d\tau + p(x)$$

and

$$|y(x) - y_0(x)| \le \frac{\varphi(x)}{1 - M_0(\beta_1 + \beta_2)}, \forall x \in [a,\ \infty).$$

According to these outcomes, depending upon the conditions above, the Hammerstein IE (5) *has the stability in the sense of the HUR.*

Proof. For any $n \in \mathbb{N}$, we define $I_n = [a,\ a+n]$. By virtue of Theorem 3, there exists a unique function $y_{0,n} \in C[I_n, \mathbb{C}]$, which is bounded, such that

$$y_{0,n}(x) = p(x) + f(x,y_{0,n}(x),y_{0,n}(\tau(x))) \int_a^x g(x,\tau)h(\tau,y_{0,n}(\tau),y_{0,n}(\vartheta(\tau)))d\tau$$

and

$$|y(x) - y_{0,n}(x)| \le \frac{\varphi(x)}{1 - M_0(\beta_1 + \beta_2)},$$

for all $x \in I_n$. If $x \in I_n$, then the uniqueness of $y_{0,n}$ implies that

$$y_{0,n}(x) = y_{0,n+1}(x) = y_{0,n+2}(x) = \dots.$$

For any $x \in [a,\infty)$, we define $n(x) \in \mathbb{N}$ by $n(x) = \min\{n \in \mathbb{N} : x \in I_n\}$. Next, we will also define a function $y_0 : [a,\infty) \to \mathbb{C}$ by

$$y_0(x) = y_{0,n(x)}(x).$$

Hence, we claim that y_0 is continuous. Following similar mathematical calculations as in Castro and Simões ([11], Theorem 5), it can be shown that $y_0(x) = y_{0,n_1+1}(x)$ for all $x \in (x_1 - \varepsilon,\ x_1 + \varepsilon)$ and since y_{0,n_1+1} is continuous at x_1, (see, Theorem 3), y_0 is also continuous at $x_1,\ x_1 \in \mathbb{R}$.

We will now prove that y_0 is a solution of

$$y_0(x) = f(x, y_0(x), y_0(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_0(\tau), y_0(\vartheta(\tau)))d\tau + p(x)$$

and

$$|y(x) - y_0(x)| \leq \frac{1}{1 - M_0(\beta_1 + \beta_2)}\varphi(x),$$

for all $x \in [a,\ \infty)$.

For an arbitrary $x \in [a,\ \infty)$, we choose $n(x)$ such that $x \in I_{n(x)}$.

Then, in light of the results above, we have

$$\begin{aligned} y_0(x) =& y_{0,n(x)}(x) \\ =& f(x, y_{0,n(x)}(x), y_{0,n(x)}(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_{0,n(x)}(\tau), y_{0,n(x)}(\vartheta(\tau)))d\tau + p(x) \\ =& f(x, y_0(x), y_0(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_0(\tau), y_0(\vartheta(\tau)))d\tau + p(x). \end{aligned}$$

Since $n(s) \ \leq \ n(x)$ for all $x \in I_{n(x)}$, we derive that

$$y_0(x) = y_{0,n(x)}(x) = y_{0,n(x)}(s).$$

Hence, the equality

$$y_0(x) = f(x, y_0(x), y_0(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_0(\tau), y_0(\vartheta(\tau)))d\tau + p(x).$$

holds true. Thus, according to (As3), (As4) and (6), it follows that

$$|y(x) - y_0(x)| = \left|y(x) - y_{0,n(x)}(x)\right| \leq \frac{1}{1 - M_0(\beta_1 + \beta_2)}\varphi(x),$$

for all $x \in [a, \infty)$.

