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Multi-Attribute Group Decision Making Based on Spherical Fuzzy Zagreb Energy

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Abstract: Based on picture fuzzy sets (PFSs), we use a mathematical model to tackle such types of problems when a person has opinions like yes, no, abstain, and refusal. The spherical fuzzy model is more flexible and practical than the picture fuzzy model, as it enhances the space of uncertainty. It broadens the space of vague information evaluated by decision makers since graphs are the pictorial representation of information. Graphs are a tool to represent a network. To handle some real-world problems, spherical fuzzy graphs can be used more effectively as compared to picture fuzzy graphs (PFGs). In this article, we expand the notion of fuzzy Zagreb indices of the fuzzy graph to the spherical fuzzy Zagreb indices of the spherical fuzzy graph (SFG). The spherical fuzzy Zagreb matrix of SFG and Zagreb energy of SFG are defined with examples. Additionally, we develop several lower and upper bounds of the spherical Zagreb energy of SFG. In addition, we present an application of SFG by computing its Zagreb energy in the decision-making problem of choosing the best location for business purposes.

Keywords: spherical fuzzy graph; spherical fuzzy Zagreb indices; matrix; graph energy; decision making

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1. Introduction

In 1965, the notion of fuzzy set theory, which is the generalization of classical set theory, was established by Zadeh [1]. Enormous applications of fuzzy sets are found in telecommunication, control engineering, decision theory, expert systems, logic, management science, operation research, and others, which are all based on Zadeh's exceptional idea. One of the limitations of fuzzy set (FS) is that it deals only with the membership degree ($\breve{\alpha}$). To overcome this deficiency, Atanassov [2] in 1983 initiated the concept of the intuitionistic fuzzy set (IFS), which is the generalization of the fuzzy set (FS) theory. He expanded the notion of FSs by declaring the degree of truthiness ($\breve{\alpha}$) besides the degree of falseness ($\breve{\gamma}$) with the constraint $0 \leq \breve{\alpha} + \breve{\gamma} \leq 1$. On emphasizing the notion of IFS, Yager presented the Pythagorean fuzzy set (PyFS), which enlarged the space with the new constraint, $0 \leq \breve{\alpha}^2 + \breve{\beta}^2 \leq 1$. The situation when a person has an opinion like yes, abstain, no, and refusal can be handled with the picture fuzzy set (PFS) that was initiated by Cuong [3,4] which is an abstraction of IFS. It gives grades to the three parameters, named the degree of truthiness $\breve{\alpha} : R \longrightarrow [0,1]$, degree of abstinence$\breve{\beta} : R \longrightarrow [0,1]$, and the degree of falseness $\breve{\gamma} : R \longrightarrow [0,1]$ with the constraint $0 \leq \breve{\alpha} + \breve{\beta} + \breve{\gamma} \leq 1$, where $\breve{\nabla} = 1 - (\breve{\alpha} + \breve{\beta} + \breve{\gamma})$ is the degree of refusal. Garg [5] proposed some picture fuzzy aggregation operators. The notion of PFS was extended by Gundogdu and Kahraman [6] namely as the spherical fuzzy set (SFS). It gives more space to the degree of truthiness $\breve{\alpha} : R \longrightarrow [0,1]$, degree of abstinence $\breve{\beta} : R \longrightarrow [0,1]$, and degree of falseness $\breve{\gamma} : R \longrightarrow [0,1]$ with the constraint $0 \leq \breve{\alpha}^2 + \breve{\beta}^2 + \breve{\gamma}^2 \leq 1$, where

$\check{\nabla} = \sqrt{1 - (\check{\alpha}^2 + \check{\beta}^2 + \check{\gamma}^2)}$ is the degree of refusal. Ashraf et al. [7] explored the idea of SFSs with an application in multi-criteria decision-making (MCDM) problems using weighted averaging and weighted geometric aggregation operators. Further, Ashraf et al. [8] presented some spherical fuzzy Dombi aggregation operators and their applications in the MCDM problem. Mahmood et al. [9] presented the concept of the spherical fuzzy set and T-spherical fuzzy set as the generalization of the fuzzy set, intuitionistic fuzzy set, and Pythagorean fuzzy set. Some operations of SFSs and T-SFSs were introduced. They utilized these in MCDM of medical diagnostics problems. The q-rung orthopair fuzzy sets (q-ROFs) are the generalized form of IFS and PyFS and provide a more flexible way to handle uncertain information [10,11]. Then, Li et al. [12] introduced the idea of the q-Rung picture fuzzy set (q-RPFS), which is the generalization of SFSs. It provides huge space to the conditions of PFS as well as SFS with $0 \leq \check{\alpha}^q + \check{\beta}^q + \check{\gamma}^q \leq 1, \;\; q \geq 1$. By this, we can obtain more accurate outcomes as we increase the q-rungs.

In mathematics, graphs are a way to formally represent a network, which is basically just a collection of objects that are all interconnected. The notion of fuzzy graphs (FGs) based on fuzzy relation is presented by Kaufmann [13] and by some other experts who contributed to fuzzy graphs [14–25]. In [26], a MCDM problem is tackled with the help of a certain form of Pythagorean fuzzy graphs. Akram et al. [27] presented an idea of spherical fuzzy graph (SFG) as well as defining some operations on SFGs, namely symmetric difference and rejection. Akram [28] also presented MCDM models involving SFG. Then, Guleria and Bajaj [29] introduced a generalized version of SFGs using T-spherical fuzzy sets.

Topological indices are arithmetical quantities for structural molecular graphs. In 1972, Gutman [30] introduced the first Zagreb index. In 1978, Gutman [31] introduced the idea of the energy of a graph as the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph. Gutman and Zhou [32] presented the term Laplacian energy of a graph as the sum of the absolute values of the differences of the average vertex degree of graph and the Laplacian eigenvalues of the graph.

The research study of topological fuzzy indices (TFIs) seems to be advantageous for multi-criteria decision-making (MCDM) problems and many connected fuzzy networks. Kalathian et al. [33] defined many TFIs for FGs, including the Gutman index, hyper Wiener index, Schultz index, Zagreb indices, Randic index, modified Wiener index, and harmonic index, and they established bounds for few of the indices as well. Islam and Pal [34] investigated the first Zagreb index for various FGs, like path, star, cycle, and fuzzy subgraphs, as well as proving several results. Moreover, a MCDM technique was elaborated that uses the first Zegrab index of a FG for evaluating the best employee in a company. The fuzzy Zagreb index is a degree-based index. Ahmad and Nasir [35] presented Randic and harmonic indices for FGs and derived some upper and lower bounds for different types of products, such as the cross product, Cartesian product and lexicographic product. These fuzzy indices were then utilized to handle a cybercrime problem as well. The Wiener index (based on geodesic distances between the vertices) for directed rough fuzzy graph was defined by Ahmad and Iqra [36], with its application to human trafficking. Some TIs in FGs, which are based on the degree and distance between the vertices, were explored in [37] and used in a multi-criteria decision-making problem.

Anjali and Mathew [38] elaborated the energy for a FG as the sum of the modulus values of the eigenvalues corresponding to the adjacency matrix of FG. The Laplacian energy of FG has defined in [39]. Kale and Minirani [40] presented the fuzzy Zagreb matrix and derived some bounds for fuzzy Zagreb energy. Praba et al.[41] extended the idea of energy of FG to the energy of an IFG and derived several lower and upper bounds. Akram and Naz [42] presented the energy of an IFG to Pythagorean fuzzy graphs (PyFGs) and explored the concepts of energy and Laplacian energy of PyFGs, as well as the energy and Laplacian energy of Pythagorean fuzzy digraphs. Akram et al. [43] introduced the notion of energy of double dominating bipolar fuzzy graphs. Shi et al. [44] extended the notion of the energy of the Pythagorean fuzzy graph to the energy of picture fuzzy graphs (PFGs) and presented the types of energy, including Laplacian energy as well as skew-Laplacian energy in both PFGs and picture fuzzy digraphs. They ranked the suitable locations for business purposes by making use of the picture fuzzy graph and its related energies. Yahya and Mohamed [45] explored the energy of SFG and extracted some bounds of the energy of SFG.

The following points influenced us to write this article:

- Due to the enormous applications of TIs, including Zagreb indices for FGs in distinct decision-making problems, it also seems advantageous to expand the notion of Zagreb indices in SFG.
- There are numerous applications of the spectrum of fuzzy graph theory in solving linear systems, computer science, chemistry, and others.
- The spectrum of the graph plays a crucial role in combinatorial optimization problems in mathematics.
- Moreover, the Zagreb energy of SFG has not yet been discussed and studied in the literature; therefore, we expanded the notion of the energy of SFG to the Zagreb energy of SFG.

The subscription of our proposed research work is given as follows:

- The aim of this research study is to establish the notion of the first and second Zagreb indices of spherical fuzzy graphs.
- We introduce the concept of spherical fuzzy Zagreb matrices of SFGs and corresponding spectra.
- We define the Zagreb energy of SFG and establish the lower and upper bounds for the Zagreb energy of SFGs and some of their results.
- Finally, we utilize the idea of Zagreb energy of SFG in a MCDM problem. In particular, we determine the best place to start a certain business.

This paper is organized as follows.

In Section 2, some definitions are recalled for better understanding. In Section 3, we define fuzzy Zagreb indices of SFG. In Section 4, we discuss the lower and upper bounds of Zagreb energy and some of the main results as well. In addition, in Section 5, we present an application of SFG by computing Zagreb energy in a multi-attribute decision-making problem. Finally, we conclude our proposed work.

Some of the symbols used in this article are mentioned in Table 1.

Table 1. List of symbols.

Symbols	Description	Symbols	Description
$\check{\alpha}_R(v)$	truthiness membership of v_i	s_t	$\max\{\check{\alpha}_\mathcal{Y}(v_i)\}$
$\check{\beta}_R(v)$	abstinence membership of v_i	s_a	$\max\{\check{\beta}_\mathcal{Y}(v_i)\}$
$\check{\gamma}_R(v)$	falseness membership of v_i	s_f	$\max\{\check{\gamma}_\mathcal{Y}(v_i)\}$
g_t	$\min\{\check{\alpha}_\mathcal{Y}(v_i)\}$	g_a	$\min\{\check{\beta}_\mathcal{Y}(v_i)\}$
g_f	$\max\{\check{\gamma}_\mathcal{Y}(v_i)\}$	Δ_t	$\max\{d_t(v_i)\}$
Δ_a	$\max\{d_a(v_i)\}$	Δ_f	$\max\{d_f(v_i)\}$
δ_t	$\min\{d_t(v_i)\}$	δ_a	$\min\{d_a(v_i)\}$
δ_f	$\min\{d_f(v_i)\}$	ZE	Zagreb energy of matrix
SZM	spherical fuzzy Zagreb matrix	$SFM_1(G)$	Zagreb first index of SFG
$SFM_2(G)$	Zagreb second index of SFG	SZE	spherical fuzzy Zagreb energy
e	fuzzy size	e^*	crisp size

2. Preliminaries

Definition 1 ([27]). *Let $\check{V}$ be a nonempty underlying set of vertices. A spherical fuzzy set R over $\check{V}$ is defined as*

$$R = \{(v,\ \breve{\alpha}_R(v),\ \breve{\beta}_R(v),\ \breve{\gamma}_R(v))\ |v \in \check{V}\},$$

where $\breve{\alpha}_R(v)$, $\breve{\beta}_R(v)$, and $\breve{\gamma}_R(v)$ denote the degree of truthiness of v in R, degree of abstinence of v in R, and degree of falseness of v in R, respectively. Moreover, $\breve{\alpha}(v),\ \breve{\beta}(v),\ \breve{\gamma}(v) \in [0,1]$, such that $0 \leq \breve{\alpha}^2_R(v) + \breve{\beta}^2_R(v) + \breve{\gamma}^2_R(v) \leq 1$. Further, $\check{\nabla}_R(v) = \sqrt{1 - (\breve{\alpha}^2_R(v) + \breve{\beta}^2_R(v) + \breve{\gamma}^2_R(v))}$ for all $v \in \check{V}$ is called the degree of refusal of membership of v in R.

