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Solitary Solutions for the Stochastic Fokas System Found in Monomode Optical Fibers

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Abstract: The stochastic Fokas system (SFS), driven by multiplicative noise in the Itô sense, was investigated in this study. Novel trigonometric, rational, hyperbolic, and elliptic stochastic solutions are found using a modified mapping method. Because the Fokas system is used to explain nonlinear pulse propagation in monomode optical fibers, the solutions provided may be utilized to analyze a broad range of critical physical phenomena. In order to explain the impacts of multiplicative noise, the dynamic performances of the different found solutions are illustrated using 3D and 2D curves. We conclude that multiplicative noise eliminates the symmetry of the solutions of the SFS and stabilizes them.

Keywords: multiplicative noise; stochastic Fokas system; modified mapping method; optical solitons

MSC: 35A20; 60H10; 60H15; 35Q51; 83C15

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1. Introduction

Nonlinear evolution equations (NLEEs) play an essential part in the comprehension of numerous significant phenomena and dynamic processes in engineering technology science, life science, geoscience, mechanics, and physics due to the development of nonlinear science. It is crucial to investigate the exact explicit solutions of NLEEs in order to gain a deeper understanding of the phenomena described by NLEEs. In recent years, quite a few techniques, including the first-integral method [1], sine–cosine procedure [2], exp-function method [3], Jacobi elliptic function expansion [4], mapping method [5], auxiliary equation scheme [6], extended tanh function method [7], generalized Kudryashov approach [8], $exp(-\phi(\varsigma))$-expansion method [9], F-expansion approach [10], Taylor's power series expansion [11], q-homotopy analysis transform method [12], bifurcation analysis [13,14], and (G'/G)-expansion [15,16], have been proposed for solving NLEEs. Moreover, the Lie symmetry method [17] is the most significant method for developing analytical solutions for nonlinear NLEEs.

Until the 1950s, deterministic models of differential equations played an important role in the study of natural phenomena across a vast array of physical sciences. Nevertheless, it is evident that the phenomena that occur in the world today are not deterministic in nature. For these NLEEs, it is crucial to take random effects into account. Equations that take into account random fluctuations are called stochastic NLEEs. These kinds of equations are increasingly being utilized in the climate, biophysics, condensed matter, electrical engineering, information systems, finance, materials sciences, and other disciplines to develop mathematical models of complex processes [18,19]. In recent years, analytical solutions for some stochastic NLEEs, for example [20–22], have been discovered.

In this paper, we consider the Fokas system, which is one of the most significant of the NLEEs, forced by multiplicative noise:

$$\begin{aligned} &i\mathcal{G}_t + \varepsilon_1 \mathcal{G}_{xx} + \varepsilon_2 \mathcal{G}\mathcal{W} = i\delta\mathcal{G}\mathcal{B}_t, \\ &\varepsilon_3 \mathcal{W}_y = \varepsilon_4 (|\mathcal{G}|^2)_x, \end{aligned} \tag{1}$$

where $\mathcal{G} = \mathcal{G}(x, y, t)$ and $\mathcal{W} = \mathcal{W}(x, y, t)$ are complex functions, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 are arbitrary constants, δ is the noise strength. $\mathcal{B} = \mathcal{B}(t)$ is the Brownian motion, $\mathcal{B}_t = \frac{\partial \mathcal{B}}{\partial t}$. If we set $\delta = 0$, then we have the deterministic Fokas system (DFS):

$$i\mathcal{G}_t + \varepsilon_1 \mathcal{G}_{xx} + \varepsilon_2 \mathcal{G}\mathcal{W} = 0, \ \varepsilon_3 \mathcal{W}_y = \varepsilon_4 (|\mathcal{G}|^2)_x. \tag{2}$$

Fokas [23] and Shulman [24] introduced the DFS (2) for examining NLSE in (2 + 1) dimensions. Because of the crucial role of DFS (2), numerous authors have attained the exact solutions by employing numerous techniques, such as the exp-function method [25], Jacobi elliptic function expansion [26], He's frequency formulation method, the extended rational sinh–cosh and sine–cosine [27], the variational method and simplified extended tanh-function [28], Hirota's bilinear method [29], and the generalized Kudryashov method [30]. The stochastic Fokas system (SFS) (1) has not been studied until now.

The objective of this study is to acquire the analytical stochastic solutions of the SFS (1). We use a modified mapping method to obtain the stochastic solutions of SFS (1) in the form of rational, elliptic, hyperbolic, and trigonometric functions. Since the Fokas system is used to explain nonlinear pulse propagation in monomode optical fibers, the obtained solutions can be applied to the analysis of a wide variety of crucial physical phenomena. The dynamic performances of the various obtained solutions are depicted using 3D and 2D curves in order to interpret the effects of multiplicative noise.

The paper has the following structure: In the next section, we derive the wave equation for the SFS (1). In Section 3, the modified mapping technique is employed to derive the exact solutions of SFS (1). In Section 4, we analyze the impact of the Brownian motion on the solution of SFS (1). Finally, the conclusion of the paper is provided.

2. Traveling Wave Equation for SFS

We utilize

$$\mathcal{G}(x, y, t) = \mathcal{U}(\xi) e^{i\omega + \delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t}, \ \mathcal{W}(x, y, t) = \mathcal{V}(\xi) e^{\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t},$$

with

$$\omega = \omega_1 x + \omega_2 y + \omega_3 t \ \text{ and } \ \xi = \xi_1 x + \xi_2 y + \xi_3 t, \tag{3}$$

to obtain the wave equation for SFS (1), where $\mathcal{U}$ is a deterministic and real function, ρ_1, ρ_2, ξ_1, and ξ_2 are non-zero constants. We note that

$$\begin{aligned} \mathcal{G}_t &= [\xi_3 \mathcal{U}' + i\omega_3 \mathcal{U} + \delta\mathcal{U}\mathcal{B}_t - \frac{1}{2}\delta^2\mathcal{U} + \frac{1}{2}\delta^2\mathcal{U}] e^{i\omega + \delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t} \\ &= [\xi_3 \mathcal{U}' + i\omega_3 \mathcal{U} + \delta\mathcal{U}\mathcal{B}_t] e^{i\omega + \delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t}, \end{aligned} \tag{4}$$

where $\frac{1}{2}\delta^2\mathcal{U}$ is the Itô correction term, and

$$\begin{aligned} \mathcal{G}_x &= (\xi_1 \mathcal{U}' + i\omega_1 \mathcal{U}) e^{i\omega + \delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t}, \ (|\mathcal{G}|^2)_x = \xi_1 (\mathcal{U}^2)' e^{2\delta\mathcal{B}(t) - \delta^2 t}, \\ \mathcal{G}_{xx} &= (\xi_1^2 \mathcal{U}'' + 2i\omega_1 \xi_1 \mathcal{U}' - \omega_1^2 \mathcal{U}) e^{i\omega + \delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t}, \ \mathcal{W}_y = \xi_2 \mathcal{V}' e^{\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \end{aligned} \tag{5}$$

