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# Total Perfect Roman Domination

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**Abstract:** A total perfect Roman dominating function (TPRDF) on a graph $G = (V, E)$ is a function $f$ from $V$ to $\{0, 1, 2\}$ satisfying (i) every vertex $v$ with $f(v) = 0$ is a neighbor of exactly one vertex $u$ with $f(u) = 2$; in addition, (ii) the subgraph of $G$ that is induced by the vertices with nonzero weight has no isolated vertex. The weight of a TPRDF $f$ is $\sum_{v\in V} f(v)$. The total perfect Roman domination number of $G$, denoted by $\gamma^p_{tR}(G)$, is the minimum weight of a TPRDF on $G$. In this paper, we initiated the study of total perfect Roman domination. We characterized graphs with the largest-possible $\gamma^p_{tR}(G)$. We proved that total perfect Roman domination is NP-complete for chordal graphs, bipartite graphs, and for planar bipartite graphs. Finally, we related $\gamma^p_{tR}(G)$ to perfect domination $\gamma^p(G)$ by proving $\gamma^p_{tR}(G) \leq 3\gamma^p(G)$ for every graph $G$, and we characterized trees $T$ of order $n \geq 3$ for which $\gamma^p_{tR}(T) = 3\gamma^p(T)$. This notion can be utilized to develop a defensive strategy with some properties.

**Keywords:** vertex domination; total Roman domination; perfect Roman domination; NP-completeness

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**Citation:** Almulhim, A. Total Perfect Roman Domination. *Symmetry* **2023**, *15*, 1676. https://doi.org/10.3390/sym15091676

Academic Editors: Abel Cabrera Martínez and Alejandro Estrada-Moreno

Received: 4 August 2023
Revised: 24 August 2023
Accepted: 25 August 2023
Published: 31 August 2023



## 1. Introduction

Roman domination was sparked by defensive measures taken to defend the Roman dynasty. Constantine, who was born in 272 and died in 337 AD, decreed that every city in the empire should be stationed by at most two legions. Furthermore, every city without a legion must be close to a city with two armies, so if a city with no army was attacked, then the city with two armies could send one of its armies to the city under attack. Stewart [1] and ReVelle and Rosing [2,3] discussed Roman domination as a mathematical concept, which was developed later by Cockayne et al. [4]. It is clear that the above defending strategy is not enough to protect the empire if more than one attack occurs at the same time. This raised the need to a stronger and more-efficient defending strategy, and it motivated researchers to introduce and investigate different variants of Roman domination. Since then, over 100 papers on Roman domination and its variants have been published. Perfect Roman domination [5,6], Italian domination [7,8], perfect Italian domination [9], double-Roman domination [10], perfect double-Roman domination [11], double-Italian domination [12], total Roman domination [13,14], vertex–edge Roman domination [15], and vertex–edge perfect Roman domination [16] are a few examples of Roman domination variants.

This work was motivated by two previously introduced variants, namely perfect Roman domination and total Roman domination. Our variant, total perfect Roman domination, combines those two variants, and it gives a stronger defending strategy than perfect Roman domination does with less than the expected extra cost. Total perfect Roman domination gives extra security compared to what perfect Roman domination does by requiring every vertex (city) to be secured by a neighbor with a legion, so if multiple attacks occur on the city, then its neighbor can send a legion to it.

All graphs in this work are finite, simple, and undirected. We say that the vertex $v$ is a *neighbor* of a vertex $u$ or $v$ and $u$ are *adjacent* if $vu \in E$. For $v \in V(G)$ and a subset $X \subseteq V(G)$, we denote the set of all edges $vx$ with $x \in X$ by $E(v, X)$. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N(v) := \{u \in V | vu \in E(G)\}$, and the *closed neighborhood* of $v$ is the set $N[v] := N(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $|N(v)|$, and it is denoted by $d_G(v)$ or $d(v)$ if $G$ is known. The *maximum degree* of $G$ is $\Delta(G) := \max_{v\in V(G)} d(v)$. The *minimum degree* of $G$ is denoted by $\delta(G)$ and defined as the minimum degree of a vertex in $G$, i.e., $\delta(G) = \min_{v \in V(G)} \mathrm{d}(v)$. The *length* of a path $P$ is $|E(P)|$. A graph is *connected* if, for any two vertices in the graph, there exists a path between them. The length of a shortest path between two vertices $u$ and $v$ in a connected graph $G$ is the *distance* between them, and it is denoted by $\mathrm{dist}_G(u, v)$. Let $G$ be a connected graph. The *diameter* of $G$ is $\mathrm{diam}(G) := \max\{\mathrm{dist}(u, v) | (u, v) \in V(G) \times V(G)\}$. A path $P$ in $G$ is called a *diametral path* if it is a shortest path between its ends and the length of $P$ is equal to $\mathrm{diam}(G)$.

A vertex $v \in V(G)$ is a *leaf* if it is a neighbor for exactly one vertex in $G$. A vertex in $G$ is called a *support* vertex if it is adjacent to a leaf. A vertex in $G$ is called a *strong support* vertex if it is a support vertex and adjacent to at least two leaves, and it is called a *weak support* vertex if it is a support vertex and adjacent to exactly one leaf. An edge $uv$ is a *pendant edge* if $u$ or $v$ is a leaf. An *isolated vertex* vertex $v$ is a vertex with $\mathrm{d}(v) = 0$. A cycle with $n$ edges is denoted by $C^n$. A graph is called a *star* if it is connected and contains exactly one non-leaf vertex. A *subdivided star* is a graph obtained from a star graph by subdividing each edge once. A *double-star* is a connected graph with exactly two vertices with degree greater than one (those two vertices are necessarily adjacent). The graph that is obtained from a graph $G$ by adding a pendant edge to each vertex in $G$ is called the *corona* of $G$, and it is denoted by $\mathrm{corona}(G)$. A cycle $C$ in a graph $G$ has a *chord* if there is an edge joining non-consecutive vertices of $C$. A graph $G$ is *chordal* if every cycle of length four or more has a chord. A subset $A \subseteq V(G)$ is called a *clique* if, for every $v, u \in A$, $vu \in E(G)$. The *chromatic number* of a graph $G$ is the minimum number of colors needed to color the vertices of $G$ such that no adjacent vertices receive the same color. A graph $G$ is called *perfect* if the chromatic number equals the cardinality of a maximum clique for every induced subgraph of $G$. A subset $A \subseteq V(G)$ is called *packing* if $\mathrm{dist}_G(u, v) \geq 3$ for any distinct vertices $u, v \in A$.

Any function $f$ from $V(G)$ to $\{0, 1, 2\}$ is represented by the ordered partition $(V_0^f, V_1^f, V_2^f)$, where $V_j^f = \{u \in V | f(u) = j\}$, $j \in \{0, 1, 2\}$. We sometimes omit $f$ and write $(V_0, V_1, V_2)$ if $f$ is known from the context. The *weight* $w(f)$ of the function $f$ is the sum $\sum_{v \in V(G)} f(v)$. If $H$ is a subgraph of $G$, we denote the sum $\sum_{v \in V(H)} f(v)$ by $f(H)$, and it is called the *restriction of $f$ on $H$*.

A function $f : V \longrightarrow \{0, 1, 2\}$ is a *Roman dominating function* on $G$, abbreviated as the RD-function, if for every $v \in V_0$, $N(v) \cap V_2 \neq \emptyset$. The *Roman domination number* of $G$ is $\gamma_R(G) := \min\{w(f) | f \text{ is an RD-function on} G\}$.

A set $B \subseteq V(G)$ is called a *perfect dominating set*, abbreviated as PDS, of $G$, if for every $v \in V(G) \backslash B$, $|N(v) \cap B| = 1$. The *perfect domination number* of $G$ is $\gamma^p(G) := \min\{|B| \mid B \text{ is a PDS of } G\}$. A PDS $S$ of $G$ with $|S| = \gamma^p(G)$ is denoted by the $\gamma^p(G)$-set. Perfect domination was investigated under a variety of terminology. Perhaps Biggs [17] was the first one who studied perfect domination in graphs, which Biggs called *perfect code*.

A function $f$ from $V(G)$ to $\{0, 1, 2\}$ is a *perfect Roman dominating function* on $G$, abbreviated as the PRD-function, if for all vertices $v \in V_0$, $|N(v) \cap V_2| = 1$. The *perfect Roman domination number* of $G$, introduced in [5], is $\gamma_R^p(G) := \min\{w(f) | f \text{ is a PRD-function on} G\}$. We refer the reader to [18,19] for recent work on perfect Roman domination.

A function $f$ from $V(G)$ to $\{0, 1, 2\}$ is a *total Roman dominating function* on $G$, abbreviated as the TRD-function, if for all $v \in V_0$, $N(v) \cap V_2 \neq \emptyset$ and $\delta(G[V_1 \cup V_2]) \geq 1$. The *total Roman domination number* of $G$ is $\gamma_{tR}(G) := \min\{w(f) | f \text{ is a TRD-function on } G\}$. Total Roman domination was introduced in [13] as a special case of the more general setting introduced in [20]. For recent work on total Roman domination, we refer the reader to [21–23].