Definition 2 ([45]). *Let V be an underlying set of vertices. A spherical fuzzy graph $G = (R, S)$ is a pair of mappings such that $R : V \longrightarrow [0,1]$ and $S : V \times V \longrightarrow [0,1]$, where R is a spherical fuzzy set with constraint $0 \leq \breve{\alpha}^2 + \breve{\beta}^2 + \breve{\gamma}^2 \leq 1$ for all $v \in V$ and S is spherical fuzzy relation on $V \times V$, such that $\breve{\alpha}_S(v_1, v_2) \leq min\{\breve{\alpha}_R(v_1), \breve{\alpha}_R(v_2)\}$, $\breve{\beta}_S(v_1, v_2) \leq min\{\breve{\beta}_R(v_1), \breve{\beta}_R(v_2)\}$, and $\breve{\gamma}_S(v_1, v_2) \leq max\{\breve{\gamma}_R(v_1), \breve{\gamma}_R(v_2)\}$, where $\breve{\alpha}$, $\breve{\beta}$, and $\breve{\gamma}$ denote the degree of truthiness, degree of abstinence, and degree of falseness, respectively, satisfying the following condition:*

$$0 \leq \breve{\alpha}^2_S(v_1, v_2) + \breve{\beta}^2_S(v_1, v_2) + \breve{\gamma}^2_S(v_1, v_2) \leq 1$$

for all $(v_1, v_2) \in \check{V} \times \check{V}$.

Example 1. *The graph $L^* = (V, E)$ is defined on an underlying set of vertices $V = \{\acute{v}_1, \acute{v}_2, \acute{v}_3, \acute{v}_4\}$ and $E = \{\acute{v}_1\acute{v}_2, \acute{v}_1\acute{v}_3, \acute{v}_3\acute{v}_4\} \subseteq V \times V$. Consider the graph $\mathcal{C} = (\mathcal{Y}, \mathcal{Z})$, where $\mathcal{Y}$ is a spherical fuzzy subset of V and $\mathcal{Z}$ is a spherical fuzzy subset of E defined as in Tables 2 and 3, respectively. The graph $\mathcal{C} = (\mathcal{Y}, \mathcal{Z})$ is shown in Figure 1.*

Table 2. Table for $\mathcal{Y}$.

$\mathcal{Y}$	$\acute{v}_1$	$\acute{v}_2$	$\acute{v}_3$	$\acute{v}_4$
$\breve{\alpha}_{\mathcal{Y}}$	0.2	0.3	0.4	0.4
$\breve{\beta}_{\mathcal{Y}}$	0.3	0.5	0.8	0.2
$\breve{\gamma}_{\mathcal{Y}}$	0.4	0.7	0.3	0.6

Table 3. Table for $\mathcal{Z}$.

$\mathcal{Z}$	$\acute{v}_1\acute{v}_2$	$\acute{v}_1\acute{v}_3$	$\acute{v}_3\acute{v}_4$
$\breve{\alpha}_{\mathcal{Z}}$	0.1	0.2	0.2
$\breve{\beta}_{\mathcal{Z}}$	0.2	0.3	0.1
$\breve{\gamma}_{\mathcal{Z}}$	0.6	0.4	0.5

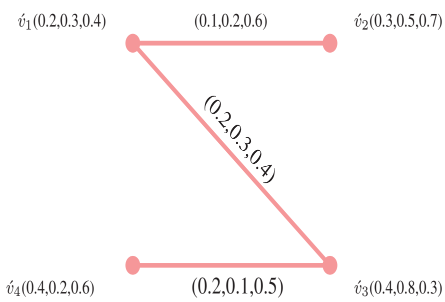

Figure 1. A SFG $\mathcal{C}$.

Definition 3 ([27]). *Consider a SFG $G = (R, S)$ defined on $G^* = (\check{V}, E)$. The degree of a vertex v of G ($d_G(v) = (d_{\check{\alpha}}(v), d_{\check{\beta}}(v), d_{\check{\gamma}}(v))$) is expressed as*

$$d_G(v) = \left(\sum_{v_1 \neq v_2} \check{\alpha}_S(v_1, v_2), \sum_{v_1 \neq v_2} \check{\beta}_S(v_1, v_2), \sum_{v_1 \neq v_2} \check{\gamma}_S(v_1, v_2) \right)$$

for all $(v_1, v_2) \in \check{V} \times \check{V}$.

Definition 4. *Using OWA operator weights for preference aggregation, let wj be the relative importance weight attached to the jth ranking place and vij be the vote candidate i receives in the j-th ranking place. The total score of each candidate is defined as [36], which is a linear function of the relative importance weights. Although the weights are not required to be summed to one, we assume them to be normalized in this paper. That is, once the weights are determined, candidates can be ranked in terms of their total.*

3. Spherical Fuzzy Zagreb Indices

Several topological indices are defined in a crisp graph, but in real life, many issues cannot be tackled by these indices. So, there is a need to define these indices for the fuzzy graph. One of the most famous and useful indices is the Zagreb index, which can be used in several fields of mathematics and chemistry. It is a degree-based topological index. The strength of vertices plays a vital role in fuzzy graph theory. Fuzzy Zagreb indices can be used to measure the connectivity of edges. These can be used to evaluate how much the fuzzy graph is strengthened. In spherical fuzzy Zagreb indices, we involve the membership values of vertices as well as the degree of vertices in terms of truthiness, abstinence, and falseness, which help to encounter the connectivity of neighborhoods of the vertices.

Definition 5. *Let $G = (R, S)$ be a spherical fuzzy graph. Then, the first Zagreb index denoted as $(SFM_1(G) = (SFM_1(G))_t, (SFM_1(G))_a, (SFM_1(G))_f)$ of SFG is defined by*

$$\begin{aligned} SFM_1(G) = \sum_{v_i v_j \in E(G)\ i \neq j} & [\check{\alpha}(v_i) d_{\check{\alpha}}(v_i) + \check{\alpha}(v_j) d_{\check{\alpha}}(v_j),\ \check{\beta}(v_i) d_{\check{\beta}}(v_i) + \check{\beta}(v_j) d_{\check{\beta}}(v_j), \\ & \check{\gamma}(v_i) d_{\check{\gamma}}(v_i) + \check{\gamma}(v_j) d_{\check{\gamma}}(v_j)], \end{aligned} \tag{1}$$

Here , $\check{\alpha}(v_i)$, $\check{\beta}(v_i)$, $\check{\gamma}(v_i)$, $d_{\check{\alpha}}(v_i)$, $d_{\check{\beta}}(v_i)$, and $d_{\check{\gamma}}(v_i)$ are defined in Table 1.

Definition 6. *Let $G = (R, S)$ be a spherical fuzzy graph. Then the second Zagreb index denoted as $(SFM_2(G) = (SFM_2(G))_t, (SFM_2(G))_a, (SFM_2(G))_f)$ of the spherical fuzzy graph is defined as*

$$\begin{aligned} SFM_2(G) = \sum_{v_i v_j \in E(G) i \neq j} & [\check{\alpha}(v_i) d_{\check{\alpha}}(v_i) \check{\alpha}(v_j) d_{\check{\alpha}}(v_j),\ \check{\beta}(v_i) d_{\check{\beta}}(v_i) \\ & \check{\beta}(v_j) d_{\check{\beta}}(v_j),\ \check{\gamma}(v_i) d_{\check{\gamma}}(v_i) \check{\gamma}(v_j) d_{\check{\gamma}}(v_j)], \end{aligned} \tag{2}$$

Here, $\check{\alpha}(v_i)$, $\check{\beta}(v_i)$, $\check{\gamma}(v_i)$, $d_{\check{\alpha}}(v_i)$, $d_{\check{\beta}}(v_i)$, and $d_{\check{\gamma}}(v_i)$ are defined in Table 1.

Remark 1. *From Definitions 5 and 6, it is noted that $(SFM_1(G))_t \leq (SFM_2(G))_t$, $SFM_1(G))_a \leq (SFM_2(G))_a$ and $SFM_1(G))_f \leq (SFM_2(G))_f$.*

Example 2. *Consider the spherical fuzzy graph $J = (D, F)$ as shown in Figure 2.*

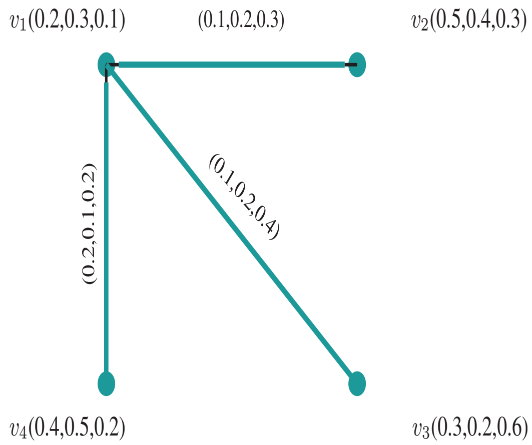

Figure 2. A SFG J.

By using (1), the first Zagreb index of SFG J is calculated as

$$SFM_1(G) = \sum_{v_iv_j\in E(G)\ i\neq j} [\breve{\alpha}(v_i)d_{\breve{\alpha}}(v_i) + \breve{\alpha}(v_j)d_{\breve{\alpha}}(v_j), \breve{\beta}(v_i)d_{\breve{\beta}}(v_i) + \breve{\beta}(v_j) d_{\breve{\beta}}(v_j), \breve{\gamma}(v_i)d_{\breve{\gamma}}(v_i) + \breve{\gamma}(v_j)d_{\breve{\gamma}}(v_j)].$$

$$\begin{aligned} SFM_1(G) =& [\breve{\alpha}(v_1)d_{\breve{\alpha}}(v_1) + \breve{\alpha}(v_2)d_{\breve{\alpha}}(v_2), \breve{\beta}(v_1)d_{\breve{\beta}}(v_1) + \breve{\beta}(v_2)d_{\breve{\beta}}(v_2), \breve{\gamma}(v_1) \\ & d_{\breve{\gamma}}(v_1) + \breve{\gamma}(v_2)d_{\breve{\gamma}}(v_2)] + [\breve{\alpha}(v_1)d_{\breve{\alpha}}(v_1) + \breve{\alpha}(v_3)d_{\breve{\alpha}}(v_3), \breve{\beta}(v_1) \\ & d_{\breve{\beta}}(v_1) + \breve{\beta}(v_3)d_{\breve{\beta}}(v_3), \breve{\gamma}(v_1)d_{\breve{\gamma}}(v_1) + \breve{\gamma}(v_3)d_{\breve{\gamma}}(v_3)] + [\breve{\alpha}(v_1) \\ & d_{\breve{\alpha}}(v_1) + \breve{\alpha}(v_4)d_{\breve{\alpha}}(v_4), \breve{\beta}(v_1)d_{\breve{\beta}}(v_1) + \breve{\beta}(v_4)d_{\breve{\beta}}(v_4), \breve{\gamma}(v_1)d_{\breve{\gamma}}(v_1) \\ & + \breve{\gamma}(v_4)d_{\breve{\gamma}}(v_4)]. \\ =& [(0.2)(0.4) + (0.5)(0.1), (0.3)(0.5) + (0.4)(0.2), (0.1)(0.9)+ \\ & (0.3)(0.3)] + [(0.2)(0.4) + (0.3)(0.1), (0.3)(0.5) + (0.2)(0.2), \\ & (0.1)(0.9) + (0.6)(0.4)] + [(0.2)(0.4) + (0.4)(0.2), (0.3)(0.5)+ \\ & (0.5)(0.1), (0.1)(0.9) + (0.2)(0.2)]. \\ =& (0.4, 0.62, 0.64). \end{aligned}$$

