symmetry

Article

Super-Connectivity of the Folded Locally Twisted Cube

Lantao You [1,2,3,4], Yuejuan Han [4,*] and Jianfeng Jiang [1]

1 School of Information Engineering, Suzhou Industrial Park Institute of Services Outsourcing, Suzhou 215123, China; yoult@siso.edu.cn (L.Y.); jiangjf@siso.edu.cn (J.J.)
2 Suzhou Industrial Park Human Resources Development Co., Ltd., Suzhou 215028, China
3 Provincial Key Laboratory for Computer Information Processing Technology, Soochow University, Suzhou 215006, China
4 School of Computer Science and Technology, Soochow University, Suzhou 215006, China
* Correspondence: hyj@suda.edu.cn

Abstract: The hypercube Q_n is one of the most popular interconnection networks with high symmetry. To reduce the diameter of Q_n, many variants of Q_n have been proposed, such as the n-dimensional locally twisted cube LTQ_n. To further optimize the diameter of LTQ_n, the n-dimensional folded locally twisted cube $FLTQ_n$ is proposed, which is built based on LTQ_n by adding 2^{n-1} complementary edges. Connectivity is an important indicator to measure the fault tolerance and reliability of a network. However, the connectivity has an obvious shortcoming, in that it assumes all the adjacent vertices of a vertex will fail at the same time. Super-connectivity is a more refined index to judge the fault tolerance of a network, which ensures that each vertex has at least one neighbor. In this paper, we show that the super-connectivity $\kappa^{(1)}(FLTQ_n) = 2n$ for any integer $n \geq 6$, which is about twice $\kappa(FLTQ_n)$.

Keywords: super-connectivity; folded locally twisted cube; fault tolerance; interconnection network; reliability

check for updates

Citation: You, L.; Han, Y.; Jiang, J. Super-Connectivity of the Folded Locally Twisted Cube. *Symmetry* **2023**, *15*, 1349. https://doi.org/10.3390/sym15071349

Academic Editors: Dongqin Cheng, Jou-Ming Chang and Chengkuan Lin

Received: 19 June 2023
Revised: 28 June 2023
Accepted: 29 June 2023
Published: 2 July 2023

1. Introduction

High-performance computers can be widely used in many fields thanks to the development of high performance computing technology. The topological properties of interconnection networks are very important for high-performance computers. One typically uses an undirected graph $G = (V(G), E(G))$ to model the topology of a multiprocessor system H, where the processor set of H is represented by $V(G)$ and the link set of H is represented by $E(G)$.

Interconnection networks have many important properties, one of which is the connectivity denoted by $\kappa(G)$. A graph's connectivity is the minimum number of vertices whose removal makes the graph disconnected or trivial [1]. Connectivity is an important indicator to measure the fault tolerance and reliability of a network. In a large interconnection network, each vertex has a large number of neighbors. This property has an obvious deficiency, in that it assumes that all the adjacent vertices of a vertex will fail at the same time. However, this situation does not happen frequently in real networks. To address this deficiency, Esfahanian et al. [2] introduced the concept of restricted connectivity by imposing additionally restricted conditions on a network. Super-connectivity is a special case of restricted connectivity. When determing the super-connectivity of a network, one needs to ensure that each vertex has at least one neighbor. Hence, super-connectivity is a more refined index to judge the fault tolerance of a network.

Let K be a subset of $V(G)$. $G \setminus K$ (or $G - K$) denotes a graph obtained by removing all the vertices in K and edges incident to at least one vertex in K from G. If $G \setminus K$ is disconnected and each component of $G \setminus K$ has at least two vertices, then K is called a super vertex cut. Let S be a subset of $E(G)$. If $G \setminus S$ is disconnected and each component of $G \setminus S$ has at least two vertices, then S is called a super edge cut. The super-connectivity of G (or, respectively, the super edge connectivity), denoted by $\kappa^{(1)}(G)$ (or $\lambda^{(1)}(G)$), is the minimum cardinality of all super vertex cuts (or super edge cuts) in G, if any exist. Many relevant results have been obtained regarding super-connectivity and super edge connectivity [3–16].

The hypercube Q_n has become one of the most popular interconnection networks, because of its many attractive properties, such as its regularity and symmetry. Q_n is a Cayley graph and hence vertex-transitive and edge-transitive. However, the diameter of Q_n is not optimal. In order to enhance the hypercube, researchers have proposed many variants, such as crossed cubes [17], locally twisted cubes [18], and spined cubes [19]. The n-dimensional locally twisted cube LTQ_n was proposed by Yang et al. [18], whose diameter was only about half that of Q_n. Many research results have been published on the properties of LTQ_n [20–25]. LTQ_n is vertex-transitive if and only if $n \leq 3$, and it is edge-transitive if and only if $n = 2$ [25]. To further enhance the hypercube, inspired by the folded cube [26], Peng et al. [27] proposed a new network topology called the folded locally twisted cube $FLTQ_n$. So far there, no work has been reported on the super-connectivity of $FLTQ_n$. In this work, we studied the super-connectivity of $FLTQ_n$ and obtained the result that the super-connectivity $\kappa^{(1)}(FLTQ_n)$ is $2n$ for $n \geq 6$, which is about twice $\kappa(FLTQ_n)$.

2. Preliminaries

In this paper, we use the terms vertex and node interchangeably. We also use (x, y) to denote an edge between vertices x and y. For any vertex $x \in V(G)$, the neighboring set of x is denoted by $N_G(x) = \{y | (x, y) \in E(G)\}$ (or $N(x)$ for short). Let $S \subset V(G)$. The neighboring set of S is defined as $N_G(S) = (\bigcup_{x \in S} N(x)) \setminus S$ (or $N(S)$ for short). We define $N_G[S] = \bigcup_{x \in S} N(x)$ and $N_G[x] = N_G(x) \cup \{x\}$. We use $x_n x_{n-1} \cdots x_2 x_1$ to represent a binary string μ of length n, where $x_i \in \{0, 1\}$ for $1 \leq i \leq n$ is a part of μ. x_1 is the first part of μ, and x_n is the nth part of μ. The symbol $\bar{x}_i$ is used to represent the complement of x_i. As a variant of Q_n, LTQ_n has the same number of vertices as Q_n. Each vertex of LTQ_n is denoted by a unique binary string of length n. The definition of LTQ_n is given below.

