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Islands in Generalized Dilaton Theories

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Abstract: This work systematically studies the island formula in the general asymptotically flat eternal black holes in generalized dilaton gravity theories or higher-dimensional spherical black holes. Under some reasonable and mild assumptions, we prove that (the boundary of) the island always appears barely outside the horizon in the late time of Hawking radiation, so the information paradox is resolved. In particular, we find a proper island in the Liouville black hole that solves the previous the puzzle.

Keywords: information problem; two-dimensional gravity

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1. Introduction

Over the past five years, significant progress has been made in exploring the quantum aspects of black holes [1], particularly the information paradox [2,3]. Assuming that black hole evaporation is unitary, the von Neumann entropy of Hawking radiation should initially rise and then fall, following the Page curve [4]. However, this conflicts with the thermal properties of Hawking radiation. Recently, this tension has been resolved with the introduction of a new rule, the island formula [5–7], for the computation of the entanglement entropy of Hawking radiation. (It is argued in [8] that the island proposal may be inconsistent in standard theories of massless gravity. The crucial point is that it is not possible to localize an operator and its dressing entirely to an island. A possible loophole is that there may exist other dressing schemes.) The island formula mimics the quantum extremal surface (QES) prescription [9] for generalized entanglement entropy in a system coupled with gravity. The von Neumann entropy of a subsystem of Hawking radiation (denoted by Rad) is calculated by extremizing and then minimizing the following functional (in this paper, we will use $S_{\text{island}} = \frac{A(\partial I)}{4G_N} + S_{\text{semi-cl}}[\text{Rad} \cup I]$ to denote the entropy functional):

$$S_{\text{Rad}} = \min_I \left\{ \text{ext}_I \left[\frac{A(\partial I)}{4G_N} + S_{\text{semi-cl}}[\text{Rad} \cup I] \right] \right\}, \tag{1}$$

where I denotes the island, typically a codimension-one region mostly situated in the interior of the black hole, and $A(\partial I)$ is the area of its boundary ∂I, which is the QES. A remarkable feature of this formula is that the right-hand side depends only on semiclassical physics, making it computable with standard methods. This formula can be justified using replica tricks in the gravitational Euclidean path integral [10,11] or by combining the AdS/BCFT correspondence and brane world holography [12–15].

The island formula has been successfully applied to black holes in various gravitational theories [16–38]. However, there is a notable counterexample [39] in which it is claimed that the island formula cannot solve the information paradox of black holes in Liouville theory because the island cannot exist. More puzzlingly, this claim seems inconsistent with the systematic analysis of QES in general D-dimensional asymptotically flat (or AdS) eternal black holes performed in [38]. In this work, we show that Liouville theory contains another family of black hole solutions for which the island exists. We also conduct a systematic analysis of islands in asymptotically flat eternal black holes in generalized dilaton gravity theories, which include almost all interesting two-dimensional quantum gravity models. As higher-dimensional spherical black holes can be effectively described by generalized dilaton theory (GDT) after taking the s-wave approximation, our analysis is also valid in these cases. We find that our results agree with those obtained in [38]: (the boundary of) the island is outside the horizon.

2. Two-Dimensional Generalized Dilaton Gravity Theory

As the exact computation of the entanglement entropy in higher-dimensional theories is exceedingly challenging, almost all the island formula computations found in the literature are either restricted to two-dimensional theories or are reduced to two dimensions through the s-wave limit. A vast class of two-dimensional gravity models can be expressed as 2D GDTs. Due to their exceptional solvability, 2D GDTs also function as quantum gravity toy models, providing a platform for the investigation of the characteristics of black holes. The general action of 2D GDTs is

$$S[g_{\mu\nu}, X] = \frac{1}{2\pi}\int \mathrm{d}^2 x\sqrt{-g}\Big[XR - U(X)(\nabla X)^2 - 2V(X)\Big]. \tag{2}$$

The scalar field X is related to the dilaton field ϕ via $X = e^{-2\Phi}$. In terms of the dilaton field Φ, the action takes a more recognizable form,

$$S_{\rm dil} = \frac{1}{2\pi}\int \mathrm{d}^2 x\sqrt{-g}e^{-2\Phi}\Big[R - \tilde{U}(\Phi)(\nabla\Phi)^2 - 2\tilde{V}(\Phi)\Big] \tag{3}$$

with the identifications

$$\tilde{U} = 4e^{-2\Phi}U, \quad \tilde{V} = e^{2\Phi}V. \tag{4}$$

Remarkably, all the classical solutions of (2) can be found in closed form. As we review in the Appendix A, the general solution of (2) is given by the following (here, we only consider the interesting linear dilaton vacua) [40–43]:

$$\begin{aligned} \mathrm{d}s^2 &= 2e^Q\,\mathrm{d}v(\mathrm{d}X + (w(X) - C_0)\,\mathrm{d}v) \\ &= 2\,\mathrm{d}v\,\mathrm{d}\tilde{X} + \xi(\tilde{X})\,\mathrm{d}v^2, \end{aligned} \tag{5}$$

where we have introduced

$$\mathrm{d}\tilde{X} = \mathrm{d}Xe^Q, \tag{6}$$
$$\xi(\tilde{X}) = 2e^Q(w - C_0), \tag{7}$$
$$Q = \int^X U(y)\,\mathrm{d}y, \tag{8}$$
$$w = \int^X e^Q V(y)\,\mathrm{d}y. \tag{9}$$

The solution is parameterized by a constant C_0 that is usually related to the mass of the black hole. The metric (5) can be transformed into the Schwarzschild gauge

$$\mathrm{d}s^2 = \xi\,\mathrm{d}t^2 - \frac{1}{\xi}\mathrm{d}r^2, \tag{10}$$

by introducing the coordinates $r = \tilde{X}$, $t = v + \int^t \xi^{-1}$. In this paper, because we focus on asymptotically flat solutions, we require that ξ approaches some negative constant

$$\lim_{r\to\infty}\xi = -\xi_0^2 < 0, \tag{11}$$

such that

$$\mathrm{d}\,s^2 = \frac{\xi}{\xi_0^2}\mathrm{d}\,t'^2 - \frac{\xi_0^2}{\xi}\mathrm{d}\,r'^2 \tag{12}$$

$$\lim_{r'\to\infty} \mathrm{d}\,s^2 \to -\mathrm{d}\,t'^2 + \mathrm{d}\,r'^2, \tag{13}$$

where we have rescaled the coordinates as

$$t = \frac{t'}{\xi_0}, \quad r = \xi_0 r'. \tag{14}$$

The location of the (outer) horizon of the black hole is where $\xi(r'_H) = 0$. Once the horizon of the black hole is identified, it is possible to calculate the temperature and the Bekenstein–Hawking entropy using the standard formula [44]

$$T_{\mathrm{BH}} = \partial_{r'}\xi/\xi_0^2\Big|_{r'=r'_H}, \quad S_{\mathrm{BH}} = 2X(r'_H). \tag{15}$$

3. General Results

3.1. Setting Up the Calculation

Our goal is to evaluate the entanglement entropy of the Hawking radiation emitted by the eternal black hole, utilizing the island Formula (1). In the context of the 2d GDT, the first term in Formula (1) is determined by the dilaton field value at the boundary of the island, namely

$$\frac{A(\partial I)}{4G_N} = 2X(\partial I), \tag{16}$$

as established in [44]. On the other hand, $S_{\text{semi-cl}}[\mathrm{Rad} \cup I]$ represents the entanglement entropy of the Hawking radiation present in the joint domain of $\mathrm{Rad} \cup I$. To fully cover the area of eternal black holes, we introduce the Kruskal coordinates:

$$\mathrm{d}\,s^2 = -e^{2\rho(y^+,y^-)}\,\mathrm{d}\,y^+\,\mathrm{d}\,y^-, \quad -\infty < y^\pm < \infty. \tag{17}$$

