symmetry

Article

Development of Fixed Point Results for α_Γ-F-Fuzzy Contraction Mappings with Applications

Salvatore Sessa [1,*,†], Fahad Jahangeer [2,†], Doha A. Kattan [3,†] and Umar Ishtiaq [4,*,†]

1 Dipartimento di Architettura, Università Dinapoli Federico II, Via Toledo 403, 80121 Napoli, Italy
2 Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad 44000, Pakistan; fahad.msma719@iiu.edu.pk
3 Department of Mathematics, Faculty of Sciences and Arts, King Abdulaziz University, Rabigh 21589, Saudi Arabia; dakattan@kau.edu.sa
4 Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore 54770, Pakistan
* Correspondence: salvasessa@gmail.com (S.S.); umarishtiaq000@gmail.com (U.I.)
† These authors contributed equally to this work.

Abstract: This manuscript contains several fixed point results for α_Γ-F-fuzzy contractive mappings in the framework of orthogonal fuzzy metric spaces. The symmetric property guarantees that the distance function is consistent and does not favour any one direction in orthogonal fuzzy metric spaces. No matter how the points are arranged, it enables a fair assessment of the separations between all of them. In fixed point results, the symmetry condition is preserved for several types of contractive self-mappings. Moreover, we provide several non-trivial examples to show the validity of our main results. Furthermore, we solve non-linear fractional differential equations, the Atangana–Baleanu fractional integral operator and Fredholm integral equations by utilizing our main results.

Keywords: fixed point; fuzzy metric spaces; integral equations; Atangana–Baleanu fractional integral operator

check for updates

Citation: Sessa, S.; Jahangeer, F.; Kattan, D.A.; Ishtiaq, U. Development of Fixed Point Results for α_Γ-F-Fuzzy Contraction Mappings with Applications. *Symmetry* **2023**, *15*, 1300. https://doi.org/10.3390/sym15071300

Academic Editor: Hongkun Xu

Received: 25 May 2023
Revised: 13 June 2023
Accepted: 20 June 2023
Published: 22 June 2023

1. Introduction

Fixed point theory plays an important role in the field of mathematics and has received popularity due to its ability to deal with a wide range of mathematical equations in a straight forward manner. As a result, the concrete solution of such equations is taken into account in fixed point theory. Fixed point theory is currently one of the most attractive fields in the area of nonlinear analysis and even in mathematics in general. Its results can be applied to an extensive set of a distinct type of equations (integral, differential, matricial, etc.) in order to prove the existence and uniqueness of several classes of nonlinear problems.

Zadeh [1] produced the idea of fuzzy sets. Then Schweizer and Sklar [2] produced the idea of t-norm. This increased the range of mathematics. First of all, fuzzy metric space was introduced by Karamosil and Michlek [3]. Then many researchers worked on it and developed many ideas to do with fuzzy metric space. So fuzzy metric space is very famous nowadays. The validity of fixed point theorem was proved by Gregory and Sapena [4] in a fuzzy metric space. The fuzzy distance function in a fuzzy metric space meets a number of criteria, including the symmetric property. With the symmetric property, any two points in the fuzzy metric space can be separated from one another regardless of their consideration order.

The classes of fuzzy contractive conditions were defined by Trade [5], Greogre and Sapena [4] and Mihet [6]. Hung [7] introduced F-contraction in the framework of fuzzy metric spaces. Zhou et al. [8] introduced some fixed point results for contraction mappings in the context of fuzzy metric space. Hezarjaribi [9] introduced the notion of an orthogonal fuzzy metric space and proved the validity of some fixed point theorems in it. Schweizer and Sklar [2] introduced the concept of statistical metric space and validated several fixed point results with applications. Panda et al. [10] introduced an extended F-metric space and validated related fixed point results. Some fixed point results were validated with a new type of fuzzy contractive mapping, and also the Γ function family was defined by Patel et al. [11]. Nazam et al. [12] introduced (Ψ, Φ) orthogonal interpolative contraction and validated fixed point results. Moreover, several related fixed point results were validated in [13,14]. The authors in [15–17] worked on complex valued metric spaces and complex valued fuzzy metric spaces, and utilized several generalized contraction mappings to validate fixed point results.

Inspired by the work of [11], we introduce α_{Γ}-F-orthogonal fuzzy contractive mappings and give examples to verify our main result. We divide this paper into four main parts. In the first part, we present some basic definitions and in the second part we validate main theorems and corollaries for fixed point results and also give an example to fulfill the conditions of our main result. In the third part, we provide several applications of our main results, in which we utilize the Atangana–Baleanu fractional integral operator and Fredholm integral equations to show the validity of the main results, and we present the conclusion in the fourth part of the paper.

2. Preliminaries

In this section, we provide several definitions that are helpful for readers with respect to understanding the main section.

Definition 1 ([2])**.** *A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm (in short, ctn) if it satisfies the following axioms:*
(T1) $\breve{a} * \bar{e} = \bar{e} * \breve{a}$ and $\breve{a} * (\bar{e} * \hat{c}) = (\breve{a} * \bar{e}) * \hat{c}$ for all $\breve{a}, \bar{e}, \hat{c} \in I$;
(T2) $*$ is continuous;
(T3) $\breve{a} * 1 = \breve{a}$ for all $\breve{a} \in I$;
(T4) $\breve{a} * \bar{e} \leq \hat{c} * \tilde{\imath}$ when $\breve{a} \leq c$ and $\bar{e} \leq \tilde{\imath}$, with $\breve{a}, \bar{e}, \hat{c}, \tilde{\imath} \in I$.

Example 1. *(i) $\breve{a} * \bar{e} = \min\{\breve{a}, \bar{e}\}$*
(ii) $\breve{a} * \bar{e} = \breve{a}\bar{e}$
(iii) $\breve{a} * \bar{e} = \max\{0, \breve{a} + \bar{e} - 1\}$.

Definition 2 ([8])**.** *A mapping $\sigma : \mathcal{A} \times \mathcal{A} \times (0, +\infty) \to [0,1]$ and $*$ is ctn; then $(\mathcal{A}, \sigma, *)$ is called fuzzy metric space (in short, FMS) if it satisfies the following axioms for all $\jmath, \varrho, w \in \mathcal{A}$ and $\rho, \varpi > 0$:*
C1: $\sigma(\jmath, \varrho, \rho) > 0$;
C2: $\sigma(\jmath, \varrho, \rho) = 1$ iff $\jmath = \varrho$;
C3: $\sigma(\jmath, \varrho, \rho) = \sigma(\varrho, \jmath, \rho)$;
C4: $\sigma(\jmath, w, \rho + \varpi) \geq \sigma(\jmath, \varrho, \rho) * \sigma(\varrho, w, \varpi)$;
C5: $\sigma(\jmath, v, .) : (0, +\infty) \to [0,1]$ is continuous.
If we replace (C4) by
C6: $\sigma(\jmath, w, \max\{\rho, \varpi\}) \geq \sigma(\jmath, \varrho, \rho) * \sigma(\varrho, w, \varpi)$,
then $(\mathcal{A}, \sigma, *)$ is said to be a non-Archimedean fuzzy metric space (in short, NAFMS). Note that, Since (C6) implies (C4), each non-Archimedean fuzzy metric space is a fuzzy metric space.

Definition 3 ([9])**.** *Suppose $(\mathcal{A}, \sigma, *)$ is an FMS and $\perp \in \mathcal{A} \times \mathcal{A}$ is a binary relation. Assume that there exists $\jmath_0 \in \mathcal{A}$ s.t $\jmath_0 \perp \jmath$ for all $\jmath \in \mathcal{A}$. Then, it is called an orthogonal fuzzy metric space (in short OFMS). We denote that OFMS is $(\mathcal{A}, \sigma, *, \perp)$.*

Definition 4 ([9])**.** *Suppose $(\mathcal{A}, \sigma, *, \perp)$ is an OFMS. A sequence $\{\jmath_r\}_{r \in \mathbb{N}}$ is called O-sequence if $\jmath_r \perp \jmath_{r+1}$ for all $r \in \mathbb{N}$. Moreover, $\Lambda : \mathcal{A} \to \mathcal{A}$ is $\perp$-continuous in $\jmath \in \mathcal{A}$ if for each O- sequence (in short, OS) $\{\jmath_r\}_{r \in \mathbb{N}}$ in $\mathcal{A}$ if $\lim_{r \to +\infty} \sigma(\jmath_r, \jmath, \rho) = 1$ for all $\rho > 0$, then $\lim_{r \to +\infty} \sigma(\Lambda \jmath_r, \Lambda \jmath, \rho) = 1$ for all $\rho > 0$. Furthermore, Λ is $\perp$-continuous if Λ is $\perp$-continuous in each $\jmath \in \mathcal{A}$. Moreover, we* *say that Λ is $\perp$-preserving if $\Lambda\jmath \perp \Lambda\varrho$ whence $\jmath \perp \varrho$. Finally, $\mathcal{A}$ is orthogonally complete (in brief, OC) if every O Cauchy sequence (in short, OCS) is convergent.*