Definition 1 ([18])**.** *For $n \geq 2$, an n-dimensional locally twisted cube, LTQ_n, is defined recursively as follows:*

(1) LTQ_2 is a graph consisting of four nodes labeled with 00, 01, 10, and 11, which are connected by four edges, (00, 01), (00, 10), (01, 11), and (10, 11).

(2) For $n \geq 3$, LTQ_n is built from two disjointed copies of LTQ_{n-1} named LTQ^0_{n-1} and LTQ^1_{n-1}. Let LTQ^0_{n-1} (or, respectively, LTQ^1_{n-1}) be the graph obtained by prefixing the label of each node of one copy of LTQ_{n-1} with 0 (or with 1); each node $x = 0x_{n-1}x_{n-2} \cdots x_2 x_1$ of LTQ^0_{n-1} is connected to the node $1(x_{n-1} + x_1)x_{n-2} \cdots x_2 x_1$ of LTQ^1_{n-1} with an edge, where $'+'$ represents modulo 2 addition.

LTQ_3 and LTQ_4 are demonstrated in Figure 1. Each node in LTQ^0_{n-1} has only one adjacent node in LTQ^1_{n-1}. The set of edges between LTQ^0_{n-1} and LTQ^1_{n-1} is called a perfect matching M of LTQ_n. Hence, we can write $LTQ_n = G(LTQ^0_{n-1}, LTQ^1_{n-1}, M)$. In [18], Yang et al. also provided a non-recursive definition of LTQ_n.

Definition 2 ([18])**.** *Let $\mu = x_n x_{n-1} \cdots x_1$ and $\nu = y_n y_{n-1} \cdots y_1$ be any two distinct vertices of LTQ_n for $n \geq 2$. μ and ν are connected if and only if one of the following conditions is satisfied:*

1. There is an integer $3 \leq k \leq n$ such that
(a) $x_k = \bar{y}_k$;
(b) $x_{k-1} = y_{k-1} + x_1$ ($'+'$ represents modulo 2 addition);
(c) all the remaining bits of μ and ν are the same.
2. There is an integer $1 \leq k \leq 2$ such that μ and ν only differ in the kth bit.

Let $\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 x_1$ be any vertex of LTQ_n. By Definition 2, all the n neighbors of μ are listed as follows:

$\mu_1 = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 \bar{x}_1$;
$\mu_2 = x_n x_{n-1} x_{n-2} \ldots x_3 \bar{x}_2 x_1$;
$\mu_3 = x_n x_{n-1} x_{n-2} \ldots \bar{x}_3 (x_2 + x_1) x_1$;
$\ldots$
$\mu_{n-1} = x_n \bar{x}_{n-1} (x_{n-2} + x_1) x_{n-3} \ldots x_2 x_1$;
$\mu_n = \bar{x}_n (x_{n-1} + x_1) x_{n-2} \ldots x_3 x_2 x_1$.

We call μ_i the ith dimensional neighbor of μ for $1 \leq i \leq n$.

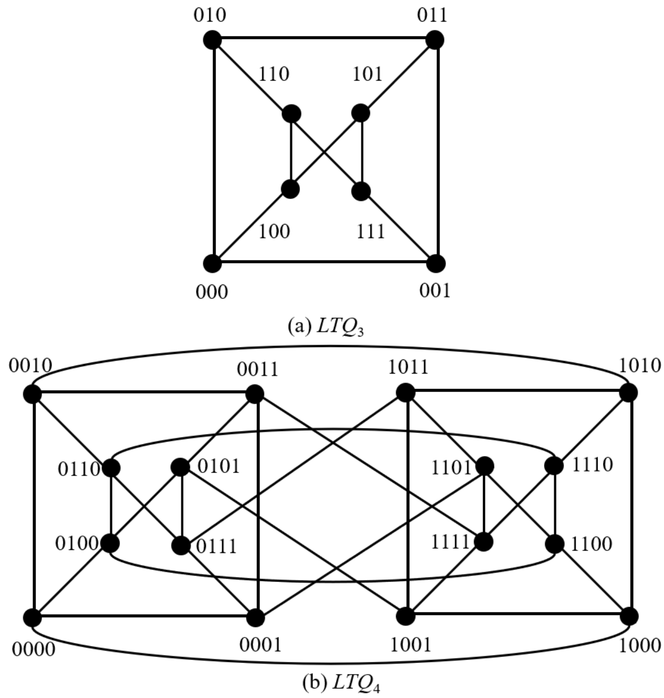

Figure 1. (**a**) The three-dimensional locally twisted cube LTQ_3; (**b**) the four-dimensional locally twisted cube LTQ_4.