**Definition 1.** *Let $G = (V, E)$ be a graph with no vertex $v$ with $\mathrm{d}(v) = 0$. A function $f : V \longrightarrow \{0, 1, 2\}$ is a total perfect Roman dominating function, abbreviated as TPRDF, if every vertex $v \in V_0$ is adjacent to exactly one neighbor $u \in V_2$ and the induced subgraph $G[V_1 \cup V_2]$ has no vertex $v$* with $\mathrm{d}(v) = 0$*. The total perfect Roman domination number of $G$ is* $\gamma_{tR}^p(G) := \min\{w(f) | f \text{ is a TPRDF on} G\}$.

This paper includes several symmetrical graphs; some symmetrical graphs, such as paths, cycles, and some corona graphs, attain the largest possible total perfect Roman domination number. Other symmetrical graphs, namely the complete multipartite graphs with a big enough number of vertices, are used to show that the inequality $\gamma_{tR}(G) < \gamma_R^p(G) < \gamma_{tR}^p(G)$ is valid for some graphs. Due to the importance of symmetrical graphs in practical problems, we believe that this work will attract researchers who investigate different graph parameters on symmetrical graphs. See, for example, [24,25].

In Figure 1, the Roman Empire regions are shown along with two labeling functions, one in blue and the other in red. Formally, let $f$ be the map defined by setting $f(Britain) = f(Gaul) = f(Rome) = f(North\ Africa) = f(Constantinople) = f(Egypt) = 0$ and $f(Iberia) = f(Asia\ Minor) = 2$; let $g$ be the map defined by setting $g(Britain) = g(Iberia) = g(North\ Africa) = g(Egypt) = g(Asia\ Minor) = 0$, $g(Rome) = 1$ and $g(Constantinople) = g(Gaul) = 2$. Then $f$ is a RD-function, which is also an PRD-function, but it is not a TPRDF. The map $g$ is a TPRDF.

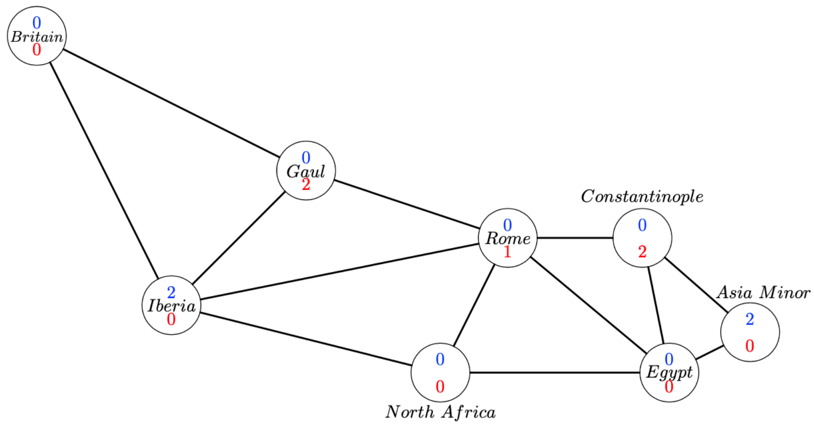


**Figure 1.** Roman, perfect Roman, and total perfect Roman dominating functions for the Roman Empire.

It is clear that $\gamma_{tR}^p(G) \leq n$ for any graph $G$ of order $n$, as the function that assigns a value of 1 to every vertex in $G$ is a TPRDF on $G$. It is also clear that every TPRDF on $G$ is a PRD-function and a TRD-function.

**Proposition 1.** *Let G be a graph. Then:*

(1) $\gamma_R^p(G) \leq \gamma_{tR}^p(G)$;
(2) $\gamma_{tR}(G) \leq \gamma_{tR}^p(G)$.

If $G = C^n$, where $n = 3k$ and $k \in \mathbb{Z}^+$, then $\gamma_R^p(C^n) = \lceil \frac{2n}{3} \rceil = \frac{2n}{3}$ [26], and $\gamma_{tR}(C^n) = n$ [13]. As $\gamma_{tR}(C^n) \leq \gamma_{tR}^p(C^n) \leq n$, $\gamma_{tR}^p(C^n) = n$. Thus, $\gamma_R^p(C^n) < \gamma_{tR}^p(C^n)$.

We will give an example of a graph $G$ such that $\gamma_{tR}(G) < \gamma_R^p(G) < \gamma_{tR}^p(G)$, but we first need the following proposition.

Let $K_{m_1,m_2,\cdots,m_r}$ be the complete $r$-partite graph with parts $A_1, A_2, \cdots, A_r$. Fix the notation so that $m_1 \leq m_2 \leq \cdots \leq m_r$.

**Proposition 2.** *Let* $G = K_{m_1,m_2,\cdots,m_r}$ *and* $r \geq 2$.

(1) *If* $m_1 = 1$, *then* $\gamma_{tR}^p(K_{m_1,m_2,\cdots,m_r}) = 2$ *when* $n = 2$ *and* $\gamma_{tR}^p(K_{m_1,m_2,\cdots,m_r}) = 3$ *otherwise.*
(2) *If* $m_1 \geq 2$, *then*

$$\gamma_{tR}^p(G) = \begin{cases} 4, & \text{if } r = 2, \\ m_1 + 2, & \text{if } r \geq 3. \end{cases}$$

**Proof.** (1) If $n = 2$, then $G = P_2$ and $\gamma^p_{tR}(P_2) = 2$. Otherwise, assign 2 to the unique vertex in $A_1$, assign 1 to one vertex in $A_2$, and assign 0 to the other vertices. Therefore, $\gamma^p_{tR}(K_{m_1,m_2,\cdots,m_r}) \leq 3$. It is straightforward to check that $\gamma^p_{tR}(K_{m_1,m_2,\cdots,m_r}) \geq 3$. Thus, the equality holds.

(2) Assume that $r = 2$. Assign 2 to one vertex in $A_1$; assign 2 to one vertex in $A_2$; assign 0 to the other vertices of $G$. This is a TPRDF of weight 4, so $\gamma^p_{tR}(G) \leq 4$. Assume there is a TPRDF $f$ on $G$ with $w(f) \leq 3$. Then, there exists a vertex $v \in V(G)$ such that $f(v) = 0$. We can assume that $v \in A_1$. Then, there is $u \in A_2$ with $f(u) = 2$. Let $x$ be a vertex in $A_2 \backslash \{u\}$. This vertex exists as $2 \leq m_1 \leq m_2$. If $f(x) = 0$, then there exists a vertex in $A_1$ labeled 2, which contradicts the assumption that $w(f) < 4$. Thus, $f(x) \geq 1$. As $w(f) < 4$, all vertices in $A_1$ are labeled 0. Thus, $G[V_1 \cup V_2]$ has a vertex $v$ with $\mathrm{d}(v) = 0$, a contradiction. Therefore, $\gamma^p_{tR}(G) = 4$.

Assume that $r \geq 3$. Assign 2 to one vertex in $A_1$; assign 1 to the other vertices in $A_1$; assign 1 to one vertex in $A_2$; assign 0 to the other vertices of $G$. This is a TPRDF of weight $m_1 + 2$. Thus, $\gamma^p_{tR}(G) \leq m_1 + 2$. Assume that there exists a TPRDF $f$ with $w(f) < m_1 + 2$. As $|G| > m_1 + 2$, there exists a vertex $v \in A_i$ with $f(v) = 0$ for some $i \in [r]$. Therefore, there exists $u \in A_j$, with $f(u) = 2$ for some $j \neq i$. Assume that $A_j$ contains a vertex $x$ with $f(x) = 0$, then there exists $l \in [r] \backslash \{j\}$ and $y \in A_l$ for which $f(y) = 2$. If $l = i$, then for all $z \in V(G) \backslash (A_i \cup A_j)$, $f(z) \geq 1$, a contradiction with the assumption that $w(f) < m_1 + 2$. Therefore, $l \neq i$, a contradiction, as $v$ is adjacent to two vertices labeled 2. Thus, for all $x \in A_j$, $f(x) \geq 1$, but this is a contradiction with the assumption that $w(f) < m_1 + 2$. Hence, $\gamma^p_{tR}(G) = m_1 + 2$. $\square$