Definition 5 ([11]). *Suppose $(\mathcal{A}, \sigma, *, \perp)$ is an OFMS and $F \in \triangle_F$. The mapping $\Lambda : \mathcal{A} \to \mathcal{A}$ is said to be fuzzy F-contraction if there exists $\tau \in (0,1)$ such that*

$$\tau.F(\sigma(\Lambda\jmath, \Lambda\varrho, \rho)) \geq F(\sigma(\jmath, \varrho, \rho)).$$

Definition 6 ([11]). *Let $\Lambda : \mathcal{A} \to \mathcal{A}$ and $\alpha : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$ be a function. We say that Λ is an α-admissible mapping if*

$$\jmath, \varrho \in \mathcal{A}, \alpha(\jmath, \varrho) \geq 1 \ \Rightarrow \alpha(\Lambda\jmath, \Lambda\varrho) \geq 1.$$

Definition 7 ([11]). *Suppose $\Lambda : \mathcal{A} \to \mathcal{A}$ and two functions $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$. We say that Λ is an α-admissible mapping with respect to η if*

$$\jmath, \varrho \in \mathcal{A}, \alpha(\jmath, \varrho) \geq \eta(\jmath, \varrho) \ \Rightarrow \alpha(\Lambda\jmath, \Lambda\varrho) \geq \eta(\Lambda\jmath, \Lambda\varrho).$$

Definition 8 ([12]). *The OFMS $(\mathcal{A}, \sigma, *, \perp)$ is called $\perp$-regular if for any orthogonal sequence $\{\jmath_r\} \subseteq \mathcal{A}$ converging to $\jmath$, we have either $\jmath_r \perp \jmath$, or $\jmath \perp \jmath_r$ for all $r \in \mathbb{N}$.*

Definition 9 ([12]). *Let $\Lambda : \mathcal{A} \to P(\mathcal{A})$ and $\perp \subset \mathcal{A} \times \mathcal{A}$ be an orthogonal relation. The mapping Λ is called $\perp$-preserving if for each $l \in \mathcal{A}$ and $u \in \Lambda(l)$ such that $l \perp u$ or $u \perp l$, there is $w \in \Lambda(u)$ satisfying $u \perp w$ or $w \perp u$.*

Definition 10 ([11]). *Let $(\mathcal{A}, \sigma, *, \perp)$ be an OFMS and a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$. Suppose two functions $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$. A Λ is called an α_Γ-F-fuzzy contractive mapping, if for $\jmath, \varrho \in \mathcal{A}$ and $\eta(\jmath, \Lambda\jmath) \leq \alpha(\jmath, \varrho)$ and $\sigma(\Lambda\jmath, \Lambda\varrho, \rho) > 0$, we have*

$$\eta(\sigma(\jmath, \Lambda\jmath, \rho), \sigma(\varrho, \Lambda\varrho, \rho), \sigma(\varrho, \Lambda\jmath, \rho)).F(\sigma(\Lambda\jmath, \Lambda\varrho, \rho)) \geq F(\sigma(\jmath, \varrho, \rho)),$$

where $\Gamma \in \triangle_\Gamma$ and $F \in \triangle_F$.

Definition 11 ([11]). *Suppose $(\mathcal{A}, \sigma, *, \perp)$ is an OFMS and $\alpha - \eta : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$ and $\Lambda : \mathcal{A} \to \mathcal{A}$ is a function. Λ is called an $\alpha - \jmath$-continuous mapping on an OFMS. If, for a given $\jmath \in \mathcal{A}$ and O-sequence $\{\jmath_r\}$*

$$\lim_{r \to +\infty} \sigma(\jmath_r, \jmath, \rho) \to 1, \alpha(\jmath_r, \jmath_{r+1}) \geq \eta(\jmath_r, \jmath_{r+1})$$

for all $r \in \mathbb{N}$, then $\lim_{r \to +\infty} \sigma(\Lambda\jmath_r, \Lambda\jmath, \rho) \to 1$.

3. Fixed Point Theorems for α_Γ-F-Orthogonal Fuzzy Contraction

In this section, we prove several fixed point results for contraction mappings.

The set of all the continuous function $\triangle_\Gamma$ is $\Gamma : (\mathbb{R}^+)^4 \to \mathbb{R}$ satisfying the following:

1. for all $\rho_1, \rho_2, \rho_3, \rho_4 \in \mathbb{R}^+$ with $\max(\rho_1, \rho_2, \rho_3, \rho_4) = 1$, there exists $\tau \in (0,1)$ such that $\Gamma(\rho_1, \rho_2, \rho_3, \rho_4) = \tau$.

We have the following examples:

1′. $\Gamma(\rho_1, \rho_2, \rho_3, \rho_4) = \tau + L. \log_e \max(\rho_1, \rho_2, \rho_3, \rho_4)$, where $L \in \mathbb{R}^+$.

2′. $\Gamma(\rho_1, \rho_2, \rho_3, \rho_4) = \frac{\tau}{\max(\rho_1, \rho_2, \rho_3, \rho_4)}$.

3′. $\Gamma(\rho_1, \rho_2, \rho_3, \rho_4) = \frac{e.\tau}{e^{\max(\rho_1, \rho_2, \rho_3, \rho_4)}}$.

Here, $\tau \in (0,1)$ and then $\Gamma \in \triangle_\Gamma$ and $F \in \triangle_F$.

Definition 12. *Let $(\mathcal{A}, \sigma, *, \perp)$ be an OFMS and a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$. Furthermore, suppose that $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$. A mapping is called $\alpha_{\jmath} - F$-OF contractive mapping on $\mathcal{A}$, if for $\jmath, \varrho \in \mathcal{A}$ with $\jmath \perp \varrho$ or $\varrho \perp \jmath$ and $\eta(\jmath, \Lambda\jmath) \leq \alpha(\jmath, \varrho)$ and $\sigma(\Lambda\jmath, \Lambda\varrho, \rho) > 0$, we have*

$$\eta(\sigma(\jmath, \Lambda\jmath, \rho), \sigma(\varrho, \Lambda\varrho, \rho), \sigma(\varrho, \Lambda\jmath, \rho)).F(\sigma(\Lambda\jmath, \Lambda\varrho, \rho)) \geq F(\sigma(\jmath, \varrho, \rho)), \tag{1}$$

where $\Gamma \in \triangle_\Gamma$ and $F \in \triangle_F$.

Definition 13. *Let $(\mathcal{A}, \sigma, *, \perp)$ be an OFMS and $\alpha - \eta : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$ and $\Lambda : \mathcal{A} \to \mathcal{A}$ be a function. We say Λ is an $\alpha - \jmath$-continuous mapping on an OFMS. If, for a given $\jmath \in \mathcal{A}$ and sequence $\{\jmath_r\}$ such that $\jmath_r \perp \jmath$ or $\jmath \perp \jmath_r$,*

$$\lim_{r \to +\infty} \sigma(\jmath_r, \jmath, \rho) \to 1, \alpha(\jmath_r, \jmath_{r+1}) \geq \eta(\jmath_r, \jmath_{r+1}),$$

for all $r \in \mathbb{N}$, then $\lim_{r \to +\infty} \sigma(\Lambda\jmath_r, \Lambda\jmath, \rho) \to 1$.

Lemma 1. *Let $(\mathcal{A}, \sigma, *, \perp)$ be an OFMS and $\{\jmath_r\}$ be an O-sequence in $\mathcal{A}$ such that for each $r \in \mathbb{N}$, $\jmath_r \perp \jmath_{r+1}$ or $\jmath_{r+1} \perp \jmath_r$*

$$\lim_{\rho \to 0^+} \sigma(\jmath_r, \jmath_{r+1}, \rho) > 0,$$

and for any $\rho > 0$,

$$\lim_{r \to +\infty} \sigma(\jmath_r, \jmath_{r+1}, \rho) = 1.$$

If $\{\jmath_r\}$ is not an OCS in $\mathcal{A}$, then there exists $\varepsilon \in (0, 1)$, $\rho_0 > 0$ and two O-sequences of positive integers $\{r_k\}, \{u_k\}, r_k > u_k > k, k \in \mathbb{N}$, such that the following

$$\begin{aligned}&\{\sigma(\jmath_{u_k}, \jmath_{r_k}, \rho_0)\}, \{\sigma(\jmath_{u_k}, \jmath_{r_k+1}, \rho_0)\}, \{\sigma(\jmath_{u_k-1}, \jmath_{r_k}, \rho_0)\},\\ &\{\sigma(\jmath_{u_k-1}, \jmath_{r_k+1}, \rho_0)\}, \{\sigma(\jmath_{u_k+1}, \jmath_{r_k+1}, \rho_0)\}\end{aligned}$$

tend to $1 - \varepsilon$ as $k \to +\infty$.

Now, we are ready to validate our main results.