Definition 3 ([27])**.** *For any integer $n \geq 2$, an n-dimensional folded locally twisted cube, denoted by $FLTQ_n$, is a graph constructed based on LTQ_n by adding all complementary edges. Each vertex $x = x_n x_{n-1} \ldots x_1$ in LTQ_n is incident to another vertex $\overline{x} = \overline{x}_n \overline{x}_{n-1} \ldots \overline{x}_1$ through a complementary edge, where $\overline{x}_i = 1 - x_i$.*

We call the added complementary edges c-links. $FLTQ_n$ has 2^{n-1} c-links, and each vertex $\mu = x_n x_{n-1} \ldots x_1$ is connected to a complementary vertex $\mu_c = \overline{x}_n \overline{x}_{n-1} \ldots \overline{x}_1$ by a c-link. The set of complementary edges between LTQ^0_{n-1} and LTQ^1_{n-1} is a perfect matching C of $FLTQ_n$. Hence, we can write $FLTQ_n = G(LTQ^0_{n-1}, LTQ^1_{n-1}, M, C)$ or $G(LTQ_n, C)$. Each node $\mu \in V(FLTQ_n)$ in LTQ^0_{n-1} (or, respectively, LTQ^1_{n-1}) has two neighbors, μ_n and μ_c, in LTQ^1_{n-1} (or LTQ^0_{n-1}) for $n \geq 3$. Compared with LTQ_n, each vertex in $FLTQ_n$ has one more neighbor. Then, the node degree of $FLTQ_n$ is $n+1$ and $\kappa(FLTQ_n) = n+1$ [27]. Figure 2 demonstrates $FLTQ_3$ and $FLTQ_4$, respectively, and Figure 3 demonstrates $FLTQ_5$.

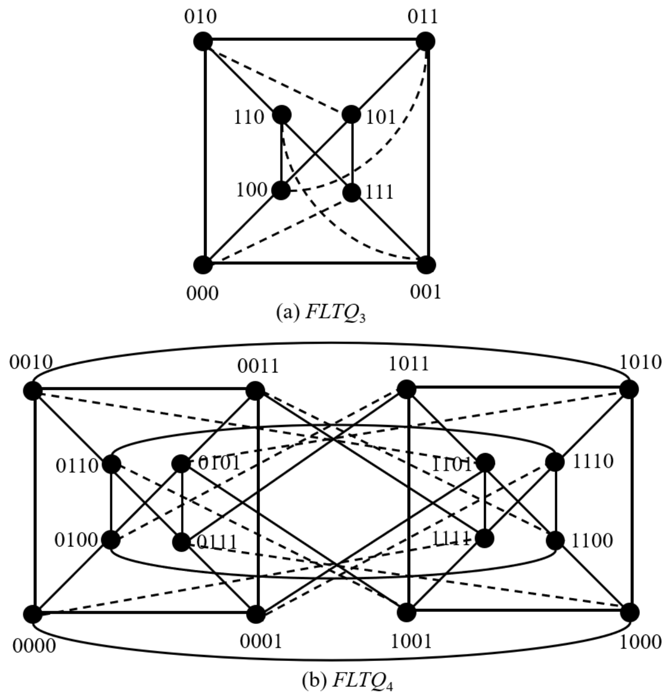

Figure 2. (**a**) The three-dimensional folded locally twisted cube $FLTQ_3$; (**b**) the four-dimensional folded locally twisted cube $FLTQ_4$.

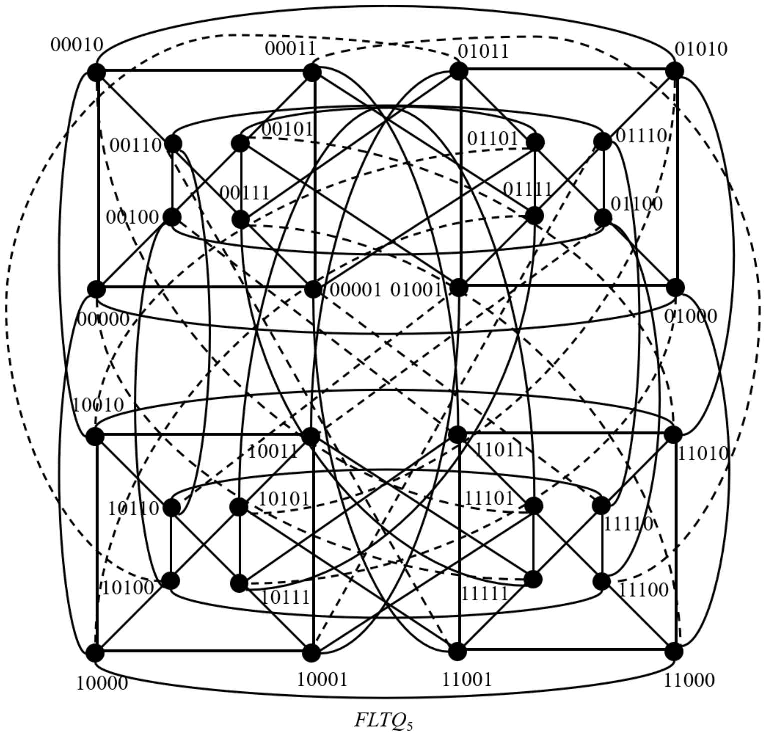

Figure 3. The five-dimensional folded locally twisted cube $FLTQ_5$.

3. Super Connectivity of $FLTQ_n$

In this section, we study the super connectivity of $FLTQ_n$ for any integer $n \geq 6$. Since $FLTQ_n$ is composed of LTQ_n and the complementary edge set C, we can use some properties of LTQ_n to prove the super-connectivity property of $FLTQ_n$.

Lemma 1 ([18])**.** *For $n \geq 2$, $\kappa(LTQ_n) = \lambda(LTQ_n) = n$.*

Lemma 2 ([28]). *For any two vertices $\mu, \nu \in V(LTQ_n)(n \geq 2)$, we have $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| \leq 2$.*

Lemma 3 ([28]). *Let μ and ν be any two distinct vertices in $LTQ_n(n \geq 4)$ such that $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| = 2$.*

(1) If $\mu \in V(LTQ^0_{n-1})$ and $\nu \in V(LTQ^1_{n-1})$, then the one common neighbor is in LTQ^0_{n-1}, and the other one is in LTQ^1_{n-1}.

(2) If $\mu, \nu \in V(LTQ^0_{n-1})$ or $V(LTQ^1_{n-1})$, then the two common neighbors are in LTQ^0_{n-1} or LTQ^1_{n-1}.