Now, by using (2), the second Zagreb index of SFG J is computed as

$$SFM_2(G) = \sum_{v_iv_j\in E(G)\ i\neq j} [\breve{\alpha}(v_i)d_{\breve{\alpha}}(v_i) \cdot \breve{\alpha}(v_j)d_{\breve{\alpha}}(v_j), \breve{\beta}(v_i)d_{\breve{\beta}}(v_i) \cdot \breve{\beta}(v_j)d_{\breve{\beta}}(v_j), \breve{\gamma}(v_i)d_{\breve{\gamma}}(v_i) \cdot \breve{\gamma}(v_j)d_{\breve{\gamma}}(v_j)].$$

to remove numbering (before each equation)

$$\begin{aligned} SFM_2(G) =& [\breve{\alpha}(v_1)d_{\breve{\alpha}}(v_1) \cdot \breve{\alpha}(v_2)d_{\breve{\alpha}}(v_2), \breve{\beta}(v_1)d_{\breve{\beta}}(v_1) \cdot \breve{\beta}(v_2)d_{\breve{\beta}}(v_2), \breve{\gamma}(v_1)d_{\breve{\gamma}}(v_1) \\ & \cdot \breve{\gamma}(v_2)d_{\breve{\gamma}}(v_2)] \cdot [\breve{\alpha}(v_1)d_{\breve{\alpha}}(v_1) \cdot \breve{\alpha}(v_3)d_{\breve{\alpha}}(v_3), \beta(v_1)d_{\beta}(v_1) \cdot \beta(v_3)d_{\beta}(v_3), \\ & \breve{\gamma}(v_1)d_{\breve{\gamma}}(v_1) \cdot \breve{\gamma}(v_3)d_{\breve{\gamma}}(v_3)] \cdot [\breve{\alpha}(v_1)d_{\breve{\alpha}}(v_1) \cdot \breve{\alpha}(v_4)d_{\breve{\alpha}}(v_4), \breve{\beta}(v_1) \\ & d_{\breve{\beta}}(v_1) \cdot \breve{\beta}(v_4)d_{\breve{\beta}}(v_4), \breve{\gamma}(v_1)d_{\breve{\gamma}}(v_1) \cdot \breve{\gamma}(v_4)d_{\breve{\gamma}}(v_4)]. \\ =& [(0.2)(0.4) \cdot (0.5)(0.1), (0.3)(0.5) \cdot (0.4)(0.2), (0.1)(0.9) \cdot (0.3)(0.3)]+ \\ & [(0.2)(0.4) \cdot (0.3)(0.1), (0.3)(0.5) \cdot (0.2)(0.2), (0.1)(0.9) \cdot (0.6)(0.4)]+ \\ & [(0.2)(0.4) \cdot (0.4)(0.2), (0.3)(0.5) \cdot (0.5)(0.1), (0.1)(0.9) \cdot (0.2)(0.2)]. \\ =& (0.0128, 0.0255, 0.0333). \end{aligned}$$

Definition 7. *Consider a SFG $G = (R, S)$ and $\check{V} = \{v_1, v_2, v_3, \ldots, v_n\}$. Then the spherical fuzzy Zagreb matrix ($\mathtt{SZM}$) is defined as*

$$[\mathtt{SZM}]_{ij} = ([ZM\theta_1]_{ij}, [ZM\theta_2]_{ij}, [ZM\theta_3]_{ij}),$$

where

$$[ZM\theta_1]_{ij} = \begin{cases} \check{\alpha}(v_i)d_{\check{\alpha}}(v_i) + \check{\alpha}(v_j)d_{\check{\alpha}}(v_j), & v_iv_j \in \check{V}; \\ 0, & v_iv_j \notin \check{V}; \\ 0, & v_i = v_j. \end{cases}$$

$$[ZM\theta_2]_{ij} = \begin{cases} \check{\beta}(v_i)d_{\check{\beta}}(v_i) + \check{\beta}(v_j)d_{\check{\beta}}(v_j), & v_iv_j \in \check{V}; \\ 0, & v_iv_j \notin \check{V}; \\ 0, & v_i = v_j. \end{cases}$$

and,

$$[ZM\theta_3]_{ij} = \begin{cases} \check{\gamma}(v_i)d_{\check{\gamma}}(v_i) + \check{\gamma}(v_j)d_{\check{\gamma}}(v_j), & v_iv_j \in \check{V}; \\ 0, & v_iv_j \notin \check{V}; \\ 0, & v_i = v_j. \end{cases}$$

Definition 8. *The spectrum of $\mathtt{SZM}$ of SFG is expressed as $(\omega^{(1)}, \omega^{(2)}, \omega^{(3)})$, where $\omega^{(1)}$, $\omega^{(2)}$, and $\omega^{(3)}$ are the sets of eigenvalues of $ZM\theta_1$, $ZM\theta_2$, and $ZM\theta_3$, respectively.*

Definition 9. *The spherical fuzzy Zagreb energy ($\mathtt{SZE}$) of SFG is defined as*

$$\mathtt{SZE}(G) = \Big(ZE(ZM\theta_1),\ ZE(ZM\theta_2),\ ZE(ZM\theta_3)\Big).$$

$$= \left(\sum_{i=1}^{n} |\xi_i|, \sum_{i=1}^{n} |\eta_i|, \sum_{i=1}^{n} |\zeta_i|\right), \tag{3}$$

where ξ_i, η_i, and ζ_i are the eigenvalues of $ZM\theta_1$, $ZM\theta_2$, and $ZM\theta_3$, respectively. Further, ZE denotes the fuzzy Zagreb energy.

Example 3. *Consider the SFG `W=(P, Q)` as shown in Figure 3.*

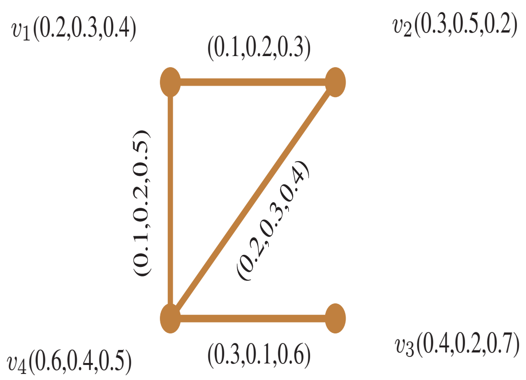

Figure 3. Spherical fuzzy graph `W=(P, Q)`.

The spherical fuzzy Zagreb matrix is obtained according to the definition mentioned as above:

$$\mathtt{SZM} = \begin{bmatrix} (0,0,0) & (0.13,0.37,0.46) & (0,0,0) & (0.4,0.36,1.07) \\ (0.13,0.37,0.46) & (0,0,0) & (0,0,0) & (0.45,0.49,0.89) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0.48,0.26,1.17) \\ (0.4,0.36,1.07) & (0.45,0.49,0.89) & (0.48,0.26,1.17) & (0,0,0) \end{bmatrix}.$$

The set of eigenvalues of $ZM\theta_1$, $ZM\theta_2$, and $ZM\theta_3$ is given as

$$\begin{aligned}\omega^{(1)} &= (-0.7340, -0.1296, 0.0503, 0.8133),\\ \omega^{(2)} &= (-0.5566, -0.3461, 0.0568, 0.8459),\end{aligned}$$

and,

$$\omega^{(3)} = (-1.7040, -0.4565, 0.1889, 1.9716),$$

respectively. Also, by using Definition 8, we compute that

$$\begin{aligned}Spectrum\ \ of\, ZM\theta_1 = &\quad \{-0.7340, -0.1296, 0.0503, 0.8133\},\\ Spectrum\ \ of\, ZM\theta_2 = &\quad \{-0.5566, -0.3461, 0.0568, 0.8459\},\\ and,\quad Spectrum\ \ of\, ZM\theta_3 = &\quad \{-1.7040, -0.4565, 0.1889, 1.9716\}.\end{aligned}$$

Now, by using (3), the spherical fuzzy Zagreb energy of SFG is calculated as

$$\mathcal{SZE}(G) = \left(\sum_{i=1}^{n} |\xi_i|, \sum_{i=1}^{n} |\eta_i|, \sum_{i=1}^{n} |\zeta_i|\right).$$

By putting the values, we obtain

$$\mathcal{SZE}(G) = (1.7272, 1.8054, 4.321).$$

4. Main Results

In this section, we present some upper and lower bounds related to the maximum eigenvalue and spherical fuzzy energy of SFG. The obtained bounds are then illustrated with the help of examples to show the validity of the results.

Theorem 1. *Let $G = (R, S)$ with order n and size m be a spherical fuzzy undirected and connected graph defined on $G^* = (\check{V}, \check{E})$. Then,*

$$2g_t\delta_t e^* \leq SFM_1(G) \leq 2s_t\Delta_t e^*,$$

$$2g_a\delta_a e^* \leq SFM_1(G) \leq 2s_a\Delta_a e^*,$$

$$2g_f\delta_f e^* \leq SFM_1(G) \leq 2s_f\Delta_f e^*,$$

where these symbols s_t, s_a, s_f, g_t, g_a, g_f, Δ_t, Δ_a Δ_f, δ_a, δ_f and e^* are described in Table 1. In particular, the equalities in (5) hold for SFG such that $\breve{\alpha}_\mathcal{Y}$, $\breve{\beta}_\mathcal{Y}$ and $\breve{\gamma}_\mathcal{Y}$ are constant functions, and each vertex has the same spherical fuzzy degree.

Proof.

$$\begin{aligned}\sum_{v_iv_j\in\check{E}(G)} (\breve{\alpha}_\mathcal{Y}(v_i)d_t(v_i) + \breve{\alpha}_\mathcal{Y}(v_j)d_t(v_j) \quad &\leq \sum_{v_iv_j\in\check{E}(G)} [s_t d_t(v_i) + s_t d_t(v_j)]\\ &\leq s_t \sum_{v_iv_j\in\check{E}(G)} [d_t(v_i) + d_t(v_j)]\\ &\leq s_t \sum_{v_iv_j\in\check{E}(G)} 2\Delta_t\\ &\leq 2s_t\Delta_t \sum_{v_iv_j\in\check{E}(G)} 1\\ &\leq 2s_t\Delta_t e^*.\end{aligned}$$

It implies that

$$(SFM_1(G))_t \leq 2s_t\Delta_t e^*.$$

Similarly, we can prove other bounds. $\square$

Theorem 2. *Let $G = (R, S)$ with order n and size m be a spherical fuzzy undirected, and connected graph defined on $G^* = (\check{V}, \check{E})$, and let $\mathtt{SZM} = (ZM\theta_1, ZM\theta_2, ZM\theta_3)$ be its spherical fuzzy Zagreb matrix. If $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$, $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$, and $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_n$ are the eigenvalues of $ZM\theta_1$, $ZM\theta_2$, and $ZM\theta_3$, respectively. Then*

$$8e^*(\delta_t g_t)^2 \leq \sum_{i=1}^{n} \xi_i^2 \leq 8e^*(\Delta_t s_t)^2,$$

$$8e^*(\delta_a g_a)^2 \leq \sum_{i=1}^{n} \eta_i^2 \leq 8e^*(\Delta_a s_a)^2,$$

and

$$8e^*(\delta_f g_f)^2 \leq \sum_{i=1}^{n} \zeta_i^2 \leq 8e^*(\Delta_f s_f)^2.$$

where $s_t = \max\{\breve{\alpha}_{\mathcal{Y}}(v_i)\}$, $s_a = \max\{\breve{\beta}_{\mathcal{Y}}(v_i)\}$, and $s_f = \max\{\breve{\gamma}_{\mathcal{Y}}(v_i)\}$. $g_t = \min\{\breve{\alpha}_{\mathcal{Y}}(v_i)\}$, $g_a = \min\{\breve{\beta}_{\mathcal{Y}}(v_i)\}$, $g_f = \min\{\breve{\gamma}_{\mathcal{Y}}(v_i)\}$. Further, $\Delta_t = \max\{d_t(v_i)\}$, $\Delta_a = \max\{d_a(v_i)\}$ $\Delta_f = \max\{d_f(v_i)\}$, $\delta_t = \min\{d_t(v_i)\}$, $\delta_a = \min\{d_a(v_i)\}$, $\delta_f = \min\{d_f(v_i)\}$ and e^* denotes the number of edges in SFG. In particular, the equalities in (5) hold for SFG such that $\breve{\alpha}_{\mathcal{Y}}$, $\breve{\beta}_{\mathcal{Y}}$ and $\breve{\gamma}_{\mathcal{Y}}$ are constant functions, and each vertex has the same spherical fuzzy degree.