Plugging Equations (4) and (5) into Equation (1), for the real part, we have

$$(\varepsilon_1 \xi_1^2)\mathcal{U}'' + (-\omega_3 - \varepsilon_1 \omega_1^2)\mathcal{U} + \varepsilon_2 \mathcal{U}\mathcal{V} e^{\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t} = 0, \ \varepsilon_3 \xi_2 \mathcal{V}' = \varepsilon_4 \xi_1 (\mathcal{U}^2)' e^{\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t} \tag{6}$$

and for the imaginary part:
$$(\varepsilon_1\omega_1\xi_1+\xi_3)\mathcal{U}'=0. \tag{7}$$

Setting that
$$\xi_3=-\varepsilon_1\omega_1\xi_1.$$

Then, Equation (7) vanishes. Now, we take the expectations on both sides into Equation (6) to obtain

$$(\varepsilon_1\xi_1^2)\mathcal{U}''+(-\omega_3-\varepsilon_1\omega_1^2)\mathcal{U}+\varepsilon_2\mathcal{U}\mathcal{V}e^{-\frac{1}{2}\delta^2 t}\mathbb{E}(e^{\delta\mathcal{B}(t)})=0, \tag{8}$$

and
$$\varepsilon_3\xi_2\mathcal{V}'=\varepsilon_4\xi_1(\mathcal{U}^2)'e^{-\frac{1}{2}\delta^2 t}\mathbb{E}(e^{\delta\mathcal{B}(t)}). \tag{9}$$

Since $\mathcal{B}(t)$ is the normal process, then $\mathbb{E}(e^{\delta\mathcal{B}(t)})=e^{\frac{1}{2}\delta^2 t}$ for any real number δ. Hence, Equations (8) and (9) become

$$(\varepsilon_1\xi_1^2)\mathcal{U}''+(-\omega_3-\varepsilon_1\omega_1^2)\mathcal{U}+\varepsilon_2\mathcal{U}\mathcal{V}=0, \tag{10}$$

and
$$\varepsilon_3\xi_2\mathcal{V}'=\varepsilon_4\xi_1(\mathcal{U}^2)'. \tag{11}$$

Integrating (11) once, we obtain

$$\mathcal{V}=\frac{\varepsilon_4\xi_1}{\varepsilon_3\xi_2}\mathcal{U}^2+C, \tag{12}$$

where C is the constant of the integration. Substituting Equation (12) into Equation (10), we have
$$\mathcal{U}''+\mathcal{H}_1\mathcal{U}+\mathcal{H}_2\mathcal{U}^3=0, \tag{13}$$

where
$$\mathcal{H}_1=\frac{-\omega_3-\varepsilon_1\omega_1^2+C\varepsilon_2}{\varepsilon_1\xi_1^2}, \text{ and } \mathcal{H}_2=\frac{\varepsilon_2\varepsilon_4}{\varepsilon_3\varepsilon_1\xi_1\xi_2}.$$

3. Exact Solutions of SFS

Here, the modified mapping method described in [31] is applied. Let the solutions of Equation (13) have the following form:

$$\mathcal{U}(\xi)=\sum_{i=0}^{K}\ell_i u^i(\xi)+\sum_{i=1}^{K}\varpi_i u^{-i}(\xi), \tag{14}$$

where ℓ_i and ϖ_i are undetermined constants to be calculated, and u solves

$$u'=\sqrt{\hbar_1 u^4+\hbar_2 u^2+\hbar_3}, \tag{15}$$

where $\hbar_1,\hbar_2$, and $\hbar_3$ are real constants.

First, let us balance $\mathcal{U}''$ with $\mathcal{U}^3$ in Equation (13) to find the parameter K as

$$K+2=3K\Longrightarrow K=1.$$

With $K=1$, Equation (15) has the form

$$\mathcal{U}(\xi)=\ell_0+\ell_1 u(\xi)+\frac{\varpi_1}{u(\xi)}. \tag{16}$$

Differentiating Equation (16) twice and using (15), we obtain

$$\mathcal{U}''=\ell_1(\hbar_2 u+2\hbar_1 u^3)+\varpi_1(\hbar_2 u^{-1}+2\hbar_3 u^{-3}). \tag{17}$$

Putting Equation (16) and Equation (17) into Equation (13), we obtain

$$(2\ell_1\hbar_1 + \mathcal{H}_2\ell_1^3)u^3 + 3\mathcal{H}_2\ell_0\ell_1^2u^2 + (\ell_1\hbar_2 + 3\mathcal{H}_2\ell_0^2\ell_1 + 3\mathcal{H}_2\varpi_1\ell_1^2 + \ell_1\mathcal{H}_1)u$$
$$+(\mathcal{H}_1\ell_0 + \mathcal{H}_2\ell_0^3 + 6\mathcal{H}_2\ell_0\ell_1\varpi_1) + (\mathcal{H}_1\varpi_1 + \varpi_1\hbar_2 + 3\mathcal{H}_2\ell_0^2\varpi_1 + 3\mathcal{H}_2\ell_1\varpi_1^2)u^{-1}$$
$$+3\mathcal{H}_2\varpi_1^2u^{-2} + (2\hbar_3\varpi_1 + \mathcal{H}_2\varpi_1^3)u^{-3} = 0.$$