The Kruskal coordinates $y^\pm_{R(L)} = y^0_{R(L)} \pm y^1_{R(L)}$ are related to the two copies of Schwarzschild coordinates (12) via

$$e^{lx_R^+} = ly_R^+, \quad e^{-lx_R^-} = -ly_R^-, \quad e^{-lx_L^+} = -ly_L^+, \quad e^{lx_L^-} = ly_L^-, \tag{18}$$

$$y_R^+ \geq 0, \quad y_R^- \leq 0, \quad y_L^+ \leq 0, \quad y_L^- \geq 0, \tag{19}$$

and

$$x^\pm_{R(L)} = t'_{R(L)} \pm x^*_{R(L)}, \quad x^*_{R(L)} = \int \frac{\mathrm{d}\,r'}{-\xi/\xi_0^2}, \tag{20}$$

where y_R and y_L denote the right and left patches of Kruskal spacetime, respectively, and l is some convenient constant. Using the transformations (as in x_R), we identify that

$$e^{2\rho} = \frac{\xi(y^+y^-)}{\xi_0^2 l^2 y^+ y^-}. \tag{21}$$

In order to calculate the second term in the island Formula (1), we shall utilize a probe conformal field theory with a central charge $c << 1/G_N$ to model the Hawking radiation, allowing us to employ the known semiclassical formula to compute the entanglement entropy $S_{\text{semi-cl}}(\mathrm{Rad} \cup I)$. To simplify the model, we can ensure that the subsystem Rad consists of two symmetric intervals $[y_{L\infty}, y_{-a}] \cup [y_a, y_{R\infty}]$, with the respective endpoint coordinates being

$$y_{L\infty} = (y_a^0, -\infty), \quad y_{R\infty} = (y_a^0, \infty), \quad y_{-a} = (y_a^0, -y_a^1), \quad y_a = (y_a^0, y_a^1), \tag{22}$$

where y_a^0 labels the Cauchy slice. This symmetric choice makes it reasonable to anticipate that the island $[y_{-d}, y_d]$ (we posit the dominance of the one-interval island configuration) will also reflect such symmetry, with the coordinates of its endpoints being

$$y_{-d} = (y_d^0, -y_d^1), \quad y_d = (y_d^0, y_d^1). \tag{23}$$

3.2. Entanglement Entropy without Islands

During the early stages of Hawking radiation, there are insufficient Hawking quanta present within the interiors of black holes to support an island configuration. As an increasing number of Hawking quanta escape to infinity, the asymptotic observer should observe a growing entanglement entropy as predicted by the Page curve. Assuming that the probe 2d CFT is in a pure state, then the entanglement entropy on the interval $[y_{L\infty}, y_{-a}] \cup [y_a, y_{R\infty}]$ is equal to the entanglement entropy on the complementary region $[y_{-a}, y_a]$. In the metric (17), it is equal to [6]

$$\begin{aligned} S_{\text{Rad}}([y_{-a}, y_a]) &= \frac{c}{6}\log\Big(|(y_a - y_{-a})^+(y_a - y_{-a})^-|e^{\rho(y_a)}e^{\rho(y_{-a})}\Big) \qquad (24)\\ &= \frac{c}{6}\log\Big(|2y_a^1|^2 e^{\rho(y_a^+, y_a^-)}e^{\rho(y_a^-, y_a^+)}\Big), \\ &= \frac{c}{3}\log\big(2\cosh(lt_a')\big) + \frac{c}{6}\log\left(-\frac{\xi(y_a^+ y_a^-)}{\xi_0^2 l^2}\right), \qquad (25) \end{aligned}$$

where $t_a' = -\frac{1}{2l}(\log\frac{y_a^+}{y_a^-})$ is the Schwarzschild time of the Cauchy slice. The entanglement entropy will eventually reach the entropy bound, which is twice the black hole Bekenstein–Hawking entropy, at a specific moment defined as the Page time. This moment is dependent on both the metric and the dilaton field and can be expressed as

$$t_p = \frac{1}{l}\text{arccosh}\left(\frac{1}{2}\exp\left(\frac{12X(r_H')}{cG_N}\right)\sqrt{-\frac{\xi_0^2 l^2}{\xi(y_a^+ y_a^-)}}\right). \tag{26}$$

After the Page time, this entanglement entropy (25) will surpass the entropy bound, thus presenting the information paradox and consequentially leading one to anticipate the emergence of the island.

3.3. Entanglement Entropy with Islands

After the Page time, the island located at $[y_{-d}, y_d]$ should be included; therefore, the Rad subsystem consists of two intervals: $[y_{-a}, y_{-d}] \cup [y_d, y_a]$. Note that since we consider eternal black holes, the entanglement entropy will not fall down but saturate the entropy band at late time [11]. In the semiclassical regime, the first term in the island Formula (1) dominates over the second term, yielding

$$S_{\text{Rad}} \approx 2X(\partial I) = S_{BH} = 2X(r_H'), \tag{27}$$

where (15) and (16) have been used. This implies that the location of ∂I is very close to the horizon, namely $y_d^- = y_d^0 - y_d^1 \approx 0$, resulting in y_d^1 being exceptionally large at late time. Consequently, the two intervals are well separated, and the entanglement entropy $S_{\text{island}}([y_{-a}, y_{-d}] \cup [y_d, y_a])$ can be approximated by $2S_{\text{island}}([y_d, y_a])$. Thus, we can express the entropy functional as follows (note that since the island possesses two boundaries, the factor in the first term is 4):

$$S_{\text{island}}([y_d, y_a]) = \frac{4}{G_N} X(y_d) + \frac{c}{3} \log\Big(|(y_a - y_d)^+ (y_a - y_d)^- | e^{\rho(y_a)} e^{\rho(y_d)}\Big), \tag{28}$$

where the introduction of Newton's constant G_N serves to emphasize that we are within the semiclassical domain. Differentiating Equation (28) with respect to $y_d^{\pm}$ yields two extremal conditions expressed as

$$\frac{4}{G_N} \frac{\mathrm{d}\, X}{\mathrm{d}\, y_d^+} + \frac{c}{3} \left(\frac{1}{y_d^+ - y_a^+} + \frac{\mathrm{d}\, \rho}{\mathrm{d}\, y_d^+} \right) = 0, \tag{29}$$

$$\frac{4}{G_N} \frac{\mathrm{d}\, X}{\mathrm{d}\, y_d^-} + \frac{c}{3} \left(\frac{1}{y_d^- - y_a^-} + \frac{\mathrm{d}\, \rho}{\mathrm{d}\, y_d^-} \right) = 0. \tag{30}$$

Recalling that $\tilde{X} = r$, the first term of these equations can be evaluated as

$$\frac{\mathrm{d}\, X}{\mathrm{d}\, d_R^{\pm}} = \frac{\mathrm{d}\, X}{\mathrm{d}\, \tilde{X}} \frac{\mathrm{d}\, r}{\mathrm{d}\, y_d^{\pm}} = e^{-Q} \frac{\xi_0\, \mathrm{d}\, r'}{\mathrm{d}\, y_d^{\pm}} = -e^{-Q} \frac{\xi\, \mathrm{d}\, x^*}{\xi_0\, \mathrm{d}\, y_d^{\pm}} = -e^{-Q} \frac{\xi}{2l\xi_0 y_d^{\pm}} \tag{31}$$

where $\xi(z)$ and $Q(z)$ should be understood as functions of $y_d^+ y_d^- \equiv z$. Using expression (21), the second term can be computed as

$$\frac{\mathrm{d}\, \rho}{\mathrm{d}\, y_d^{\pm}} = \frac{1}{2y_d^{\pm}} \left(\frac{\xi'}{\xi} \frac{\mathrm{d}\, z}{y_d^{\pm}} - \frac{1}{y_d^{\pm}} \right) = \frac{1}{2y_d^{\pm}} \left(\frac{\xi' z}{\xi} - 1 \right), \tag{32}$$

where ξ' is $d\xi/dz$. By combining these two evaluations, Equations (29) and (30) can be rewritten as

$$\frac{1}{3} \frac{1}{y_d^{\pm} - y_a^{\pm}} - \frac{2\xi e^{-Q}}{cG_N l \xi_0} \frac{1}{y_d^{\pm}} + \frac{1}{3} \frac{\xi - z\xi'}{2\xi} \frac{1}{y_d^{\pm}} = 0, \tag{33}$$

which yields the relation

$$\frac{y_d^+}{y_a^+ - y_d^+} = \frac{y_d^-}{y_a^- - y_d^-}, \quad \rightarrow \quad y_a^+ y_d^- = y_a^- y_d^+, \quad \text{or} \quad \frac{y_a^+}{y_a^-} = \frac{y_d^+}{y_d^-}. \tag{34}$$

In the semiclassical regime, where $cG_N \ll 1$, the final term in Equation (33) can be disregarded. Thus, we need only consider the following equations:

$$\frac{z}{y_d^- y_a^+ - z} = -\frac{6\xi e^{-Q}}{cG_N l \xi_0}, \quad \frac{z}{y_d^+ y_a^- - z} = -\frac{6\xi e^{-Q}}{cG_N l \xi_0}. \tag{35}$$

These equations yield a *single* equation for z:

$$z \left(1 - \frac{\epsilon}{\xi(z) e^{-Q(z)}} \right)^2 = y_a^+ y_a^-, \quad \epsilon \equiv \frac{cG_N l \xi_0}{6}. \tag{36}$$

By solving for z and utilizing Equation (34), we can determine the position $y_d^{\pm}$ of the island. Equation (36) is very general and relies upon abstract functions ξ and Q, which define various GDT models. Nevertheless, for the metric (12) to accurately depict a black hole, these functions must possess specific analytical properties. The properties can be most easily described using Schwarzschild coordinates.