Proof. By the trace properties of a matrix, we have

$$\begin{aligned} tr(ZM\theta_1)^2 &= \sum_{i=1}^{n} \xi_i^2, \\ \text{where } tr(ZM\theta_1)^2 =& (0 + (\breve{\alpha}_{\mathcal{Y}}(v_1)d_t(v_1) + \breve{\alpha}_{\mathcal{Y}}(v_2)d_t(v_2))^2 + \cdots + (\breve{\alpha}_{\mathcal{Y}}(v_1) \\ & d_t(v_1) + \breve{\alpha}_{\mathcal{Y}}(v_n)d_t(v_n))^2) + ((\breve{\alpha}_{\mathcal{Y}}(v_2)d_t(v_2) + \breve{\alpha}_{\mathcal{Y}}(v_1) \\ & d_t(v_1))^2 + \cdots + (\breve{\alpha}_{\mathcal{Y}}(v_2)d_t(v_2) + \breve{\alpha}_{\mathcal{Y}}(v_n)d_t(v_n))^2) \\ & + \cdots + (0 + (\breve{\alpha}_{\mathcal{Y}}(v_n)d_t(v_n) + \breve{\alpha}_{\mathcal{Y}}(v_1)d_t(v_1))^2 \\ & + (\breve{\alpha}_{\mathcal{Y}}(v_n)d_t(v_n) + \breve{\alpha}_{\mathcal{Y}}(v_2)d_t(v_2))^2 + \cdots + 0). \\ =& 2 \sum_{v_i v_j \in \check{E}(G)} (\breve{\alpha}_{\mathcal{Y}}(v_i)d_t(v_i) + \breve{\alpha}_{\mathcal{Y}}(v_j)d_t(v_j))^2. \end{aligned}$$

Hence, we have

$$\sum_{i=1}^{n} \xi_i^2 = 2 \sum_{v_i v_j \in \check{E}(G)} (\breve{\alpha}_{\mathcal{Y}}(v_i)d_t(v_i) + \breve{\alpha}_{\mathcal{Y}}(v_j)d_t(v_j))^2. \quad (4)$$

Let $s_t = \max\{\breve{\alpha}_{\mathcal{Y}}(v_i)\}$ and $\Delta_t = \max\{d_t(v_i)\}$. Now consider,

$$\begin{aligned}\sum_{v_iv_j\in\check{E}(G)}(\check{\alpha}_{\mathcal{Y}}(v_i)d_t(v_i)+\check{\alpha}_{\mathcal{Y}}(v_j)d_t(v_j))^2 &\leq \sum_{v_iv_j\in\check{E}(G)}[s_td_t(v_i)+s_td_t(v_j)]^2\\ &\leq s_t^2\sum_{v_iv_j\in\check{E}(G)}[d_t(v_i)+d_t(v_j)]^2\\ &\leq s_t^2\sum_{v_iv_j\in\check{E}(G)}(2\Delta_t)^2\\ &\leq 4s_t^2\Delta_t^2\sum_{v_iv_j\in\check{E}(G)}1\\ &\leq 4s_t^2\Delta_t^2e^*\\ &\leq (2\Delta s_t)^2e^*. \end{aligned} \tag{5}$$

From (4) and (5), we have

$$\sum_{i=1}^{n}\xi_i^2=2\sum_{v_iv_j\in\check{E}(G)}(\check{\alpha}_{\mathcal{Y}}(v_i)d_t(v_i)+\check{\alpha}_{\mathcal{Y}}(v_j)d_t(v_j))^2\leq 8(\Delta_ts_t)^2e^*.$$

Hence,

$$\sum_{i=1}^{n}\xi_i^2\leq 8e^*(\Delta_ts_t)^2. \tag{6}$$

Similarly, we have

$$\sum_{i=1}^{n}\eta_i^2\leq 8e^*(\Delta_as_a)^2,$$

and

$$\sum_{i=1}^{n}\zeta_i^2\leq 8e^*(\Delta_fs_f)^2.$$

For the lower bound, let $\delta_t=\min\{d_t(v_i)\}$, $g_t=\min\{\check{\alpha}_{\mathcal{Y}}(v_i)\}$. Now using the same arguments as those used to prove (5), we have

$$\begin{aligned}\sum_{v_iv_j\in\check{E}(G)}(\check{\alpha}_{\mathcal{Y}}(v_i)d_t(v_i)+\check{\alpha}_{\mathcal{Y}}(v_j)d_t(v_j))^2 &\geq \sum_{v_iv_j\in\check{E}(G)}[g_td_t(v_i)+g_td_t(v_j)]^2.\\ &\geq 4(\delta_tg_t)^2e^*.\end{aligned}$$

Hence,

$$\sum_{i=1}^{n}|\xi_i|^2\geq 8e^*(\delta_tg_t)^2. \tag{7}$$

Similarly,

$$\sum_{i=1}^{n}|\eta_i|^2\geq 8e^*(\delta_ag_a)^2.$$

$$\sum_{i=1}^{n}|\zeta_i|^2\geq 8e^*(\delta_fg_f)^2.$$

□

Theorem 3. *Consider a SFG G=(R,S) having n number of vertices and*

$$\mathtt{SZM}(G)=\Big(ZM\theta_1,\ ZM\theta_2,\ ZM\theta_3\Big)$$

as the corresponding $\mathtt{SZM}$ *of* G. *Then,*

$$\sqrt{8e^*(\delta_t g_t)^2 + n(n-1)|det(ZM\theta_1)|^{\frac{2}{n}}} \leq \mathtt{ZE}(ZM\theta_1) \leq 2\sqrt{2ne^*}(s_t\Delta_t).$$

$$\sqrt{8e^*(\delta_a g_a)^2 + n(n-1)|det(ZM\theta_2)|^{\frac{2}{n}}} \leq \mathtt{ZE}(ZM\theta_2) \leq 2\sqrt{2ne^*}(s_a\Delta_a).$$

and

$$\sqrt{8e^*(\delta_f g_f)^2 + n(n-1)|det(ZM\theta_3)|^{\frac{2}{n}}} \leq \mathtt{ZE}(ZM\theta_3) \leq 2\sqrt{2ne^*}(s_f\Delta_f).$$

Proof. By considering two sequences (1, 1, 1, ..., 1) and $(|\xi_1|, |\xi_2|, |\xi_3|, ..., |\xi_n|)$ and applying the Cauchy–Schwarz inequality, we have

$$\sum_{i=1}^{n} |\xi_i| \leq \sqrt{n}\sqrt{\sum_{i=1}^{n} |\xi_i|^2} \tag{8}$$

By using (6) from Theorem 2,

$$\sum_{i=1}^{n} |\xi_i|^2 \leq 8e^*(\Delta_t s_t)^2$$

Equation (8) implies

$$\begin{aligned} \sum_{i=1}^{n} |\xi_i| &\leq \sqrt{n}\sqrt{8e^*(\Delta_t s_t)^2} \\ \sum_{i=1}^{n} |\xi_i| &\leq 2\sqrt{2ne^*}(s_t\Delta_t) \end{aligned}$$

Now, by using (3), it follows that

$$\mathtt{ZE}(ZM\theta_1) \leq 2\sqrt{2ne^*}(s_t\Delta_t).$$

Similarly, we have

$$\begin{aligned} \mathtt{ZE}(ZM\theta_2) &\leq 2\sqrt{2ne^*}(s_a\Delta_a), \\ \mathtt{ZE}(ZM\theta_3) &\leq 2\sqrt{2ne^*}(s_f\Delta_f). \end{aligned}$$

Now, for the lower bound, consider

$$\begin{aligned} (\mathtt{ZE}(ZM\theta_1))^2 &= (\sum_{i=1}^{n} |\xi_i|)^2. \\ &= \sum_{i=1}^{n} |\xi_i|^2 + 2\sum_{1\leq i<j\leq n} |\xi_i\xi_j|. \\ &= \sum_{i=1}^{n} |\xi_i|^2 + \frac{2n(n-1)}{2} AM\{|\xi_i\xi_j|\}, \end{aligned}$$

where AM represents the arithmetic mean of the terms $\{|\xi_i\xi_j|\}$. As we know that the arithmetic mean (AM) of the terms is always greater than the geometric mean (GM) of these terms $\{|\xi_i\xi_j|\}$ i.e., AM $\{|\xi_i\xi_j|\} \geq GM\{|\xi_i\xi_j|\}, 1 \leq i < j \leq n$; therefore, it follows that

$$\mathtt{ZE}(ZM\theta_1) \geq \sqrt{\sum_{i=1}^{n} |\xi_i|^2 + n(n-1)GM\{|\xi_i\xi_j|\}}.$$

Now from Theorem 2, we have

$$\mathtt{ZE}(ZM\theta_1) \geq \sqrt{8e^*(\delta_t g_t)^2 + n(n-1)GM\{|\xi_i\xi_j|\}}. \tag{9}$$

Also,

$$\begin{aligned} GM\{|\xi_i\xi_j|\} &= \left(\prod_{1\leq i<j\leq n} |\xi_i\xi_j|\right)^{\frac{2}{n(n-1)}}. \\ &= \left(\prod_{i=1}^{n} |\xi_i|^{n-1}\right)^{\frac{2}{n(n-1)}}. \\ &= \left(\prod_{i=1}^{n} |\xi_i|\right)^{\frac{2}{n}}. \\ &= \left|det(ZM\theta_1)\right|^{\frac{2}{n}}. \end{aligned} \tag{10}$$

Therefore, by substituting (10) in (9), we obtain

$$\mathtt{ZE}(ZM\theta_1) \geq \sqrt{8e^*(\delta_t g_t)^2 + n(n-1)|det(ZM\theta_1)|^{\frac{2}{n}}}.$$

Thus,

$$\sqrt{8e^*(\delta_t g_t)^2 + n(n-1)|det(ZM\theta_1)|^{\frac{2}{n}}} \leq \mathtt{ZE}(ZM\theta_1) \leq 2\sqrt{2ne^*}(s_t\Delta_t).$$

Similarly, we have

$$\sqrt{8e^*(\delta_a g_a)^2 + n(n-1)|det(ZM\theta_2)|^{\frac{2}{n}}} \leq \mathtt{ZE}(ZM\theta_2) \leq 2\sqrt{2ne^*}(s_a\Delta_a).$$

and

$$\sqrt{8e^*(\delta_f g_f)^2 + n(n-1)|det(ZM\theta_3)|^{\frac{2}{n}}} \leq \mathtt{ZE}(ZM\theta_3) \leq 2\sqrt{2ne^*}(s_f\Delta_f).$$