Theorem 1. *Suppose $(\mathcal{A}, \sigma, *, \perp)$ is an OFMS. A mapping $\Lambda : \mathcal{A} \to \mathcal{A}$ satisfies the following conditions:*

1. *Λ is an α-admissible mapping with respect to η;*
2. *For each $\jmath_0 \in \mathcal{A}$, there is $\jmath_1 = \Lambda(\jmath_0)$ such that $\jmath_1 \perp \jmath_0$ or $\jmath_0 \perp \jmath_1$;*
3. *Λ is an $\alpha_\Gamma - \eta - F$-orthogonal fuzzy contractive mapping;*
4. *$\exists\ \jmath_0 \in \mathcal{A}$ such that $\alpha(\jmath_0, \Lambda\jmath_0) \geq \eta(\jmath_0, \Lambda\jmath_0)$;*
5. *Λ is an $\alpha - \eta$-continuous map;*
6. *Λ is orthogonal preserving mapping.*

Then, Λ has an FP. Moreover, Λ has a UFP whenever $\alpha(\jmath, \varrho) \geq \eta(\jmath, \jmath)$ for all $\jmath, \varrho \in Fix(\Lambda)$.

Proof. Suppose $\jmath_0 \in \mathcal{A}\ \exists\ \jmath_1 \in \mathcal{A}$ s.t $\jmath_0 \perp \jmath_1$ or $\jmath_1 \perp \jmath_0$ s.t $\alpha(\jmath_0, \Lambda\jmath_0) \geq \eta(\jmath_0, \Lambda\jmath_0)$. Then by using the $\perp$-preserving nature of Λ for each $\jmath_0 \in \mathcal{A}$, we define the OS $\{\jmath_r\}$ such that $\jmath_r \perp \jmath_{r+1}$ or $\jmath_{r+1} \perp \jmath_r$ by $\jmath_{r+1} = \Lambda^r(\jmath_0) = \Lambda\jmath_r$ for all $r \in \mathbb{N}$. Now, Since by (1) Λ,

$$\alpha(\jmath_0, \jmath_1) = \alpha(\jmath_0, \Lambda\jmath_0) \geq \eta(\jmath_0, \Lambda\jmath_0 = \eta(\jmath_0, \jmath_1)),$$

by taking this process continuously, we have

$$\eta(\jmath_{r-1}, \jmath_r) \leq \alpha(\jmath_{r-1}, \jmath_r)$$

for all $r \in \mathbb{N}$.

Furthermore, suppose $r_0 \in \mathbb{N}$ such that $\jmath_{r_0} = \Lambda\jmath_{r_0}$, then $\jmath_{r_0}$ is FP of Λ and there is nothing to prove.

Let us assume $\jmath_r \neq \jmath_{r+1}$ or $\sigma(\jmath_r, \jmath_{r+1}, \rho) \in (0,1)$ for all $r \in \mathbb{N}$. With conditions (3) and (1) where $\jmath_r \perp \jmath_{r-1}$ or $\jmath_{r-1} \perp \jmath_r$, we obtain,

$$\begin{aligned} & \Gamma(\sigma(\jmath_{r-1}, \Lambda\jmath_{r-1}, \rho), \sigma(\jmath_r, \Lambda\jmath_r, \rho), \sigma(\jmath_{r-1}, \Lambda\jmath_r, \rho), \sigma(\jmath_r, \Lambda\jmath_{r-1}, \rho)). \\ F(\sigma(\Lambda\jmath_{r-1}, \Lambda\jmath_r, \rho)) \geq & F(\sigma(\jmath_{r-1}, \jmath_r, \rho)), \end{aligned}$$

which implies

$$\begin{aligned} & \Gamma(\sigma(\jmath_{r-1}, \jmath_r, \rho), \sigma(\jmath_r, \jmath_{r+1}, \rho), \sigma(\jmath_{r-1}, \jmath_{r+1}, \rho), \sigma(\jmath_r, \jmath_r, \rho)). \\ F(\sigma(\jmath_r, \jmath_{r+1}, \rho)) \geq & F(\sigma(\jmath_{r-1}, \jmath_r, \rho)). \end{aligned}$$

Since $\max(\sigma(\jmath_{r-1}, \jmath_r, \rho), \sigma(\jmath_r, \jmath_{r+1}, \rho), \sigma(\jmath_{r-1}, \jmath_{r+1}, \rho), \sigma(\jmath_r, \jmath_r, \rho)) = 1$, with the Γ-function $\exists$ $\tau \in (0,1)$ we have

$$\Gamma(\sigma(\jmath_{r-1}, \jmath_r, \rho), \sigma(\jmath_r, \jmath_{r+1}, \rho), \sigma(\jmath_{r-1}, \jmath_{r+1}, \rho), \sigma(\jmath_r, \jmath_r, \rho)) = \tau.$$

Therefore,

$$\tau.F(\sigma(\jmath_r, \jmath_{r+1}, \rho)) \geq F(\sigma(\jmath_{r-1}, \jmath_r, \rho)).$$

We have

$$F(\sigma(\jmath_r, \jmath_{r+1}, \rho)) > \tau.F(\sigma(\jmath_r, \jmath_{r+1}, \rho)) \geq F(\sigma(\jmath_{r-1}, \jmath_r, \rho)). \tag{2}$$

This is because F is a strictly increasing function (in short, SIF).

$$\sigma(\jmath_r, \jmath_{r+1}, \rho) > \sigma(\jmath_{r-1}, \jmath_r, \rho).$$

Thus, the sequence $\{\sigma(\jmath_r, \jmath_{r+1}, \rho)\}$ $(\rho > 0)$ is an SIF-bounded form as above, and thus sequence $\{\sigma(\jmath_r, \jmath_{r+1}, \rho)\}$ $(\rho > 0)$ is convergent. Λ is orthogonal preserving mapping. So, there exists $\alpha(\rho) \in I$ such that

$$\lim_{r \to +\infty} \sigma(\jmath_r, \jmath_{r+1}, \rho) = \alpha(\rho), \tag{3}$$

for any $\rho > 0$ and $r \in \mathbb{N}$. It follows that

$$\sigma(\jmath_r, \jmath_{r+1}, \rho) < \alpha(\rho), \tag{4}$$

by (3) and (4), for any $\rho > 0$; we have

$$\lim_{r \to +\infty} F(\sigma(\jmath_r, \jmath_{r+1}, \rho)) = F(\alpha(\rho) - 0). \tag{5}$$

We have to show that $\alpha(\rho) = 1$. Assume that $\alpha(\rho) < 1$ for some $\rho > 0$ and $r \to +\infty$ in (2); using (5), we obtain

$$F(\alpha(\rho) - 0) \geq \tau.F(\alpha(\rho) - 0) \geq F(\alpha(\rho) - 0).$$

This is a contradiction with $F(\alpha(\rho) - 0) > 0$. Therefore,

$$\lim_{r \to +\infty} \sigma(\jmath_r, \jmath_{r+1}, \rho) = 1. \tag{6}$$

Next, we have to show that $\{\jmath_r\}$ is an OCS. Let $\{\jmath_r\}$ not be an OCS. By using the Lemma 1, $\exists\, \varepsilon \in (0,1), \rho_0 > 0$ and sequences $\{\jmath_{u_k}\}$ and $\{\jmath_{r_k}\}$ such that $\jmath_{u_k} \perp \jmath_{r_k}$ or $\jmath_{r_k} \perp \jmath_{u_k}$

$$\lim_{k \to +\infty} \sigma(\jmath_{u_k}, \jmath_{r_k}, \rho_0) = 1 - \varepsilon. \tag{7}$$

Again, in (1) we have

$$
\begin{aligned}
& \Gamma\big(\sigma\big(\jmath_{u_k},\jmath_{u_k+1},\rho\big),\sigma\big(\jmath_{r_k},\jmath_{r_k+1},\rho\big),\sigma\big(\jmath_{u_k},\jmath_{r_k+1},\rho\big),\sigma\big(\jmath_{r_k},\jmath_{u_k+1}\big)\big) \\
.F\big(\sigma\big(\jmath_{u_k+1},\jmath_{r_k+1},\rho\big)\big) \quad \geq \quad & F\big(\sigma\big(\jmath_{u_k},\jmath_{r_k},\rho\big)\big).
\end{aligned}
$$

Let the limit $k \to +\infty$; we have

$$
\begin{aligned}
\lim_{k\to+\infty}[\Gamma\big(\sigma\big(\jmath_{u_k},\jmath_{u_k+1},\rho\big),\sigma\big(\jmath_{r_k},\jmath_{r_k+1},\rho\big),\sigma\big(\jmath_{u_k},\jmath_{r_k+1},\rho\big),\sigma\big(\jmath_{r_k},\jmath_{u_k+1}\big)\big) \\
.F\big(\sigma\big(\jmath_{u_k+1},\jmath_{r_k+1},\rho\big)\big)] \geq \lim_{k\to+\infty} F\big(\sigma\big(\jmath_{u_k},\jmath_{r_k},\rho\big)\big).
\end{aligned} \tag{8}
$$

This implies

$$
\begin{aligned}
& \Gamma\left(1,1,\lim_{k\to+\infty}\sigma\big(\jmath_{u_k},\jmath_{r_k+1},\rho\big),\lim_{k\to+\infty}\sigma\big(\jmath_{r_k},\jmath_{u_k+1}\big)\right). \\
.F\big(\sigma\big(\jmath_{u_k+1},\jmath_{r_k},\rho\big)\big) \quad \geq \quad & \lim_{k\to+\infty} F\big(\sigma\big(\jmath_{u_k},\jmath_{r_k},\rho\big)\big).
\end{aligned}
$$

Since $\max\big(1,1,\sigma\big(\jmath_{u_k},\jmath_{r_k+1},\rho\big),\sigma\big(\jmath_{r_k},\jmath_{u_k+1}\big)\big) = \tau$.