For $k = 3, 2, 1, 0$, we balance each coefficient of u^k and u^{-k} with 0 to have

$$2\ell_1\hbar_1 + \mathcal{H}_2\ell_1^3 = 0,$$
$$3\mathcal{H}_2\ell_0\ell_1^2 = 0,$$
$$\ell_1\hbar_2 + 3\mathcal{H}_2\ell_0^2\ell_1 + 3\mathcal{H}_2\varpi_1\ell_1^2 + \ell_1\mathcal{H}_1 = 0,$$
$$\mathcal{H}_1\ell_0 + \mathcal{H}_2\ell_0^3 + 6\mathcal{H}_2\ell_0\ell_1\varpi_1 = 0,$$
$$\mathcal{H}_1\varpi_1 + \varpi_1\hbar_2 + 3\mathcal{H}_2\ell_0^2\varpi_1 + 3\mathcal{H}_2\ell_1\varpi_1^2 = 0,$$
$$3\mathcal{H}_2\ell_0\varpi_1^2 = 0$$

and

$$2\hbar_3\varpi_1 + \mathcal{H}_2\varpi_1^3 = 0.$$

We obtain three distinct sets when we solve these equations:
First set:

$$\ell_0 = 0,\ \ \ell_1 = \pm\sqrt{\frac{-2\hbar_1}{\mathcal{H}_2}},\ \varpi_1 = 0,\ \ \hbar_2 = -\mathcal{H}_1. \tag{18}$$

Second set:

$$\ell_0 = 0,\ \ \ell_1 = 0,\ \varpi_1 = \pm\sqrt{\frac{-2\hbar_3}{\mathcal{H}_2}},\ \ \hbar_2 = -\mathcal{H}_1. \tag{19}$$

Third set:

$$\ell_0 = 0,\ \ \ell_1 = \pm\sqrt{\frac{-2\hbar_1}{\mathcal{H}_2}},\ \varpi_1 = \pm\sqrt{\frac{-2\hbar_3}{\mathcal{H}_2}},\ \ \hbar_2 = 6\sqrt{\hbar_1\hbar_3} - \mathcal{H}_1. \tag{20}$$

First set: By utilizing Equations (12), (13), (16) and (18), the solution of Equation (1) is

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2\hbar_1}{\varepsilon_2\varepsilon_4}}u(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}, \tag{21}$$

and

$$\mathcal{W}(x,y,t) = [\frac{-2\varepsilon_1\xi_1^2\hbar_1}{\varepsilon_2}u^2(\xi) + C]e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}. \tag{22}$$

Many cases rely on $\hbar_1$:
Case 1-1: If $\hbar_1 = m^2$, $\hbar_2 = -(1+m^2)$ and $\hbar_3 = 1$, then $u(\xi) = sn(\xi)$. Thus, using Equations (21) and (22), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm m\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}sn(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}, \tag{23}$$

$$\mathcal{W}(x,y,t) = [\frac{-2\varepsilon_1\xi_1^2m^2}{\varepsilon_2}sn^2(\xi) + C]e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}. \tag{24}$$

If $m \to 1$, then Equation (23) becomes

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}\tanh(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{25}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\tanh^2(\xi)+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{26}$$

Case 1-2: If $\hbar_1 = 1$, $\hbar_2 = 2m^2-1$ and $\hbar_3 = -m^2(1-m^2)$, then $u(\xi) = ds(\xi)$. Thus, using Equations (21) and (22), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}ds(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{27}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}ds^2(\xi)+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{28}$$

If $m \to 1$, Equation (27) becomes

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}\operatorname{csch}(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{29}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\operatorname{csch}^2(\xi)+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{30}$$

If $m \to 0$, Equation (27) becomes

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}\csc(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{31}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\csc^2(\xi)+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{32}$$

Case 1-3: If $\hbar_1 = 1$, $\hbar_2 = 2-m^2$ and $\hbar_3 = (1-m^2)$, then $u(\xi) = cs(\xi)$. Thus, using Equations (21) and (22), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}cs(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{33}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}cs^2(\xi)+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{34}$$

If $m \to 1$, Equation (33) becomes

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}\operatorname{csch}(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{35}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\operatorname{csch}^2(\xi)+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{36}$$

If $m \to 0$, Equation (33) becomes

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}\cot(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{37}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\cot^2(\xi)+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t},\tag{38}$$

where $\xi=\xi_1x+\xi_2y+\xi_3t$.

Case 1-4: If $\hbar_1=\frac{m^2}{4}$, $\hbar_2=\frac{(m^2-2)}{2}$, and $\hbar_3=\frac{1}{4}$, then $u(\xi)=\frac{sn(\xi)}{1+dn(\xi)}$. Thus, using Equations (21) and (22), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t)=\pm m\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}\frac{sn(\xi)}{1+dn(\xi)}e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{39}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-\varepsilon_1\xi_1^2m^2}{2\varepsilon_2}[\frac{sn(\xi)}{1+dn(\xi)}]^2+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{40}$$

If $m\to1$, then Equation (39) tends to

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}\frac{\tanh(\xi)}{1+\operatorname{sech}(\xi)}e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{41}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-\varepsilon_1\xi_1^2\hbar_1}{2\varepsilon_2}[\frac{\tanh(\xi)}{1+\operatorname{sech}(\xi)}]^2+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{42}$$

Case 1-5: If $\hbar_1=\frac{(1-m^2)^2}{4}$, $\hbar_2=\frac{(1-m^2)^2}{2}$ and $\hbar_3=\frac{1}{4}$, then $u(\xi)=\frac{sn(\xi)}{dn(\xi)+cn(\xi)}$. Thus, using Equations (21) and (22), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t)=\pm(1-m^2)\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\frac{sn(\xi)}{dn(\xi)+cn(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{43}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-\varepsilon_1\xi_1^2(1-m^2)^2}{2\varepsilon_2}[\frac{sn(\xi)}{dn(\xi)+cn(\xi)}]^2+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{44}$$

When $m\to0$, Equation (43) tends to

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\frac{\sin(\xi)}{1+\cos(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{45}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2}[\frac{\sin(\xi)}{1+\cos(\xi)}]^2+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{46}$$

Case 1-6: If $\hbar_1=\frac{1-m^2}{4}$, $\hbar_2=\frac{(1-m^2)}{2}$ and $\hbar_3=\frac{1-m^2}{4}$, then $u(\xi)=\frac{cn(\xi)}{1+sn(\xi)}$. Thus, using Equations (21) and (22), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-(1-m^2)\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\frac{cn(\xi)}{1+sn(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{47}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-\varepsilon_1\xi_1^2(1-m^2)}{2\varepsilon_2}[\frac{cn(\xi)}{1+sn(\xi)}]^2+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t}.\tag{48}$$

When $m\to0$, Equation (47) tends to

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\frac{\cos(\xi)}{1+\sin(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2t},\tag{49}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-\varepsilon_1 \xi_1^2}{2\varepsilon_2} [\frac{\cos(\xi)}{1+\sin(\xi)}]^2 + C \right) e^{\delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}, \tag{50}$$

where $\xi = \xi_1 x + \xi_2 y + \xi_3 t$.