Lemma 4. *Let μ and ν be any two distinct vertices in the same LTQ^i_{n-1} for $0 \leq i \leq 1$ and $n \geq 6$. If $\mu_n = \nu_c$ or $\mu_c = \nu_n$, then $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| = 1$.*

Proof. Without loss of generality, we suppose that $\mu, \nu \in V(LTQ^0_{n-1})$, and $\mu_n = \nu_c$. Then, μ_n is the common neighbor for μ and ν. Let $\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 x_1$ and $X = FLTQ_n \setminus \{\mu_n\}$. Next, we consider the neighbors of μ and ν in X according to different values of the first part x_1 of μ.

Case 1. $x_1 = 0$.

$\mu_n = \bar{x}_n x_{n-1} x_{n-2} \ldots x_3 x_2 0 = \nu_c$ and $\nu = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$. We list $N_X(\mu)$ and $N_X(\nu)$ separately in Table 1.

Table 1. The neighbors of μ and ν in X, where $x_1 = 0$.

$N_X(\mu)$	$N_X(\nu)$
$\mu_1 = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 1$	$\nu_1 = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$
$\mu_2 = x_n x_{n-1} x_{n-2} \ldots x_3 \bar{x}_2 0$	$\nu_2 = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 x_2 1$
$\mu_3 = x_n x_{n-1} x_{n-2} \ldots \bar{x}_3 x_2 0$	$\nu_3 = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots x_3 x_2 1$
$\ldots$	$\ldots$
$\mu_{n-1} = x_n \bar{x}_{n-1} x_{n-2} \ldots x_3 x_2 0$	$\nu_{n-1} = x_n x_{n-1} x_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$
$\mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$	$\nu_n = \bar{x}_n x_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$

It is obvious that $|N_X(\mu) \cap N_X(\nu)| = 0$.

Case 2. $x_1 = 1$.

$\mu_n = \bar{x}_n \bar{x}_{n-1} x_{n-2} \ldots x_3 x_2 1 = \nu_c$ and $\nu = x_n x_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$. We list $N_X(\mu)$ and $N_X(\nu)$ separately in Table 2.

Table 2. The neighbors of μ and ν in X, where $x_1 = 1$.

$N_X(\mu)$	$N_X(\nu)$
$\mu_1 = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 0$	$\nu_1 = x_n x_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$
$\mu_2 = x_n x_{n-1} x_{n-2} \ldots x_3 \bar{x}_2 1$	$\nu_2 = x_n x_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 x_2 0$
$\mu_3 = x_n x_{n-1} x_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$	$\nu_3 = x_n x_{n-1} \bar{x}_{n-2} \ldots x_3 \bar{x}_2 0$
$\ldots$	$\ldots$
$\mu_{n-1} = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots x_3 x_2 1$	$\nu_{n-1} = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$
$\mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$	$\nu_n = \bar{x}_n x_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$

It is obvious that $|N_X(\mu) \cap N_X(\nu)| = 0$.

Hence, μ and ν have only one common neighbor in $FLTQ_n$ and $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| = 1$. □

Lemma 5. *Let μ be any node in $FLTQ_n$, where $n \geq 6$ and $X = FLTQ_n \setminus \{\mu\}$. Then, $|N_X(\mu_n) \cap N_X(\mu_c)| = 0$.*

Proof. Let $\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 x_1$. We consider the different values of the first part x_1 of μ.

Case 1. $x_1 = 0$.

Let $\alpha = \mu_n = \bar{x}_n x_{n-1} x_{n-2} \dots x_3 x_2 0$ and $\beta = \mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 \bar{x}_2 1$. All the neighbors of α and β in X are listed separately in Table 3.

Table 3. The neighbors of α and β in X, where $x_1 = 0$.

$N_X(\alpha)$	$N_X(\beta)$
$\alpha_1 = \bar{x}_n x_{n-1} x_{n-2} \dots x_3 x_2 1$	$\beta_1 = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 \bar{x}_2 0$
$\alpha_2 = \bar{x}_n x_{n-1} x_{n-2} \dots x_3 \bar{x}_2 0$	$\beta_2 = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 x_2 1$
$\alpha_3 = \bar{x}_n x_{n-1} x_{n-2} \dots \bar{x}_3 x_2 0$	$\beta_3 = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \dots x_3 x_2 1$
$\dots$	$\dots$
$\alpha_{n-1} = \bar{x}_n \bar{x}_{n-1} x_{n-2} \dots x_3 x_2 0$	$\beta_{n-1} = \bar{x}_n x_{n-1} x_{n-2} \dots \bar{x}_3 \bar{x}_2 1$
$\alpha_c = x_n \bar{x}_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 \bar{x}_2 1$	$\beta_n = x_n x_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 \bar{x}_2 1$

It is obvious that $N_X(\alpha) \cap N_X(\beta) = \emptyset$.

Case 2. $x_1 = 1$.

Let $\alpha = \mu_n = \bar{x}_n \bar{x}_{n-1} x_{n-2} \dots x_3 x_2 1$ and $\beta = \mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 \bar{x}_2 0$. All the neighbors of α and β in X are listed separately in Table 4.

Table 4. The neighbors of α and β in X, where $x_1 = 1$.