Using (7) and (8),

$$\tau.F((1-\varepsilon)-0) \geq F((1-\varepsilon)-0).$$

This is a contradiction with $F((1-\varepsilon)-0) > 0$. Thus, The sequence $\{\jmath_r\}$ is an OCS in $\mathcal{A}$. Since OFMS $(\mathcal{A},\sigma,*,\perp)$ is complete, there exists $\jmath^* \in \mathcal{A}$ such that $\jmath^* \perp \jmath_r$ or $\jmath_r \perp \jmath^*$

$$\lim_{r\to+\infty} \jmath_r = \jmath^*.$$

Suppose $\jmath^*$ is an FP of Λ. With condition (5) and $\eta(\jmath_{r-1},\jmath_r) \leq \alpha(\jmath_{r-1},\jmath_r)$ for all $r \in \mathbb{N}$, $\lim_{r\to+\infty}\sigma(\Lambda\jmath_r,\Lambda\jmath^*,\rho) = 1$ implies $\sigma(\jmath^*,\Lambda\jmath^*,\rho) = 1$; that is, $\jmath^* = \Lambda\jmath^*$.

Let $\jmath,\varrho \in Fix(\Lambda)$ such that $\jmath \perp \varrho$ or $\varrho \perp \jmath$; by (1),

$$\Gamma(\sigma(\jmath,\jmath,\rho),\sigma(\varrho,\varrho,\rho),\sigma(\jmath,\varrho,\rho),\sigma(\varrho,\jmath,\rho)).F(\sigma(\jmath,\varrho,\rho)) \geq F(\sigma(\jmath,\varrho,\rho))$$

$$\Gamma(1,1,\sigma(\jmath,\varrho,\rho),\sigma(\varrho,\jmath,\rho)).F(\sigma(\jmath,\varrho,\rho)) \geq F(\sigma(\jmath,\varrho,\rho)).$$

Since $\max(1,1,\sigma(\jmath,\varrho,\rho),\sigma(\varrho,\jmath,\rho)).F(\sigma(\jmath,\varrho,\rho)) = 1$, there exists $\tau \in (0,1)$ such that $\Gamma(1,1,\sigma(\jmath,\varrho,\rho),\sigma(\varrho,\jmath,\rho)).F(\sigma(\jmath,\varrho,\rho)) = \tau$. Hence we show that

$$\tau.F(\sigma(\jmath,\varrho,\rho)) \geq F(\sigma(\jmath,\varrho,\rho)).$$

This implies that

$$F(\sigma(\jmath,\varrho,\rho)) > \tau.F(\sigma(\jmath,\varrho,\rho)) \geq F(\sigma(\jmath,\varrho,\rho)),$$

which is contradiction. Thus, Λ has a UFP. □

Corollary 1. *Let $(\mathcal{A},\sigma,*,\perp)$ be an OCFMS. Suppose a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$ which satisfies the following conditions:*

1. *Λ is an α-admissible mapping with respect to $\mathbf{\jmath}$;*
2. *For each $\jmath_0 \in \mathcal{A}$, there is $\jmath_1 = \Lambda(\jmath_0)$ such that $\jmath_1 \perp \jmath_0$ or $\jmath_0 \perp \jmath_1$;*
3. *For $\jmath,\varrho \in \mathcal{A}$ such that $\jmath \perp \varrho$ or $\varrho \perp \jmath$ with $\alpha(\jmath,\varrho) \geq \eta(\jmath,\Lambda\jmath)$ and $\sigma(\Lambda\jmath,\Lambda\varrho,\rho) > 0$, we have*

$$\tau.F(\sigma(\Lambda\jmath,\Lambda\varrho,\rho)) \geq F(\sigma(\jmath,\varrho,\rho)),$$

where $\jmath \neq \varrho$, $\tau \in (0,1)$ and $F \in \triangle_F$;

4. *There exists $\jmath_0 \in \mathcal{A}$ such that $\alpha(\jmath_0,\Lambda\jmath_0) \geq \eta(\jmath_0,\Lambda\jmath_0)$;*
5. *Λ is an $\alpha-\mathbf{\jmath}$-continuous map;*
6. *Λ is orthogonal preserving mapping.*

Then, Λ has an FP. Moreover, Λ has a UFP whenever $\alpha(\jmath,\varrho) \geq \eta(\jmath,\jmath)$ for all $\jmath,\varrho \in Fix(\Lambda)$.

Example 2. *Suppose $\mathcal{A} = [0, +\infty)$ and t-norm is defined by $\rho(\breve{a}, \bar{e}) = \breve{a}\bar{e}$ for all $\breve{a}, \bar{e} \in I$. Define a fuzzy set $\sigma : \mathcal{A} \times \mathcal{A} \times (0, 1) \to I$ such that*

$$\sigma(\jmath, \varrho, \rho) = e^{\frac{-|\jmath-\varrho|}{\rho}}$$

for all $\jmath, \varrho \in \mathcal{A}$ and for all $\rho > 0$.

Define the relation $\perp \subset \mathcal{A}^2$ as

$$\ell \perp w \text{ if } \ell w \leq \{\ell, w\}.$$

Thus, $(\mathcal{A}, \sigma, *, \perp)$ is an OCFMS.

Define $\Lambda : \mathcal{A} \to \mathcal{A}$ such that

$$\Lambda(\jmath) = \left\{ \begin{array}{c} \frac{1}{2}\jmath^2, \text{if } \jmath \in [0, 1], \\ 3\jmath, \text{ if } \jmath \in (1, +\infty). \end{array} \right\}$$

Let $\mathbf{J}-\alpha$: $\mathcal{A} \times \mathcal{A} \to [0, +\infty]$ defined by $\mathbf{J}(\jmath, \varrho) = \frac{1}{4}$ for all $\jmath, \varrho \in \mathcal{A}$ and

$$\alpha(\jmath, \varrho) = \left\{ \begin{array}{c} \frac{1}{2} \text{ if } \jmath, \varrho \in [0, 1] \\ \frac{1}{9} \text{ otherwise.} \end{array} \right\}$$

Let $F(\jmath) = \log_e \jmath$ be any SIF and consider Γ function $\Gamma : (\mathbb{R}^+)^4 \to \mathbb{R}$ defined by $\Gamma(\rho_1, \rho_2, \rho_3, \rho_4) = \tau$, where $\tau \in (0, 1)$.

1. Let $\alpha(\jmath, \varrho) \geq \eta(\jmath, \varrho)$, then $\jmath, \varrho \in [0, 1]$; on the other hand, $\Lambda(\jmath) \in [0, 1]$ for all $\jmath, \varrho \in [0, 1]$, then $\alpha(\Lambda\jmath, \Lambda\varrho) \geq \eta(\Lambda\jmath, \Lambda\varrho)$ (or $\alpha(\Lambda\jmath, \Lambda\varrho) = \alpha\left(\frac{1}{2}\jmath^2, \frac{1}{2}\varrho^2\right) = \frac{1}{2} > \frac{1}{4}$). So, condition (1) of Theorem 3.4 is satisfied.

2. $\exists\, \jmath_0 \in \mathcal{A}$, so condition (4) is also satisfied.

3. Let $\jmath_r = \frac{1}{r} \to 0, \alpha(\jmath_r, \jmath_{r+1}) = \alpha\left(\frac{1}{r}, \frac{1}{r+1}\right) = \frac{1}{2} \geq \eta\left(\frac{1}{r}, \frac{1}{r+1}\right) = \frac{1}{4}$ for all $r \in \mathbb{N}$. This implies $\Lambda(\jmath_r) = \Lambda\left(\frac{1}{r}\right) \to 0 = \Lambda(0)$. Thus, condition (5) is satisfied. Similarly, all other conditions are satisfied.

Now, let $\jmath \perp \varrho$; then

$$\begin{aligned} \tau.F(\sigma(\Lambda\jmath, \Lambda\varrho, \rho)) &= \tau.F\left(e^{-\left(\frac{1}{2\rho}|\jmath^2-\varrho^2|\right)}\right) \\ &= \tau.\log\left\{e^{-\left(\frac{1}{2\rho}|\jmath^2-\varrho^2|\right)}\right\} \\ &= -\tau.\frac{1}{2\rho} \mid \jmath^2 - \varrho^2 \mid \\ &\geq -\frac{\mid \jmath - \varrho \mid}{\rho} \\ &= \log e^{-\frac{|\jmath-\varrho|}{\rho}} \\ &= F(\sigma(\jmath, \varrho, \rho)). \end{aligned}$$

Hence, all the conditions of Theorem 1 hold. Thus, $\jmath = 0$ is an FP for the self map Λ. Now, given $\jmath = 2$, $\varrho = 3$ and $\tau \in (0, 1)$, the conditions of Theorem 1 are not satisfied without orthogonality.