Case 1-7: If $\hbar_1 = 1$, $\hbar_2 = 0$ and $\hbar_3 = 0$, then $u(\xi) = \frac{c}{\xi}$. Thus, using Equations (21) and (22), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm \sqrt{\frac{-2\varepsilon_3 \varepsilon_1 \xi_1 \xi_2}{\varepsilon_2 \varepsilon_4}} [\frac{c}{(\xi_1 x + \xi_2 y + \xi_3 t)}] e^{i\omega + \delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}, \tag{51}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1 \xi_1^2}{\varepsilon_2} [\frac{c}{(\xi_1 x + \xi_2 y + \xi_3 t)}]^2 + C \right) e^{\delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \tag{52}$$

Second set: By using Equations (16) and (19), the solution of Equation (13) is

$$\mathcal{G}(x,y,t) = \pm \sqrt{\frac{-2\varepsilon_3 \varepsilon_1 \xi_1 \xi_2 \hbar_3}{\varepsilon_2 \varepsilon_4}} \frac{1}{u(\xi)} e^{i\omega + \delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}, \tag{53}$$

and

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1 \xi_1^2 \hbar_3}{\varepsilon_2} \frac{1}{u^2(\xi)} + C \right) e^{\delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \tag{54}$$

Many cases rely on $\hbar_3$:

Case 2-1: If $\hbar_1 = m^2$, $\hbar_2 = -(1+m^2)$ and $\hbar_3 = 1$, then $u(\xi) = sn(\xi)$. Thus, using Equations (53) and (54), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm \sqrt{\frac{-2\varepsilon_3 \varepsilon_1 \xi_1 \xi_2}{\varepsilon_2 \varepsilon_4}} \frac{1}{sn(\xi)} e^{i\omega + \delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \tag{55}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1 \xi_1^2}{\varepsilon_2} \frac{1}{sn^2(\xi)} + C \right) e^{\delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \tag{56}$$

If $m \to 1$, Equation (55) tends to

$$\mathcal{G}(x,y,t) = \pm \sqrt{\frac{-2\varepsilon_3 \varepsilon_1 \xi_1 \xi_2}{\varepsilon_2 \varepsilon_4}} \coth(\xi) e^{i\omega + \delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \tag{57}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1 \xi_1^2}{\varepsilon_2} \coth^2(\xi) + C \right) e^{\delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \tag{58}$$

Case 2-2: If $\hbar_1 = 1$, $\hbar_2 = 2 - m^2$ and $\hbar_3 = (1-m^2)$, then $u(\xi) = cs(\xi)$. Thus, using Equations (53) and (54), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm \sqrt{\frac{-2(1-m^2)\varepsilon_3 \varepsilon_1 \xi_1 \xi_2 \hbar_3}{\varepsilon_2 \varepsilon_4}} \frac{1}{cs(\xi)} e^{i\omega + \delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \tag{59}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2(1-m^2)\varepsilon_1 \xi_1^2}{\varepsilon_2} \frac{1}{cs^2(\xi)} + C \right) e^{\delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \tag{60}$$

When $m \to 0$, Equation (59) tends to

$$\mathcal{G}(x,y,t) = \pm \sqrt{\frac{-2\varepsilon_3 \varepsilon_1 \xi_1 \xi_2}{\varepsilon_2 \varepsilon_4}} \tan(\xi) e^{i\omega + \delta \mathcal{B}(t) - \frac{1}{2}\delta^2 t}. \tag{61}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2} \tan^2(\xi) + C \right) e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{62}$$

where $\xi = \xi_1 x + \xi_2 y + \xi_3 t$.

Case 2-3: If $\hbar_1 = -m^2$, $\hbar_2 = 2m^2 - 1$ and $\hbar_3 = (1 - m^2)$, then $u(\xi) = cn(\rho)$. Thus, using Equations (53) and (54), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2(1-m^2)\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}} \frac{1}{cn(\xi)} e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{63}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2(1-m^2)\varepsilon_1\xi_1^2\hbar_3}{\varepsilon_2} \frac{1}{cn^2(\xi)} + C \right) e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{64}$$

When $m \to 0$, Equation (67) tends to

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}} \sec(\xi) e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{65}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-2\varepsilon_1\xi_1^2\hbar_3}{\varepsilon_2} \sec^2(\xi) + C \right) e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{66}$$

Case 2-4: If $\hbar_1 = \frac{m^2}{4}$, $\hbar_2 = \frac{(m^2-2)}{2}$ and $\hbar_3 = \frac{1}{4}$, then $u(\xi) = \frac{sn(\xi)}{1+dn(\xi)}$. Thus, using Equations (53) and (54), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}} \frac{1+dn(\xi)}{sn(\xi)} e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{67}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2} [\frac{1+dn(\xi)}{sn(\xi)}]^2 + C \right) e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{68}$$

When $m \to 1$, Equation (67) tends to

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}} [\coth(\xi) + \operatorname{csch}(\xi)] e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{69}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2} [\coth(\xi) + \operatorname{csch}(\xi)]^2 + C \right) e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{70}$$

Case 2-5: If $\hbar_1 = \frac{1-m^2}{4}$, $\hbar_2 = \frac{(1-m^2)}{2}$ and $\hbar_3 = \frac{1-m^2}{4}$, then $u(\xi) = \frac{cn(\xi)}{1+sn(\xi)}$. Thus, using Equations (53) and (54), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-(1-m^2)\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}} [\frac{1+sn(\xi)}{cn(\xi)}] e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{71}$$

$$\mathcal{W}(x,y,t) = \left(\frac{-(1-m^2)\varepsilon_1\xi_1^2}{2\varepsilon_2} [\frac{1+sn(\xi)}{cn(\xi)}]^2 + C \right) e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{72}$$

When $m \to 0$, Equation (71) tends to

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}} [\sec(\xi) + \tan(\xi)] e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{73}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}[\sec(\xi)+\tan(\xi)]^2+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t},\tag{74}$$

where $\xi=\xi_1x+\xi_2y+\xi_3t$.