To proceed, let us transition to the Schwarzschild coordinates. The Kruskal coordinates $y_a^\pm$ and $y_d^\pm$ can be related to their Schwarzschild counterparts (t_a, a) and (t_d, d) via

$$\frac{1}{l}Ye^{lt_a} = y_a^+, \quad \frac{1}{l}Ye^{-lt_a} = -y_a^-, \quad Y = e^{ly^*}, \quad y^* = \int^a \frac{\mathrm{d}\, r'}{-\xi/\xi_0^2}, \tag{37}$$

$$\frac{1}{l}De^{lt_d} = y_d^+, \quad \frac{1}{l}De^{-lt_d} = -y_d^-, \quad D = e^{ld^*}, \quad d^* = \int^d \frac{\mathrm{d}\, r'}{-\xi/\xi_0^2}, \tag{38}$$

thus, the entanglement entropy (28) can be expressed as

$$S_{\text{island}} = \frac{4X(d)}{G_N} + \frac{c}{3}\log\frac{1}{l^2}\Big(Y^2 + D^2 - 2YD\cosh l(t_a - t_d)\Big) + \frac{c}{3}(\rho(d) + \rho(a)). \tag{39}$$

Varying (39) with respect to t_d implies $t_d = t_a$; therefore, the extremum of S_{island} is time-independent, as expected. Assuming that the black hole (12) is not extremal, then ξ should have a single zero at the horizon $r' = r'_H$. This implies that $d^*(d)$ as a function of d has a logarithmic singularity at $d = r'_H$ according to the definition of d^* (38), so we can rewrite

$$d^* = f(d) + \frac{1}{2l}\log(d - r'_H) \tag{40}$$

where $f(d)$ is regular at r'_H. Assuming that the dilaton field is well defined at the horizon, then e^{-Q} is regular and not vanishing at $d = r'_H$ according to (6), so we can rewrite

$$\xi e^{-Q} = (d - r'_H)g(d), \tag{41}$$

where $g(d)$ is regular at r'_H. Differentiating Equation (39) with respect to parameter d should result in an equation that is equivalent to (36)

$$z\Big(1 - \frac{\epsilon l^2 e^{2lf(r'_H)}}{zg(d)}\Big)^2 \approx z + \frac{1}{z}\left(\frac{\epsilon l^2 e^{2lf(r'_H)}}{g(r'_H)}\right)^2 - 2\frac{\epsilon l^2 e^{2lf(r'_H)}}{g(r'_H)} = y_a^+ y_a^- \equiv y^2. \tag{42}$$

Here, we employ the relationship between d^* and z given by

$$e^{2ld^*} = -\frac{1}{l^2}y_d^+ y_d^- = -\frac{1}{l^2}z = e^{2lf(d)}(d - r'_H), \tag{43}$$

to express d in terms of z. Then, we expand d into a series of powers of z as

$$d = r'_H - e^{-2lf(r'_H)}\frac{1}{l^2}z + \mathcal{O}(z^2), \tag{44}$$

where we have used the assumption that the island is very close to the horizon $d \approx r'_H$ or equivalently $z \approx 0$. It is found that indeed (42) has one solution that satisfies our assumption $z \approx 0$:

$$z = \frac{\beta^2}{y^2} + \mathcal{O}(\beta^3), \quad \beta = \frac{\epsilon l^2 e^{2lf(r'_H)}}{g(r'_H)}, \tag{45}$$

$$y_d^+ = -\frac{\beta}{y_a^-}, \quad y_d^- = -\frac{\beta}{y_a^+}, \tag{46}$$

which yields

$$S_{\text{Rad}} = \frac{4X(r'_H)}{G_N} + \frac{c}{3}\log\Big(|y_a^+ y_a^-| e^{\rho(y_a^\pm)}\Big). \tag{47}$$

This is the main result of this paper: under some reasonable assumptions for a general asymptotically flat eternal black hole in GDT, we can find an island such that the (generalized) entanglement entropy of Hawking radiation follows the Page curve (of the eternal black hole), which resolves the information paradox. In the rest of this paper, we will examine certain models in detail.

4. Examples

4.1. CGHS Model

The most extensively investigated GDT that admits an asymptotically flat black hole solution is the Callan–Giddings–Harvey–Strominger model (CGHS model) [45]. Islands in this model have been discovered in [20,37]. In this section, we will re-derive the island using our general procedures to confirm the validity of our overall analysis. The action of the CGHS model is

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 x\sqrt{-g}\Big[e^{-2\phi}(R+4(\nabla\phi)^2+4\lambda^2)\Big]. \tag{48}$$

4.1.1. The Geometry

Comparing (48) with (3), we note that

$$U = -\frac{1}{X}, \quad V = -2\lambda^2 X\,, \tag{49}$$

Consequently, we can compute the following quantities based on our general discussion:

$$e^Q = \frac{1}{X}, \quad X = \exp\tilde{X}, \quad w = -2\lambda^2 X = -2\lambda^2 e^{\tilde{X}} \tag{50}$$

$$\xi = -2C_0 e^{-\tilde{X}} - 4\lambda^2, \quad \xi_0 = 2\lambda. \tag{51}$$

Thus, the Schwarzschild metric takes the form

$$\mathrm{d}s^2 = -\left(\frac{C_0}{2\lambda^2}e^{-2\lambda r'}+1\right)\mathrm{d}t'^2 + \frac{\mathrm{d}r'^2}{\left(\frac{C_0}{2\lambda^2}e^{-2\lambda r'}+1\right)}. \tag{52}$$

It is evident that the horizon and curvature singularity are located at

$$r'_H = -\frac{1}{2\lambda}\log\left(-\frac{2\lambda^2}{C_0}\right), \quad r'_s = -\infty. \tag{53}$$

Therefore, to have a well-defined horizon, we can choose $\lambda^2 > 0$ and $C_0 \equiv -C < 0$. Introducing the new variable (20), we define the Kruskal coordinates $y^\pm$ as

$$e^{\lambda x^+} = \lambda y^+, \quad e^{-\lambda x^-} = -\lambda y^- \tag{54}$$

which results in the metric and dilaton

$$\mathrm{d}s^2 = -\frac{\mathrm{d}y^+\,\mathrm{d}y^-}{C-\lambda^2 y^+ y^-}, \quad e^{2\rho} = \frac{1}{C-\lambda^2 y^+ y^-}, \tag{55}$$

$$X = \frac{1}{2\lambda^2}\Big(C-\lambda^2 y^+ y^-\Big). \tag{56}$$

The horizon is located at $y^+y^- = 0$ and the singularity is located at $y^+y^- = C$. As we have shown in the general discussion, without including the island, the entanglement entropy is given by the general Formula (25). Let us focus on the derivation of the island.

4.1.2. The Derivation of the Island

In the presence of the island, the entropy functional is given by (28):

$$S_{\text{island}} = \frac{1}{G_N}\frac{4}{2\lambda^2}\left(C - \lambda^2 y_d^+ y_d^-\right) + \frac{c}{3}\log\left(\frac{|(y_a - y_d)^+(y_a - y_d)^-|}{\sqrt{C - \lambda^2 y_a^+ y_a^-}\sqrt{C - \lambda^2 y_d^+ y_d^-}}\right). \tag{57}$$

Differentiating the functional with respect to y_d^- and y_d^+, we obtain the equations

$$\frac{\lambda^2 y_d^+}{6(C - \lambda^2 y_d^+ y_d^-)} + \frac{1}{3(y_d^- - y_a^-)} - \frac{2y_d^+}{cG_N} = 0, \tag{58}$$

$$\frac{\lambda^2 y_d^-}{6(C - \lambda^2 y_d^+ y_d^-)} + \frac{1}{3(y_d^+ - y_a^+)} - \frac{2y_d^-}{cG_N} = 0. \tag{59}$$

The precise solutions can be easily derived, but they are highly complex. To extract relevant information, we shall once again consider the semiclassical regime where $G_N \to 0$. In this limit, we find that the non-trivial solutions are

$$y_d^+ = -\frac{cG_N}{6y_a^-}, \quad y_d^- = -\frac{cG_N}{6y_a^+} \tag{60}$$

and the resulting entanglement entropy is

$$S_{\text{Rad}} = 2\frac{1}{G_N}\frac{C}{\lambda^2} + \frac{c}{3}\log\left(\frac{|y_a^+ y_a^-|}{\sqrt{C(C - \lambda^2 y_a^+ y_a^-)}}\right) \tag{61}$$

which is time-independent and coincides with the results in [20,37].

Alternatively, we can utilize our general Equation (36):

$$z(1 - \frac{\epsilon}{\xi e^{-Q}})^2 = y^2 \quad \to \quad z(1 - \frac{\epsilon}{2\lambda^2 z})^2 = y^2, \quad \epsilon = \frac{cG_N\lambda^2}{3} \tag{62}$$

which has two solutions

$$z_1 = y^2 + \frac{\epsilon}{\lambda^2} - \frac{\epsilon^2}{4\lambda^4 y^2} + \mathcal{O}(\epsilon^3), \quad z_2 = \frac{\epsilon^2}{4y^2\lambda^4} + \mathcal{O}(\epsilon^3). \tag{63}$$

The solution z_1 yields a trivial solution, while the solution z_2 leads to (60).