$N_X(\alpha)$	$N_X(\beta)$
$\alpha_1 = \bar{x}_n \bar{x}_{n-1} x_{n-2} \dots x_3 x_2 0$	$\beta_1 = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 \bar{x}_2 1$
$\alpha_2 = \bar{x}_n \bar{x}_{n-1} x_{n-2} \dots x_3 \bar{x}_2 1$	$\beta_2 = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 x_2 0$
$\alpha_3 = \bar{x}_n \bar{x}_{n-1} x_{n-2} \dots \bar{x}_3 \bar{x}_2 1$	$\beta_3 = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \dots x_3 \bar{x}_2 0$
$\dots$	$\dots$
$\alpha_{n-1} = \bar{x}_n x_{n-1} \bar{x}_{n-2} \dots x_3 x_2 1$	$\beta_{n-1} = \bar{x}_n x_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 \bar{x}_2 0$
$\alpha_c = x_n x_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 \bar{x}_2 0$	$\beta_n = x_n \bar{x}_{n-1} \bar{x}_{n-2} \dots \bar{x}_3 \bar{x}_2 0$

It is obvious that $N_X(\alpha) \cap N_X(\beta) = \emptyset$.

Hence, $|N_X(\mu_n) \cap N_X(\mu_c)| = 0$. □

Lemma 6. *Let $\mu, \nu \in V(FLTQ_n)$ where $n \geq 6$. Then $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| \leq 2$.*

Proof. Since $FLTQ_n$ is constructed from LTQ_n by adding the complementary edge set C, we can study this lemma based on LTQ_n.

Case 1. μ, ν are in the same LTQ^i_{n-1} for $0 \leq i \leq 1$.

Without loss of generality, we suppose that $\mu, \nu \in V(LTQ^0_{n-1})$. According to Lemmas 2 and 3, $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| \leq 2$ for $n \geq 6$, and the two common neighbors are in LTQ^0_{n-1}. According to the definition of $FLTQ_n$, we have $N_{LTQ^1_{n-1}}(\mu) = \{\mu_n, \mu_c\}$, $N_{LTQ^1_{n-1}}(\nu) = \{\nu_n, \nu_c\}$, $\mu_n \neq \nu_n$, and $\mu_c \neq \nu_c$. If $\mu_c \neq \nu_n$ and $\mu_n \neq \nu_c$, then μ and ν do not have the same neighbors in LTQ^1_{n-1}. Hence, $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| \leq 2$. According to Lemma 4, if $\mu_c = \nu_n$ or $\mu_n = \nu_c$, then μ and ν have only one common neighbor in $FLTQ_n$ and $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| = 1 \leq 2$.

Case 2. μ and ν are in a different LTQ^i_{n-1} for $0 \leq i \leq 1$.

Without loss of generality, we suppose that $\mu \in V(LTQ^0_{n-1})$ and $\nu \in V(LTQ^1_{n-1})$. According to Lemma 2, $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| \leq 2$. Based on the definition of $FLTQ_n$, we have $N_{LTQ^1_{n-1}}(u) = \{u_n, u_c\}$ and $N_{LTQ^0_{n-1}}(v) = \{v_n, v_c\}$. According to Lemma 5, $|N_{FLTQ_n \setminus \{\mu\}}(\mu_n) \cap N_{FLTQ_n \setminus \{\mu\}}(\mu_c)| = 0$ and $|N_{FLTQ_n \setminus \{\nu\}}(\nu_n) \cap N_{FLTQ_n \setminus \{\nu\}}(\nu_c)| = 0$. Hence, we cannot find a vertex $\mu' \in V(LTQ^1_{n-1})$, where μ' and $\mu \in V(LTQ^0_{n-1})$ have two common neighbors, nor can we find a vertex $\nu' \in V(LTQ^0_{n-1})$, where ν' and $\nu \in V(LTQ^1_{n-1})$ have two common neighbors. Then, u and v cannot have three or four common neighbors in $FLTQ_n$. Hence, $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| \leq 2$. □

Lemma 7 ([28])**.** *If μ and ν are two vertices of LTQ_n and $(\mu,\nu) \in E(LTQ_n)$, where $n \geq 2$, then $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| = 0$.*

Lemma 8. *If μ and ν are two vertices of $FLTQ_n$ and $(\mu,\nu) \in E(FLTQ_n)$, where $n \geq 3$, then $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| = 0$.*

Proof. According to the position of μ and ν, we consider two cases.

Case 1. μ and ν are in the same LTQ^i_{n-1} for $0 \leq i \leq 1$.

Without loss of generality, we assume that $\mu,\nu \in V(LTQ^0_{n-1})$. According to Lemma 7, $|N_{LTQ^0_{n-1}}(\mu) \cap N_{LTQ^0_{n-1}}(\nu)| = 0$. We have $N_{LTQ^1_{n-1}}(\mu) = \{\mu_n, \mu_c\}$ and $N_{LTQ^1_{n-1}}(\nu) = \{\nu_n, \nu_c\}$. If $N_{LTQ^1_{n-1}}(\mu) \cap N_{LTQ^1_{n-1}}(\nu) = \emptyset$, then $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| = 0$. Otherwise, if $\mu_n = \nu_c$ or $\mu_c = \nu_n$, then we let $\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 x_1$. All the possible values of μ and ν are listed in Table 5.

Table 5. The possible values of μ and ν.

$\mu_n = \nu_c$	$\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 0$ $\mu_n = \bar{x}_n x_{n-1} x_{n-2} \ldots x_3 x_2 0 = \nu_c$ $\nu = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$	$x_1 = 0$
$\mu_n = \nu_c$	$\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 1$ $\mu_n = \bar{x}_n \bar{x}_{n-1} x_{n-2} \ldots x_3 x_2 1 = \nu_c$ $\nu = x_n x_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$	$x_1 = 1$
$\mu_c = \nu_n$	$\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 0$ $\mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1 = \nu_n$ $\nu = x_n x_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$	$x_1 = 0$
$\mu_c = \nu_n$	$\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 1$ $\mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0 = \nu_n$ $\nu = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$	$x_1 = 1$

It is obvious that $(\mu,\nu) \notin E(FLTQ_n)$; then, we reach a contradiction, and all these values of μ and ν are impossible. Hence, $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| = 0$.

Case 2. μ and ν are in a different LTQ^i_{n-1} for $0 \leq i \leq 1$.