Case 2-6: If $\hbar_1=\frac{(1-m^2)^2}{4}$, $\hbar_2=\frac{(1-m^2)^2}{2}$, and $\hbar_3=\frac{1}{4}$, then $u(\xi)=\frac{sn(\xi)}{dn(\xi)+cn(\xi)}$. Thus, using Equations (53) and (54), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\frac{dn(\xi)+cn(\xi)}{sn(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\tag{75}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2}[\frac{dn(\xi)+cn(\xi)}{sn(\xi)}]^2+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\tag{76}$$

If $m\to 0$, then Equation (75) becomes

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\csc(\xi)+\cot(\xi)]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t},\tag{77}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2}[\csc(\xi)+\cot(\xi)]^2+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\tag{78}$$

If $m\to 1$, then Equation (75) becomes

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}\operatorname{csch}(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t},\tag{79}$$

$$\mathcal{W}(x,y,t)=\left(\frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2}\operatorname{csch}^2(\xi)+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\tag{80}$$

Third set: By using Equations (16) and (20), the solution of Equation (13) is

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\sqrt{\hbar_1}u(\xi)+\sqrt{\hbar_3}\frac{1}{u(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\tag{81}$$

$$\mathcal{W}(x,y,t)=\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\left(\hbar_1u^2(\xi)+\frac{\hbar_3}{u^2(\xi)}+2\sqrt{\hbar_1\hbar_3}+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\tag{82}$$

Many cases rely on $\hbar_1$ and $\hbar_3$:

Case 3-1: If $\hbar_1=m^2$, $\hbar_2=-(1+m^2)$ and $\hbar_3=1$, then $u(\xi)=sn(\xi)$. Thus, using Equations (81) and (82), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[msn(\xi)+\frac{1}{sn(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t},\tag{83}$$

$$\mathcal{W}(x,y,t)=\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\left(m^2sn^2(\xi)+\frac{1}{sn^2(\xi)}+2m+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\tag{84}$$

When $m\to 1$, Equation (83) tends to

$$\mathcal{G}(x,y,t)=\pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\tanh(\xi)+\coth(\xi)]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\tag{85}$$

$$\mathcal{W}(x,y,t)=\frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\left(\tanh^2(\xi)+\coth^2(\xi)+2+C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\tag{86}$$

Case 3-2: If $\hbar_1 = 1$, $\hbar_2 = 2 - m^2$ and $\hbar_3 = (1 - m^2)$, then $u(\xi) = cs(\xi)$. Thus, using Equations (81) and (82), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}[cs(\xi) + \sqrt{(1-m^2)}\frac{1}{cs(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{87}$$

$$\mathcal{W}(x,y,t) = \frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\left(cs^2(\xi) + \frac{(1-m^2)}{cs^2(\xi)} + 2\sqrt{(1-m^2)} + C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{88}$$

where $\xi = \xi_1 x + \xi_2 y + \xi_3 t$. When $m \to 0$, Equation (87) tends to

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}[\cot(\xi) + \tan(\xi)]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{89}$$

$$\mathcal{W}(x,y,t) = \frac{-2\varepsilon_1\xi_1^2}{\varepsilon_2}\left(\cot^2(\xi) + \tan^2(\xi) + C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{90}$$

Case 3-3: If $\hbar_1 = \frac{m^2}{4}$, $\hbar_2 = \frac{(m^2-2)}{2}$ and $\hbar_3 = \frac{1}{4}$, then $u(\xi) = \frac{sn(\xi)}{1\pm dn(\xi)}$. Thus, using Equations (81) and (82), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[m\frac{sn(\xi)}{1+dn(\xi)} + \frac{1+dn(\xi)}{sn(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{91}$$

$$\mathcal{W}(x,y,t) = \frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2}\left(m(\frac{sn(\xi)}{1+dn(\xi)})^2 + (\frac{1+dn(\xi)}{sn(\xi)})^2 + \frac{m^2}{4} + C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{92}$$

where $\xi = \xi_1 x + \xi_2 y + \xi_3 t$. When $m \to 1$, Equation (91) tends to

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}\coth(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{93}$$

$$\mathcal{W}(x,y,t) = \frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2}\left(\coth^2(\xi) + C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{94}$$

Case 3-4: If $\hbar_1 = \frac{1-m^2}{4}$, $\hbar_2 = \frac{(1-m^2)}{2}$ and $\hbar_3 = \frac{1-m^2}{4}$, then $u(\xi) = \frac{cn(\xi)}{1+sn(\xi)}$. Thus, using Equations (81) and (82), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-(1-m^2)\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\frac{cn(\xi)}{1+sn(\xi)} + \frac{1+sn(\xi)}{cn(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{95}$$

$$\mathcal{W}(x,y,t) = \frac{-(1-m^2)\varepsilon_1\xi_1^2}{2\varepsilon_2}\left((\frac{cn(\xi)}{1+sn(\xi)})^2 + (\frac{1+sn(\xi)}{cn(\xi)})^2 + C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{96}$$

where $\xi = \xi_1 x + \xi_2 y + \xi_3 t$. When $m \to 0$, Equation (95) tends to

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}\sec(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \tag{97}$$

$$\mathcal{W}(x,y,t) = \frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2}\left(\sec^2(\xi) + C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \tag{98}$$

Case 3-5: If $\hbar_1 = \frac{(1-m^2)^2}{4}$, $\hbar_2 = \frac{(1-m^2)^2}{2}$ and $\hbar_3 = \frac{1}{4}$, then $u(\xi) = \frac{sn(\xi)}{dn(\xi)+cn(\xi)}$. Thus, using Equations (81) and (82), the solutions of SFS (1) are