5. Liouville Gravity

A particular generalization of the CGHS model is the one with exponential potential. The action is

$$S = \frac{1}{2\pi}\int d^2x\sqrt{-g}\left[RX + \sum_i 4\alpha_i^2 e^{\beta_i X}\right], \tag{64}$$

with

$$U(X) = 0, \quad V(X) = -2\sum_i \alpha_i^2 e^{\beta_i X}. \tag{65}$$

For simplicity, let us take $k = 1$, and the model is called Liouville gravity. There are some interesting reasons to consider such exponential potentials. It is shown in [46] that this type of model admits extra (conformal) symmetries. If we add $2X$ in the potential, this type of model, as deformations of JT gravity, is shown to have a matrix model dual [47].

Liouville gravity also has asymptotically flat black hole solutions. Surprisingly, [39] claimed that the information paradox of Liouville gravity, based on the black hole solution, which was discovered in [46,48], cannot be resolved using the island formula. In this section, we will use the general solution of GDT to derive an alternative solution, such that the island formula successfully resolves the information paradox.

5.1. The Geometry

Given the potentials (65), we can compute the following quantities:

$$Q = 0, \quad X = \tilde{X} = r, \quad w = -\frac{2\alpha^2 e^{\beta X}}{\beta}, \tag{66}$$

$$\xi = 2(w - C_0^L) = -2(\frac{2\alpha^2 e^{\beta X}}{\beta} + C_0^L), \quad \xi_0 = \sqrt{2C_0^L}, \tag{67}$$

where we choose $\beta < 0$. The corresponding Schwarzschild metric is

$$\mathrm{d}\, s^2 = -(1 + \frac{2\alpha^2 e^{\beta\sqrt{2C_0^L}r'}}{C_0^L \beta})\mathrm{d}\, t'^2 + \frac{1}{1 + \frac{2\alpha^2 e^{\beta\sqrt{2C_0^L}r'}}{C_0^L\beta}}\mathrm{d}\, r'^2. \tag{68}$$

Therefore, the horizon is at

$$r'_H = \frac{1}{\beta}\log\left(\frac{-C_0^L\beta}{2\alpha^2}\right), \tag{69}$$

and the Ricci scalar is

$$R = -4\beta e^{\beta\sqrt{2C_0^L}r'}\alpha^2. \tag{70}$$

In order to ensure asymptotic flatness, we can make the following choices:

$$\beta < 0, \quad C_0^L > 0, \quad \alpha^2 > 0. \tag{71}$$

Note that the metric (68) is equivalent to (52) if we define

$$\beta = -\frac{2\lambda}{\sqrt{2C_0^L}}, \quad \alpha^2 = \frac{C}{2\lambda}\sqrt{\frac{C_0^L}{2}}; \tag{72}$$

however, there are differences in the dilaton fields between these two models. Therefore, we can employ the Kruskal coordinates (54), and the resulting solution is as follows (we have confirmed that these solutions satisfy the equations of motion in the second-order formalism):

$$\mathrm{d}^2\, s = -\frac{\mathrm{d}\, y^+\, \mathrm{d}\, y^-}{C - \lambda^2 y^+ y^-}, \tag{73}$$

$$X = -\frac{1}{\beta}\log\left[\frac{1}{2\lambda^2}\left(C - \lambda^2 y^+ y^-\right)\right]. \tag{74}$$

As the geometry remains unchanged, the entanglement entropy in the absence of an island is identical to that in the CGHS black hole. Let us focus on the entanglement entropy of Hawking radiation in the presence of islands.

5.2. The Derivation of the Island

In the presence of the island, the entropy functional is given by (28)

$$S_{\text{island}} = \frac{4}{G_N}\frac{\sqrt{2C_0^L}}{2\lambda}\log\left[\frac{1}{2\lambda^2}\left(C-\lambda^2 y_d^+ y_d^-\right)\right] + \frac{c}{3}\log\left(\frac{|(y_a-y_d)^+(y_a-y_d)^-|}{\sqrt{C-\lambda^2 y_a^+ y_a^-}\sqrt{C-\lambda^2 y_d^+ y_d^-}}\right).$$

The extremal conditions are

$$-\frac{4}{G_N}\frac{\sqrt{2C_0^L}}{2\lambda}\frac{\lambda^2 y_d^+}{C-\lambda^2 y_d^+ y_d^-} + \frac{c}{3(y_d^- - y_a^-)} + \frac{c y_d^+}{6(C-\lambda^2 y_d^+ y_d^-)} = 0, \tag{75}$$

$$-\frac{4}{G_N}\frac{\sqrt{2C_0^L}}{2\lambda}\frac{\lambda^2 y_d^-}{C-\lambda^2 y_d^+ y_d^-} + \frac{c}{3(y_d^+ - y_a^+)} + \frac{c y_d^-}{6(C-\lambda^2 y_d^+ y_d^-)} = 0\,. \tag{76}$$

These equations are quadratic, so they can be easily solved. In the limit $G_N \to 0$, the two solutions behave as

$$y_d^- = \frac{cG_N C\beta}{12\lambda^2 y_a^+} = -\frac{cG_N C}{6\sqrt{2C_0^L}\,y_a^+\lambda}, \quad y_d^+ = \frac{cG_N C\beta}{12\lambda^2 y_a^-} = -\frac{cG_N C}{6\sqrt{2C_0^L}\,y_a^-\lambda}, \tag{77}$$

$$y_d^- = y_a^- + \mathcal{O}(G_N), \quad y_d^+ = y_a^+ + \mathcal{O}(G_N). \tag{78}$$

Let us now attempt to derive these solutions directly from our general Equation (36):

$$z(1-\frac{\epsilon}{\xi e^{-Q}})^2 = y^2 \quad\to\quad z\left((1+\frac{\epsilon}{2C_0^L}) - \frac{\epsilon C}{2C_0^L\lambda^2 z}\right)^2 = y^2, \tag{79}$$

with

$$\epsilon = \frac{cG_N\lambda\sqrt{2C_0^L}}{6}. \tag{80}$$

Equation (79) is also quadratic with solutions to be

$$z_1 = \frac{C^2\epsilon^2}{4{C_0^L}^2\lambda^4 y^2} + \mathcal{O}(\epsilon^3), \quad z_2 = y^2 - \frac{\epsilon(\lambda^2 y^2 - C)}{C_0^L\lambda^2} + \mathcal{O}(\epsilon^2) \tag{81}$$

which will correspond to (77) and (78), respectively. The first solution (77) is the non-trivial one, which gives the generalized entanglement entropy

$$S_{\text{Rad}} = 2\frac{1}{G_N}\frac{\sqrt{2C_0^L}}{\lambda}\log\left(\frac{C}{2\lambda^2}\right) + \frac{c}{3}\log\left(\frac{|y_a^+ y_a^-|}{\sqrt{C(C-\lambda^2 y_a^+ y_a^-)}}\right). \tag{82}$$

Thus, we have derived the Page curve for the Liouville black hole. This success can be attributed to the fact that we have derived another black hole solution with parameters opposite to those used in [39] or derived in [46,48]. Let us revisit the black hole geometry used in [39].

5.3. The Other Black Geometry

To obtain the solution, we begin with (68) and apply the transformation of reversing the radial coordinate, given by

$$r' \to -r', \tag{83}$$

which results in the metric

$$\mathrm{d}\, s^2 = -(1 + \frac{2\alpha^2 e^{-\beta\sqrt{2C_0^L} r'}}{C_0^L \beta})\mathrm{d}\, t'^2 + \frac{1}{1 + \frac{2\alpha^2 e^{-\beta\sqrt{2C_0^L} r'}}{C_0^L \beta}} \mathrm{d}\, r'^2. \tag{84}$$

The position of the event horizon and the Ricci scalar is

$$r'_H = \frac{1}{\sqrt{2C_0^L}\beta} \log\left(-\frac{2\alpha^2}{C_0^L \beta}\right), \quad R = -4\beta\alpha^2 e^{-\sqrt{2C_0^L}\beta r'}. \tag{85}$$

The requirement of asymptotic flatness as $r' \to \infty$ necessitates the choice of $\beta > 0$. The requirement for a well-defined horizon mandates $\alpha^2 < 0$. With these selections, the solution in Kruskal coordinates remains as given in Equation (73):

$$\mathrm{d}^2\, s = -\frac{\mathrm{d}\, y^+ \,\mathrm{d}\, y^-}{C - \lambda^2 y^+ y^-}, \tag{86}$$

$$X = -\frac{1}{\beta} \log\left[\frac{1}{2\lambda^2}\left(C - \lambda^2 y^+ y^-\right)\right]. \tag{87}$$

but with different identification

$$\beta = \frac{2\lambda}{\sqrt{2C_0^L}}, \quad \alpha^2 = -\frac{C}{2\lambda}\sqrt{\frac{C_0^L}{2}}. \tag{88}$$

However, in this black hole solution, the position of the island is at

$$y_d^- = \frac{cG_N C\beta}{12\lambda^2 y_a^+} = \frac{cG_N C}{6\sqrt{2C_0^L} y_a^+ \lambda}, \quad y_d^+ = \frac{cG_N C\beta}{12\lambda^2 y_a^-} = \frac{cG_N C}{6\sqrt{2C_0^L} y_a^- \lambda}, \tag{89}$$

which is in the left Kruskal patch. This contradicts the assumption that y_d is in the right patch, and this is why [39] claims the failure of the island formula. However, we have shown that this is only because the "wrong" solution is used.