Without loss of generality, we assume that $\mu \in V(LTQ^0_{n-1})$ and $\nu \in V(LTQ^1_{n-1})$. Since $(\mu,\nu) \in E(FLTQ_n)$, ν should be μ_n or μ_c. If $\mu_n = \nu$, let $K = \{\mu, \nu, \mu_c, \nu_c\}$. Otherwise, If $\mu_c = \nu$, let $K = \{\mu, \nu, \mu_n, \nu_n\}$. Let $\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 x_1$. All the possible values of K are listed in Table 6.

Since $(\mu_c,\nu), (\mu,\nu_c) \notin E(FLTQ_n)$ when $\mu_n = \nu$ and $(\mu_n,\nu), (\mu,\nu_n) \notin E(FLTQ_n)$, when $\mu_c = \nu$, μ and ν do not have common neighbors. Hence, $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| = 0$. □

Lemma 9. *Let μ be any node in LTQ_n for any integer $n \geq 3$. Then, $LTQ_n \setminus N_{LTQ_n}[\mu]$ is connected.*

Proof. We use mathematical induction on n to prove this lemma. According to Lemma 1, we know that this lemma obviously holds when $n = 3$. Suppose that this lemma holds for $n \leq k (k \geq 3)$. Let μ be any node in LTQ_{k+1}. Without loss of generality, we suppose that $\mu \in V(LTQ^0_k)$. Then, by the induction hypothesis, $LTQ^0_k \setminus N_{LTQ^0_k}[\mu]$ is connected. Since $N_{LTQ^1_k}(\mu) = \{\mu_{k+1}\}$, according to Lemma 1, $LTQ^1_k \setminus \{\mu_{k+1}\}$ is connected. Since each node in LTQ^0_k is connected to a node in LTQ^1_k, $LTQ^0_k \setminus N_{LTQ^0_k}[\mu]$ is connected to $LTQ^1_k \setminus \{\mu_{k+1}\}$. Then, $LTQ_{k+1} \setminus N_{LTQ_{k+1}}[\mu]$ is connected. Hence, this lemma holds. □

Table 6. The possible values of K.

$\mu_n = \nu$	$\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 0 = \nu_n$ $\mu_n = \bar{x}_n x_{n-1} x_{n-2} \ldots x_3 x_2 0 = \nu$ $\nu_c = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$ $\mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$	$x_1 = 0$
$\mu_n = \nu$	$\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 1 = \nu_n$ $\mu_n = \bar{x}_n \bar{x}_{n-1} x_{n-2} \ldots x_3 x_2 1 = \nu$ $\nu_c = x_n x_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$ $\mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$	$x_1 = 1$
$\mu_c = \nu$	$\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 0 = \nu_c$ $\mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1 = \nu$ $\nu_n = x_n x_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 1$ $\mu_n = \bar{x}_n x_{n-1} x_{n-2} \ldots x_3 x_2 0$	$x_1 = 0$
$\mu_c = \nu$	$\mu = x_n x_{n-1} x_{n-2} \ldots x_3 x_2 1 = \nu_c$ $\mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0 = \nu$ $\nu_n = x_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_3 \bar{x}_2 0$ $\mu_n = \bar{x}_n \bar{x}_{n-1} x_{n-2} \ldots x_3 x_2 1$	$x_1 = 1$

Since $\kappa^{(1)}(FLTQ_n)$ is the minimum cardinality of all super vertex cuts in $FLTQ_n$, to obtain the upper bound of $\kappa^{(1)}(FLTQ_n)$, we just need to find a super vertex cut F. Then, we have $\kappa^{(1)}(FLTQ_n) \leq |F|$. If we can prove that $FLTQ_n$ is connected after removing $|F| - 1$ vertices, then we have the lower bound $\kappa^{(1)}(FLTQ_n) \geq |F|$. With these two results, we can obtain $\kappa^{(1)}(FLTQ_n) = |F|$. In the following, we will present two important lemmas to prove the upper bound and lower bound of $\kappa^{(1)}(FLTQ_n)$.

Lemma 10. *$\kappa^{(1)}(FLTQ_n) \leq 2n$ for any integer $n \geq 6$.*

Proof. Consider an edge $(x, y) \in E(FLTQ_n)$. Let $F = \{x, y\}$. Then, $FLTQ_n - N_{FLTQ_n}(F)$ is disconnected, and the edge (x, y) is one component of $FLTQ_n - N_{FLTQ_n}(F)$. According to Lemma 8, $|N_{FLTQ_n}(F)| = (n+1) + (n+1) - 2 = 2n$. Let $K = FLTQ_n - N_{FLTQ_n}[F]$. To prove that $N_{FLTQ_n}(F)$ is a super vertex cut, we need to show that each vertex $\alpha \in V(K)$ has at least one neighbor. According to Lemma 6, $|N_{FLTQ_n}(\alpha) \cap N_{FLTQ_n}(x)| \leq 2$ and $|N_{FLTQ_n}(\alpha) \cap N_{FLTQ_n}(y)| \leq 2$. Since $\kappa(FLTQ_n) = n + 1$ and $n + 1 - 2 - 2 \geq 1$ for $n \geq 6$, α has at least one neighbor in K. Hence, $N_{FLTQ_n}(F)$ is a super vertex cut and $\kappa^{(1)}(FLTQ_n) \leq 2n$ for $n \geq 6$. □

Lemma 11. *$\kappa^{(1)}(FLTQ_n) \geq 2n$ for $n \geq 6$.*

Proof. Suppose that F is a super vertex cut of $FLTQ_n$. Then, $FLTQ_n \backslash F$ is disconnected, and each vertex in $FLTQ_n \backslash F$ has at least one neighbor. To prove $\kappa^{(1)}(FLTQ_n) \geq 2n$, we will show that $FLTQ_n \backslash F$ is connected when $|F| \leq 2n - 1$. Let $F_i = F \cap LTQ_{n-1}^i$ for $0 \leq i \leq 1$, $K_0 = LTQ_{n-1}^0 \setminus F_0$, and $K_1 = LTQ_{n-1}^1 \setminus F_1$. Without loss of generality, we suppose that $|F_0| \geq |F_1|$. Then, $|F_1| \leq n - 1$.