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-\varepsilon_3\varepsilon_1\xi_1\xi_2}{2\varepsilon_2\varepsilon_4}}[\frac{(1-m^2)sn(\xi)}{dn(\xi)+cn(\xi)} + \frac{dn(\xi)+cn(\xi)}{sn(\xi)}]e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \quad (99)$$

$$\mathcal{W}(x,y,t) = \frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2}\left((\frac{(1-m^2)sn(\xi)}{dn(\xi)+cn(\xi)})^2 + (\frac{dn(\xi)+cn(\xi)}{sn(\xi)})^2 + C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \quad (100)$$

When $m \to 0$, Equation (75) tends to

$$\mathcal{G}(x,y,t) = \pm\sqrt{\frac{-2\varepsilon_3\varepsilon_1\xi_1\xi_2}{\varepsilon_2\varepsilon_4}}\csc(\xi)e^{i\omega+\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}. \quad (101)$$

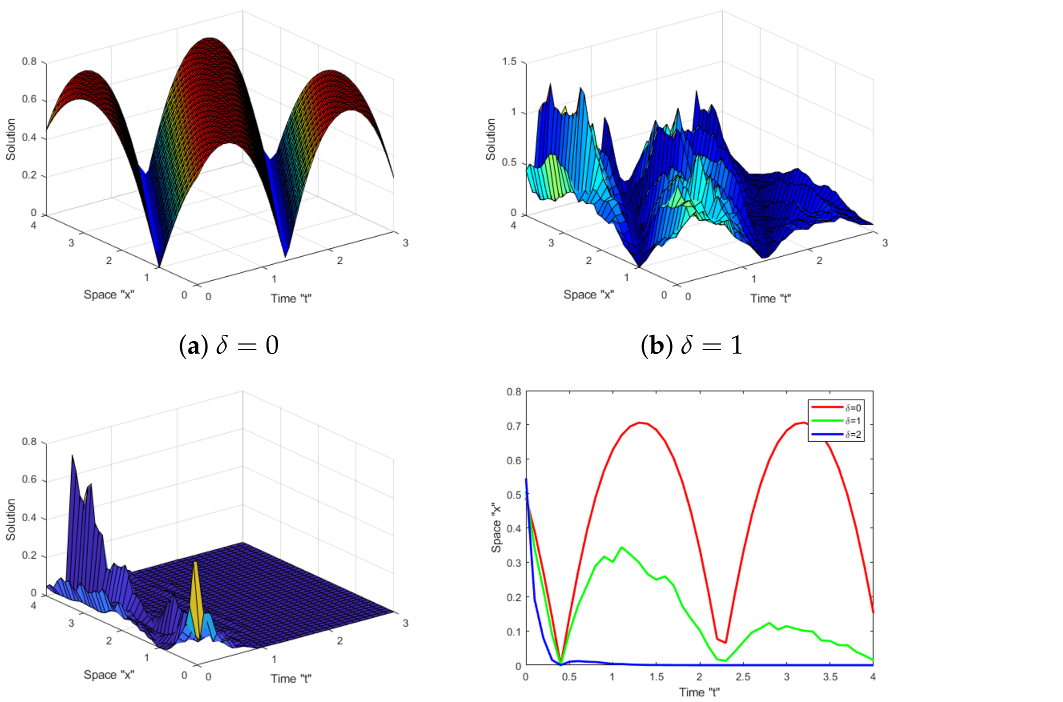
$$\mathcal{W}(x,y,t) = \frac{-\varepsilon_1\xi_1^2}{2\varepsilon_2}\left(\csc^2(\xi) + C\right)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}, \quad (102)$$

where $\xi = \xi_1 x + \xi_2 y + \xi_3 t$.

4. Impacts of Noise

In this section, we address the effect of Brownian motion on the exact solution of the SFS (1). Several diagrams are supplied to show the behaviors of some obtained solutions, such as (23)–(26), (47) and (48). Let us fix the parameters $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$, $\xi_2 = -1$, $\xi_3 = -2, C = 0$, $y = 0$, $x \in [0,4]$ and $t \in [0,3]$ to simulate these diagrams.

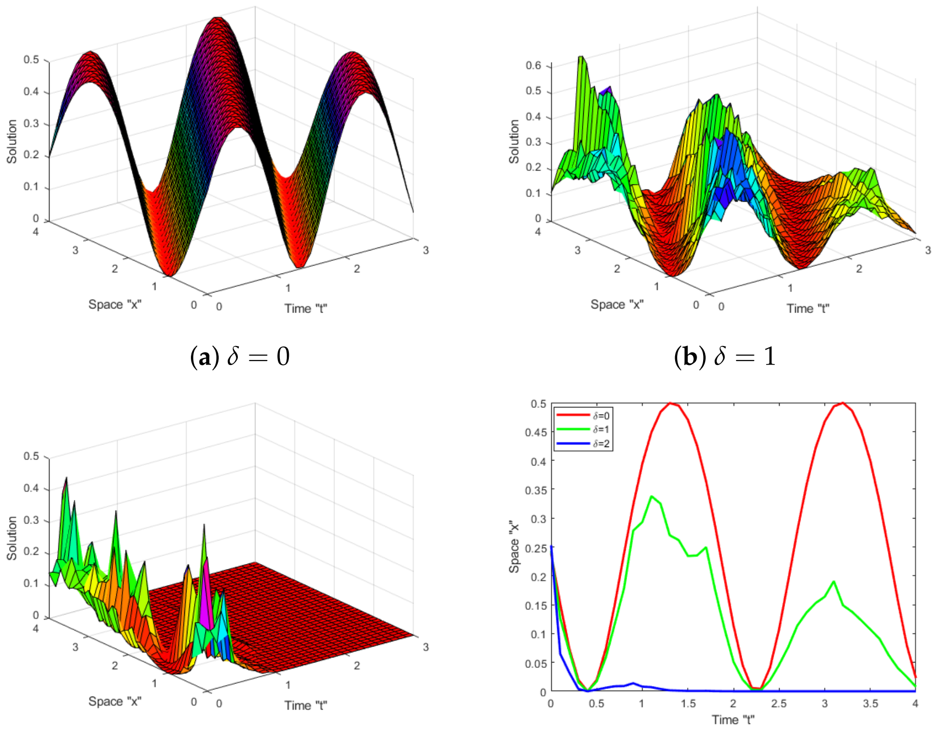
Now, we can see from Figures 1–6 that when the Brownian motion is ignored (i.e., when $\delta = 0$), there are many other kinds of solutions, including periodic solutions, kink solutions, and so on. After short transit patterns, the surface becomes flatter when noise is incorporated and its amplitude is increased by $\delta = 1,2$. This demonstrates that Brownian motion stabilizes the SFS solutions, maintaining them around zero.