To summarize, the island can ***save*** the information paradox of Liouville gravity.

6. Ab-Family

In this section, we examine a broad class of dilaton gravity theories characterized by the following potentials:

$$U(X) = -\frac{a}{X}, \quad V(X) = -\frac{B}{2} X^{a+b}. \tag{90}$$

This family is commonly referred to as the ab-family as it contains two free parameters [49]. From our general discussion, the classical solution is

$$\mathrm{d}^2\, s = 2X^{-a}\,\mathrm{d}\, X \,\mathrm{d}\, v - X^{-2}(2C_0 + \frac{BX^{b+1}}{b+1})\,\mathrm{d}^2\, v. \tag{91}$$

To obtain asymptotically flat black hole solutions, we have to choose [49]

$$b = a - 1, \quad a \in (0, 1). \tag{92}$$

This choice can be understood from the expression of the Ricci scalar (here, we consider the case $b \neq -1$):

$$R = -2aC_0X^{a-2} + \frac{bB(a-b-1)}{b+1}X^{a+b-1}. \tag{93}$$

Note that C_0 is related to the mass of the black hole. Thus, the solution with $C_0 = 0$ should represent the Minkowski spacetime, which corresponds to $b = a - 1$. Additionally, the asymptotic flatness condition $\lim_{X\to\infty} R = 0$ restricts $a \in (0,1)$. Following the general discussion, we can compute the following quantities:

$$Q = -a\log X, \quad X = (1-a)^{\frac{1}{1-a}}\tilde{X}^{\frac{a}{1-a}}, \quad w = -\frac{B}{2a}(1-a)^{\frac{a}{1-a}}\tilde{X}^{\frac{a}{1-a}}, \tag{94}$$

$$\xi = -\frac{B}{a} - \frac{2C_0^f}{(1-a)^{\frac{a}{1-a}}\tilde{X}^{\frac{a}{1-a}}}, \quad \xi_0 = \sqrt{\frac{B}{a}}. \tag{95}$$

Thus, the corresponding Schwarzschild metric is

$$\mathrm{d}s^2 = -(1 - \frac{1}{2\lambda r'^{\frac{a}{1-a}}})\mathrm{d}t'^2 + \frac{1}{1-\frac{1}{2\lambda r'^{\frac{a}{1-a}}}}\mathrm{d}r'^2, \tag{96}$$

where

$$\frac{1}{2\lambda} = -\frac{2C_0^f a}{B}\left(\sqrt{\frac{B}{a}}(1-a)\right)^{\frac{a}{a-1}}, \quad C_0^f < 0, \quad \lambda > 0. \tag{97}$$

Thus, the horizon is located at

$$r'_H = (2\lambda)^{\frac{a-1}{a}}. \tag{98}$$

Next, we can transform it into the Kruskal coordinates by introducing

$$x^* = \int \frac{\mathrm{d}r'}{1-\frac{1}{2\lambda r'^{\frac{a}{1-a}}}} = 2(a-1)\lambda r'^{\frac{1}{1-a}}\,{}_2F_1\left(1, \frac{1}{a}; 1+\frac{1}{a}; 2r'^{\frac{a}{1-a}}\lambda\right) + c_1, \tag{99}$$

where c_1 is a constant that can be chosen for our convenience. The hypergeometric function generally cannot be inverted to write $r'(x^*)$ as a function of x^* explicitly. However, for the special case of $a = 1/2$, we can invert the function with the product logarithm:

$$x^* = r' + \frac{\log(2\lambda r' - 1)}{2\lambda} + c_1, \qquad \rightarrow \tag{100}$$

$$r' = \frac{1}{2\lambda} + \frac{W_0\left(e^{2\lambda(x^*-c_1)-1}\right)}{2\lambda}, \tag{101}$$

where W_0 is the principle branch of the Lambert W function or product logarithm. Thus, we can define the Kruskal coordinates via (20) and the result is

$$e^{\pm\lambda x^{\pm}} = \pm\lambda y^{\pm}, \quad c_1 = -\frac{1}{2\lambda}, \tag{102}$$

$$\mathrm{d}s^2 = -e^{2\rho}\,\mathrm{d}y^+\,\mathrm{d}y^-, \quad e^{2\rho} = \frac{1}{e^{W_0(-\lambda^2y^2)} - \lambda^2y^2}, \tag{103}$$

$$X = \sqrt{\frac{B}{2}}\frac{1}{4\lambda}\left(1 + W_0(-\lambda^2y^2)\right), \quad y^2 = y^+y^-. \tag{104}$$

Equation (36) for the determination of the position of the island becomes

$$z\left(\frac{\epsilon 2^{1/4}\sqrt{\lambda}\sqrt{W_0(\lambda^2(-z))+1}}{B^{5/4}W_0(\lambda^2(-z))}+1\right)^2=y^2,\quad \epsilon=\frac{cG_N\lambda}{6}\sqrt{\frac{B}{a}}. \tag{105}$$

Assuming $\epsilon\to 0$, we can expand the left-hand side to the first order of z. The resulting quadratic equation has two solutions:

$$z_1=y^2+\frac{2^{1/4}\alpha(2-3y^2\lambda^2)}{B^{5/4}\lambda^{3/2}}+\mathcal{O}(\epsilon^2),\quad z_2=\frac{\beta^2}{y^2},\quad \beta^2=\frac{\sqrt{2}\epsilon^2}{B^{5/2}\lambda^3}. \tag{106}$$

Thus, the physical solution, which satisfies the assumption $z\approx 0$, is

$$y_d^+=-\frac{\beta}{y_a^-},\quad y_d^-=-\frac{\beta}{y_a^+}, \tag{107}$$

which leads to

$$S_{\rm Rad}=\sqrt{\frac{B}{2}}\frac{1}{G_N\lambda}+\frac{c}{3}\log\frac{|y_a^+y_a^-|}{\sqrt{e^{W_0(-\lambda^2y^2)}-\lambda^2y^2}}+\mathcal{O}(G_N). \tag{108}$$

For generic a, we observe that x^* can be always decomposed into

$$x^*=f(r')+\frac{1-a}{a}(2\lambda)^{\frac{a-1}{a}}\log(r'-r'_H), \tag{109}$$

where $f(r')$ is regular at $r'=r'_H$, as we expect in (40). Here, we omit the further analysis since it is very similar to the general result.

7. Reissner–Nordstrom

Islands in higher-dimensional black holes have been extensively explored in the recent literature, including studies such as [27–36]. While the focus is on higher-dimensional black holes, the model effectively reduces to two dimensions through the dimensional reduction of the sphere and the s-wave approximation. As such, these studies can all be described by the GDT. For example, the higher-dimensional Schwarzschild black hole can be reduced to the GDT with the following potentials: $U=-1/(2X)$, $V=-\lambda^2$. This characteristic feature explains the universality of the island formula and why the influence of higher dimensions appears negligible. However, distinct higher-dimensional black holes are characterized by different GDTs, resulting in variation in the location of the island and the Page time. As a simple example, let us consider the Reissner–Nordstrom black holes.