**Proposition 3.** *There are graphs G for which* $\gamma_{tR}(G) < \gamma^p_R(G) < \gamma^p_{tR}(G)$.

**Proof.** Let $G$ be the complete tripartite graph $K_{m,m,m}$ where $m \geq 5$. Denote the parts of $G$ by $A, B$, and $C$; see Figure 2. Define a TRD-function $f$ on G by assigning 2 to an arbitrary vertex in $A$, assigning 2 to an arbitrary node in $B$, and assigning 0 to other nodes of $G$. One can check that $f$ gives the best-possible weight, so $\gamma_{tR}(G) = 4$. Define a PRD-function $f$ on G by assigning 2 to an arbitrary vertex in $A$, assigning 1 to the other vertices in $A$, and assigning 0 to all vertices in $B \cup C$. Then, $\gamma^p_R(G) \leq m + 1$. Assume there exists a PRD-function $g$ on G with $w(g) < m + 1$. Assume that every part contains a vertex labeled 0, say $a_1 \in A$, $b_1 \in B$, $c_1 \in C$. Then, $a_1$ is adjacent to a vertex in $B \cup C$, say in $B$, labeled 2. Similarly, $b_1$ has a neighbor in $A \cup C$, say in A, labeled 2. Now, $c_1$ is adjacent to at least two vertices labeled 2, a contradiction. Thus, $V^g_0 \cap D = 0$ for some $D \in \{A, B, C\}$. As $w(g) < m + 1$, $g(v) = 1$ for every $v \in D$ and $g(v) = 0$ to every $v \in V \backslash D$, a contradiction. Thus, $\gamma^p_R(G) = m + 1$. We show in Proposition 2 that $\gamma^p_{tR}(G) = m + 2$. $\square$

**Proposition 4.** *For any connected graph G of order* $n \geq 2$, *we have* $\gamma^p_R(G) \leq \gamma^p_{tR}(G) \leq 2\gamma^p_R(G) - 1$. *Moreover, for a graph G of order* $n \geq 3$, $\gamma^p_{tR}(G) = 2\gamma^p_R(G) - 1$ *if and only if* $\Delta(G) = n - 1$.

**Proof.** The lower bound follows from Proposition 1. Let $f$ be a PRD-function with $w(f) = \gamma^p_R(G)$. Choose $f$ such that $|V^f_2|$ is the maximum possible. If $V^f_2 = \emptyset$, then $V^f_0 = \emptyset$, and thus, $V(G) = V^f_1$ and $\gamma^p_R(G) = n$. Therefore, $\gamma^p_{tR}(G) = n \leq 2n - 2 = 2\gamma^p_R - 2$ as $n \geq 2$. Now, assume that $V^f_2 \neq \emptyset$. Starting with the vertex set $V^f_1$, if $v$ is an isolated vertex in $G[V^f_1 \cup V^f_2]$, perform the following. Pick an arbitrary neighbor $u$ of $v$ in $V^f_0$. Let $f_1$ be the mapping obtained from $f$ by changing the labeling of $u$ to 1. Repeat if $G[V^{f_1}_1 \cup V^{f_1}_2]$ has a vertex $v$ with $\mathrm{d}(v) = 0$. This procedure will stop after a finite number of steps, and the output is a TPRDF $g$. Now,

$$\gamma^p_{tR}(G) \leq w(g) \leq w(f) + |V^f_1| + |V^f_2| < w(f) + |V^f_1| + 2|V^f_2| = 2\gamma^p_R(G).$$

Therefore, $\gamma_{tR}^{p}(G) \leq 2\gamma_{R}^{p}(G) - 1$. Assume that $\gamma_{tR}^{p}(G) = 2\gamma_{R}^{p}(G) - 1$, then $|V_2^f| = 1$. Let $V_2^f = \{w\}$. Therefore, all vertices in $V_0^f$ are dominated by $w$. If $V_1^f = \emptyset$, then $\mathrm{d}(w) = n - 1 = \Delta(G)$. Assume $V_1^f \neq \emptyset$, and let $x \in V_1^f$. We claim that $x$ is a neighbor for a vertex in $V_0^f$, so assume not. As $G$ is connected, $x$ is a neighbor for a vertex $y \in V_1^f \cup V_2^f$; if $y = w$, we can change the labeling of $x$ to 0 and obtain a PRD-function $h$ with $w(h) < w(f)$, a contradiction; thus, $xw \notin E(G)$ for all $x \in V_1^f$, if $y \in V_1^f$, then we can reassign a value of 2 to $x$ and reassign a value of 0 to $y$ to obtain a PRD-function $q$ with $w(q) = w(f)$ and $|V_2^q| > |V_2^f|$, a contradiction. Thus, the claim holds. Therefore, every $v \in V_1^f$ is a neighbor of some $u \in V_0^f$. Now, $\gamma_{tR}^{p}(G) \leq w(g) \leq w(f) + |V_1^f| = 2w(f) - 2 = 2\gamma_R^p - 2 < 2\gamma_R^p - 1$, a contradiction. Thus, if $\gamma_{tR}^{p}(G) = 2\gamma_{R}^{p}(G) - 1$, then $\Delta(G) = n - 1$. Conversely, assume that $n \geq 3$ and $\Delta(G) = n - 1$. Then, $\gamma_{R}^{p}(G) = 2$ and $\gamma_{tR}^{p}(G) = 3$. Therefore, $\gamma_{tR}^{p}(G) = 2\gamma_{R}^{p}(G) - 1$. $\square$

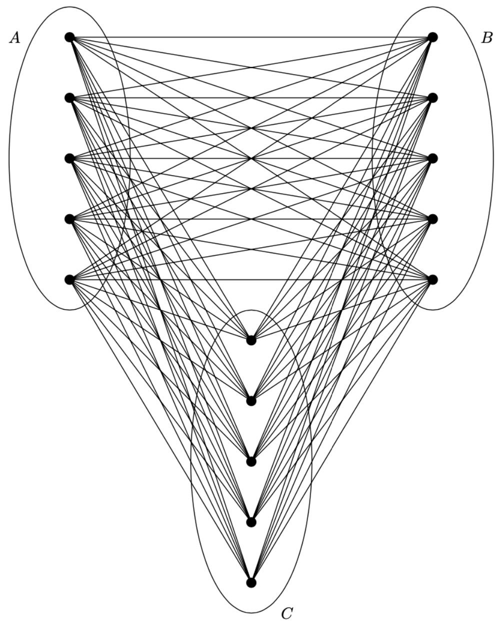


**Figure 2.** The complete tripartite graph $K_{5,5,5}$.

## 2. Graphs with Largest-Possible $\gamma_{tR}^{p}(G)$

In this section, we characterize graphs $G$ with the greatest-possible $\gamma_{tR}^{p}(G)$. It is clear that $\gamma_{tR}^{p}(G) \leq n$ as we can simply assign a value of 1 to every vertex in the graph. We characterize graphs attaining this upper bound.

**Theorem 1.** *Let $G$ be a graph. Then, $\gamma_{tR}^{p}(G) = |G|$ if and only if $\gamma_{tR}(G) = |G|$.*

**Proof.** The sufficient condition is a direct as $\gamma_{tR}(G) \leq \gamma_{tR}^{p}(G) \leq n$ for any graph $G$. Now, assume that $\gamma_{tR}^{p}(G) = n$, and assume for contradiction that $\gamma_{tR}(G) < n$. Let $f$ be a TRD-function on $G$ such that $|V_0|$ is as minimum as possible. As $\gamma_{tR}^{p}(G) = n$, $f$ is not a TPRDF. Therefore, there exists $v \in V(G)$ such that $f(v) = 0$ and $|N(v) \cap V_2| \geq 2$, say $N(v) \cap V_2 = \{u_1, \cdots, u_k\}, k \geq 2$. Observe that $N(u_i) \cap (V_1 \cup V_2) \neq \emptyset$ as $f$ is a TRD-function. $\square$

**Claim 1.** *For all $i \in [k]$, $|N(u_i) \cap V_0| \geq 3$.*

**Proof.** Let $i \in [k]$. Assume that $|N(u_i) \cap V_0| = 1$. Then, $N(u_i) \cap V_0 = \{v\}$. Define a TRD-function $g$ by setting $g(u_i) = 1$ and $g(x) = f(x)$ for every $x \in V \backslash \{u_i\}$. Then, $w(g) < w(f)$, which is a contradiction. Thus, $|N(u_i) \cap V_0| \geq 2$.

Assume that $|N(u_i) \cap V_0| = 2$. Then, $N(u_i) \cap V_0 = \{v, y_i\}$ for some $y_i \in V \backslash \{v\}$. Define a TRD-function $h$ by setting $h(u_i) = h(y_i) = 1$ and $h(x) = f(x)$ for every $x \in V \backslash \{u_i, y_i\}$. Then, $h$ and $f$ have the same weight and $|V_0^h| < |V_0|$, a contradiction. Thus, $|N(u_i) \cap V_0| \geq 3$. □

Fix any $i \in [k]$. Define a TPRDF $q$ on $G$ by setting $q(u_i) = f(u_i) = 2$ and $q(x) = f(x) = 0$ for every $x \in N(u_i) \cap V_0^f$, and set $q(x) = 1$ for the other vertices of $G$. Now, every vertex in $V_0^q$ is adjacent to exactly one neighbor in $V_2^q$, and $G[V_1^q \cup V_2^q]$ has no vertex $v$ with $\mathrm{d}(v) = 0$. Therefore, $w(q) < n$, a contradiction. Therefore, $\gamma_{tR}(G) = n$, as desired.