Case 1. K_1 is connected.

Let α be any node in K_0. We have $N_{LTQ_{n-1}^1}(\alpha) = \{\alpha_n, \alpha_c\}$. If $|N_{LTQ_{n-1}^1}(\alpha) \cap F_1| \leq 1$, then α is connected to K_1. Since K_1 is connected, then $K_0 \cup K_1$ is connected, which means that $FLTQ_n \setminus F$ is connected. Otherwise, since each vertex in $FLTQ_n \setminus F$ has at least one neighbor, there must be a vertex $\beta \in K_0$ such that $(\alpha, \beta) \in E(K_0)$. We have $N_{LTQ_{n-1}^1}(\beta) = \{\beta_n, \beta_c\}$. If $|N_{LTQ_{n-1}^1}(\beta) \cap F_1| \leq 1$, then α can be connected to K_1 through vertex β, and $FLTQ_n \setminus F$ is connected. Otherwise, we have $\{\alpha_n, \alpha_c, \beta_n, \beta_c\} \in F_1$, $|F_1| \geq 4$, and $|F_0| \leq 2n - 5$. Let $Y = N_{LTQ_{n-1}^0}(\alpha) \cup N_{LTQ_{n-1}^0}(\beta) \setminus \{\alpha, \beta\}$. According to Lemma 8,

$|Y| = (n-1) + (n-1) - 2 = 2n-4$. Since $|F_0| \leq 2n-5$, we can find at least one vertex $\gamma \in Y$ such that α and β are connected to K_1 through γ. Hence, $FLTQ_n \setminus F$ is connected.

Case 2. K_1 is disconnected.

According to Lemma 1, we have $\kappa(LTQ_{n-1}) = n-1$. Since K_1 is disconnected, then $|F_1| = n-1$ and $|F_0| = n$. There should be an isolated vertex ω in K_1 and $F_1 = N_{LTQ^1_{n-1}}(\omega)$. According to Lemma 9, $LTQ^1_{n-1} \setminus N_{LTQ^1_{n-1}}[\omega]$ is connected. For any vertex α in K_0 where $(\alpha, \omega) \in E(FLTQ_n)$, based on Lemma 8, α and ω do not have common neighbors. Then, there exists a neighbor α' of α in LTQ^1_n such that $\alpha' \notin N_{LTQ^1_{n-1}}[\omega]$. Hence, α is connected to $LTQ^1_{n-1} \setminus N_{LTQ^1_{n-1}}[\omega]$ through α'. For any vertex α in K_0 where $(\alpha, \omega) \notin E(FLTQ_n)$, there must exist a neighbor β in K_0. Let $Y = N_{LTQ^0_{n-1}}(\alpha) \cup N_{LTQ^0_{n-1}}(\beta)$. According to Lemma 8, $|Y| = (n-1) + (n-1) = 2n-2$. Since $|F_0| = n$, we can find at least $n-2$ vertices in Y connected to LTQ^1_{n-1}. Since there exist two neighbors in LTQ^1_{n-1} for each vertex in Y and $2n-4 > n-1$ when $n \geq 6$, we can find a vertex γ in Y such that α and β are connected to $LTQ^1_{n-1} \setminus N_{LTQ^1_{n-1}}[\omega]$ through γ. Hence, $FLTQ_n - F$ is connected.

Thus, $FLTQ_n \setminus F$ is connected when $|F| \leq 2n-1$ and $\kappa^{(1)}(FLTQ_n) \geq 2n$ for any integer $n \geq 6$. □

According to Lemmas 10 and 11, we obtain the following result:

Theorem 1. *$\kappa^{(1)}(FLTQ_n) = 2n$ for $n \geq 6$.*

4. Conclusions

The folded locally twisted cube $FLTQ_n$ was introduced based on the locally twisted cube LTQ_n and the folded hypercube FQ_n. In this paper, we studied the super-connectivity of folded locally twisted cubes, which is an important indicator to measure the fault tolerance and reliability of a network. The main contribution of this work was that we addressed the super-connectivity of $FLTQ_n$. We proved that $\kappa^{(1)}(FLTQ_n) = 2n$ for any integer $n \geq 6$. Independent spanning trees and mesh embedding could be considered as future research directions. Independent spanning trees could be applied to reliable communication protocols, reliable broadcasting, and so on [29]. Meshes are fundamental guest graphs on which many algorithms, such as linear algebra algorithms and combinatorial algorithms, can be efficiently performed [30]. The results of independent spanning trees and mesh embedding for $FLTQ_n$ could be compared with the results of LTQ_n [31,32].

Author Contributions: Conceptualization, L.Y.; methodology, Y.H.; investigation, J.J.; writing—original draft preparation, L.Y. and J.J.; writing—review and editing, Y.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Research Project of Suzhou Industrial Park Institute of Services Outsourcing (No. SISO-ZD202202) and sponsored by the Qing Lan Project.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. West, D.B. *Introduction to Graph Theory*; Prentice Hall Publishers: Hoboken, NJ, USA , 2001.
2. Esfahanian, A.H.; Hakimi, S.L. On computing a conditional edge-connectivity of a graph. *Inf. Process. Lett.* **1988**, *27*, 195–199. [CrossRef]
3. Xu, J.-M.; Xu, M.; Zhu, Q. The super connectivity of shuffle-cubes. *Inf. Process. Lett*. **2005**, *96*, 123–127. [CrossRef]
4. Zhu, Q.; Xu, J.-M.; Hou, X.; Xu, M. On reliability of the folded hypercubes. *Inf. Sci*. **2007**, *177*, 1782–1788. [CrossRef]
5. Guo, L.; Qin, C.; Guo, X. Super connectivity of Kronecker products of graphs. *Inf. Process. Lett*. **2010**, *110*, 659–661. [CrossRef]