□

Example 4. *Consider the SFG* `W=(P, Q)` *as shown in Figure 3. The value of* `ZE` *from Example 3 is given as* $\mathtt{ZE}(ZM\theta_1) = \sum_{i=1}^{n}|\xi_i| = 1.7272$, $\mathtt{ZE}(ZM\theta_2) = \sum_{i=1}^{n}|\eta_i| = 1.8054$, *and* $\mathtt{ZE}(ZM\theta_3) = \sum_{i=1}^{n}|\zeta_i| = 4.321$. *Now from Theorem 3, the lower bound for* $\mathtt{ZE}(ZM\theta_1) = \sqrt{8e^*(\delta_t g_t)^2 + n(n-1)|det(ZM\theta_1)|^{\frac{2}{n}}}$, *and the upper bound for* $\mathtt{ZE}(ZM\theta_1) = 2\sqrt{2ne^*}(\Delta_t s_t)$. *Here,* $n = 4$, $e^* = 4$, $\delta_t = 0.2$, $g_t = 0.2$, $det(ZM\theta_1) = 0.0039$, $\Delta_t = 0.6$, *and* $s_t = 0.6$. *Thus, we obtain*

$$\begin{aligned} \textit{Lower bound for } \mathtt{ZE}(ZM\theta_1) &= \sqrt{8e^*(\delta_t g_t)^2 + n(n-1)|det(ZM\theta_1)|^{\frac{2}{n}}} \\ &= \sqrt{8*4(0.2*0.2)^2 + 4(3)(0.0039)^{\frac{2}{4}}} = 0.8947624. \\ \textit{Upper bound for } \mathtt{ZE}(ZM\theta_1) &= 2\sqrt{2ne^*}(\Delta_t s_t) \\ &= 2\sqrt{2*4*4}(0.6*0.6) = 4.07293506. \end{aligned}$$

Thus,

$$0.8947624 < ZE(ZM\theta_1) = 1.7272 < 4.07293506.$$

Similarly, we compute that

$$1.08168299 < ZE(ZM\theta_2) = 1.8054 < 3.3941125,$$

and

$$2.6305235 < ZE(ZM\theta_3) = 4.321 < 11.87939392.$$

This shows that the values of spherical Zagreb fuzzy energies lie within the bounds given in Theorem 3.

Remark 2. *In particular, if we take a SFG such that the membership values of the vertices are constant, the spherical fuzzy degree of each vertex is also the same. Then, the bounds given in Theorem 3 seem to be sharp.*

Remark 2 can be explained with the following example:

Example 5. *Let D be a SFG as depicted in Figure 4. Since, $ZE(ZM\theta_1) = \sum_{i=1}^{n} |\xi_i| = 0.3200$, $ZE(ZM\theta_2) = \sum_{i=1}^{n} |\eta_i| = 0.9600$, and $ZE(ZM\theta_3) = \sum_{i=1}^{n} |\zeta_i| = 3.200$. Now from Theorem 3, Lower bound for $ZE(ZM\theta_1) = \sqrt{8e^*(\delta_t g_t)^2 + n(n-1)|det(ZM\theta_1)|^{\frac{2}{n}}}$, Upper bound for $ZE(ZM\theta_1) = 2\sqrt{2ne^*}(\Delta_t s_t)$. Here, $n = 4$, $e^* = 3$, $\delta_t = 0.2$, $g_t = 0.2$, $det(ZM\theta_1) = 0.0010$, $\Delta_t = 0.2$, and $s_t = 0.2$. Thus, we obtain*

$$\begin{aligned} \textit{Lower bound for } ZE(ZM\theta_1) = \sqrt{8e^*(\delta_t g_t)^2 + n(n-1)|det(ZM\theta_1)|^{\frac{2}{n}}} \\ \sqrt{8*3(0.2*0.2)^2 + 3(2)(0.0010)^{\frac{2}{3}}} &= 0.3181. \\ \textit{Upper bound for } ZE(ZM\theta_1) = 2\sqrt{2ne^*}(\Delta_t s_t) \\ 2\sqrt{2*3*3}(0.2*0.2) &= 0.33941. \end{aligned}$$

Thus,

$$0.3181 < ZE(ZM\theta_1) = 0.3200 < 0.33941.$$

Similarly, we compute that

$$0.9459 < ZE(ZM\theta_2) = 0.9600 < 1.0182,$$

and

$$3.1519 < ZE(ZM\theta_3) = 3.200 < 3.3941.$$

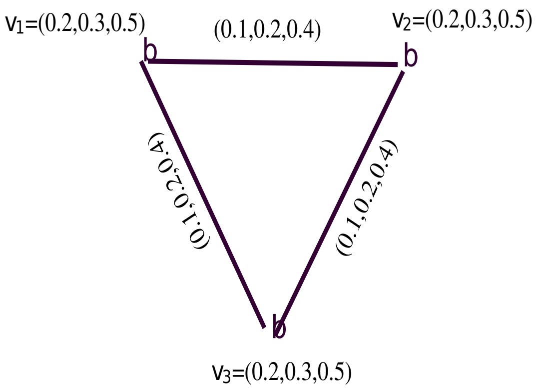

Figure 4. The SFG D.

Lemma 1. *If $G = (R, S)$ is a SFG with $\mathtt{SZM}(G) = (ZM\theta_1, ZM\theta_2, ZM\theta_3)$ is its corresponding $\mathtt{SZM}$ such that ξ_1 is the maximum eigenvalue of $ZM\theta_1$, η_1 is the maximum eigenvalue of $ZM\theta_2$, and ζ_1 is the maximum eigenvalue of $ZM\theta_3$, then*

$$\xi_1 \leq 2s_t\Delta_t\Delta^*, \eta_1 \leq 2s_a\Delta_a\Delta^*, \zeta_1 \leq 2s_f\Delta_f\Delta^*,$$

where Δ^* is a maximum crisp degree.

Proof. Given that ξ_1 is the maximum eigenvalue, let T be the eigenvector that corresponds to ξ_1. Then, vividly,

$$\xi_1 T = ZM\theta_1 T.$$

If the vector T has maximum entry $t_0 \geq 0$ corresponding to the vertex v_0 and $[ZM\theta_1]$ has corresponding row $[ZM\theta_1]_{v_0}$, then we obtain

$$\xi_1 t_0 = [ZM\theta_1]_{v_0} v_0 \leq \left(\sum_{v_0 v_j \in \check{E}(G)} \breve{\alpha}\mathcal{Y}(v_0) d_t(v_0) + \breve{\alpha}\mathcal{Y}(v_0) d_t(v_j) \right) t_0$$

By using the same arguments, as we prove (2) in Theorem 2, we have

$$\xi_1 t_0 \leq 2s_t\Delta_t\Delta^* t_0.$$

Hence,

$$\xi_1 \leq 2s_t\Delta_t\Delta^*. \tag{11}$$

Similarly, for $ZM\theta_2$ and $ZM\theta_3$, we have

$$\begin{aligned} \eta_1 &\leq 2s_a\Delta_a\Delta^*, \\ \zeta_1 &\leq 2s_f\Delta_f\Delta^*. \end{aligned}$$

□

Theorem 4. *For a SFG $G = (R, S)$ with n vertices and*

$$\mathtt{SZM}(G) = (ZM\theta_1, ZM\theta_2, ZM\theta_3),$$

$$\mathtt{ZE}(ZM\theta_1) \leq 2s_t\Delta_t\Delta^* + \sqrt{(n-1)\{8e^*(\Delta_t s_t)^2 - \left(\frac{2g_t\delta_t e^*}{n}\right)^2\}}.$$

Similarly,

$$\mathtt{ZE}(ZM\theta_2) \leq 2s_a\Delta_a\Delta^* + \sqrt{(n-1)\{8e^*(\Delta_a s_a)^2 - \left(\frac{2g_a\delta_a e^*}{n}\right)^2\}},$$

$$\mathtt{ZE}(ZM\theta_3) \leq 2s_f\Delta_f\Delta^* + \sqrt{(n-1)\{8e^*(\Delta_f s_f)^2 - \left(\frac{2g_f\delta_f e^*}{n}\right)^2\}}.$$

Proof. Since $ZM\theta_1$, $ZM\theta_2$ and $ZM\theta_3$ are symmetric matrices with zero trace, so we have

$$\xi_{max} \geq \frac{2\sum_{1\leq i\leq j\leq n}[ZM\theta_1]_{ij}}{n}, \ \eta_{max} \geq \frac{2\sum_{1\leq i\leq j\leq n}[ZM\theta_2]_{ij}}{n}, \ \zeta_{max} \geq \frac{2\sum_{1\leq i\leq j\leq n}[ZM\theta_3]_{ij}}{n}$$

where ξ_{max}, η_{max} and ζ_{max} are the greatest eigenvalues of $ZM\theta_1$, $ZM\theta_2$, and $ZM\theta_3$, respectively. Let ξ_{max} be the maximum eigenvalue of the matrix $ZM\theta_1$ and by using Definition 7, we can write

$$\xi_{max} \geq \frac{2\sum_{v_i v_j \in E(G)}[\breve{\alpha}\mathcal{Y}(v_i) d_t(v_i) + \breve{\alpha}\mathcal{Y}(v_j) d_t(v_j)]}{n}$$

By using Definition 5 , we can write

$$\geq \frac{2(SFM_1(G))_t}{n}$$

Using Theorem 1,

$$\begin{aligned}
\xi_{max} &\geq \frac{2(2g_t\delta_t e_t^*)}{n}. \\
\xi_{max} &\geq \frac{4g_t\delta_t e_t^*}{n}.
\end{aligned} \tag{12}$$

Moreover, from Theorem 2, we have $\xi_1 = \xi_{max}$ such that

$$\sum_{i=1}^{n} \xi_i^2 \leq 8e^*(\Delta_t s_t)^2.$$

$$\sum_{i=2}^{n} \xi_i^2 \leq 8e^*(\Delta_t s_t)^2 - \xi_{max}^2. \tag{13}$$

By using (3), we can write

$$\begin{aligned}
\mathtt{ZE}(ZM\theta_1) &= \sum_{i=1}^{n} |\xi_i|. \\
&= \sum_{i=2}^{n} |\xi_i| - \xi_{max}.
\end{aligned}$$

It implies that

$$\mathtt{ZE}(ZM\theta_1) - \xi_{max} = \sum_{i=2}^{n} |\xi_i|. \tag{14}$$

By considering two sequences $(1, 1, 1, \cdots, 1)$ and $(|\xi_1|, |\xi_2|, \cdots, |\xi_n|)$ and then applying the Cauchy–Schwarz inequality with $n-1$ entries, we have

$$\sum_{i=2}^{n} |\xi_i| \leq \sqrt{(n-1)\sum_{i=2}^{n} |\xi_i|^2}. \tag{15}$$

From (14) and (15), we have

$$\mathtt{ZE}(ZM\theta_1) - \xi_{max} \leq \sqrt{(n-1)\sum_{i=2}^{n} |\xi_i|^2}. \tag{16}$$

By substituting (13) in (16), we attain

$$\mathtt{ZE}(ZM\theta_1) - \xi_{max} \leq \sqrt{(n-1)(8e^*(\Delta_t s_t)^2 - \xi_{max}^2)}.$$

$$\mathtt{ZE}(ZM\theta_1) \leq \xi_{max} + \sqrt{(n-1)(8e^*(\Delta_t s_t)^2 - \xi_{max}^2)}. \tag{17}$$

Using Lemma 1, $\xi_1 = \xi_{max} \leq 2s_t\Delta_t\Delta^*$ and from (12), $-\xi_{max} \leq -\frac{4g_t\delta_t e_t^*}{n}$. Hence,

$$\mathtt{ZE}(ZM\theta_1) \leq 2s_t\Delta_t\Delta^* + \sqrt{(n-1)\{8e^*(\Delta_t s_t)^2 - \left(\frac{2g_t\delta_t e^*}{n}\right)^2\}}.$$