**Remark 1.** *We proved in Claim 1 that $|N(u_i) \cap V_0| \geq 3$ for all $i \in [k]$, but it is enough to show that $|N(u_i) \cap V_0| \geq 2$.*

Let $\mathcal{G}$ be the family of graphs that is obtained from a four-cycle $v_1v_2v_3v_4v_1$ by adding $k_1$ leaves to $v_3$ and $k_2$ leaves to $v_4$, where $k_1 + k_2 \geq 1$, then subdividing each pendant edge one time. Let $\mathcal{H}$ be the family of graphs that is obtained from a double-star by subdividing the non-pendant edge $t \geq 0$ times and subdividing each pendant edge once. Let $\mathcal{C}$ be the family of graphs that satisfy one of the following conditions:

(1) $G$ is a cycle or a path;
(2) $G = \mathrm{corona}(F)$ for some graph $F$;
(3) $G$ is a subdivided star;
(4) $G \in \mathcal{G} \cup \mathcal{H}$.

**Theorem 2** ([13])**.** *Let $G$ be a connected graph of order $n \geq 2$. Then, $\gamma_{tR}(G) = n$ if and only if $G \in \mathcal{C}$.*

From Theorem 2 and Theorem 1, we obtain the following result.

**Theorem 3.** *Let $G$ be a connected graph of order $n \geq 2$. Then, $\gamma_{tR}^p(G) = n$ if and only if $G \in \mathcal{C}$.*

We give examples of classes of graphs where the tight upper bound of $\gamma_{tR}^p(G)$ is $n$.

A subset $A \subseteq V(G)$ is called an *independent set* if no two vertices in $A$ are adjacent. A graph $G$ is called a *split* graph if its vertices can be partitioned into a clique and an independent set.

**Corollary 1.** *For every positive even integer $n$, there exists a split graph $G$ of order $n$ such that $\gamma_{tR}^p(G) = n$.*

**Proof.** Let $k := \frac{n}{2}$. Let $G$ be the split graph obtained from the disjoint union of a complete graph with the set of vertices $\{v_1, v_2, \cdots, v_k\}$ and an empty graph with the set of vertices $\{u_1, u_2, \cdots, u_k\}$ by adding the set of edges $\{v_iu_i | i \in [k]\}$. Observe that $G = \mathrm{corona}(K_k)$. From Theorem 3, $\gamma_{tR}^p(G) = n$. □

Since split graphs are chordal and the latter graph is perfect, $n$ is a tight upper bound for split graphs, chordal graphs, and perfect graphs.

## 3. Complexity

In this section, we prove that total perfect Roman domination for chordal graphs and that for planar bipartite graphs are NP-complete. We define the following decision problem.

TPRD
**Instance:** Graph $G = (V, E)$ and a positive integer $k \leq |V|$.
**Question:** Does $G$ have a TPRDF $f$ with $w(f) \leq k$?

The following decision problem is a well-known NP-complete problem [27].

Exact cover (XC)

**Instance:** A set $X$ of size $q$ and a collection C of subsets of $X$.
**Question:** Is there a sub-collection C´ of C such that every element of X appears in exactly one element of C´?

We transform the XC to TPRD.

**Theorem 4.** *TPRD is NP-complete for chordal graphs.*

**Proof.** If $f : V(G) \longrightarrow \{0,1,2\}$ is a function and $k \le |V|$ is an integer, we can check in polynomial time if $f$ is a TPRDF and $w(f) \le k$. Therefore, the TPRD problem is in the NP class.

Let $(X, C)$ be an arbitrary instance of the XC, where $X = \{x_1, \cdots, x_q\}$ and $C = \{C_1, \cdots, C_t\}$. Let $H$ be a clique graph with $V(H) = X$. Let $Q_i$ be the graph obtained from the cycle $s_i t_i c_i s_i$ by adding a leaf $c_i'$ adjacent to $c_i$ and a leaf $t_i'$ adjacent to $t_i$. Let $Q$ be the disjoint union of $Q_i$, $i \in [t]$. Finally, let $G$ be the graph obtained from $H$ and $Q$ by adding the set of edges $c_i x_j$ if $x_j \in C_i$; see Figure 3. It is simple to see that $G$ is a chordal graph. Let $k = 4t$. We show that $(X, C)$ has an exact cover if and only if $G$ admits a TPRDF $f$ with $w(f) \le k$.

Assume that $(X, C)$ has an exact cover $C'$. Define $f : V(G) \longrightarrow \{0,1,2\}$ as follows. Set $f(x_i) = 0$ for every $i \in [q]$; if $C_i \in C'$; set $f(c_i) = 2$, $f(t_i) = f(t_i') = 1$, and $f(s_i) = f(c_i') = 0$; if $C_i \notin C'$, set $f(c_i) = f(c_i') = 1$, $f(t_i) = 2$, and $f(t_i') = f(s_i) = 0$. Since $C'$ is an exact cover, each vertex $x_i$ is adjacent to exactly one vertex labeled 2, and it easy to check that every vertex $u \in Q$ with $f(u) = 0$ is adjacent to exactly one vertex labeled 2, while $G[V_1 \cup V_2]$ has no vertex $v$ with $\mathrm{d}(v) = 0$. Thus, $f$ is a TPRDF on $G$ with $w(f) = k$.

Now, assume that $G$ admits a TPRDF $f$ with $w(f) \le k$. Observe that $f(Q_i) \ge 4$ for all $i \in [t]$, so $f(Q) \ge 4t$. as $w(f) \le 4t$, then $w(f) = 4t$ and $f(Q_i) = 4$ for all $i \in [t]$, so $f(x_i) = 0$. Thus, for every $i \in [q]$, $x_i$ is adjacent to exactly one vertex labeled 2, and this neighbor is obviously in the set $\{c_1, \cdots, c_t\}$. Let $C' = \{C_i \in C | f(c_i) = 2\}$. Then, $C'$ is an exact cover. $\square$

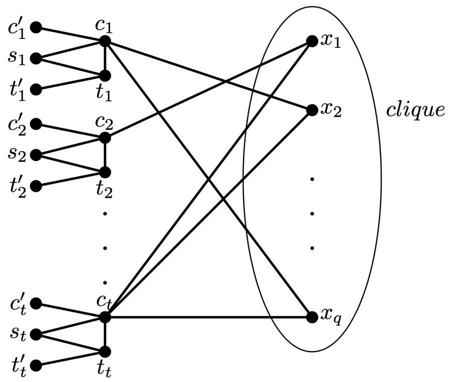


**Figure 3.** The chordal graph $G$.

**Theorem 5.** *TPRD is NP-complete for bipartite graphs.*

**Proof.** We have seen that TPRD is in the NP class. Let $X = \{x_1, \cdots, x_q\}$ be a set and $C = \{C_1, \cdots, C_t\}$ be a collection of subsets of $X$. Let $H$ be the graph with $V(H) = \{x_1, \cdots, x_q, b_1, \cdots, b_q, d_1, \cdots, d_4, a\}$, and $E(H) = \{x_i b_i, b_i a, d_j a | i \in [q], j \in [4]\}$. For every $i \in [t]$, let $Q_i$ be the graph with $V(Q_i) = \{c_i, s_i, s_i', t_i, u_i\}$ and $E(Q_i) = \{c_i s_i, s_i s_i', s_i t_i, t_i u_i, u_i c_i\}$. Let $Q$ be the disjoint union of $Q_i$, $i \in [t]$. Let $G$ be the graph obtained from $H$ and $Q$ by adding the set of edges $x_i c_j$ if and only if $x_i \in C_j$; see Figure 4. It is simple to see that $G$ is a bipartite graph. Set $k = 4t + 3$. We show that $(X, C)$ has an exact cover if and only if $G$ admits a TPRDF $f$ with $w(f) \le k$.

Assume that $(X, C)$ has an exact cover $C'$. Define a TPRDF $f$ on $G$ by setting $f(x_i) = f(b_i) = 0$ for all $i \in [q]$; set $f(a) = 2$, set $f(d_4) = 1$, and set $f(d_i) = 0$ for all $i \in [3]$.

If $C_i \in C'$, set $f(c_i) = f(s_i) = 2$ and set $f(s'_i) = f(t_i) = f(u_i) = 0$. If $C_i \notin C'$, set $f(c_i) = f(s'_i) = 0$, set $f(s_i) = 2$, and set $f(t_i) = f(u_i) = 1$. Since $C'$ is an exact cover, every vertex $x_i$, $i \in [q]$, is adjacent to exactly one neighbor labeled 2. It easy to check that every vertex labeled 0 is adjacent to exactly one neighbor labeled 2, and $G[V_1 \cup V_2]$ has no vertex $v$ with $\mathrm{d}(v) = 0$. Thus, $f$ is a TPRDF of weight $w(f) = 4t + 3 = k$.