If we let $\eta(\jmath, \varrho) = 1$, with respect to Theorem 1 and Corrolary 1, we have the following.

Definition 14. *Suppose $(\mathcal{A}, \sigma, *, \perp)$ is an OFMS and a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$. Furthermore, suppose two functions $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$. Then, a mapping Λ is called $\alpha_\Gamma-F$-orthogonal* *fuzzy contractive if for $\jmath, \varrho \in \mathcal{A}$ with $\jmath \perp \varrho$ or $\varrho \perp \jmath$ and $\alpha(\jmath, \varrho) \geq 1$ and $\sigma(\Lambda\jmath, \Lambda\varrho, \rho) > 0$, we have*

$$\Gamma(\sigma(\jmath, \Lambda\jmath, \rho), \sigma(\varrho, \Lambda\varrho, \rho), \sigma(\varrho, \Lambda\jmath, \rho)).F(\sigma(\Lambda\jmath, \Lambda\varrho, \rho)) \geq F(\sigma(\jmath, \varrho, \rho)), \tag{9}$$

where $\Gamma \in \triangle_\Gamma$ and $F \in \triangle_F$.

Theorem 2. *Suppose $(\mathcal{A}, \sigma, *, \perp)$ is an OCFMS. Suppose a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$ satisfies the following conditions:*

1. *Λ is an α-admissible mapping with respect to $\mathbf{J}$;*
2. *For each $\jmath_0 \in \mathcal{A}$, there is $\jmath_1 = \Lambda(\jmath_0)$ such that $\jmath_1 \perp \jmath_0$ or $\jmath_0 \perp \jmath_1$;*
3. *Λ is an $\alpha_\Gamma - F$-orthogonal fuzzy contractive mapping;*
4. *$\exists\ \jmath_0 \in \mathcal{A}$ such that $\alpha(\jmath_0, \Lambda\jmath_0) \geq 1$;*
5. *Λ is an α-continuous map;*
6. *Λ is orthogonal preserving mapping.*

Then, Λ has an FP. Moreover, Λ has a UFP whenever $\alpha(\jmath, \varrho) \geq 1$ for all $\jmath, \varrho \in Fix(\Lambda)$.

Proof. Proceeding as in the proof of Theorem 1, we have the following. □

Corollary 2. *Suppose $(\mathcal{A}, \sigma, *, \perp)$ is an OCFMS. Suppose a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$ satisfies the following conditions:*

1. *Λ is an α-admissible mapping with respect to $\mathbf{J}$;*
2. *For each $\jmath_0 \in \mathcal{A}$, there is $\jmath_1 = \Lambda(\jmath_0)$ such that $\jmath_1 \perp \jmath_0$ or $\jmath_0 \perp \jmath_1$;*
3. *If for $\jmath, \varrho \in \mathcal{A}$ such that $\jmath \perp \varrho$ or $\varrho \perp \jmath$ with $\alpha(\jmath, \varrho) \geq 1$ and $\sigma(\Lambda\jmath, \Lambda\varrho, \rho) > 0$, we have*

$$\tau.F(\sigma(\Lambda\jmath, \Lambda\varrho, \rho)) \geq F(\sigma(\jmath, \varrho, \rho)),$$

where $\jmath \neq \varrho$, $\tau \in (0,1)$ and $F \in \triangle_F$;

4. *$\exists\ \jmath_0 \in \mathcal{A}$ such that $\alpha(\jmath_0, \Lambda\jmath_0) \geq 1$;*
5. *Λ is an α-continuous map;*
6. *Λ is orthogonal preserving mapping.*

Then, Λ has an FP. Moreover, Λ has a UFP whenever $\alpha(\jmath, \varrho) \geq 1$ for all $\jmath, \varrho \in Fix(\Lambda)$.

If we take $\alpha(\jmath, \varrho) = 1$ in Corollary 2 for all $\jmath, \varrho \in \mathcal{A}$, we obtain the following result.

Corollary 3. *Let $(\mathcal{A}, \sigma, *, \perp)$ be an OCFMS such that*

$$\lim_{\rho \to 0^+} \sigma(\jmath, \varrho, \rho) > 0$$

for all $\jmath, \varrho \in \mathcal{A}$. If $\Lambda : \mathcal{A} \to \mathcal{A}$ is a continuous OF F-contraction, then Λ has a UFP in $\mathcal{A}$.

In the next theorem, if we omit condition (5) from theorem then, we have following result.

Theorem 3. *Suppose $(\mathcal{A}, \sigma, *, \perp)$ is an OCFMS. Suppose a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$ satisfies the following conditions:*

1. *Λ is an α-admissible mapping with respect to $\mathbf{J}$;*
2. *For each $\jmath_0 \in \mathcal{A}$, there is $\jmath_1 = \Lambda(\jmath_0)$ such that $\jmath_1 \perp \jmath_0$ or $\jmath_0 \perp \jmath_1$;*
3. *Λ is an $\alpha_\Gamma - F$-orthogonal fuzzy contractive mapping;*
4. *$\exists\ \jmath_0 \in \mathcal{A}$ such that $\alpha(\jmath_0, \Lambda\jmath_0) \geq \eta(\jmath_0, \Lambda\jmath_0)$;*
5. *If $\{\jmath_r\}$ is an OS in $\mathcal{A}$ such that $\jmath_r \perp \jmath_{r+1}$ or $\jmath_{r+1} \perp \jmath_r$ and $\alpha(\jmath_r, \jmath_{r+1}) \geq \eta(\jmath_r, \jmath_{r+1})$ with $\jmath_r \to \jmath$ as $r \to +\infty$, then*

$$\mathbf{J}\left(\Lambda\jmath_r, \Lambda^2\jmath_r\right) \leq \alpha(\Lambda\jmath_r, \jmath) \text{ or } \eta\left(\Lambda^2\jmath_r, \Lambda^3\jmath_r\right) \leq \alpha\left(\Lambda^2\jmath_r, \jmath\right)$$

holds for all $r \in \mathbb{N}$.

6. *Λ is orthogonal preserving mapping.*

Then, Λ has an FP. Moreover, Λ has a UFP whenever $\alpha(\jmath,\varrho) \geq 1$ for all $\jmath,\varrho \in Fix(\Lambda)$.

Proof. Let $\jmath_0 \in \mathcal{A} \ \exists \ \jmath_1 \in \mathcal{A}$ such that $\jmath_0 \perp \jmath_1$ or $\jmath_1 \perp \jmath_0$ such that $\alpha(\jmath_0,\Lambda\jmath_0) \geq \mathbf{J}(\jmath_0,\Lambda\jmath_0)$. Proceeding as in the proof of Theorem 1, we conclude that

$$\alpha(\jmath_r,\jmath_{r+1}) \geq \eta(\jmath_r,\jmath_{r+1}) \text{ and } \jmath_r \to \jmath^* \text{ as } r \to +\infty$$

where, $\jmath_{r+1} = \Lambda\jmath_r$. By assumption (5), either

$$\eta\left(\Lambda\jmath_r,\Lambda^2\jmath_r\right) \leq \alpha(\Lambda\jmath_r,\jmath^*) \text{ or } \eta\left(\Lambda^2\jmath_r,\Lambda^3\jmath_r\right) \leq \mathbf{ff}\left(\Lambda^2\jmath_r,\jmath^*\right)$$

is satisfied for all $r \in \mathbb{N}$. Equivalently, $\exists$ is a subsequence $\{\jmath_{r_k}\}$ of $\{\jmath_r\}$ such that $\jmath_{r_k} \perp \jmath_{r_k+1}$ or $\jmath_{r_k+1} \perp \jmath_{r_k}$,

$$\eta(\jmath_{r_k},\jmath_{r_k+1}) \leq \alpha(\jmath_{r_k},\jmath^*),$$

and by (1), we obtain

$$\begin{aligned} &\Gamma(\sigma(\jmath_{r_k},\Lambda\jmath_{r_k},\rho),\sigma(\jmath^*,\Lambda\jmath^*,\rho),\sigma(\jmath_{r_k},\Lambda\jmath^*,\rho),\sigma(\jmath^*,\Lambda\jmath_{r_k},\rho)).F(\sigma(\Lambda\jmath_{r_k},\Lambda\jmath^*,\rho)) \\ \geq\ & F(\sigma(\jmath_{r_k},\jmath^*,\rho)), \end{aligned}$$

which implies for any $\rho > 0$,

$$F(\sigma(\Lambda\jmath_{r_k},\Lambda\jmath^*,\rho)) > \tau.F(\sigma(\Lambda\jmath_{r_k},\Lambda\jmath^*,\rho)) \geq F(\sigma(\jmath_{r_k},\jmath^*,\rho)).$$