Similarly,

$$\texttt{ZE}(ZM\theta_2) \leq 2s_a\Delta_a\Delta^* + \sqrt{(n-1)\{8e^*(\Delta_a s_a)^2 - \left(\frac{2g_a\delta_a e^*}{n}\right)^2\}}.$$

$$\texttt{ZE}(ZM\theta_3) \leq 2s_f\Delta_f\Delta^* + \sqrt{(n-1)\{8e^*(\Delta_f s_f)^2 - \left(\frac{2g_f\delta_f e^*}{n}\right)^2\}}.$$

□

The above Theorem 4 can be explained by the following example:

Example 6. *Consider the SFG* `W=(P, Q)` *as shown in Figure 3. The value of* `ZE` *from Example 3 is given as* $\texttt{ZE}(ZM\theta_1) = \sum_{i=1}^{n}|\xi_i| = 1.7272$, $\texttt{ZE}(ZM\theta_2) = \sum_{i=1}^{n}|\eta_i| = 1.8054$, *and* $\texttt{ZE}(ZM\theta_3) = \sum_{i=1}^{n}|\zeta_i| = 4.321$. *Now from Theorem 4, the upper bound for* $\texttt{ZE}(ZM\theta_1)$ $2s_t\Delta_t\Delta^* + \sqrt{(n-1)\{8e^*(\Delta_t s_t)^2 - \left(\frac{2g_t\delta_t e^*}{n}\right)^2\}}$. *Here,* $n = 4$, $e^* = 4$, $\delta_t = 0.2$, $g_t = 0.2$, $\Delta_t = 0.6$, $\Delta^* = 3$ *and* $s_t = 0.6$. *Thus upper bound for textttZE*$(ZM\theta_1)$ *is* 2.97. *Similarly, we compute the upper bounds for* $\texttt{ZE}(ZM\theta_2)$ *as 5.53 and for* $\texttt{ZE}(ZM\theta_3)$ *as 14.37. Thus,*

$$\texttt{ZE}(ZM\theta_1) = 1.7272 < 2.97,$$

$$1.08168299 < \texttt{ZE}(ZM\theta_2) = 1.8054 < 5.53,$$

and

$$2.6305235 < \texttt{ZE}(ZM\theta_3) = 4.321 < 14.37.$$

This shows that the values of the spherical Zagreb fuzzy energies lie within the bounds given in Theorem 4.

Remark 3. *It is observed from Examples 4 and 6 that the upper bound of* $\texttt{ZE}(ZM\theta_1)$ *obtained from Theorem 4 is closer to the exact value than the upper bound obtained from Theorem 1, whereas the upper bounds of* $\texttt{ZE}(ZM\theta_2)$ *and* $\texttt{ZE}(ZM\theta_3)$ *given in Theorem 1 are closer to the exact value as compared to the upper bounds obtained in Theorem 4. This indicates that together with both bounds obtained from Theorems 1 and 4, we can better estimate the exact value of the spherical Zagreb fuzzy energies of SFG.*

5. Application

In this section, we present an application related to the selection of the best location to start a new business to describe the suitability of *SFGs*. Further, we illustrate a comparative analysis with the existing technique.

5.1. Selection of Best Location for Business Purpose

Business depends on profitability and loss; thus, it has a ratio of success or failure with the wide range of variations. Usually, the term business is figured out as a 'company', but it has broad meanings. Business is not just the ownership of a multinational company; the work of a street peddler is a business too. Some of the factors which enhance the profitability of business include high skills of management, mental peace, a broad mental vision , teamwork, patience, and good work performance. But one of the most important factors is location to upgrade the level of business. Since the growth of a business is not certain, a businessman should be mentally ready to accept the decline of a business at any stage of its life. But, if proper consultation is given by an expert, then a business can achieve success at a large scale.

Therefore, we consider five different locations L_i, $(i = 1,2,3,4,5)$ for a new business and taking a panel of four experts U_j, $(j = 1,2,3,4)$ from real estate, finance, law and business communities to decide the most preferable location for the required purpose. The following are the some factors that affect the location.

1. Proximity to target customers.
2. Competitors' locations.
3. Taxes (utilities and other costs).
4. Government laws and policies.
5. Infrastructure and accessibility.
6. Safety.
7. Parking facility.

Each consultant makes distinct judgments based on their experiences between the two different locations. Their opinions depend on three factors, including neutral, appropriate and inappropriate.

An Algorithm 1 to describe our proposed technique is given below:

Algorithm 1: Algorithm to find the best place for business purpose.

INPUT: A discrete set of locations $L = \{\check{l}_1, \check{l}_2, \check{l}_3, \check{l}_4, \check{l}_5\}$, a set of decision-makers $U = \{u_1, u_2, u_3, u_4\}$ in order to attain the goal and fashioning of spherical fuzzy preference relations (SFPRs) $\mathtt{M}_{\check{a}}$, $\check{a} = (1, 2, 3, 4)$ for each consultant.

OUTPUT: The nomination of best location for business.

1. Consider four SFGs corresponding to four spherical fuzzy preference relations. Make spherical fuzzy Zagreb matrices $(\mathtt{SZM})_i$ $(i = 1, 2, 3, 4)$ as defined in Definition 7 corresponding to the given SFGs.
2. Calculate the $\mathtt{SZE}$ of all $(\mathtt{SZM})_i$, (i=1, 2, 3, 4) denoted as

$$\mathtt{SZE}(G) = \Big(ZE(ZM\theta_1),\ ZE(ZM\theta_2),\ ZE(ZM\theta_3)\Big).$$

which can be calculated by adding the absolute values of eigenvalues of matrices, i.e.,

$$= \left(\sum_{i=1}^{n} |\xi_i|, \sum_{i=1}^{n} |\eta_i|, \sum_{i=1}^{n} |\zeta_i|\right),$$

where $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$, $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$, and $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_n$ are the eigenvalues of $ZM\theta_1,\ ZM\theta_2$, and $ZM\theta_3$, respectively.

3. Determine the weight vector by using:

$$w_i = ((w_\alpha)_i, (w_\beta)_i, (w_\gamma)_i).$$

$$= \left(\frac{\mathtt{ZE}((\mathtt{SZM})_{\check{\alpha}})_i}{\sum_{j=1}^{4} \mathtt{ZE}((\mathtt{SZM})_{\check{\alpha}})_j}, \frac{\mathtt{ZE}((\mathtt{SZM})_{\check{\beta}})_i}{\sum_{j=1}^{4} \mathtt{ZE}((\mathtt{SZM})_{\check{\beta}})_j}, \frac{\mathtt{ZE}((\mathtt{SZM})_{\check{\gamma}})_i}{\sum_{j=1}^{4} \mathtt{ZE}((\mathtt{SZM})_{\check{\gamma}})_j}\right). \tag{18}$$

4. Now to aggregate the spherical Zagreb preference matrices obtained in step 2, we have to apply one of the suitable aggregation operators to merge the multi-agent information. For this purpose, we apply a well- known simple ordered averaging operator (OWA) [46] given as $\mathtt{SZM} = \sum_{i=1}^{4}(w_i)((\mathtt{SZM})_i)$, where w_i are given from (18) obtained in step 4 to form the combined spherical fuzzy Zagreb preference relation $(SFZPR)$ by aggregating the matrices $(SFZPRs)$ obtained in step 2.
5. Plot an impact model related to collective SFZPR.
6. Plot the partial impact model associated with collective SFZPR using constraint $\alpha_{ij} \geq 0.5$. Determine the degrees of all vertices (alternatives) that appear in the model of SFZPR.
7. All the alternatives are ranked according to the truthiness degree of vertices.
8. *Output*: The best alternative is nominated based on the largest truthiness degree of the vertex.

The flowchart corresponding to the proposed technique is exhibited in Figure 5:

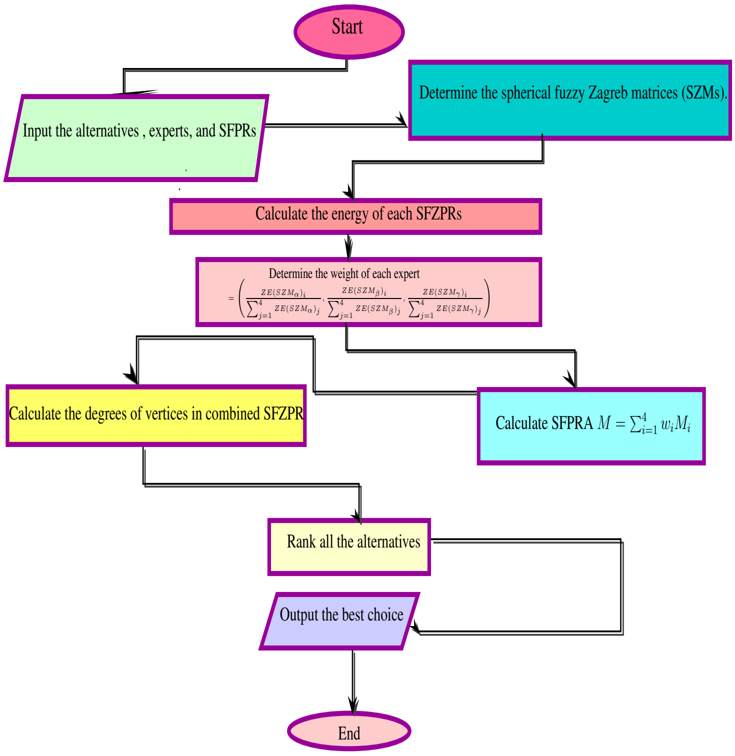

Figure 5. Flowchart for algorithm.

The spherical fuzzy preference relations ($SFPRs$) $\mathtt{M}_{\breve{a}}$, $\breve{a} = (1,2,3,4)$ are given in Tables 4–7:

Table 4. SFPRs of the first decision-making expert.

$\mathtt{M}_1$	$\breve{l}_1$	$\breve{l}_2$	$\breve{l}_3$	$\breve{l}_4$	$\breve{l}_5$
$\breve{l}_1$	(0.3, 0.6, 0.5)	(0.1, 0.3, 0.6)	(0.3, 0.1, 0.5)	(0.3, 0.1, 0.3)	(0.3, 0.2, 0.4)
$\breve{l}_2$	(0.1, 0.3, 0.6)	(0.2, 0.4, 0.7)	(0.1, 0.1, 0.5)	(0.2, 0.1, 0.6)	(0.1, 0.2, 0.5)
$\breve{l}_3$	(0.3, 0.1, 0.5)	(0.1, 0.1, 0.5)	(0.3, 0.1, 0.6)	(0.2, 0.1, 0.5)	(0.3, 0.1, 0.5)
$\breve{l}_4$	(0.3, 0.1, 0.3)	(0.2, 0.1, 0.6)	(0.2, 0.1, 0.5)	(0.4, 0.2, 0.4)	(0.3, 0.1, 0.4)
$\breve{l}_5$	(0.3, 0.2, 0.4)	(0.1, 0.2, 0.5)	(0.3, 0.1, 0.5)	(0.3, 0.1, 0.4)	(0.5, 0.2, 0.3)

Table 5. SFPRs of the second decision-making expert.

$\mathtt{M}_2$	$\breve{l}_1$	$\breve{l}_2$	$\breve{l}_3$	$\breve{l}_4$	$\breve{l}_5$
$\breve{l}_1$	(0.3, 0.2, 0.5)	(0.2, 0.1, 0.6)	(0.2, 0.1, 0.4)	(0.2, 0.1, 0.4)	(0.1, 0.1, 0.4)
$\breve{l}_2$	(0.2, 0.1, 0.6)	(0.4, 0.3, 0.7)	(0.4, 0.2, 0.6)	(0.3, 0.2, 0.6)	(0.1, 0.2, 0.5)
$\breve{l}_3$	(0.2, 0.1, 0.4)	(0.4, 0.2, 0.6)	(0.5, 0.6, 0.2)	(0.3, 0.4, 0.3)	(0.1, 0.3, 0.5)
$\breve{l}_4$	(0.2, 0.1, 0.4)	(0.3, 0.2, 0.6)	(0.3, 0.4, 0.3)	(0.4, 0.5, 0.3)	(0.1, 0.3, 0.4)
$\breve{l}_5$	(0.1, 0.1, 0.4)	(0.1, 0.2, 0.5)	(0.1, 0.3, 0.5)	(0.1, 0.3, 0.4)	(0.1, 0.4, 0.6)

Table 6. SFPRs of the third decision-making expert.