Conversely, assume that $G$ admits a TPRDF $f$ with $w(f) \leq k$. Observe that $f(H) \geq 3$ and $f(Q_i) \geq 4$ for all $i \in [t]$, so $w(f) \geq k$. Then, $w(f) = k$, and therefore, $f(H) = 3$ and $f(Q_i) = 4$ for all $i \in [t]$. As $4 > 3$, there exists $i \in [4]$ such that $f(d_i) = 0$, so $f(a) = 2$. As $G[V_1 \cup V_2]$ has no vertex $v$ with $\mathrm{d}(v) = 0$, $a$ is adjacent to a vertex labeled 1. Therefore, $f(x_i) = 0$ for all $i \in [q]$ and $f(b_i) \neq 2$ for all $i \in [q]$. Thus, for all $i \in [q]$, $x_i$ is adjacent to exactly one vertex labeled 2, and this vertex is in the set $\{c_1, \cdots, c_t\}$. Then, $C' := \{C_i \in C | f(c_i) = 2\}$ is a solution. $\square$

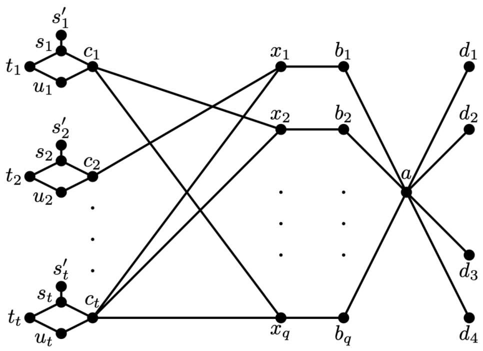


**Figure 4.** The bipartite graph $G$.

Exact Three-Cover (X3C)
**Instance:** A set $X$ of size $3q$ and a collection C of three-element subsets of $X$.
**Question:** Is there a sub-collection C´ of C such that every element of X appears in exactly one element of C´?

The exact three-cover is NP-complete [28].

Every instance of X3C corresponds to a bipartite graph $B$ with the parts $X$ and $C$, where $x_iC_j \in E(B)$ if and only if $x_i \in C_j$. The instance $(X, C)$ is called *planar* if $B$ is a planar graph. The Planar Exact Three-Cover ( P-X3C) is NP-complete [29].

P-X3C
**Instance:** A set $X$ of size $3q$, a collection C of three-element subsets of $X$; the associated graph $B$ is planar.
**Question:** Is there a sub-collection C´ of C such that every element of X appears in exactly one element of C´?

If we use an instance of P-X3C in the proof of Theorem 5, the constructed graph $G$ is a planar bipartite graph. A similar argument can be used to show that the instance of P-X3C has an exact cover if and only if $G$ admits a TPRDF $f$ with $w(f) \leq 4t + 3$. Thus, we have the following corollary.

**Corollary 2.** *TPRD is NP-complete for planar bipartite graphs.*

## 4. $\gamma^p_{tR}(G)$ and $\gamma^p(G)$

In this section, we relate $\gamma^p_{tR}(G)$ to $\gamma^p(G)$, and we characterize trees $T$ for which $\gamma^p_{tR}(T) = 3\gamma^p(T)$.

**Proposition 5.** *Let $G$ be a graph with $\delta(G) \geq 1$. Then, $\gamma^p_{tR}(G) \leq 3\gamma^p(G)$. Moreover, if $\gamma^p_{tR}(G) = 3\gamma^p(G)$, then every PDS $D$ with $|D| = \gamma^p(G)$ is a packing in $G$.*

**Proof.** Let $D$ be a PDS with $|D| = \gamma^p(G)$. Assign a value of 2 to every vertex in $D$. For every vertex $v \in D$ with $\mathrm{d}(v) = 0$, assign a value of 1 to an arbitrary neighbor $u \in V(G) \backslash D$

of $v$, and assign a value of 0 to the other vertices of $G$. The result is a TPRDF $f$ with $w(f) \leq 2|D| + |D| = 3\gamma^p(G)$. Thus, $\gamma^p_{tR}(G) \leq 3\gamma^p(G)$.

Now, assume $\gamma^p_{tR}(G) = 3\gamma^p(G)$ and $D$ is not a packing in $G$. Then, there exist distinct vertices $u, v \in D$ such that $\mathrm{dist}_G(u,v) \leq 2$. Choose $u, v \in D$ such that $\mathrm{dist}_G(u,v)$ is as small as possible. If $\mathrm{dist}_G(u,v) = 2$, then the common neighbor of $u$ and $v$ is not in $D$ and it is dominated by two vertices in $D$, a contradiction. Therefore, $u$ and $v$ are neighbors, and thus, $u$ and $v$ are not isolated vertices in $D$. We can use the same argument in the above paragraph to find a TPRDF $f$ with $w(f) \leq 2|D| + |D| - 2 < 3\gamma^p(G)$, which contradicts the assumption. Thus, the second statement holds. □

The upper bound in Proposition 5 is sharp. If $G$ is a star graph $K_{1,t}$ with $t \geq 2$, then $\gamma^p_{tR}(G) = 3 = 3\gamma^p(G)$.

Let $D := \{v_1, \cdots, v_k\}$ be a PDS of a graph $G$ with $|D| = \gamma^p(G)$. For every $i \in [k]$, let $V_i = N_G[v_i]$. Every vertex in $V \backslash D$ is adjacent to exactly one vertex in $D$, and from Proposition 5, $D$ is a packing in $G$. Thus, $\{V_1, \cdots, V_k\}$ is a partition of $V$. We call it *the partition associated with D*.

**Theorem 6.** *Let G be a connected graph of order $n \geq 3$ with $\delta(G) \geq 1$. If $\gamma^p_{tR}(G) = 3\gamma^p(G)$ and D is a PDS with $|D| = \gamma^p(G)$, then the partition $\{V_1, V_2, \cdots, V_k\}$ associated with D satisfies the following statements:*

(a) *For all $i \in [k]$, $|V_i| \geq 3$;*
(b) *If a vertex v in $V_i$ is adjacent to parts $V'_1, \cdots V'_m$, where $m \geq 2$, $|V'_j| = 3$ for all $j \in [m]$ and $V_i \notin \{V'_1, \cdots, V'_m\}$, then $\sum(|E(v, V_j)| - 1) \geq m - 1$, where the sum is taken over all parts $V_j$ satisfying $|E(v, V_j)| \geq 2$ and $|V_j| \geq 4$.*

**Proof.** Assume that $\gamma^p_{tR}(G) = 3\gamma^p(G)$, and assume that $D$ is a perfect domination set with $|D| = \gamma^p(G)$. Observe that, for every $i \in [k]$, $|V_i| \neq 1$ as $G$ is connected. Therefore, $|V_i| \geq 2$ for all $i \in [k]$. Assume that there exists $i \in [k]$ such that $|V_i| = 2$. We define a TPRDF $f$ on $G$. For each $i \in [k]$ with $|V_i| = 2$, assign 1 to every vertex in $V_i$. For each $i \in [k]$ with $|V_i| \geq 3$, assign 2 to $v_i$, assign 1 to an arbitrary neighbor of $v_i$, and assign 0 to the other vertices in $V_i$. Thus, $\gamma^p_{tR}(G) \leq w(f) \leq 2 + 3(k-1) < 3|D| = 3\gamma^p(G)$, a contradiction. Therefore, $|V_i| \geq 3$ for all $i \in [k]$, and thus, the first condition holds.