The Reissner–Nordstrom black hole in the four-dimensional Maxwell–Einstein gravity theory corresponds to the GDT with the potentials

$$U(X)=-\frac{1}{2X},\quad V(X)=-\lambda^2+\frac{A}{X}, \tag{110}$$

which lead to the following quantities:

$$e^Q=\frac{1}{\sqrt{X}},\quad X=\frac{\tilde{X}^2}{4},\quad w=-\frac{2(A+\lambda^2X)}{\sqrt{X}}=-\lambda^2\tilde{X}-\frac{4A}{\tilde{X}}, \tag{111}$$

$$\xi=-4\lambda^2-\frac{4C_0^R}{\tilde{X}}-\frac{16A}{\tilde{X}^2},\quad \xi_0=2\lambda. \tag{112}$$

Thus, the metric and dilaton field are given by

$$\mathrm{d}\,s^2 = -(1+\frac{C_0^R}{2r'\lambda^3}+\frac{A}{r'^2\lambda^4})\mathrm{d}\,t'^2 + \frac{1}{1+\frac{C_0^R}{2r'\lambda^3}+\frac{A}{r'^2\lambda^4}}\mathrm{d}\,r'^2, \tag{113}$$

$$X = \lambda^2 r'^2. \tag{114}$$

Compared with the standard Reissner–Nordstrom solution, we can identify the following parameters:

$$C_0^R = -4M\lambda^3, \quad A = \lambda^4 Q_c^2, \quad \lambda = 1, \tag{115}$$

$$M = \frac{r_+ + r_-}{2}, \quad Q_c = \sqrt{r_+ r_-}, \tag{116}$$

where M and Q_c are the mass and charge of the black hole and $r_\pm$ are the positions of the outer $(+)$ and inner $(-)$ horizons. By first introducing the new variable

$$x^* = \int \frac{\mathrm{d}\,r'}{1-\frac{2M}{r'}+\frac{Q_c^2}{r'^2}} = r' + \frac{r_+^2\log(r'-r_+) - r_-^2\log(r'-r_-^2)}{r_+ - r_-}, \tag{117}$$

$$\exp(2x^*) = e^{2r'}(r'-r_+)^{\frac{2r_+^2}{r_+-r_-}}(r'-r_-)^{-\frac{2r_-^2}{r_+-r_-}}, \tag{118}$$

we can define Kruskal coordinates via (20):

$$e^{lx^+} = ly^+, \quad e^{-lx^-} = -ly^-, \quad l = \frac{r_+ - r_-}{2r_+^2}. \tag{119}$$

Using (118), we can directly obtain

$$-l^2 z = e^{2dl}(d-r_+)(d-r_-)^{-\frac{r_-^2}{r_+^2}}, \tag{120}$$

which leads to

$$d = r_+ - e^{-2lr_+}l^2(r_+-r_-)^{\frac{r_-^2}{r_+^2}}z + \mathcal{O}(z^2)\,. \tag{121}$$

Substituting (121) into (36) with the help of (111) and (112), we obtain an equation of z. The non-trivial solution is

$$z = \frac{\beta^2}{y^2}, \quad \beta^2 = \frac{\epsilon^2 e^{4lr_+}r_+^2(r_+-r_-)^{-\frac{2r_-^2}{r_+^2}-2}}{16l^4y^2}, \quad \epsilon = \frac{cG_N l}{3}, \tag{122}$$

$$y_d^+ = -\frac{\beta}{y_a^-}, \quad y_d^- = -\frac{\beta}{y_a^+}\,. \tag{123}$$

7.1. Other Charged Dilaton Black Hole I

We can also consider islands in other charged dilaton black holes. In [30], the charged dilaton black hole has the metric

$$\mathrm{d}\,s^2 = -r^2\left(1-\frac{2M}{r^2}+\frac{Q_c^2}{4r^4}\right)\mathrm{d}\,t^2 + \left(1-\frac{2M}{r^2}+\frac{Q_c^2}{4r^4}\right)^{-1}\mathrm{d}\,r^2 + r^2(\mathrm{d}\,x^2+dy^2), \tag{124}$$

which can be reduced to 2d by identifying

$$\mathrm{d}\,s^2 = -H(r)\,\mathrm{d}\,t^2 + r^2 H(r)^{-1}\,\mathrm{d}\,r^2, \quad X = r^2, \tag{125}$$

where

$$H(r) = r^2\left(1 - \frac{2M}{r^2} + \frac{Q_c^2}{4r^4}\right). \tag{126}$$

To transform this into the Schwarzschild metric, let us introduce

$$dr' = 2r\,\mathrm{d}\,r, \quad \to, \quad r' = r^2, \tag{127}$$

such that

$$\mathrm{d}\,s^2 = -H(r')\mathrm{d}\,t'^2 + H(r')^{-1}\mathrm{d}\,r'^2, \quad t' = t/2, \tag{128}$$

$$H(r') = 4r' - 8M - \frac{Q_c^2}{r'} = \frac{4(r-r_+)(r-r_-)}{r'}, \quad r_\pm = M \pm \frac{\sqrt{4M^2 - Q_c^2}}{2}. \tag{129}$$

The geometry is asymptotically flat. The outer event horizon and curvature singularity are located at $r' = r_+$ and $r' = 0$, respectively. The solution can be embedded into dilaton gravity by choosing the following potentials:

$$U(X) = 0, \quad V(X) = -2 + \frac{Q_c^2}{2X^2}, \quad C_0 = 4M. \tag{130}$$

When $Q_c = 0$, the geometry (128) reduces to the Rindler patch. From the potential, we obtain

$$Q = 0, \quad X = \tilde{X} = r', \quad w = -2X - \frac{Q_c^2}{2X}, \quad \xi = 8M - \frac{Q_c^2}{X} - 4X. \tag{131}$$

Following the general procedure, we find that in late time, the position of the island is

$$d = r_+ - l^2(r_+ - r_-)^{\frac{r_-}{r_+}} z, \tag{132}$$

$$z = \frac{\beta^2}{y^2}, \quad \beta^2 = \frac{\epsilon^2 r_+^2 (r_+ - r_-)^{-\frac{2(r_- + r_+)}{r_+}}}{16 l^4}, \quad \epsilon = \frac{cG_N l}{6}, \tag{133}$$

$$y_d^+ = -\frac{\beta}{y_a^-}, \quad y_d^- = -\frac{\beta}{y_a^+}, \tag{134}$$

which coincides with the results in [30].

7.2. Other Charged Dilaton Black Hole II

In [29], the charged dilaton black hole has the metric

$$\mathrm{d}\,s^2 = -W(r)\,\mathrm{d}\,t^2 + W^{-1}\,\mathrm{d}\,r^2 + R(r)^2\,\mathrm{d}\,\Omega^2, \tag{135}$$

with the function

$$W(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^n, \quad R^2 = r^2\left(1 - \frac{r_-}{r}\right)^{1-n}, \quad n \in [0, 1). \tag{136}$$

The effective 2D model is

$$\mathrm{d}\,s^2 = -W(r)\,\mathrm{d}\,t^2 + W^{-1}\,\mathrm{d}\,r^2, \quad X = r^2\left(1 - \frac{r_-}{r}\right)^{1-n} \equiv f(r). \tag{137}$$

The corresponding 2d dilaton potentials can be

$$e^{Q}=\frac{\mathrm{d}\,f^{-1}(X)}{\mathrm{d}\,X}\equiv f^{-1\prime},\quad U(X)=\frac{\mathrm{d}\ln(f^{-1\prime})}{\mathrm{d}\,X},\tag{138}$$

$$V(X)=-\frac{1}{2}e^{-Q(X)}\frac{\mathrm{d}\left(e^{-Q(X)}W(f^{-1}(X))\right)}{\mathrm{d}\,X}.\tag{139}$$

In general, $f(r)$ is difficult to invert, but, to solve the island, the explicit expressions of the potentials are not needed. We only need the quantity

$$\xi e^{-Q}=-W\frac{dX}{\mathrm{d}\,r}=-\frac{(2r-(1+n)r_-)(r-r_+)}{r},\tag{140}$$

which appears in (36), and the relation between r and z,

$$\begin{aligned} x^* &= \int\frac{\mathrm{d}\,r}{W(r)}=\frac{r^n}{(r-r_-)^{n-1}}+(nr_-+r_+)B_{1-\frac{r_-}{r}}(1-n,0)\\ &- \left(\frac{r_+}{r_+-r_-}\right)^n B_t(1-n,0),\quad t=\frac{r_+}{r_+-r_-}\left(1-\frac{r_-}{r}\right),\end{aligned}\tag{141}$$

where $B_\alpha(a,b)$ is the incomplete beta function. Note that

$$\lim_{t\to1}B_t(1-n,0)=-\log(t-1)\tag{142}$$

thus, let us denote x^* as

$$x^*=\mathcal{R}+\left(\frac{r_+}{r_+-r_-}\right)^n\log(r-r_+),\tag{143}$$

$$e^{2lx^*}=e^{2l\mathcal{R}}(r-r_+),\quad \frac{1}{2l}=\left(\frac{r_+}{r_+-r_-}\right)^n.\tag{144}$$

It implies that

$$d=r_+-l^2e^{-2l\mathcal{R}(r_+)}z+\mathcal{O}(z^2).\tag{145}$$

Substituting it into (36), we can solve

$$z=\frac{\beta^2}{y^2},\quad \beta=\frac{\epsilon r_+}{e^{-2l\mathcal{R}(r_+)}(2r_+-r_--nr_-)},\quad \epsilon=\frac{cG_N}{6},\tag{146}$$

$$y_d^+=-\frac{\beta}{y_a^-},\quad y_d^-=-\frac{\beta}{y_a^+},\tag{147}$$

which is consistent with the result in [29], but our method is much simpler. Thus far, we have studied three different four-dimensional charged black hole solutions, which all can be embedded into GDTs. While the island's position in these models appears uniform, its exact location varies significantly based on the specific mass and charge configuration.