6. Chang, J.-M.; Chen, X.-R.; Yang, J.-S.; Wu, R.-Y. Locally exchanged twisted cubes: Connectivity and super connectivity. *Inf. Process. Lett.* **2016**, *116*, 460–466. [CrossRef]
7. Cai, X.; Vumar, E. The super connectivity of folded crossed cubes. *Inf. Process. Lett.* **2019**, *142*, 52–56. [CrossRef]
8. Wang, S.; Ma, X. Super connectivity and diagnosability of crossed cubes. *J. Internet Technol.* **2019**, *20*, 1287–1296.
9. Cai, X.; Ma, L. The super connectivity of exchanged folded hypercube. *J. Anhui Norm. Univ. Nat. Sci.* **2020**, *43*, 216–222.
10. Ning, W. Connectivity and super connectivity of the divide-and-swap cube. *Theor. Comput. Sci.* **2020**, *842*, 1–5. [CrossRef]
11. Guo, L.; Ekinci, G.B. Super connectivity of folded twisted crossed cubes. *Discret. Appl. Math.* **2021**, *305*, 56–63. [CrossRef]
12. Gu, M.; Chang, J.-M. A note on super connectivity of the bouwer graph. *J. Interconnect. Netw.* **2021**, *21*, 2142009. [CrossRef]
13. Ekinci, G.B.; Gauci, J.B. The super-connectivity of odd graphs and of their kronecker double cover. *Rairo-Oper. Res.* **2021**, *55*, S699–S704. [CrossRef]
14. Soliemany, F.; Ghasemi, M.; Varmazyar, R. On the super connectivity of direct product of graphs. *Rairo-Oper. Res.* **2022**, *56*, 2767–2773. [CrossRef]
15. Ning, W.; Guo, L. Connectivity and super connectivity of the exchanged 3-ary n-cube. *Theor. Comput. Sci.* **2022**, *923*, 160–166. [CrossRef]
16. Zhao, S.-L.; Chang, J.-M. Connectivity, super connectivity and generalized 3-connectivity of folded divide-and-swap cubes. *Inf. Process. Lett.* **2023**, *182*, 106377. [CrossRef]
17. Efe, K. The crossed cube architecture for parallel computation. *IEEE Trans. Parallel Distrib. Syst.* **1992**, *3*, 513–524. [CrossRef]
18. Yang, X.; Evans, D.J.; Megson, G.M. The locally twisted cubes. *Int. J. Comput. Math.* **2005**, *82*, 401–413. [CrossRef]
19. Zhou, W.; Fan, J.; Jia, X.; Zhang, S. The spined cube: A new hypercube variant with smaller diameter. *Inf. Process. Lett.* **2011**, *111*, 561–567. [CrossRef]
20. Han, Y.; Fan, J.; Zhang, S. Changing the diameter of the locally twisted cube. *Int. J. Comput. Math.* **2013**, *90*, 497–510. [CrossRef]
21. Liu, Z.; Fan, J.; Zhou, J.; Cheng, B.; Jia, X. Fault-tolerant embedding of complete binary trees in locally twisted cubes. *J. Parallel. Distrib. Comput.* **2017**, *101*, 69–78. [CrossRef]
22. Wang, S.; Ren, Y. The h-extra connectivity and diagnosability of locally twisted cubes. *IEEE Access* **2019**, *7*, 102113–102118. [CrossRef]
23. Han, Y.; You, L.; Lin, C.-K.; Fan, J. Communication performance evaluation of the locally twisted cube. *Int. J. Found. Comput. Sci.* **2020**, *31*, 233–252. [CrossRef]
24. Shang, H.; Sabir, E.; Meng, J.; Guo, L. Characterizations of optimal component cuts of locally twisted cubes. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 2087–2103. [CrossRef]
25. Chang, X.; Ma, J.; Yang, D. Symmetric property and reliability of locally twisted cubes. *Discret. Appl. Math.* **2021**, *288*, 257–269. [CrossRef]
26. El-Amawy, A.; Latifi, S. Properties and performance of folded hyper-cubes. *IEEE Trans. Parallel Distrib. Syst.* **1991**, *2*, 31–42. [CrossRef]
27. Peng, S.; Guo, C.; Yang, B. Topological properties of folded locally twisted cubes. *J. Comput. Inf. Syst.* **2015**, *11*, 7667–7676. [CrossRef]
28. Guo, L.; Su, G.; Lin, W.; Chen, J. Fault tolerance of locally twisted cubes. *Appl. Math. Comput.* **2018**, *334*, 401–406. [CrossRef]
29. Cheng, B.; Fan, J.; Jia, X.; Zhang, S.; Chen, B. Constructive Algorithm of Independent Spanning Trees on Mobius Cubes. *Comput. J.* **2013**, *56*, 1347–1362. [CrossRef]
30. Wang, X.; Fan, J.; Jia, X.; Zhang, S.; Yu, J. Embedding meshes into twisted-cubes. *Inf. Sci.* **2011**, *181*, 3085–3099.
31. Liu, Y.; Lan, J.K.; Chou, W.Y.; Chen, C. Constructing independent spanning trees for locally twisted cubes. *Theor. Comput. Sci.* **2011**, *412*, 2237–2252. [CrossRef]
32. You, L.; Han, Y. An algorithm to embed a family of node-disjoint 3D meshes into locally twisted cubes. In Proceedings of the Algorithms and Architectures for Parallel Processing 14th International Conference, Dalian, China, 24–27 August 2014; pp. 219–230. [CrossRef]