For the second condition, assume that there exists a vertex $v$ in $V_i$ such that $v$ is adjacent to $u_1 \in V'_1, \cdots, u_m \in V'_m$, where $m \geq 2$, $|V'_j| = 3$ for all $j \in [m]$ and $V_i \notin \{V'_1, \cdots, V'_m\}$. As $D$ is a packing, $\{u_1, \cdots, u_m\} \cap \{v_1, \cdots, v_k\} = \emptyset$. Let $q := |\{V_j \mid |E(v, V_j)| \geq 2 \text{ and } |V_j| \geq 4\}|$, and let $s := |\{V_j \mid |E(v, V_j)| \leq 1 \text{ and} |V_j| \geq 4\}|$. Assume for contradiction that $\sum(|E(v, V_j)| - 1) < m - 1$, where the sum is over all $V_j$ satisfying $|E(v, V_j)| \geq 2$ and $|V_j| \geq 4$. Define a TPRDF $f$ as follows. If $|V_i| = 3$, assign 2 to $v$ and 1 to the other vertices in $V_i$. If $|V_i| \geq 4$, assign 2 to $v$ and $v_i$, assign 1 to the other neighbors of $v$ in $V_i$ (if there is any), and assign 0 to the other vertices in $V_i$. If $|V_j| \geq 4$ and $E(v, V_j) = \emptyset$, then assign 2 to $v_j$ and assign a value of 1 to an arbitrary neighbor of $v_j$ and assign 0 to the other vertices of $V_j$. If $|V_j| \geq 4$ and $E(v, V_j) \neq \emptyset, j \neq i$, then assign a value of 2 to $v_j$, assign 1 to every neighbor of $v$ in $V_j$, and assign 0 to the other vertices in $V_j$. Assign 0 to every vertex in $\{u_1, \cdots, u_m\}$, and assign 1 to each of the other vertices of $\cup_{l \in [m]} V'_l$. If $|V_j| = 3, j \neq i$ and $V_j \notin \{V'_1, \cdots V'_m\}$, assign 1 to every vertex in $V_j$. Thus,

$$
\begin{aligned}
\gamma^p_{tR}(G) &\leq w(f) \\
&\leq 2m + 3s + 3q + \sum(|E(v, V_j)| - 1) + 3(k - m - q - s) + 1 \\
&< 2m + m - 1 + 3k - 3m + 1 \\
&= 3k \\
&= \gamma^p(G),
\end{aligned}
$$

where the sum $\sum(|E(v,V_j)|-1)$ is over all parts $V_j$ satisfying $|E(v,V_j)| \geq 2$ and $|V_j| \geq 4$. This contradicts the assumption that $\gamma^p_{tR}(G) = 3\gamma^p(G)$. Thus, the second condition holds. $\square$

We remark that the previous theorem is a one-way direction, and it is not reversible. Consider the graph given in Figure 5. Let $D$ be the set of all support vertices in $G$, then $D$ is a PDS. Therefore, $\gamma^p(G) \leq 7$, and it is not difficult to check that $\gamma^p(G) = 7$ and $D$ is the unique PDS with $|D| = \gamma^p(G)$. Observe that the partition associated with $D$ satisfies the two conditions in Theorem 6. The labeling in Figure 5 gives a TPRDF on $G$ with the weight equal to 20. Thus, $\gamma^p_{tR}(G) < 3\gamma^p(G)$.

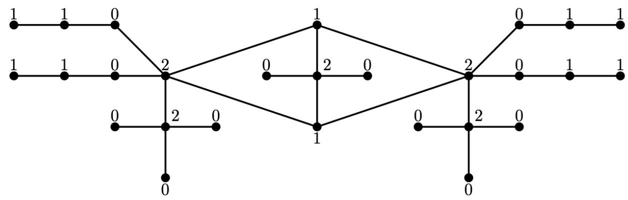

**Figure 5.** A graph $G$ with $\gamma^p_{tR}(G) < 3\gamma^p(G)$.

While we could not give a characterization for graphs $G$ for which $\gamma^p_{tR}(G) = 3\gamma^p(G)$, we give next a constructive characterization for trees $T$ for which $\gamma^p_{tR}(T) = 3\gamma^p(T)$.

**Definition 2.** *Let $G = (V,E)$ be a graph. The function $f : V \longrightarrow \{0,1,2\}$ is called* nearly TPRDF on $G$ with respect to $v \in V(G)$ *if the following three conditions hold:*

(1) *For every $u \in V \setminus \{v\}$ with $f(u) = 0$, there exists exactly one vertex $w \in N(u)$ with $f(w) = 2$;*
(2) *For every $u \in V \setminus \{v\}$ with $f(u) \geq 1$, there exists a vertex $w \in N(u)$ with $f(w) \geq 1$;*
(3) *$f(v) \geq 1$ or there exists exactly one vertex $w \in N(v)$ with $f(w) = 2$.*

*Let $\gamma^p_{tR}(G,v) := \min\{w(f)|f$ be a nearly TPRDF on $G$ with respect to $v\}$.*

Observe that every TPRDF on $G$ is a nearly TPRDF on $G$ with respect to $v$, where $v$ is any vertex in $G$. Therefore, $\gamma^p_{tR}(G,v) \leq \gamma^p_{tR}(G)$, where $v$ is any vertex in $G$. Let $W^1(G) = \{v \in V(G)|\gamma^p_{tR}(G,v) = \gamma^p_{tR}(G)\}$. Let $W^2(G) = \{x \in V(G)|x \notin D$ for some $\gamma^p(G)$-set D$\}$.

**Definition 3.** *Let $G$ be a graph, and $v \in V(G)$. We say that $v$ has property $A$ in $G$ if there exists a TPRDF $f$ on $G$ such that $w(f) = \gamma^p_{tR}(G)$ and $f(v) = 2$. Let $W^3(G) = \{v \in V(G)|v$ not have Property $A$ in $G\}$.*

Let $\mathcal{T}$ be the family of trees $T_k$ that can be constructed from a sequence of trees $T_1, \cdots, T_k$ $(k \geq 1)$, where $T_1 = P_3$, and if $k \geq 2$, $T_{i+1}$ is obtained from $T_i$ by one of the following three operations, where $1 \leq i \leq k-1$:

**Operation 1.** Attaching a new vertex $y$ to a strong support vertex $x \in T_i$.
**Operation 2.** Adding a star $K_{1,3}$ to $T_i$ by joining a leaf in $K_{1,3}$ to a vertex $x \in W^1(T_i) \cap W^2(T_i)$.
**Operation 3.** Attaching a path $P_3$ to a vertex $x \in W^1(T_i) \cap W^3(T_i)$.

**Theorem 7.** *If $T \in \mathcal{T}$, then $\gamma^p_{tR}(T) = 3\gamma^p(T)$.*

**Proof.** Let $T_k \in \mathcal{T}$, then $T_k$ is obtained from a sequence $T_1, \cdots, T_k$ as described above. We proceed by induction on $k$. If $k = 1$, then $T_1 = P_3$ and $\gamma^p_{tR}(P_3) = 3\gamma^p(P_3)$. This establishes the base step. Assume that $k \geq 2$ and the statement holds for every $i$ where $1 \leq i \leq k-1$. Therefore, $\gamma^p_{tR}(T_{k-1}) = 3\gamma^p(T_{k-1})$. $\square$

**Claim 2.** *If $T_k$ is obtained from $T_{k-1}$ by Operation 1, then $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$.*

**Proof.** It is clear that $\gamma^p(T_k) = \gamma^p(T_{k-1})$ and $\gamma^p_{tR}(T_k) \geq \gamma^p_{tR}(T_{k-1})$. Therefore, $\gamma^p_{tR}(T_k) \geq \gamma^p_{tR}(T_{k-1}) = 3\gamma^p(T_{k-1}) = 3\gamma^p(T_k)$. Thus, $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$, and we are done. □

**Claim 3.** *If $T_k$ is obtained from $T_{k-1}$ by Operation 2, then $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$.*

**Proof.** Let $V(K_{1,3}) = \{y, y_1, y_2, y_3\}$, where $xy_1 \in E(T_k)$ and $\mathrm{d}(y) = 3$. Let $D$ be a $\gamma^p(T_{k-1})$-set such that $x \notin D$. We know that $D$ exists as $x \in W^2(T_{k-1})$. It is clear that $D \cup \{y\}$ is a PDS of $T_k$. Therefore, $\gamma^p(T_k) \leq \gamma^p(T_{k-1}) + 1$. Let $f$ be a TPRDF on $T_k$ with $w(f) = \gamma^p_{tR}(T_k)$. It is clear that $f(y) + f(y_1) + f(y_2) + f(y_3) \geq 3$. If $f(x) \geq 1$, then the restriction of $f$ on $T_{k-1}$ is a nearly TPRDF on $T_{k-1}$ with respect to $x$. As $x \in W^1(T_{k-1})$, $\gamma^p_{tR}(T_{k-1}) = \gamma^p_{tR}(T_{k-1}, x) \leq \gamma^p_{tR}(T_k) - 3$. Now,

$$\begin{aligned}\gamma^p_{tR}(T_k) - 3 &\geq \gamma^p_{tR}(T_{k-1})\\ &= 3\gamma^p(T_{k-1})\\ &\geq 3\gamma^p(T_k) - 3.\end{aligned}$$

Thus, $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$. Therefore, we may assume that $f(x) = 0$. If $f(y_1) \neq 2$, then the restriction of $f$ on $T_{k-1}$ is a nearly TPRDF on $T_{k-1}$ with respect to $x$. Similar to the above argument, we obtain $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$. Therefore, we may assume that $f(y_1) = 2$, so $f(y) + f(y_1) + f(y_2) + f(y_3) \geq 4$. Define a nearly TPRDF $g$ on $T_{k-1}$ with respect to $x$ by setting $g(x) = 1$ and $g(u) = f(u)$ for every $u \in T_{k-1} \setminus \{x\}$. Therefore, $w(g) \leq \gamma^p_{tR}(T_k) - 3$. By using an argument similar to the above, we obtain $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$, and we are done. □

**Claim 4.** *If $T_k$ is obtained from $T_{k-1}$ by Operation 3, then $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$.*

**Proof.** Let $P_3 = y_1y_2y_3$, where $xy_3 \in T_k$. Let $D$ be a $\gamma^p(T_{k-1})$-set. If $x \in D$, then $D \cup \{y_1\}$ is a PDS of $T_k$; if $x \notin D$, then $D \cup \{y_2\}$ is a PDS of $T_k$. Thus, $\gamma^p(T_k) \leq \gamma^p(T_{k-1}) + 1$.