8. Conclusions and Discussion

In this work, we have studied the island Formula (1) in the general asymptotically flat eternal black holes in GDTs. Under some reasonable and mild assumptions, we prove that the island always appears barely outside of the horizon in the late time of Hawking radiation so that the information paradox is resolved; in particular, in the Liouville gravity theory, which was reported in [39], the island proposal failed. We find that the failure is due to the use of the "wrong" black hole solution. With the help of the general construction of classical solutions of GDT, we find a different black hole solution where the island appears as expected. We further apply our general analysis to a large family of GDTs and several four-dimensional charged dilaton black holes. It is found that our procedure for the discovery of an island is much simpler. From our perspective, the universality of the island formula and the reason that the influence of higher dimensions appears negligible lie in the fact that all these models can be embedded into GDTs.

There are some possible generalizations of our analysis.

- Our general analysis should be simply generalized to the asymptotically AdS black holes in GDT by gluing a flat bath. This is because, after gluing the flat bath, the whole spacetime is similar to the asymptotically flat black hole, and cut-off surface y_a can be chosen to be the boundary of the AdS space.
- In this work, we only consider the classical solutions of GDT. It is also possible to include the quantum effect that comes from the conformal anomaly, following, for example, [21].
- It is also possible to generalize our results to the single-sided black hole and consider a truly evaporating black hole. Some examples are [21,25].

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Appendix A. Review of 2D GDT

Appendix A.1. Conventions

The local Lorentz metric and the Lorentz transformation invariant tensor are chosen to be

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon^a{}_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A1}$$

Thus, the Levi–Civita tensors are

$$\epsilon^{ac} = \epsilon^a{}_b \eta^{bc} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{ab} = \eta_{ac}\epsilon^c{}_b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A2}$$

The volume form is related to the local Lorentz basis e^a via

$$\epsilon = \frac{1}{2}\epsilon_{ab} e^a \wedge e^b = \frac{1}{2}\epsilon_{ab} e^a{}_\mu e^b{}_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}\epsilon_{ab}\left(e^a{}_1 e^b{}_0 - e^a{}_0 e^b{}_1\right) \mathrm{d}\, x^1 \wedge \mathrm{d}\, x^0 \tag{A3}$$

$$= \left(e^1{}_1 e^0{}_0 - e^1{}_0 e^0{}_1\right) \mathrm{d}\, x^1 \wedge \mathrm{d}\, x^0 = \sqrt{-g}\, \mathrm{d}\, x^1 \wedge \mathrm{d}\, x^0 \to \sqrt{-g}\, \mathrm{d}^2\, x. \tag{A4}$$

In 2d, the spin connection should be proportional to ϵ: $\omega^a{}_b = \omega\epsilon^a{}_b$ and the Ricci tensor two-form is then given by

$$R_{ab} = \mathrm{d}\,\omega\epsilon_{ab}, \quad R_{ab} = \frac{1}{2}(R_{\mu\nu})_{ab} dx^\mu \wedge dx^\nu, \quad (R_{\mu\nu})_{ab} = \epsilon_{ab}\left(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu\right). \tag{A5}$$

Thus, the Ricci scalar is

$$(R_{\mu\nu})_{ab} e^{a\mu} e^{b\nu} = \epsilon^{\mu\nu}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu) = 2\epsilon^{\mu\nu}\partial_\mu\omega_\nu = 2|e|^{-1}\tilde{\epsilon}^{\mu\nu}\partial_\mu\omega_\nu \tag{A6}$$

$$= 2|e|^{-1}(\partial_0\omega_1 - \partial_1\omega_0) \tag{A7}$$

On the other hand, we have

$$\mathrm{d}\,\omega = \partial_\mu \omega_\nu dx^\mu \wedge dx^\nu = (\partial_1\omega_0 - \partial_0\omega_1)dx^1 \wedge dx^0 = -\frac{1}{2}R\sqrt{-g}\,\mathrm{d}^2\,x. \tag{A8}$$

The torsion two-form is given by

$$T^a = (D)^a{}_b e^b \quad = \quad (\delta^a_b\,\mathrm{d} + \omega^a{}_b)e^b = \mathrm{d}\,e^a + \omega^a{}_b \wedge e^b, \tag{A9}$$

where its components are

$$\begin{aligned} T^a_{\mu\nu} &= \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + (\omega_\mu)^a{}_b e^b_\nu - (\omega_\nu)^a{}_b e^b_\mu && \text{(A10)}\\ &= D_\mu e^a_\nu - D_\nu e^a_\mu. && \text{(A11)} \end{aligned}$$

It is convenient to use the light-cone gauge:

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1), \quad x^0 = \frac{1}{\sqrt{2}}(x^+ + x^-), \quad x^1 = \frac{1}{\sqrt{2}}(x^+ - x^-). \tag{A12}$$

The Lorentz transformation connecting these gauges is

$$\Lambda^a{}_{\bar{a}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{A13}$$

Thus, we can find that

$$\eta_{\bar{a}\bar{b}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{\bar{a}\bar{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{\bar{a}}{}_{\bar{b}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A14}$$

such that the torsion form (A10) can be expressed as

$$T^\pm = (\mathrm{d} \pm \omega)e^\pm. \tag{A15}$$

Appendix A.2. The First-Order Formalism of GDT

The action (2) is equivalent to

$$I_{\text{gen}}[e_a, \omega, X, X^a] = \int \left(X\,\mathrm{d}\,\omega + X_a(\mathrm{d}\,e^a + \epsilon^a{}_b\omega \wedge e^b) + \frac{1}{2}\epsilon^{ab} e_a \wedge e_b \mathcal{V}(X, X^c X_c) \right). \tag{A16}$$

We will first solve all its classical solutions and then prove the equivalence. Varying with respect to ω gives

$$\begin{aligned} &X\,\mathrm{d}\,\delta\omega + X^a\epsilon_{ab}\delta\omega \wedge e^b = -\,\mathrm{d}\,X \wedge \delta\omega - X^a\epsilon_{ab}e^b \wedge \delta\omega \to \\ &\mathrm{d}\,X + X^a\epsilon_{ab}e^b = 0. && \text{(A17)} \end{aligned}$$

Varying with respect to e, we obtain

$$\begin{aligned} &X^a\,\mathrm{d}\,\delta e_a + X^a\epsilon_a{}^b\omega \wedge \delta e_b + \frac{1}{2}\epsilon^{ab}(\delta e_a \wedge e_b - e_a \wedge \delta e_b)\mathcal{V} \to \\ &-\mathrm{d}\,X^a \wedge \delta e_a + X^b\epsilon_b{}^a\omega \wedge \delta e_a - \epsilon^{ab}e_b \wedge \delta e_a\mathcal{V} \to \\ &dX^a - X^b\epsilon_b{}^a\omega + \epsilon^{ab}e_b\mathcal{V} = dX^a + X^b\epsilon^a{}_b\omega + \epsilon^{ab}e_b\mathcal{V} = 0, && \text{(A18)} \end{aligned}$$

where, in the last line, we use $\epsilon_b{}^a = \eta_{bc}\epsilon^c{}_d\eta^{da} = -\epsilon^a{}_b$. The other two equations of motion are

$$\mathrm{d}\,\omega + \frac{1}{2}\epsilon^{ab}e_a \wedge e_b\frac{\partial\mathcal{V}}{\partial X} = 0, \tag{A19}$$

$$\mathrm{d}\,e_a + \epsilon_a{}^b\omega \wedge e_b + \frac{1}{2}\epsilon^{ab}e_a \wedge e_b\frac{\partial\mathcal{V}}{\partial X^a}. \tag{A20}$$

In the light-cone gauge, the equations of motion become

$$\mathrm{d}\,X + X^+e^- - X^-e^+ = 0, \tag{A21}$$
$$(\mathrm{d}\pm\omega)X^\pm \pm \mathcal{V}e^\pm = 0, \tag{A22}$$
$$\mathrm{d}\,\omega + \epsilon\frac{\partial V}{\partial X} = 0, \tag{A23}$$
$$(\mathrm{d}\pm\omega)e^\pm + \epsilon\frac{\partial\mathcal{V}}{\partial X_\pm} = (\mathrm{d}\pm\omega)e^\pm - \epsilon\frac{\partial\mathcal{V}}{\partial X^\mp} = 0, \tag{A24}$$

where the volume form is $\epsilon = e^+\wedge e^-$ and, in the last line, we have used $X_\pm = -X^\mp$. From (A22), we have

$$X^-\,\mathrm{d}\,X^+ + X^+\,\mathrm{d}\,X^- + \mathcal{V}(X^-e^+ - X^+e^-) = 0, \tag{A25}$$

and, using (A21), we obtain

$$\mathrm{d}(X^-X^+) + \mathcal{V}(X^-X^+, X)\,\mathrm{d}\,X = 0. \tag{A26}$$

This equation indicates that these exists a conserved quantity defined by integrating (A26).