At the end, we have to prove that y_0 is unique. Let y_1 be another continuous and bounded function, such that it satisfies

$$|y(x) - y_0(x)| \leq \frac{\varphi(x)}{1 - M_0(\beta_1 + \beta_2)}, \forall x \in [a,\ \infty)$$

and the previous equality of y_0, for all $x \in [a, \infty)$. By the uniqueness of the solution on $I_{n(x)}$ for any $n(x) \in \mathbb{N}$, we have $y_0(x) = y_0|_{I_{n(x)}}(x)$, and $y_1(x)$ satisfies the last inequality and the previous equality of $y_0, \forall x \in I_{n(x)}$. Hence, we conclude that

$$y_0(x) = y_0|_{I_{n(x)}}(x) = y_1|_{I_{n(x)}}(x) = y_1(x).$$

This is the end of the proof. $\square$

Remark 2. *We can also proceed the proof similarly for the cases* $I = (-\infty, b], b \in \mathbb{R}$*, and* $\mathbb{R} = (-\infty, \infty)$.

Example 1 (Castro and Simões [11]). *As a particular case of the Hammerstein IE (2), we consider the following IE:*

$$y(x) = \frac{1}{2}xy(x)\int_{1}^{x}\frac{1}{x+\tau}\frac{y(\tau)}{\tau^2}d\tau + x - \frac{x}{2}\ln\left(\frac{x+1}{2}\right). \tag{7}$$

The exact solution of the IE (7) is $y_0(x) = x$.
Next, let

$$Q = \frac{1}{2}xy(x)\int_{1}^{x}\frac{1}{x+\tau}\frac{y(\tau)}{\tau^2}d\tau + x - \frac{x}{2}\ln\left(\frac{x+1}{2}\right), x \in [1,2].$$

When comparing the IEs (2) and (7) and providing some related mathematical calculations, we obtain the following estimates:

$$[a,\ b] = [1,2], x,\ \tau \in [1,2];$$

$$f(x, y(x), y(\tau(x))) = \frac{1}{2}xy(x), \tau(x) = 0,$$

$$f \in C([1,2] \times \mathbb{C}, \mathbb{C}),$$

$$\max_{x\in[a,b]} |f(x, y(x), y(\tau(x)))| = \frac{1}{2}\max_{x\in[1,2]} |xy(x)| = 2 = M_0;$$

$$g(x,\tau) = \frac{1}{x+\tau}, g \in C([1,\ 2] \times [1,\ 2], \mathbb{C}).$$

Moreover, we assume that there are $\beta_1,\ \beta_2 \in [0,1)$, $\beta_1 + \beta_2 < 1$, such that

$$\int_{a}^{x} |g(x,\tau)|\mu_1(\tau)\sigma(\tau)d\tau = \int_{1}^{x}\frac{1}{x+\tau}\frac{1}{\tau}\tau d\tau$$
$$= \ln\left(\frac{2x}{x+1}\right) \le \ln\left(\frac{4}{3}\right)x = \beta_1\sigma(x),$$

where $\mu_1(x) = x$, such that $\mu_1 \in C([1,2],[0,\infty))$, $0 < \beta_1 = \ln\left(\frac{4}{3}\right) < 1$, $\sigma(x) = x$, and such that $\sigma \in C([1,2],(0,\infty))$, $\mu_2(x) = 0$ and $\beta_2 = 0$;

$$h(\tau, y(\tau), y(\vartheta(\tau))) = \frac{y(\tau)}{\tau^2}, \vartheta(\tau) = 0,$$

$$h \in C([1,\ 2] \times \mathbb{C}, \mathbb{C}),$$

$$|h(x, u(x), u(\vartheta(x))) - h(x, v(x), v(\vartheta(x)))|$$
$$= \left|\frac{u(x)}{x^2} - \frac{v(x)}{x^2}\right| = \frac{1}{x^2}|u(x) - v(x)|$$
$$\le \frac{1}{x}|u(x) - v(x)|, \forall x \in [1,2], \mu_1(x) = x;$$

$$p(x) = x - \frac{x}{2}\ln\left(\frac{x+1}{2}\right), p \in C([1,\ 2], \mathbb{C}).$$