Let $h$ be a TPRDF on $T_k$ with $w(h) = \gamma^p_{tR}(T_k)$. It is clear that $h(y_1) + h(y_2) \geq 2$. Assume that $h(y_1) + h(y_2) + h(y_3) = 2$. Then, $y_1, y_2$ are assigned 1 under $h$, and $y_3$ is assigned 0 under $h$, so $h(x) = 2$, so the restriction of $h$ on $T_{k-1}$ is a TPRDF on $T_{k-1}$. As $x \in W^3(T_{k-1})$, $\gamma^p_{tR}(T_k) \geq \gamma^p_{tR}(T_{k-1}) + 3$. Now,

$$\begin{aligned}\gamma^p_{tR}(T_k) &\geq \gamma^p_{tR}(T_{k-1}) + 3\\ &= 3\gamma^p(T_{k-1}) + 3\\ &\geq 3\gamma^p(T_k).\end{aligned}$$

Thus, $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$, as desired. We may assume now that $h(y_1) + h(y_2) + h(y_3) \geq 3$. If $h(x) \geq 1$, then the restriction of $h$ on $T_{k-1}$ is a nearly TPRDF with respect to $x$. As $x \in W^1(T_{k-1})$, $\gamma^p_{tR}(T_k) \geq \gamma^p_{tR}(T_{k-1}) + 3$. Similar to the above, we obtain $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$. If $h(x) = 0$ and $h(y_3) \neq 2$, then the restriction of $h$ on $T_{k-1}$ is a nearly TPRDF with respect to $x$; thus $\gamma^p_{tR}(T_k) \geq \gamma^p_{tR}(T_{k-1}) + 3$. Again, we obtain $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$. If $h(x) = 0$ and $h(y_3) = 2$, then $h(y_1) + h(y_2) + h(y_3) = 4$; define a nearly TPRDF $g$ on $T_{k-1}$ with respect to $x$ by setting $g(x) = 1$ and $g(u) = h(u)$ for all $u \in T_{k-1} \setminus \{x\}$, then $\gamma^p_{tR}(T_k) \geq \gamma^p_{tR}(T_{k-1}) + 3$. Thus, we obtain $\gamma^p_{tR}(T_k) = 3\gamma^p(T_k)$, as desired. □

**Theorem 8.** *If $\gamma^p_{tR}(T) = 3\gamma^p(T)$ for some tree $T$ of order $n \geq 3$, then $T \in \mathcal{T}$.*

**Proof.** We proceed by induction on $n$. If $n = 3$, then $T = P_3$. It is clear that $\gamma^p_{tR}(P_3) = 3\gamma^p(P_3)$ and $P_3 \in \mathcal{T}$. This establishes the base step. Let $n \geq 4$, and assume that the statement holds for every $k, 3 \leq k < n$, that is if $T'$ is a tree of order $k$ and $\gamma^p_{tR}(T') = 3\gamma^p(T')$, then $T' \in \mathcal{T}$. Let $T$ be a tree of order $n$ and $\gamma^p_{tR}(T) = 3\gamma^p(T)$. Let $f$ be a TPRDF on $T$

with $w(f) = \gamma^p_{tR}(T)$. If $\mathrm{diam}(T) = 2$, then $T$ is a star graph, so $T$ is obtained from $P_3$ by iteratively applying Operation 1. Thus, $T \in \mathcal{T}$. If $\mathrm{diam}(T) = 3$, then $T$ is a double-star graph, and it is clear that $\gamma^p_{tR}(T) < 3\gamma^p(T)$, so we must have $\mathrm{diam}(T) \geq 4$. Let $P := v_1 v_2 \cdots v_r$ be a diametral path in $T$, so $r \geq 5$. Root the tree at $v_r$. $\square$

**Claim 5.** *If* $\mathrm{d}(v_2) \geq 4$, *then* $T \in \mathcal{T}$.

**Proof.** Since $P$ is a diametral path, $v_2$ is adjacent to at least three leaves, and $v_3$ is the only non-leaf neighbor of $v_2$, it is clear that $\gamma^p(T - v_1) = \gamma^p(T)$. Let $g$ be a TPRDF on $T - v_1$ with $w(g) = \gamma^p_{tR}(T - v_1)$ and $g(v_2)$ be the maximum possible. Clearly $g(v_2) = 2$. Define a TPRDF $h$ on $T$ by setting $f(v_1) = 0$ and $f(u) = g(u)$ for every $u \in T - v_1$. Then, $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T - v_1)$. Now,

$$\begin{aligned} \gamma^p_{tR}(T - v_1) &\geq \gamma^p_{tR}(T) \\ &= 3\gamma^p(T) \\ &= 3\gamma^p(T - v_1). \end{aligned}$$

Thus, $\gamma^p_{tR}(T - v_1) = 3\gamma^p(T - v_1)$. From the induction hypothesis, $T - v_1 \in \mathcal{T}$, and $T$ is obtained from $T - v_1$ by applying Operation 1. Thus, $T \in \mathcal{T}$, as desired. $\square$

**Claim 6.** *If* $\mathrm{d}(v_2) = 3$, *then* $T \in \mathcal{T}$.

**Proof.** Denote the leaf adjacent to $v_2$ that is different from $v_1$ by $y$. Assume that $\mathrm{d}(v_3) \geq 3$. Then, $v_3$ has a neighbor $v \notin \{v_2, v_4\}$. As $P$ is a diametral path, $v$ is either a support vertex or a leaf. Let $T_1 = T - \{y, v_1, v_2\}$. Assume that $v$ is a strong support vertex or a leaf. Let $D$ be a $\gamma^p(T)$-set. It is clear that $v_2 \in D$, and it is not difficult to see that $v_3 \in D$. Therefore, $D \setminus \{v_2\}$ is a PDS of $T_1$, and therefore, $\gamma^p(T_1) \leq \gamma^p(T) - 1$. Fix a TPRDF $g$ on $T_1$ with $w(g) = \gamma^p_{tR}(T_1)$ and $g(v_3)$ the maximum possible. Clearly, $g(v_3) \geq 1$. Define a TPRDF $f$ on $T$ by setting $f(v_1) = f(y) = 0$, $f(v_2) = 2$, and $f(u) = g(u)$ for every $u \in T_1$, so $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_1) + 2$. Now, $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_1) + 2 \leq 3\gamma^p(T_1) + 2 \leq 3\gamma^p(T) - 1$, a contradiction. Therefore, we may assume that $v$ is a weak support vertex. Denote the leaf adjacent to $v$ by $z$. Let $T_2 = T - \{v, z\}$. It is simple to see that $\gamma^p(T_2) \leq \gamma^p(T) - 1$. Let $g$ be a TPRDF on $T_2$ such that $w(g) = \gamma^p_{tR}(T_2)$. We can extend $g$ to a TPRDF on $T$ by setting $g(v) = g(z) = 1$, so $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_2) + 2$, which results in a contradiction, as in the previous case. Thus, we must have $\mathrm{d}(v_3) = 2$.

Let $D$ be a $\gamma^p(T)$-set such that $D$ contains as much support vertices as possible; clearly $v_2 \in D$. Let $T_3 = T - \{y, v_1, v_2, v_3\}$. Let $g$ be a TPRDF on $T_3$; define a TPRDF $f$ on $T$ by setting $f(y) = f(v_1) = 0, f(v_2) = 2, f(v_3) = 1$, and $f(u) = g(u)$ for every $u \in V(T_3)$. Therefore, $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_3) + 3$. If $v_4 \in D$ or $\{v_3, v_4\} \cap D = \emptyset$, then $D \cap V(T_3)$ is a PDS of $T_3$. Therefore, $\gamma^p(T_3) \leq \gamma^p(T) - 1$. Therefore,

$$\begin{aligned} \gamma^p_{tR}(T_3) &\geq \gamma^p_{tR}(T) - 3 \\ &= 3\gamma^p(T) - 3 \\ &\geq 3\gamma^p(T_3). \end{aligned}$$