If $X^+ \neq 0$, from (A22), we can obtain

$$\omega = -\frac{\mathrm{d}\,X^+}{X^+} - Z\mathcal{V}, \quad Z \equiv \frac{e^+}{X^+}, \tag{A27}$$

and, from (A21), we can obtain

$$e^- = -\frac{\mathrm{d}\,X}{X^+} + X^-Z. \tag{A28}$$

Substituting the expression of volume form

$$\epsilon = \frac{1}{2}\epsilon_{ab}e^a\wedge e^b = e^+\wedge e^- = \mathrm{d}\,X\wedge Z \tag{A29}$$

into (A24) gives

$$\mathrm{d}\,e^+ + \omega\wedge e^+ - \mathrm{d}\,X\wedge Z\frac{\partial\mathcal{V}}{\partial X^-} = 0 \tag{A30}$$
$$= X^+\,\mathrm{d}\,Z + \mathrm{d}\,X^+\wedge Z - \mathrm{d}\,X^+\wedge Z - \mathrm{d}\,X\wedge Z\frac{\partial\mathcal{V}}{\partial X^-} = 0. \tag{A31}$$

Therefore, we have

$$\mathrm{d}\,Z = -\frac{Z\wedge\mathrm{d}\,X}{X^+}\frac{\partial\mathcal{V}}{\partial X^-}. \tag{A32}$$

Taking the ansatz of Z as

$$Z = \mathrm{d}\,v e^{Q(X)}, \quad \mathrm{d}\,Z = e^{Q(X)}\frac{\mathrm{d}\,Q}{\mathrm{d}\,X}\mathrm{d}\,X\wedge dv, \tag{A33}$$

and substituting it into (A32) gives

$$\frac{\mathrm{d}\,Q}{\mathrm{d}\,X} = \frac{1}{X^+}\frac{\partial\mathcal{V}}{\partial X^-}, \rightarrow \tag{A34}$$
$$Q = \int^X \frac{1}{X^+}\frac{\partial\mathcal{V}}{\partial X^-}. \tag{A35}$$

Recall that the metric is

$$\mathrm{d}\, s^2 = \eta_{ab} e^a e^b = -2e^+ e^- = 2(Z\, \mathrm{d}\, X - X^+ X^- Z^2) = 2e^Q(\mathrm{d}\, v\, \mathrm{d}\, X - e^Q Y\, \mathrm{d}^2\, v), \tag{A36}$$

where $Y \equiv X^+ X^-$. Thus, all solutions (in the linear dilaton vacua) for all generalized dilaton gravity models obey a generalized Birkhoff theorem, in the sense that all solutions exhibit a Killing vector ∂_v. The solution space is parameterized by two constants of integration. The one coming from the integration of (A26) is non-trivial, while the one coming from (A35) is trivial and can be fixed by a choice of units.

Appendix A.3. Back to Second-Order Formalism

First, we separate out the torsion-free part of the spin connection. To do this, we notice

$$\star T_a = \star\left(\mathrm{d}\, e_a + \epsilon_a{}^b \omega \wedge e_b\right) = \star\, \mathrm{d}\, e_a + \epsilon_a{}^b \omega^c \star (e_c \wedge e_b). \tag{A37}$$

Using

$$e_a \wedge e_b = e_{a\mu} e_{b\nu}\, \mathrm{d}\, x^\mu \wedge \mathrm{d}\, x^\nu, \tag{A38}$$
$$\star e_a \wedge e_b = e_{a\mu} e_{b\nu} \star \mathrm{d}\, x^\mu \wedge \mathrm{d}\, x^\nu = e_{a\mu} e_{b\nu} \epsilon^{\mu\nu} = \epsilon_{ab} \tag{A39}$$

we obtain

$$\star T_a = \star\, \mathrm{d}\, e_a + \epsilon_a{}^b \omega^c \epsilon_{cb} = \star\, \mathrm{d}\, e_a - \omega_a. \tag{A40}$$

Thus, we can rewrite the spin connection as

$$\omega = \omega^a e_a = (\star\, \mathrm{d}\, e_a - \star T_a) e^a = e^a \star \mathrm{d}\, e_a - e^a \star T_a. \tag{A41}$$

Then, $\tilde{\omega} = e^a \star \mathrm{d}\, e_a$ is the torsion-free part, which, in terms of components, is given by

$$\star\, \mathrm{d}\, e_a = \partial_\mu (e_\nu)_a \epsilon^{\mu\nu}, \quad \tilde{\omega} = e^a \partial_\mu (e_\nu)_a \epsilon^{\mu\nu}. \tag{A42}$$

Recall that the action in the first formalism is

$$I_{\text{gen}} \sim \int X\, \mathrm{d}\, \omega + \epsilon \mathcal{V} + X^a T_a. \tag{A43}$$

The first term can be manipulated as

$$X\, \mathrm{d}\, \omega = -\, \mathrm{d}\, X \wedge \omega = -\, \mathrm{d}\, X \wedge (\tilde{\omega} - e^a \star T_a) = X\, \mathrm{d}\, \tilde{\omega} + \mathrm{d}\, X \wedge e^a \star T_a. \tag{A44}$$

Note that

$$\mathrm{d}\, \tilde{\omega} = \partial_\mu \omega_\nu\, \mathrm{d}\, x^\mu \wedge \mathrm{d}\, x^\nu \to -\frac{R}{2} \sqrt{-g}\, \mathrm{d}^2\, x, \tag{A45}$$

which is exactly the first term in the action (2). It is obvious that

$$\epsilon \mathcal{V}(X, X^a X_a) \to \sqrt{-g} \mathcal{V}(X, X^a X_a)\, \mathrm{d}^2\, x. \tag{A46}$$

Thus, our last task is to remove X^a with the help of equation of motion (A20),

$$T_a = -\frac{1}{2} \epsilon^{bc} e_b \wedge e_c \frac{\partial \mathcal{V}}{\partial X^a} \to \tag{A47}$$
$$\star T_a = -\frac{1}{2} \epsilon^{bc} \epsilon_{bc} \frac{\partial \mathcal{V}}{\partial X^a} = \frac{\partial \mathcal{V}}{\partial X^a} \tag{A48}$$

and (A17),

$$\partial_\mu X + X^a \epsilon_a{}^b e_{b\mu} = 0 \rightarrow \tag{A49}$$
$$\partial_\mu X + X^a e_a^\nu \epsilon_{\nu\mu} = 0 \rightarrow \tag{A50}$$
$$X^a = -e_\nu^a \epsilon^{\mu\nu} \partial_\mu X. \tag{A51}$$

Thus, the terms involved in T^a are cancelled out by each other:

$$X^a T_a = \frac{1}{2} e_\nu^a \epsilon^{\mu\nu} \partial_\mu X \epsilon^{cb} e_c \wedge e_b \frac{\partial \mathcal{V}}{\partial X^a} \rightarrow e_\nu^a \epsilon^{\mu\nu} \partial_\mu X \frac{\partial \mathcal{V}}{\partial X^a} \sqrt{-g}\, \mathrm{d}^2 x, \tag{A52}$$
$$\mathrm{d}\, X \wedge e^a \star T_a = \partial_\mu X e_\nu^a \frac{\partial \mathcal{V}}{\partial X^a} \mathrm{d}\, x^\mu \wedge \mathrm{d}\, x^\nu \rightarrow -e_\nu^a \epsilon^{\mu\nu} \partial_\mu X \frac{\partial \mathcal{V}}{\partial X^a} \sqrt{-g}\, \mathrm{d}^2 x. \tag{A53}$$

Then, we arrive at the action in the second-order formalism

$$-\frac{1}{2} \int \sqrt{-g} \Big(XR - 2\mathcal{V}(X, -(\partial X)^2) \Big), \tag{A54}$$

where we have used

$$X^a X_a = -(\partial X)^2. \tag{A55}$$

Therefore, we find that the Q function is given by

$$Q = \int^X U(y)\, \mathrm{d}\, y, \tag{A56}$$

and the conserved quantity (A26) is given by

$$C_0 = e^Q Y + w, \quad Y = X^+ X^-, \quad w = \int^X e^Q V(y)\, \mathrm{d}\, y \tag{A57}$$
$$dC_0 = e^Q\, \mathrm{d}\, Y + Y e^Q U(X)\, \mathrm{d}\, X + e^Q V\, \mathrm{d}\, X = 0. \tag{A58}$$

Using this, we can rewrite the metric as

$$\mathrm{d}\, s^2 = 2e^Q\, \mathrm{d}\, v (\mathrm{d}\, X + (w(X) - C_0)\, \mathrm{d}\, v) \tag{A59}$$
$$= 2\, \mathrm{d}\, v\, \mathrm{d}\, \tilde{X} + \xi(\tilde{X})\, \mathrm{d}\, v^2, \tag{A60}$$

where we have introduced

$$\mathrm{d}\, \tilde{X} = \mathrm{d}\, X e^Q, \quad \xi(\tilde{X}) = 2e^Q (w - C_0). \tag{A61}$$

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