Letting $y(x) = (0.3)^{-1}x$, we derive that

$$|y(x) - Q| = \left|\frac{x}{0.3} - x + \frac{x}{2}\ln\left(\frac{x+1}{2}\right) - \frac{1}{2(0.3)^2}x^2\int_{1}^{x}\frac{1}{x+\tau}\frac{1}{\tau}d\tau\right|$$

$$= \left|\frac{17}{6} - \frac{50}{9}\ln\left(\frac{x+1}{2}\right)\right|x \leq \left|\frac{17}{6} - \frac{50}{9}\ln\frac{3}{3}\right|x$$
$$\leq x = \sigma(x), \forall x \in [1,2].$$

It is also clear that $M_0\beta_1 = 2 \times \ln\frac{4}{3} \cong 0.5753641449 < 1$. *Hence, using this value,* $y_0(x) = x$ *and* $y(x) = (0.3)^{-1}x$, *we have*

$$|y(x) - y_0(x)| = \left|\frac{10}{3}x - x\right| = \frac{7}{3}x \leq \frac{x}{1 - 2\times\ln\frac{4}{3}} = \frac{\sigma(x)}{1 - M_0\beta_1}, x \in [1,2].$$

Since $x \in [1,2]$, *obviously, we also have*

$$|y(x) - y_0(x)| \leq \frac{x}{1 - 2\times\ln\frac{4}{3}} = \frac{2}{1 - M_0\beta_1} = \frac{\theta}{1 - M_0\beta_1}, \ \theta = 2.$$

Hence, the IE (7) admits HUR and HU stabilities in the finite interval case.

6. Discussion

We will now explain the new results of this study shortly.

(1^0) To our knowledge, according to the data from the literature, we did not find any result in relation to HUR and HU stabilities of nonlinear delay Hammerstein IEs. This paper is an initial work with regard to HUR and HU stabilities of nonlinear delay Hammerstein IEs. Hence, this study is a new contribution to the topic of this study.

(2^0) In this study, we established three new theorems in relation to HUR and HU stabilities of a nonlinear delay Hammerstein IE on certain finite or infinite intervals, applying the Banach FPT, the generalized and the weight Bielecki metrics. The findings of this work have essential improvements and contributions from the Hammerstein IEs without delay to the Hammerstein IEs, including variable time delays.

(3^0) In the relevant literature, HUR and HU stabilities of certain linear and nonlinear IEs without and with delays can be investigated using the Banach FPT and generalized and weight Bielecki metrics, etc., as basic and powerful tools in the study. As for possible future suggestions that would benefit from the Banach FPT and generalized and weight Bielecki metrics, etc., as basic tools, the HU stability, generalized HU stability, HUR stability, generalized HUR stability and some other Ulam-type stabilities for the Caputo fractional order Hammerstein IEs with two or three variables can be considered as open problems. For the sake of brevity, we will not provide proper fractional mathematical models here.

(4^0) In a particular case of the nonlinear delay Hammerstein IE of this work, a numerical example was provided to justify the applications of the new theorems.

7. Conclusions

In this study, the nonlinear Hammerstein IE, including variable time delays, was considered. We put together new sufficient conditions with regard to the HUR stability of the Hammerstein IE on the finite and infinite interval and the HU stability of the same IE on a finite interval. The outcomes of this study were arranged throughout three new theorems. The proofs of that theorem were accomplished using the Banach FPT and the generalized or the weight Bielecki metric. An example was presented to verify the numerical applications of the result. The findings of this study are new and provide new improvements for Ulam stabilities of Hammerstein type IEs with delays.

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Abbreviations

IE	Integral equation
Hammerstein IE	Hammerstein integral equation
HU stability	Hyers–Ulam stability
HUR stability	Hyers–Ulam–Rassias stability
Banach's FPT	Banach's fixed point theorem
FPM	Fixed point method
σ-semi HU stability	σ-semi Hyers–Ulam stability
IDEs	Integro-differential equations

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