Thus, $\gamma^p_{tR}(T_3) = 3\gamma^p(T_3)$. We may assume now that $v_3 \in D$ and $v_4 \notin D$. Clearly, $v_4$ cannot be adjacent to a leaf or a strong support vertex. If $v_4$ is adjacent to a weak support vertex $u$, then $u$ is a neighbor of a leaf $w \in D$; now, $(D \setminus \{v_2, v_3, w\}) \cup \{u\}$ is a PDS of $T_3$. Therefore, $\gamma^p(T_3) \leq \gamma^p(T) - 2$, which results in $\gamma^p_{tR}(T_3) > 3\gamma^p(T_3)$, a contradiction. If $v_4 u \in E$ for some $u$, where $T_u$ is a tree with diam $= 2$, then it is not difficult to see that there exists a PDS $D'$ of $T_3$ containing $u$ (and it does not contain $v_4$) and $|D'| \leq |D| - 1$. Therefore, again, we obtain $\gamma^p_{tR}(T_3) = 3\gamma^p(T_3)$. It remains the case that $\mathrm{d}(v_4) = 2$. We claim that this case is impossible, so assume on the contrary that $\mathrm{d}(v_4) = 2$. Let $T_4 = T - \{y, v_1, v_2, v_3, v_4\}$. As $v_3 \in D$ and $v_4 \notin D$, $D \setminus \{v_2, v_3\}$ is a PDS of $T_4$, so $\gamma^p(T_4) \leq \gamma^p(T) - 2$. Let $g$ be a TPRDF on $T_4$; define a TPRDF $f$ on $T$ by setting $f(y) = f(v_1) = 0, f(v_2) = 2, f(v_3) = f(v_4) = 1$. Then, $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_4) + 4$. Now,

$$\begin{aligned}\gamma^p_{tR}(T) &\leq \gamma^p_{tR}(T_4) + 4\\ &\leq 3\gamma^p(T_4) + 4\\ &\leq 3\gamma^p(T) - 2,\end{aligned}$$

a contradiction. Hence, $T$ is obtained from $T_3$ by joining a star graph $K_{1,3}$ to $v_4$, and $\gamma^p_{tR}(T_3) = 3\gamma^p(T_3)$. From the induction hypothesis, $T_3 \in \mathcal{T}$.

It remains to show that $v_4 \in W^1(T_3) \cap W^2(T_3)$. Assume that $v_4 \notin W^1(T_3)$. Then, there exists a nearly TPRDF $g$ on $T_3$ with respect to $v_4$ with $w(g) < \gamma^p_{tR}(T_3)$. Define a TPRDF $f$ on $T$ by setting $f(y) = f(v_1) = 0, f(v_2) = 2, f(v_3) = 1$, and $f(u) = g(u)$ for every $u \in T_3$. Therefore, $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_3) + 2 = 3\gamma^p(T_3) + 2$, which results in a contradiction. Thus, $v_4 \in W^1(T_3)$. Assume that $v_4 \notin W^2(T_3)$. Then, $v_4 \in D'$ for every $\gamma^p(T_3)$-set $D'$. Let $D$ be a $\gamma^p(T)$-set such that $D$ contains as much support vertices as possible. Clearly $v_2 \in D$. Assume that $v_4 \in D$. Then, $v_3 \in D$. Now, $D \cap V(T_3)$ is a PDS of $T_3$, so $\gamma^p(T_3) \leq \gamma^p(T) - 2$. Then,

$$\begin{aligned}3\gamma^p_{tR}(T_3) &\leq 3\gamma^p(T) - 6\\ &= \gamma^p_{tR}(T) - 6\\ &\leq \gamma^p_{tR}(T_3) - 3,\end{aligned}$$

a contradiction. Therefore, assume that $v_4 \notin D$. If $v_3 \in D$, we have seen before that there exists a PDS $D'$ of $T_3$ with $v_4 \notin D'$ and $|D'| \leq |D| - 1$; as $v_4 \notin W^2(T_3)$, we obtain $\gamma^p(T_3) \leq \gamma^p(T) - 2$, which again leads to a contradiction. If $v_3 \notin D$, then $D \cap V(T_3)$ is a PDS of $T_3$, but it is not a $\gamma^p(T_3)$-set, so $\gamma^p(T_3) \leq \gamma^p(T) - 2$, which results in a contradiction. Hence, $v_4 \in W^2(T_3)$, as desired. □

**Claim 7.** *If* $\mathrm{d}(v_2) = 2$ *then* $T \in \mathcal{T}$.

**Proof.** We show first that $\mathrm{d}(v_3) = 2$. Assume on the contrary that $\mathrm{d}(v_3) \geq 3$. Then, $v_3$ is adjacent to a vertex $v \notin \{v_2, v_4\}$, and $v$ is either a support vertex or a leaf. Let $T_1 = T - \{v_1, v_2\}$. It is not difficult to see that $\gamma^p(T_1) \leq \gamma^p(T) - 1$ and $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_1) + 2$. Now, $3\gamma^p(T_1) \leq 3\gamma^p(T) - 3 = \gamma^p_{tR}(T) - 3 \leq \gamma^p_{tR}(T_1) - 1$, a contradiction. Thus, $\mathrm{d}(v_3) = 2$.

Let $D$ be a $\gamma^p(T)$-set. We show that having $v_4 \notin D$ and $v_3 \in D$ is not possible. Therefore, assume that $v_4 \notin D$ and $v_3 \in D$. Let $T_2 = T - \{v_1, v_2\}$. It is simple to see that $\gamma^p(T_2) \leq \gamma(T) - 1$ and $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_2) + 2$, which results in a contradiction, as desired.

Let $T_3 = T - \{v_1, v_2, v_3\}$. If $v_4 \in D$, then $D \cap V(T_3)$ is a PDS of $T_3$, so $\gamma^p(T_3) \leq \gamma^p(T) - 1$. If $v_4 \notin D$, then $v_3 \notin D$; again, $D \cap V(T_3)$ is a PDS of $T_3$, so we obtain $\gamma^p(T_3) \leq \gamma^p(T) - 1$. It is not difficult to see that $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_3) + 3$. Now,

$$\begin{aligned}\gamma^p_{tR}(T) &\leq \gamma^p_{tR}(T_3) + 3\\ &\leq 3\gamma^p(T_3) + 3\\ &\leq 3\gamma^p(T);\end{aligned}$$

thus, $\gamma^p_{tR}(T_3) = 3\gamma^p(T_3)$. From the induction hypothesis, $T_3 \in \mathcal{T}$.

It remains to show that $v_4 \in W^1(T_3) \cap W^3(T_3)$. Assume on the contrary that $v_4 \notin W^1(T_3)$. Then, there exists a nearly TPRDF $g$ on $T_3$ with respect to $v_4$ with $w(g) \leq \gamma^p(T_3) - 1$. Define a TPRDF $f$ on $T$ by setting $f(v_1) = 0, f(v_2) = 2, f(v_3) = 1$, and $f(u) = g(u)$ for every $u \in V(T_3)$. Now,

$$\begin{aligned}\gamma^p_{tR}(T) &\leq w(g) + 3\\ &\leq \gamma^p_{tR}(T_3) + 2\\ &= 3\gamma^p(T_3) + 2\\ &\leq 3\gamma^p(T) - 1,\end{aligned}$$

a contradiction. Thus, $v_4 \in W^1(T_3)$. Assume on the contrary that $v_4 \notin W^3(T_3)$. Then, there exists a TPRDF $g$ on $T_3$ with $w(g) = \gamma^p_{tR}(T_3)$ and $g(v_4) = 2$. Define a TPRDF $f$ on $T$ by setting $f(v_1) = f(v_2) = 1, f(v_3) = 0$, and $f(u) = g(u)$ for every $u \in V(T_3)$. Therefore, $\gamma^p_{tR}(T) \leq \gamma^p_{tR}(T_3) + 2$, which results in a contradiction. Thus, $v_4 \in W^3(T_3)$. □

We obtain the following main result from Theorems 7 and 8.

**Theorem 9.** *Let $T$ be a tree with $|T| \geq 3$. Then, $\gamma^p_{tR}(T) = 3\gamma^p(T)$ if and only if $T \in \mathcal{T}$.*

## 5. Conclusions

In this paper, we introduced a Roman domination variant called total perfect Roman domination. This variant combines two previously introduced Roman domination variants named total Roman domination and perfect Roman domination. In this paper, we determined the tight upper bound for the total perfect Roman domination number, and we characterized graphs witnessing this bound. We also proved that the total perfect Roman domination problem is NP-complete for chordal graphs, bipartite graphs, and planar bipartite graphs. We showed that, for any graph, the total perfect Roman domination number is bounded above by three-times the perfect domination number, and we characterized trees attaining this bound.

## 6. Open Problems

We end this research with the following problems:
**Problem 1:** Characterize graphs $G$ for which $\gamma^p_{tR}(G) = 3\gamma^p(G)$.
**Problem 2:** Let $k > 1$. Characterize graphs $G$ for which $\gamma^p_{tR}(G) = 2\gamma^p_R(G) - k$.

**Funding:** The author extends her appreciation to the Deputyship for Research and Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number INST198.

**Data Availability Statement:** No new data were created or analyzed in this study.

**Conflicts of Interest:** The author declares no conflict of interest.

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