$\mathtt{M}_3$	$\breve{l}_1$	$\breve{l}_2$	$\breve{l}_3$	$\breve{l}_4$	$\breve{l}_5$
$\breve{l}_1$	(0.8, 0.4, 0.3)	(0.3, 0.3, 0.4)	(0.1, 0.2, 0.3)	(0.3, 0.2, 0.3)	(0.2, 0.3, 0.4)
$\breve{l}_2$	(0.3, 0.3, 0.4)	(0.4, 0.3, 0.5)	(0.1, 0.2, 0.4)	(0.2, 0.1, 0.4)	(0.1, 0.2, 0.4)
$\breve{l}_3$	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.4)	(0.2, 0.5, 0.6)	(0.2, 0.2, 0.5)	(0.1, 0.2, 0.5)
$\breve{l}_4$	(0.3, 0.2, 0.3)	(0.2, 0.1, 0.4)	(0.2, 0.2, 0.5)	(0.7, 0.3, 0.1)	(0.1, 0.2, 0.3)
$\breve{l}_5$	(0.2, 0.3, 0.4)	(0.1, 0.2, 0.4)	(0.1, 0.2, 0.5)	(0.1, 0.2, 0.3)	(0.2, 0.3, 0.4)

Table 7. SFPRs of the fourth decision-making expert.

$\mathtt{M}_4$	$\check{l}_1$	$\check{l}_2$	$\check{l}_3$	$\check{l}_4$	$\check{l}_5$
$\check{l}_1$	(0.2, 0.7, 0.5)	(0.3, 0.2, 0.4)	(0.1, 0.2, 0.4)	(0.1, 0.3, 0.4)	(0.1, 0.4, 0.6)
$\check{l}_2$	(0.3, 0.2, 0.4)	(0.4, 0.3, 0.1)	(0.2, 0.1, 0.2)	(0.2, 0.3, 0.5)	(0.1, 0.2, 0.6)
$\check{l}_3$	(0.1, 0.2, 0.4)	(0.2, 0.1, 0.2)	(0.5, 0.3, 0.2)	(0.2, 0.3, 0.4)	(0.1, 0.2, 0.6)
$\check{l}_4$	(0.1, 0.3, 0.4)	(0.2, 0.3, 0.5)	(0.2, 0.3, 0.4)	(0.3, 0.4, 0.6)	(0.2, 0.3, 0.5)
$\check{l}_5$	(0.1, 0.4, 0.6)	(0.1, 0.2, 0.6)	(0.1, 0.2, 0.6)	(0.2, 0.3, 0.5)	(0.2, 0.5, 0.7)

The spherical fuzzy graphs $(SFG)_j$, $j = (1,2,3,4)$ based on spherical fuzzy preference relations $(SFPRs)$ $\mathtt{M}_{\check{a}}$, $\check{a} = (1,2,3,4)$ are constructed by taking each expert's preference between two distinct alternatives as the membership values of the edges, including degree of truthiness, degree of abstinence, and degree of falseness. These spherical fuzzy graphs $(SFG)_j$, $j = (1,2,3,4)$ are depicted in Figures 6 and 7.

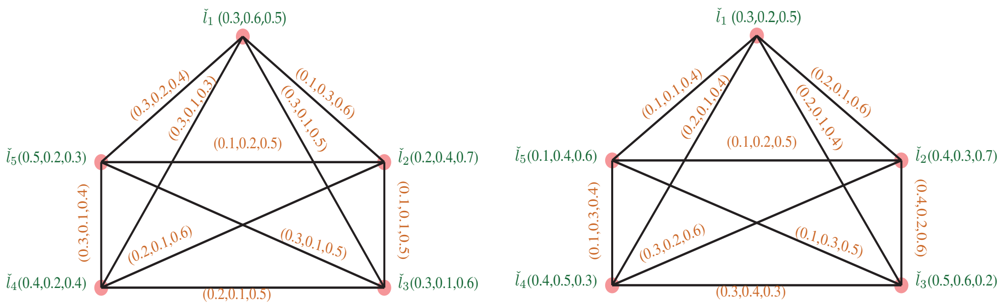

Figure 6. The graphical representations of $\mathtt{M}_{\check{1}}$ and $\mathtt{M}_{\check{2}}$.

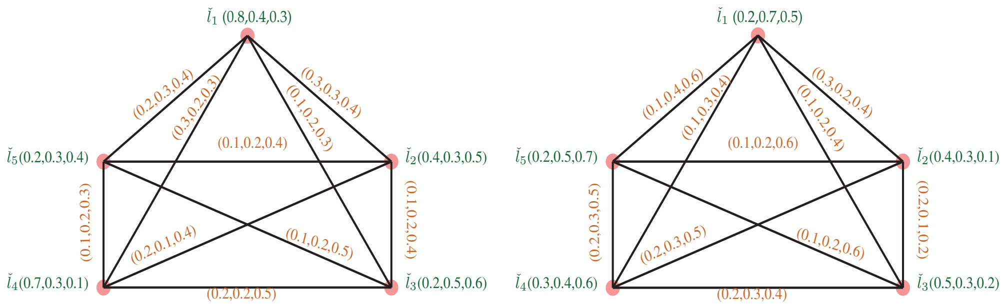

Figure 7. The graphical representations of $\mathtt{M}_{\check{3}}$ and $\mathtt{M}_{\check{4}}$.

Now, we form the spherical fuzzy Zagreb matrices $(\mathtt{SZM})_i$, $i = (1,2,3,4)$ of each $SFGs$. These matrices are based on the degree to which the vertices will give the effect of connectivity in the neighborhood. The $(\mathtt{SZM})_i$, $i = (1,2,3,4)$ of each $SFGs$ is given below:

$$(\mathtt{SZM})_1 = \begin{bmatrix} (0,0,0) & (0.4,0.7,2.44) & (0.57,0.46,2.1) & (0.7,0.5,1.62) & (0.8,0.54,1.44) \\ (0.4,0.7,2.44) & (0,0,0) & (0.37,0.32,2.74) & (0.5,0.36,2.26) & (0.6,0.4,2.08) \\ (0.57,0.46,2.1) & (0.37,0.32,2.74) & (0,0,0) & (0.67,0.12,1.92) & (0.77,0.16,1.74) \\ (0.7,0.5,1.62) & (0.5,0.36,2.26) & (0.67,0.12,1.92) & (0,0,0) & (0.9,0.2,1.26) \\ (0.8,0.54,1.44) & (0.6,0.4,2.08) & (0.77,0.16,1.74) & (0.9,0.2,1.26) & (0,0,0) \end{bmatrix}.$$

$$(\mathtt{SZM})_2 = \begin{bmatrix} (0,0,0) & (0.61,0.29,2.51) & (0.71,0.68,1.26) & (0.57,0.58,1.41) & (0.25,0.44,1.98) \\ (0.61,0.29,2.51) & (0,0,0) & (0.9,0.81,1.97) & (0.76,0.71,2.12) & (0.44,0.57,2.69) \\ (0.71,0.68,1.26) & (0.9,0.81,1.97) & (0,0,0) & (0.86,1.1,0.87) & (0.54,0.96,1.44) \\ (0.57,0.58,1.41) & (0.76,0.71,2.12) & (0.86,1.1,0.87) & (0,0,0) & (0.4,0.86,1.59) \\ (0.25,0.44,1.98) & (0.44,0.57,2.69) & (0.54,0.96,1.44) & (0.4,0.86,1.59) & (0,0,0) \end{bmatrix}.$$

$$(\mathtt{SZM})_3 = \begin{bmatrix} (0,0,0) & (1,0.64,1.22) & (0.82,0.8,1.44) & (1.28,0.61,0.57) & (0.82,0.67,1.06) \\ (1,0.64,1.22) & (0,0,0) & (0.38,0.64,1.82) & (0.84,0.45,0.95) & (0.38,0.51,1.44) \\ (0.82,0.8,1.44) & (0.38,0.64,1.82) & (0,0,0) & (0.66,0.61,1.17) & (0.2,0.67,1.66) \\ (1.28,0.61,0.57) & (0.84,0.45,0.95) & (0.66,0.61,1.17) & (0,0,0) & (0.66,0.48,0.79) \\ (0.82,0.67,1.06) & (0.38,0.51,1.44) & (0.2,0.67,1.66) & (0.66,0.48,0.79) & (0,0,0) \end{bmatrix}.$$

$$(\mathtt{SZM})_4 = \begin{bmatrix} (0,0,0) & (0.44,1.01,1.07) & (0.42,1.01,1.22) & (0.33,1.25,1.98) & (0.22,1.32,2.51) \\ (0.44,1.01,1.07) & (0,0,0) & (0.62,0.48,0.49) & (0.53,0.72,1.25) & (0.42,0.79,1.78) \\ (0.42,1.01,1.22) & (0.62,0.48,0.49) & (0,0,0) & (0.51,0.72,1.4) & (0.4,0.79,1.93) \\ (0.33,1.25,1.98) & (0.53,0.72,1.25) & (0.51,0.72,1.4) & (0,0,0) & (0.31,1.03,2.69) \\ (0.22,1.32,2.51) & (0.42,0.79,1.78) & (0.4,0.79,1.93) & (0.31,1.03,2.69) & (0,0,0) \end{bmatrix}.$$

The spherical fuzzy Zagreb relations related to $(\mathtt{SZM})_i^{\prime}s$, where i=(1, 2, 3, 4) are shown graphically in Figures 8 and 9. These figures elaborate on the concept of the connectedness of the edges. These graphs corresponding to spherical fuzzy Zagreb relations are more strengthened than the SFGs represented in Figures 6 and 7.

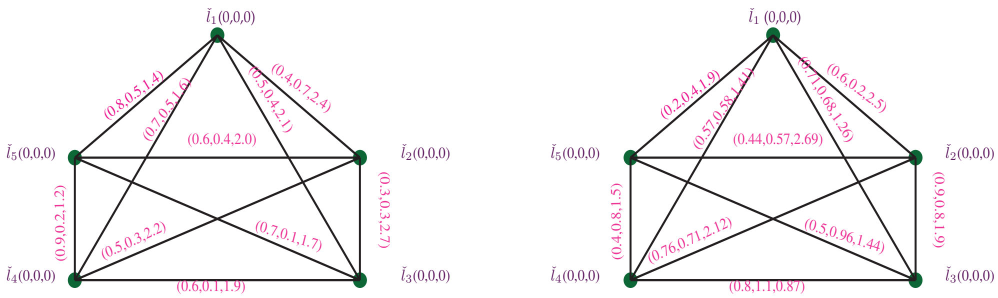

Figure 8. The graphical representation of spherical fuzzy Zagreb relations to $(\mathtt{SZM})_1$, $(\mathtt{SZM})_2$.