Since F is SIF,

$$\sigma(\Lambda\jmath_{r_k},\Lambda\jmath^*,\rho) > \sigma(\jmath_{r_k},\jmath^*,\rho).$$

$$\lim_{k\to+\infty} \sigma(\Lambda\jmath_{r_k},\Lambda\jmath^*,\rho) > \lim_{k\to+\infty} \sigma(\jmath_{r_k},\jmath^*,\rho)$$

then we obtain $\sigma(\jmath^*,\Lambda\jmath^*,\rho) = 1$, i.e, $\jmath^* = \Lambda\jmath^*$. The uniqueness of the FP is similar to Theorem 1. □

Corollary 4. *Let $(\mathcal{A},\sigma,*,\perp)$ be an OCFMS. Suppose a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$ satisfies the following conditions:*

1. *Λ is an α-admissible mapping with respect to $\mathbf{J}$;*
2. *For each $\jmath_0 \in \mathcal{A}$, there is $\jmath_1 = \Lambda(\jmath_0)$ such that $\jmath_1 \perp \jmath_0$ or $\jmath_0 \perp \jmath_1$;*
3. *If for $\jmath,\varrho \in \mathcal{A}$ such that $\jmath \perp \varrho$ or $\varrho \perp \jmath$ with $\alpha(\jmath,\varrho) \geq 1$ and $\sigma(\Lambda\jmath,\Lambda\varrho,\rho) > 0$, we have*

$$\tau.F(\sigma(\Lambda\jmath,\Lambda\varrho,\rho)) \geq F(\sigma(\jmath,\varrho,\rho)),$$

where $\jmath \neq \varrho$, $\tau \in (0,1)$ and $F \in \triangle_F$;

4. *$\exists\ \jmath_0 \in \mathcal{A}$ such that $\alpha(\jmath_0,\Lambda\jmath_0) \geq \eta(\jmath_0,\Lambda\jmath_0)$;*
5. *If $\{\jmath_r\}$ is an OS in $\mathcal{A}$ such that $\jmath_r \perp \jmath_{r+1}$ or $\jmath_{r+1} \perp \jmath_r$ and $\alpha(\jmath_r,\jmath_{r+1}) \geq \eta(\jmath_r,\jmath_{r+1})$ with $\jmath_r \to \jmath$ as $r \to +\infty$, then*

$$\eta\left(\Lambda\jmath_r,\Lambda^2\jmath_r\right) \leq \alpha(\Lambda\jmath_r,\jmath) \text{ or } \eta\left(\Lambda^2\jmath_r,\Lambda^3\jmath_r\right) \leq \alpha\left(\Lambda^2\jmath_r,\jmath\right)$$

holds for all $r \in \mathbb{N}$.

6. *Λ is orthogonal preserving mapping.*

Then, Λ has an FP. Moreover, Λ has a UFP whenever $\alpha(\jmath,\varrho) \geq 1$ for all $\jmath,\varrho \in Fix(\Lambda)$.

If we take $\eta(\jmath,\varrho) = 1$ in Theorem 3 and Corollary 4, then we obtain the following;

Theorem 4. *Suppose $(\mathcal{A},\sigma,*,\perp)$ is an OFCFMS. Suppose a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$ satisfies the following conditions:*

1. *Λ is an α-admissible mapping;*
2. *For each $\jmath_0 \in \mathcal{A}$, there is $\jmath_1 = \Lambda(\jmath_0)$ such that $\jmath_1 \perp \jmath_0$ or $\jmath_0 \perp \jmath_1$;*

3. Λ is an $\alpha_\Gamma - F$-orthogonal fuzzy contractive mapping;
4. $\exists\, \jmath_0 \in \mathcal{A}$ such that $\alpha(\jmath_0, \Lambda\jmath_0) \geq 1$;
5. If $\{\jmath_r\}$ is an OS in $\mathcal{A}$ such that $\jmath_r \perp \jmath_{r+1}$ or $\jmath_{r+1} \perp \jmath_r$ and $\alpha(\jmath_r, \jmath_{r+1}) \geq 1$ with $\jmath_r \to \jmath$ as $r \to +\infty$, then

$$\alpha(\jmath_r, \jmath) \geq 1 \text{ or } \alpha(\jmath_{r+1}, \jmath) \geq 1$$

holds for all $r \in \mathbb{N}$.
6. Λ is orthogonal preserving mapping.
Then, Λ has an FP. Moreover, Λ has a UFP whenever $\alpha(\jmath, \varrho) \geq 1$ for all $\jmath, \varrho \in Fix(\Lambda)$.

Proof. Suppose $\jmath_0 \in \mathcal{A}\ \exists\, \jmath_1 \in \mathcal{A}$ s.t $\jmath_0 \perp \jmath_1$ or $\jmath_1 \perp \jmath_0$ such that $\alpha(\jmath_0, \Lambda\jmath_0) \geq 1$. Proceeding as in the proof of theorem 1, we conclude that

$$\alpha(\jmath_r, \jmath_{r+1}) \geq 1 \text{ and } \jmath_r \to \jmath^* \text{ as } r \to +\infty$$

where, $\jmath_{r+1} = \Lambda\jmath_r$. By assumption 5, $\alpha(\Lambda\jmath_r, \jmath^*) \geq 1$ holds for all $r \in \mathbb{N}$. So, by Lemma 1 $\exists$ a subsequence $\{\jmath_{r_k}\}$ of $\{\jmath_r\}$ such $\jmath_{r_k} \perp \jmath_{r_k+1}$ or $\jmath_{r_k+1} \perp \jmath_{r_k}$, by definition of $\alpha_\Gamma - F$-OF contractive mapping; we obtain that

$$\begin{aligned} & \Gamma\big(\sigma\big(\jmath_{r_k}, \Lambda\jmath_{r_k}, \rho\big), \sigma(\jmath^*, \Lambda\jmath^*, \rho), \sigma\big(\jmath_{r_k}, \Lambda\jmath^*, \rho\big), \sigma\big(\jmath^*, \Lambda\jmath_{r_k}, \rho\big)\big).F\big(\sigma\big(\Lambda\jmath_{r_k}, \Lambda\jmath^*, \rho\big)\big) \\ \geq\ & F\big(\sigma\big(\jmath_{r_k}, \jmath^*, \rho\big)\big), \end{aligned}$$

This implies for any $\rho > 0$,

$$F\big(\sigma\big(\Lambda\jmath_{r_k}, \Lambda\jmath^*, \rho\big)\big) > \tau.F\big(\sigma\big(\Lambda\jmath_{r_k}, \Lambda\jmath^*, \rho\big)\big) \geq F\big(\sigma\big(\jmath_{r_k}, \jmath^*, \rho\big)\big).$$

Since F is SIF,

$$\sigma\big(\Lambda\jmath_{r_k}, \Lambda\jmath^*, \rho\big) > \sigma\big(\jmath_{r_k}, \jmath^*, \rho\big).$$

$$\lim_{k\to+\infty} \sigma\big(\Lambda\jmath_{r_k}, \Lambda\jmath^*, \rho\big) > \lim_{k\to+\infty} \sigma\big(\jmath_{r_k}, \jmath^*, \rho\big)$$

then we obtain $\sigma(\jmath^*, \Lambda\jmath^*, \rho) = 1$, i.e, $\jmath^* = \Lambda\jmath^*$. The uniqueness of the FP is similar to Theorem 3. $\square$

Corollary 5. *Suppose $(\mathcal{A}, \sigma, *, \perp)$ is an OCFMS. Suppose a mapping $\Lambda : \mathcal{A} \to \mathcal{A}$ satisfies the following conditions:*
1. Λ is an α-admissible mapping;
2. For each $\jmath_0 \in \mathcal{A}$, there is $\jmath_1 = \Lambda(\jmath_0)$ such that $\jmath_1 \perp \jmath_0$ or $\jmath_0 \perp \jmath_1$;
3. If for $\jmath, \varrho \in \mathcal{A}$ such that $\jmath \perp \varrho$ or $\varrho \perp \jmath$ with $\alpha(\jmath, \varrho) \geq 1$ and $\sigma(\Lambda\jmath, \Lambda\varrho, \rho) > 0$, we have

$$\tau.F(\sigma(\Lambda\jmath, \Lambda\varrho, \rho)) \geq F(\sigma(\jmath, \varrho, \rho)),$$

where $\jmath \neq \varrho$, $\tau \in (0,1)$ and $F \in \triangle_F$;
4. $\exists\, \jmath_0 \in \mathcal{A}$ such that $\alpha(\jmath_0, \Lambda\jmath_0) \geq 1$;
5. If $\{\jmath_r\}$ is an OS in $\mathcal{A}$ such that $\jmath_r \perp \jmath_{r+1}$ or $\jmath_{r+1} \perp \jmath_r$ and $\alpha(\jmath_r, \jmath_{r+1}) \geq 1$ with $\jmath_r \to \jmath$ as $r \to +\infty$, then

$$\alpha(\Lambda\jmath_r, \jmath) \geq 1 \text{ or } \alpha\Big(\Lambda^2\jmath_r, \jmath\Big) \geq 1$$

holds for all $r \in \mathbb{N}$.
6. Λ is orthogonal preserving mapping.
Then, Λ has an FP. Moreover, Λ has a UFP whenever $\alpha(\jmath, \varrho) \geq 1$ for all $\jmath, \varrho \in Fix(\Lambda)$.