(**a**) $\delta = 0$ (**b**) $\delta = 1$

(**c**) $\delta = 2$ (**d**) $\delta = 0,\ 1,\ 2$

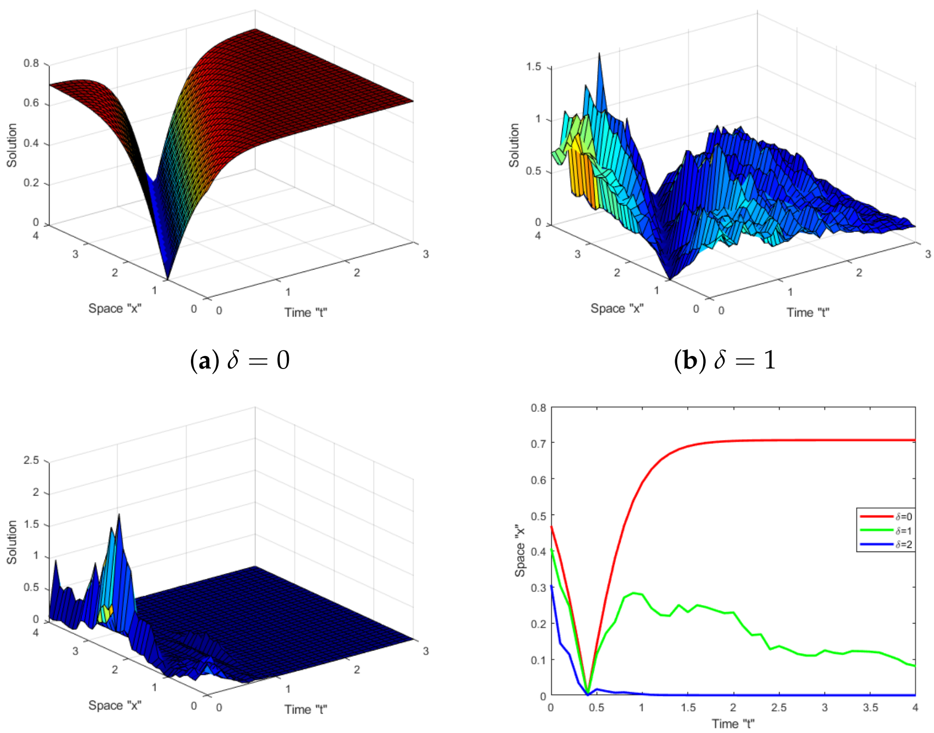
Figure 1. (**a**–**c**) The 3D style of solution $|\mathcal{G}(x,y,t)|$ in Equation (23) with $\xi_1 = 1$ and $\delta = 0,\ 1,\ 2$; (**d**) the 2D style of Equation (23) with different values of δ.



(**a**) $\delta = 0$ (**b**) $\delta = 1$

(**c**) $\delta = 2$ (**d**) $\delta = 0,\ 1,\ 2$

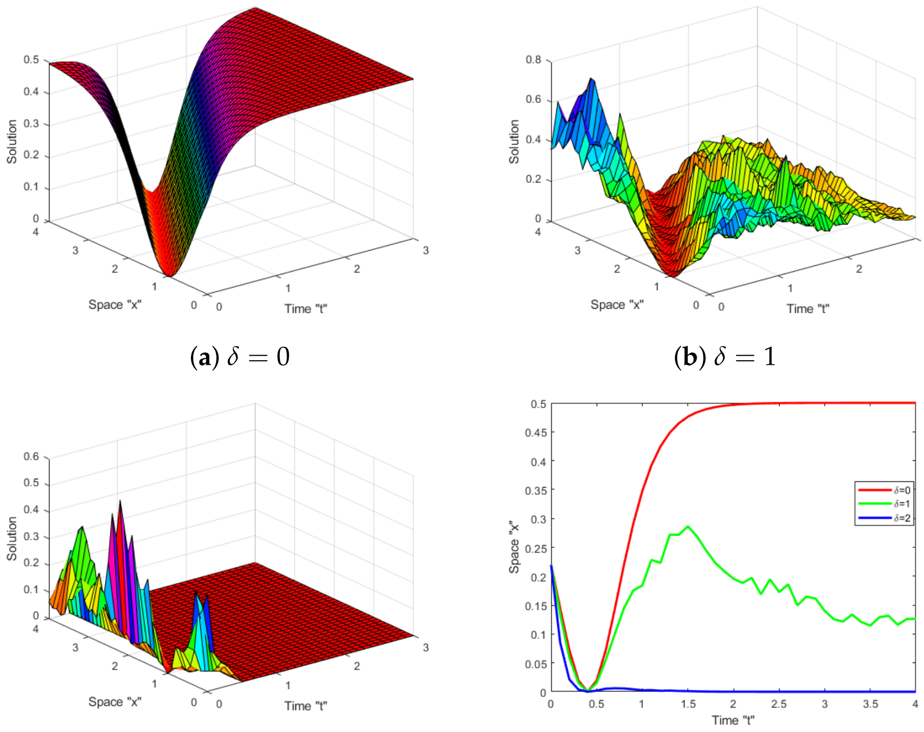
Figure 2. (**a**–**c**) The 3D style of solution $|\mathcal{G}(x,y,t)|$ in Equation (24) with $\xi_1 = 1$ and $\delta = 0,\ 1,\ 2$; (**d**) the 2D style of Equation (24) with different values of δ.