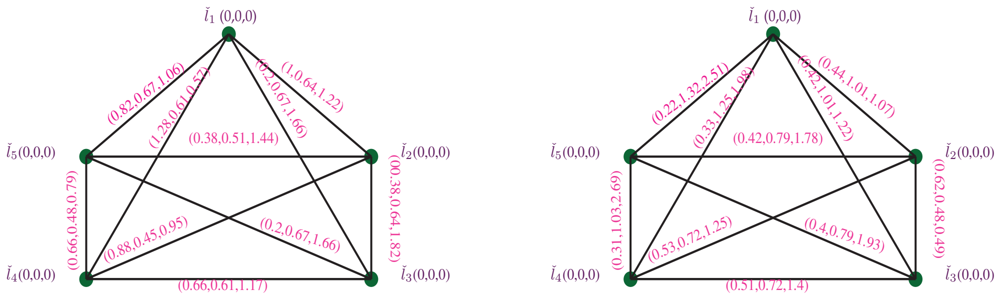

Figure 9. The graphical representation of spherical fuzzy Zagreb relations to $(\mathtt{SZM})_3$ and $(\mathtt{SZM})_4$.

The spherical fuzzy Zagreb energy of each $\mathtt{SZM}s$ is calculated as

$$\mathtt{ZE}((\mathtt{SZM})_1) = (5.1283, 3.179, 15.9035).$$

$$\mathtt{ZE}((\mathtt{SZM})_2) = (4.9912, 5.7813, 14.6477).$$

$$\mathtt{ZE}((\mathtt{SZM})_3) = (5.9254, 4.9018, 9.9656).$$

$$\mathtt{ZE}((\mathtt{SZM})_4) = (3.4282, 7.4492, 13.6328).$$

The weight of each expert is computed as

$$w_i = ((w_{\check{\alpha}})_i, (w_{\check{\beta}})_i, (w_{\check{\gamma}})_i).$$

$$w_j = \left(\frac{\mathtt{ZE}((\mathtt{SZM})_{\check{\alpha}})_j}{\sum_{s=1}^{4} \mathtt{ZE}((\mathtt{SZM})_{\check{\alpha}})_s}, \frac{\mathtt{ZE}((\mathtt{SZM})_{\check{\beta}})_j}{\sum_{s=1}^{4} \mathtt{ZE}((\mathtt{SZM})_{\check{\beta}})_s}, \frac{\mathtt{ZE}((\mathtt{SZM})_{\check{\gamma}})_j}{\sum_{s=1}^{4} \mathtt{ZE}((\mathtt{SZM})_{\check{\gamma}})_s} \right).$$

$$w_1 = (0.2633530357, 0.1491696896, 0.2936956434).$$

$$w_2 = (0.2563125542, 0.2712786174, 0.2705043066).$$

$$w_3 = (0.3042864259, 0.2300094316, 0.1840382939).$$

$$w_4 = (0.1760479841, 0.3495422616, 0.2517617859).$$

From the aggregation of four SFZPRs, the collective spherical fuzzy Zagreb preference relation (SFZPR) is calculated as

$$\mathtt{SZM} = \sum_{i=1}^{4} w_i(\mathtt{SZM})_i.$$

$$\mathtt{SZM} = \begin{bmatrix} (0,0,0) & (0.643,0.683,1.88) & (0.655,0.790,1.52) & (0.777,0.808,1.460) & (0.562,0.815,1.192) \\ (0.643,0.683,1.889) & (0,0,0) & (0.55,0.582,1.795) & (0.675,0.601,1.726) & (0.460,0.607,2.051) \\ (0.655,0.790,1.529) & (0.552,0.582,1.795) & (0,0,0) & (0.687,0.708,1.366) & (0.472,0.714,1.691) \\ (0.778,0.809,1.460) & (0.675,0.601,1.72) & (0.687,0.708,1.367) & (0,0,0) & (0.594,0.733,1.622) \\ (0.562,0.815,1.785) & (0.460,0.607,2.051) & (0.472,0.714,1.691) & (0.594,0.733,1.622) & (0,0,0) \end{bmatrix}.$$

The collective SFZPR is shown in Figure 10. In the representation of this collective SFZPR, as exhibited in Figure 10, we choose spherical fuzzy membership, whose truthiness degrees are $\breve{\alpha}_{ij} \geq 0.5$ $(i, j = 1, 2, 3, 4)$. The obtained partial model is depicted in Figure 11.

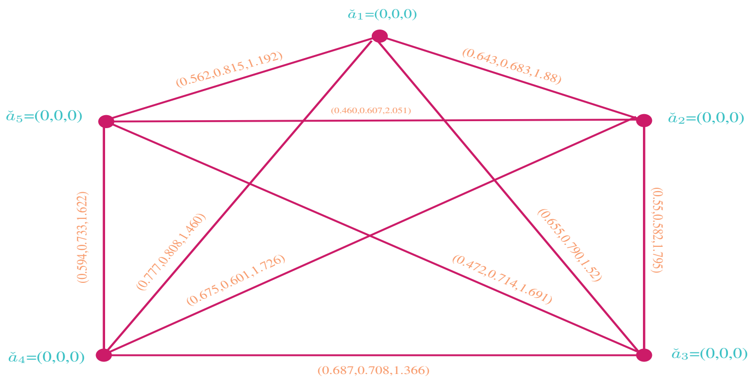

Figure 10. Model of the combined SFZPR.

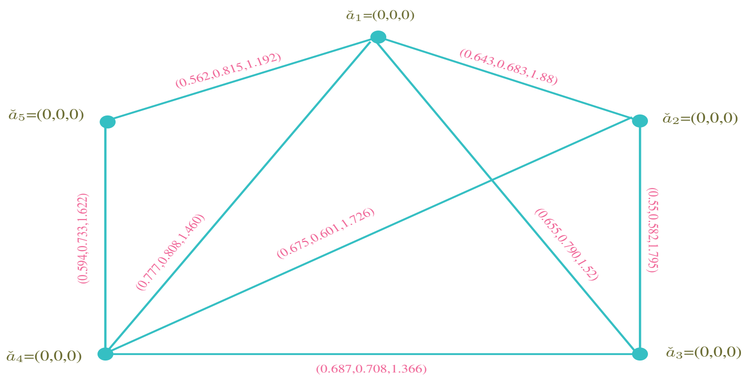

Figure 11. Partial model of the combined SFZPR.

We calculate the degrees $deg(\breve{l}_i)$ $(i = 1, 2, 3, 4, 5)$ in a partial directed model as follows: $deg(\breve{l}_1) = (2.637, 3.096, 6.052)$, $deg(\breve{l}_2) = (2.328, 2.473, 7.452)$, $deg(\breve{l}_3) = (2.364, 2.794, 6.372)$, $deg(\breve{l}_4) = (2.733, 2.85, 6.174)$, and $deg(\breve{l}_5) = (2.088, 2.869, 6.556)$.

Since $\breve{a}_4$ has the largest degree, we have the positioning of the alternatives $\breve{l}_i$ $(i = 1, 2, 3, 4, 5)$ as

$$\breve{l}_4 \succ \breve{l}_1 \succ \breve{l}_3 \succ \breve{l}_2 \succ \breve{l}_5.$$

Thus, the best choice for selecting location is $\breve{a}_4$.

5.2. Comparative Analysis

The comparative inspection with previously existing procedures is needed to examine the feasibility and validity of the proposed technique. By comparative analysis, it can be seen that one adopts whatever techniques and methods are either more accurate to one another or whose results should be the same.Here, we compare our proposed model to the already existing one. In the existing technique, which was proposed in [44], the directed spherical fuzzy graphs with their corresponding adjacency matrices are considered. The spherical fuzzy energy of all adjacency matrices is calculated by adding the absolute value of their eigenvalues. The average operator is applied to aggregate the obtained results. In that technique, the ranking of alternatives is based on the out-degree, which is greater than 0.5 in the partial model of SFGs. In our proposed model, we consider Zagreb spherical fuzzy matrices as compared to the adjacency matrices of SFGs, which give us a more reliable and feasible solution since, in our proposed technique, the concept of the degree of vertices is involved, which encounters the connectivity of the neighborhoods of the vertices. The consequences related to the methodologies are summarized in Tables 8 and 9.

Table 8. Comparison of degrees of the alternatives with existing techniques.

Techniques	Degrees of the Alternatives
Existing technique with adjacency matrices [44]	(0.78, 0.82, 1.69), (0.79, 0.85, 1.85), (0.76, 0.76, 1.7), (0.84, 0.85, 1.70), (0.58, 0.89, 1.88).
Our offered technique with SZMs	(2.63, 3.09, 6.05), (2.32, 2.47, 7.45), (2.36, 2.79, 6.37), (2.73, 2.85, 6.17), (2.08, 2.86, 6.55)

Table 9. Comparison of ranking.

Techniques	Ranking of the Alternatives
Existing technique with adjacency matrices [44]	$\breve{\alpha}_4 \succ \breve{\alpha}_2 \succ \breve{\alpha}_1 \succ \breve{\alpha}_3 \succ \breve{\alpha}_5$.
Our offered technique with SZMs	$\breve{\alpha}_4 \succ \breve{\alpha}_1 \succ \breve{\alpha}_3 \succ \breve{\alpha}_2 \succ \breve{\alpha}_5$.

Eventually, the result for the positioning of the options of the existing strategy is similar to the presented technique, which shows the accuracy and viability of the proposed procedure. Clearly, the technique introduced in this paper is more exact, adaptable, and generalized.

The comparison between the proposed and existing techniques, via finding the energy and Zagreb energy of adjacency matrices and SFGs, respectively, is graphically displayed in Figure 12 .

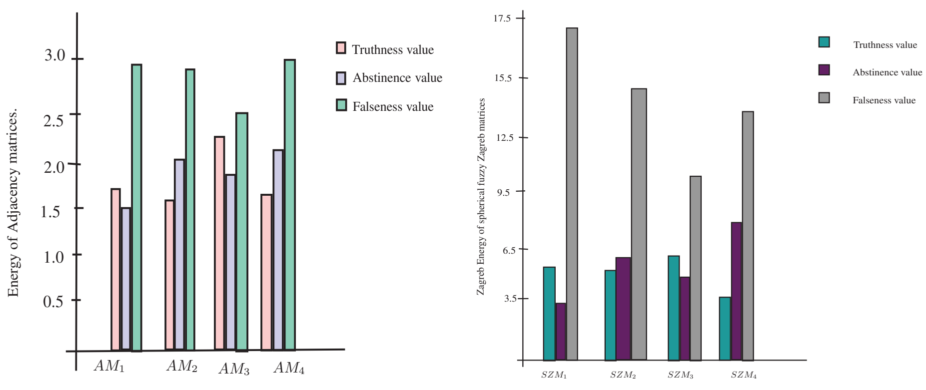

Figure 12. Comparison between energy of AMs and Zagreb energy of SZMs.

6. Conclusions

Spherical fuzzy sets are the generalization of picture fuzzy sets, as they enlarge the space of membership degrees of truthiness, abstinence, and falseness in the unit interval satisfying the condition $0 \leq \breve{\alpha}^2 + \breve{\beta}^2 + \breve{\gamma}^2 \leq 1$. Spherical fuzzy models can handle uncertainty problems more efficiently when a person has different viewpoints like yes, abstain, no, and refusal as compared to the fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set, and picture fuzzy set. Topological indices play a crucial role in fuzzy graph theory. One of the important indices are the fuzzy Zagreb index, which includes the fuzzy first, second, and hyper Zagreb index. The Zagreb index is the degree-based index, which encounters the strength of vertices as well as the strength of the fuzzy graph too.

In this research article, we discussed spherical fuzzy Zagreb indices and determined some bounds. Further, we studied how the spherical fuzzy Zagreb matrix of SFG is defined. The concept of the spectrum of SFG was introduced. In addition, we computed the spherical fuzzy Zagreb energy of the spherical fuzzy graph. Further, we extracted some bounds of spherical fuzzy Zagreb energy. Finally, we presented an application to ensure the applicability of our proposed model. In future, our goal is to extend the notion of this work to the following:

1. Hesitant SFGs.
2. SF hypergraphs.
3. Interval-valued SFGs.
4. Single-valued SFGs.
5. Complex spherical fuzzy Hamacher aggregation operators.

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