4. Applications

4.1. Application to Nonlinear Differential Equation

For the existence and uniqueness of our solution, we validate an application of the Corollary 5 for a nonlinear differential equation.

$$ {}^{C}H^{\beta}(\jmath(\rho)) = t(\rho, \jmath(\rho)) \tag{10} $$

where $(0 < \rho < 1, 1 < \beta \leq 2)$ via the integral boundary conditions

$$ \jmath(0) = 0, \jmath(1) = \int_0^{\mathbf{J}} \jmath(v)dv \quad (0 < \mathbf{J} < 1) \tag{11} $$

where ${}^{C}H^{\beta}$ denotes the Caputo fractional derivative of order β and a continous function $t : I \times \mathbb{R} \to \mathbb{R}$. Here, $(\mathcal{A}, \| \,.\, \|_{+\infty})$. Let us consider the Banach space $C(I, \mathbb{R})$ of all continuous functions defined on a real interval I (where $I = [0, 1]$)

$$ \| \jmath \|_{+\infty} = \sup_{\rho \in [0,1]} | \jmath(\rho) | \tag{12} $$

Suppose a complete OFMS $(\mathcal{A}, \sigma, *, \perp)$. The triple $(\mathcal{A}, \sigma, *_p, \perp)$ is an OFMS where the set σ is defined by

$$ \sigma(\jmath, \varrho, \rho) = e^{\frac{-|\jmath - \varrho|}{\rho}} \tag{13} $$

where $\jmath \perp \varrho$ or $\varrho \perp \jmath$ and $\jmath, \varrho \in \mathcal{A}$ and $\rho > 0$. For a continuous function $h : \mathbb{R}^+ \to \mathbb{R}$, the Caputo derivative of fractional order β is defined by

$$ {}^{C}H^{\beta}h(\rho) = \frac{1}{\Gamma(r - \beta)} \int_0^{\rho} \frac{g(v)}{(\rho - v)^{\beta - r - 1}} dv \tag{14} $$

$(r - 1 < \beta < r, r = [\beta] + 1)$, where $[\beta]$ denotes the integer part of the real number β.

Now, for the continuous function $g : \mathbb{R}^+ \to \mathbb{R}$, the Reimann–Liouville fractional derivative of order β is defined by

$$ {}^{C}H^{\beta}g(\rho) = \frac{1}{\Gamma(r - \beta)} \frac{d^r}{d\rho^r} \int_0^{\rho} \frac{g(v)}{(\rho - v)^{\beta - r - 1}} dv \tag{15} $$

$r = [\beta] + 1$; the right side is point-wise defined on $(0, +\infty)$.

Theorem 5. *Let 1.* $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a function and* $\tau \in (0, 1)$ *such that*

$$ | f(\rho, a) - f(\rho, b) | \leq \frac{\Gamma(\beta + 1)}{5} \frac{| a - b |}{\tau} \tag{16} $$

for all $\rho \in I$ *and* $a, b \in \mathbb{R}$ *with* $\zeta(a, b) > 0$;

2. $\exists\, \jmath_0 \in \mathcal{A}$ *such that* $\jmath_0 \perp \Lambda\jmath_0$ *or* $\Lambda\jmath_0 \perp \jmath_0$ *and* $\zeta(\jmath_0(\rho), \Lambda\jmath_0(\rho)) > 0$ *for all* $\rho \in [0, 1]$, *where,* $\Lambda : \mathcal{A} \to \mathcal{A}$ *is defined by*

$$ \begin{aligned} \Lambda\jmath(\rho) \;=\; & \frac{1}{\Gamma\beta} \int_0^{\rho} (\rho - v)^{\beta - 1} f(v, \jmath(v)) dv - \frac{2\rho}{(2 - \eta^2)\Gamma\beta} \int_0^{\rho} (1 - v)^{\beta - 1} f(v, \jmath(v)) dv \\ & + \frac{2\rho}{(2 - \eta^2)\Gamma\beta} \int_0^{\mathbf{J}} \left(\int_0^{v} (v - \beta)^{\beta - 1} f(u, \jmath(u)du) \right) dv \;\; (\rho \in [0, 1]); \end{aligned} $$

3. For each $\rho \in I$ *and* $\jmath, \varrho \in \mathcal{A}$ *such that* $\jmath \perp \varrho$ *or* $\varrho \perp \jmath$, $\zeta(\jmath(\rho), \varrho(\rho)) > 0$ *implies* $\zeta(\Lambda\jmath(\rho), \Lambda\varrho(\rho)) > 0$;

4. If $\{\jmath_r\}$ *is an orthogonal sequence in* $\mathcal{A}$ *such that* $\jmath_r \to \jmath$ *in* $\mathcal{A}$ *and* $\jmath_r \perp \jmath_{r+1}$ *or* $\jmath_{r+1} \perp \jmath_r$ *and* $\zeta(\jmath_r, \jmath_{r+1}) > 0$ *for all* $r \in \mathbb{N}$, *then* $\zeta(\jmath_r, \jmath) > 0$ *for all* $r \in \mathbb{N}$.

5. Λ *is orthogonal preserving mapping.*

Then, (10) has at least one solution.

Proof. Suppose $\jmath \in \mathcal{A}$ satisfies (10) if and only if it satisfies the equation

$$\begin{aligned}\jmath(\rho) \quad &= \quad \frac{1}{\Gamma\beta}\int_0^\rho (\rho-v)^{\beta-1} f(v,\jmath(v))dv - \frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^\rho (1-v)^{\beta-1} f(v,\jmath(v))dv \\ &+ \frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^1 \left(\int_0^v (v-\beta)^{\beta-1} f(u,\jmath(u)du)\right)dv\end{aligned}$$

Then problem (10) is equivalent to finding $\jmath^* \in \mathcal{A}$, which is an FP of Λ.

Now, let $\jmath, \varrho \in \mathcal{A}$ such that $\jmath \perp \varrho$ or $\varrho \perp \jmath$ and $\zeta(\jmath(\rho), \varrho(\rho)) > 0$ for all $\rho \in I$. By (1), we find

$$\begin{aligned}&\mid \Lambda\jmath(\rho) - \Lambda\varrho(\rho) \mid = \\ &\mid \frac{1}{\Gamma\beta}\int_0^\rho (\rho-v)^{\beta-1} f(v,\jmath(v))dv - \frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^\rho (1-v)^{\beta-1} f(v,\jmath(v))dv \\ &+ \frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^1 \left(\int_0^v (v-u)^{\beta-1} f(u,\jmath(u)du)\right)dv \\ &- \frac{1}{\Gamma\beta}\int_0^\rho (\rho-v)^{\beta-1} f(v,\varrho(v))dv + \frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^\rho (1-v)^{\beta-1} f(v,\varrho(v))dv \\ &- \frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^1 \left(\int_0^v (v-u)^{\beta-1} f(u,\varrho(u)du)\right)dv \mid .\end{aligned}$$

$$\begin{aligned}&\leq \quad \mid \frac{1}{\Gamma\beta}\int_0^\rho \mid \rho - v \mid^{\beta-1} \mid f(v,\jmath(v)) \\ -f(v,\varrho(v)) \quad &\mid \quad dv - \frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^\rho (1-v)^{\beta-1} \mid f(v,\jmath(v)) \\ -f(v,\varrho(v)) \quad &\mid \quad dv + \frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^1 \mid \int_0^v (v-u)^{\beta-1}(f(u,\jmath(u)du) - f(u,\varrho(u)))du \mid dv.\end{aligned}$$

$$\begin{aligned}&\leq \mid \frac{1}{\Gamma\beta}\int_0^\rho \mid \rho - v\mid^{\beta-1} \frac{\Gamma(\beta+1)}{5}\frac{\mid \varrho(v) - \jmath(v)\mid}{\tau}dv + \frac{2\rho}{(2-\eta^2)\Gamma\beta} \\ &\int_0^\rho (1-v)^{\beta-1}\frac{\Gamma(\beta+1)}{5}\frac{\mid \varrho(v) - \jmath(v)\mid}{\tau}dv + \frac{2\rho}{(2-\eta^2)\Gamma\beta} \\ &\int_0^1 \left(\int_0^v (v-u)^{\beta-1}\frac{\Gamma(\beta+1)}{5}\frac{\mid \varrho(u) - \jmath(u)\mid}{\tau}du\right)dv. \\ &\leq \frac{\Gamma(\beta+1)}{5}\frac{1}{\tau} \parallel \jmath - \varrho \parallel_{+\infty} \sup_{\rho\in(0,1)} \left(\frac{1}{\Gamma\beta}\int_0^\rho (1-v)^{\beta-1}dv\right. \\ &+ \left.\frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^\rho (1-v)^{\beta-1}dv + \frac{2\rho}{(2-\eta^2)\Gamma\beta}\int_0^1\int_0^v \mid v-u\mid^{\beta-1} dudv\right). \\ &\leq \frac{\parallel \jmath - \varrho \parallel_{+\infty}}{\tau}.\end{aligned}$$