(**a**) $\delta = 0$ (**b**) $\delta = 1$

(**c**) $\delta = 2$ (**d**) $\delta = 0,\ 1,\ 2$

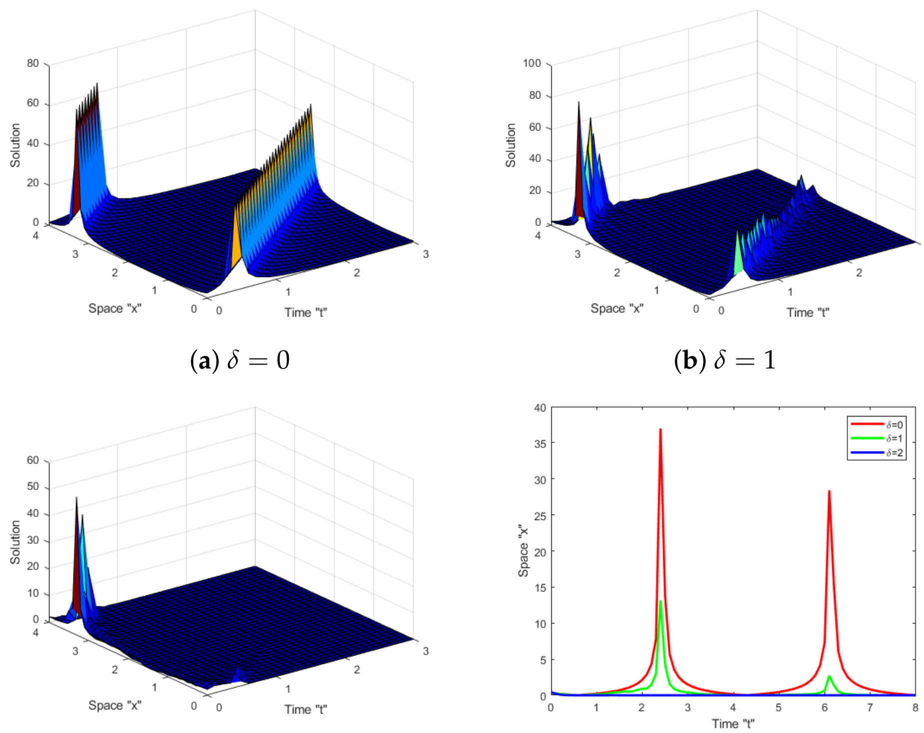
Figure 3. (**a**–**c**) The 3D style of solution $|\mathcal{G}(x,y,t)|$ in Equation (25) with $\xi_1 = 1$ and $\delta = 0,\ 1,\ 2$; (**d**) the 2D style of Equation (25) with different values of δ.



(**a**) $\delta = 0$ (**b**) $\delta = 1$

(**c**) $\delta = 2$ (**d**) $\delta = 0,\ 1,\ 2$

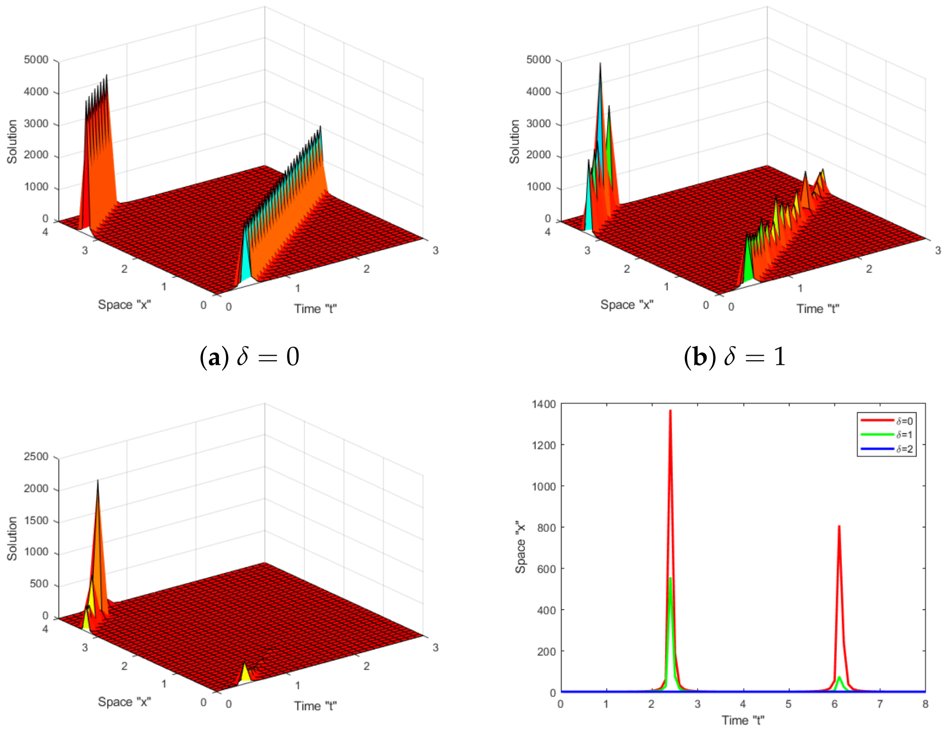
Figure 4. (**a**–**c**) The 3D style of solution $|\mathcal{G}(x, y, t)|$ in Equation (26) with $\xi_1 = 1$ and $\delta = 0,\ 1,\ 2$; (**d**) the 2D style of Equation (26) with different values of δ.



(**a**) $\delta = 0$ (**b**) $\delta = 1$

(**c**) $\delta = 2$ (**d**) $\delta = 0,\ 1,\ 2$

Figure 5. (**a**–**c**) The 3D style of solution $|\mathcal{G}(x, y, t)|$ in Equation (47) with $\xi_1 = 2$ and $\delta = 0,\ 1,\ 2$; (**d**) the 2D style of Equation (23) with different values of δ.



(**a**) $\delta = 0$ (**b**) $\delta = 1$

(**c**) $\delta = 2$ (**d**) $\delta = 0,\ 1,\ 2$

Figure 6. (**a**–**c**) The 3D style of solution $|\mathcal{G}(x, y, t)|$ in Equation (48) with $\xi_1 = 2$ and $\delta = 0,\ 1,\ 2$; (**d**) shows the 2D style of Equation (48) with different values of δ.

5. Conclusions

In this work, we considered the stochastic Fokas system (SFS) perturbed by multiplicative Brownian motion in the Itô sense. Utilizing a modified mapping method, we obtained the exact stochastic solutions. Due to the implementation of the Fokas system in explaining nonlinear pulse propagation in monomode optical fibers, these solutions can explain a wide array of fascinating and intricate physical phenomena. Using 3D and 2D curves for various values of noise strength, we depicted the dynamic behaviors of the numerous obtained solutions in order to interpret the effects of Brownian motion on these solutions. Furthermore, we established that multiplicative noise stabilizes the solutions at zero.

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