Thus, for each $\jmath, \varrho \in \mathcal{A}$ with $\zeta(\jmath(\rho) - \varrho(\rho)) > 0$ for all $\rho \in [0,1]$, we have

$$\parallel \Lambda\jmath - \Lambda\varrho \parallel_{+\infty} \leq \frac{\parallel \jmath - \varrho \parallel_{+\infty}}{\tau}$$

$$\frac{\parallel \Lambda\jmath - \Lambda\varrho \parallel_{+\infty}}{\rho} \leq \frac{\parallel \jmath - \varrho \parallel_{+\infty}}{\rho.\tau}$$

$$-\frac{\parallel \jmath - \varrho \parallel_{+\infty}}{\rho} \leq -\tau\frac{\parallel \Lambda\jmath - \Lambda\varrho \parallel_{+\infty}}{\rho}$$

$$\begin{aligned}\log_e e^{-\frac{\|\jmath-\varrho\|_{+\infty}}{\rho}} &\leq \log_e e^{-\tau\frac{\|\Lambda\jmath-\Lambda\varrho\|_{+\infty}}{\rho}} \\ \log_e e^{-\frac{\|\jmath-\varrho\|_{+\infty}}{\rho}} &\leq \tau.\log_e e^{-\frac{\|\Lambda\jmath-\Lambda\varrho\|_{+\infty}}{\rho}}.\end{aligned}$$

Now, suppose $F : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $F(\rho) = \log_e \rho$ for each $\rho > 0$ such that $F \in F$. Thus, we obtain

$$F(\sigma(\jmath, \varrho, \rho)) \leq \tau.F(\sigma(\Lambda\jmath, \Lambda\varrho, \rho))$$

for all $\jmath, \varrho \in \mathcal{A}$ with $\sigma(\Lambda\jmath, \Lambda\varrho, \rho) > 0$. Therefore, Λ is an $\alpha_\Gamma - F$ OF contractive mapping.

Next, by assumption 3 of Theorem 5, $\alpha(\jmath, \varrho) \geq 1$ implies $\zeta(\jmath(\rho), \varrho(\rho)) > 0$, which implies $\zeta(\Lambda\jmath(\rho), \Lambda\varrho(\rho)) > 0$, which implies $\alpha(\Lambda\jmath, \Lambda\varrho) \geq 1$ for all $\jmath, \varrho \in \mathcal{A}$. Hence, condition (1) of Theorem 5 holds.

From condition (2) of Theorem 5, there exists $\jmath_0 \in \mathcal{A}$ such that $\jmath_0 \perp \Lambda\jmath_0$ $\alpha(\jmath_0, \Lambda\jmath_0) \geq 1$.

Finally, from condition 4 of Theorem 5, if $\{\jmath_r\}$ is an OS in $\mathcal{A}$ such that $\jmath_r \perp \jmath_{r+1}$ or $\jmath_{r+1} \perp \jmath_r$ $\alpha(\jmath_r, \jmath_{r+1}) \geq 1$ for all $r \in \mathbb{N}$ implies $\zeta(\jmath_r, \jmath_{r+1}) > 0$ for all $r \in \mathbb{N}$, then $\zeta(\jmath_r, \jmath) > 0$ for all $r \in \mathbb{N} \Rightarrow \alpha(\jmath_r, \jmath) \geq 1$ for all $r \in \mathbb{N}$. Therefore, condition 4 of Corollary 5 holds.

By this application, we satisfy the result of Corollary 5; moreover, we show the existence of uniqueness $\jmath^* \in \mathcal{A}$ such that $\jmath^* = \Lambda\jmath^*$ which satisfies Equation (10). □

4.2. Atangana–Baleanu Fractional Integral Operator

The fractional integral of Atangana–Baleanu-type order $\propto$ of a function $z(\rho)$ is satisfied as:

$$ {}^{AB}_{v}I^{\mathbf{ff}}_{\rho}\zeta(\rho) = \frac{1-\propto}{\beta(\propto)}\zeta(\rho) + \frac{\propto}{\beta(\propto)\Gamma(\propto)}\int_0^\rho \zeta(p)(\rho-p)^{\mathbf{ff}-1}dp; \tag{17}$$

where $0 < \propto \leq 1, 0 < \rho < v$ and it is worth mentioning that $\beta(0)$ and $\beta(1)$ are equal to 1. Let $H = \mathbb{C}(I, \mathbb{R})$ be the set of all continuous functions from I into $\mathbb{R}$. Consider

$$\jmath(\rho) = \frac{1-\propto}{\beta(\mathbf{ff})}\jmath(\rho) + \frac{\propto}{\beta(\propto)\Gamma(\propto)}\int_0^\rho \jmath(p)(\rho-p)^{\mathbf{ff}-1}dp \tag{18}$$

where $\jmath(\rho) \in \mathcal{A}$.

To find the existence and uniqueness solution (18), define $\Lambda : \mathcal{A} \to \mathcal{A}$ as

$$\Lambda\jmath(\rho) = \frac{1-\propto}{\beta(\propto)}\jmath(\rho) + \frac{\propto}{\beta(\propto)\Gamma(\propto)}\int_0^\rho \jmath(p)(\rho-p)^{\mathbf{ff}-1}dp. \tag{19}$$

Define $\varepsilon : \mathcal{A} \times \mathcal{A} \to \mathbb{R}^+$ as

$$\varepsilon((\jmath(\rho), \varrho(\rho)) = \sup_{\rho\in(0,1)} \mid \jmath(\rho) - \varrho(\rho) \mid .$$

Now we will prove that the fractional integral of the Atangana–Baleanu type has a unique solution.

$$\frac{1-\alpha}{\beta(\alpha)} + \frac{v^{\mathbf{ff}}}{\beta(\alpha)\Gamma(\alpha)} < u \text{ where } u \in (0,1)$$

Consider

$$\begin{aligned} &\mid \Lambda\jmath(\rho) - \Lambda\varrho(\rho) \mid \\ &= \mid \left(\frac{1-\alpha}{\beta(\alpha)}\jmath(\rho) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_0^\rho \jmath(p)(\rho-p)^{\mathbf{ff}-1}dp\right) \\ &\quad -\left(\frac{1-\alpha}{\beta(\alpha)}\varrho(\rho) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_0^\rho \varrho(p)(\rho-p)^{\mathbf{ff}-1}dp\right) \mid . \\ &= \mid \frac{1-\alpha}{\beta(\alpha)}(\jmath(\rho) - \varrho(\rho)) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_0^\rho (\jmath(p) - \varrho(p))(\rho-p)^{\mathbf{ff}-1}dp \mid . \end{aligned}$$

$$\leq \frac{1-\alpha}{\beta(\alpha)} \mid (\jmath(\rho) - \varrho(\rho)) \mid + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)} \int_0^\rho (\rho - p)^{\text{ff}-1} dp \mid \jmath(p) - \varrho(p) \mid .$$

$$= \left(\frac{1-\alpha}{\beta(\alpha)} + \frac{\rho^{\text{ff}}}{\beta(\alpha)\Gamma(\alpha)} \right) \mid \jmath(p) - \varrho(p) \mid .$$

$$\leq \frac{\varepsilon(\jmath, \varrho)}{\tau}.$$

Thus, for each $\jmath, \varrho \in \mathcal{A}$ with $\zeta(\jmath(\rho) - \varrho(\rho)) > 0$ for all $\rho \in I$, we have

$$\begin{aligned}
\varepsilon(\Lambda\jmath(\rho), \Lambda\varrho(\rho)) &\leq \frac{\varepsilon(\jmath, \varrho)}{\tau} \\
\frac{\varepsilon(\Lambda\jmath(\rho), \Lambda\varrho(\rho))}{\rho} &\leq \frac{\varepsilon(\jmath, \varrho)}{\tau.\rho} \\
-\frac{\varepsilon(\jmath, \varrho)}{\rho} &\leq -\tau.\frac{\varepsilon(\Lambda\jmath(\rho), \Lambda\varrho(\rho))}{\rho} \\
\log_e e^{--\frac{\varepsilon(\jmath,\varrho)}{\rho}} &\leq \log_e e^{-\tau.\frac{\varepsilon(\Lambda\jmath(\rho),\Lambda\varrho(\rho))}{\rho}} \\
\log_e e^{--\frac{\varepsilon(\jmath,\varrho)}{\rho}} &\leq \tau.\log_e e^{-\frac{\varepsilon(\Lambda\jmath(\rho),\Lambda\varrho(\rho))}{\rho}}
\end{aligned}$$

Hence, the remaining proof of application is similar to the proof of Theorem 5. Therefore, Λ is an α_Γ-**F** OF contractive mapping.

Hence, the conditions of Theorem 5 and Corollary 5 are satisfied, which yields that the fractional integral of the Atangana–Baleanu type of order $\propto$ has a unique solution.

4.3. Existence of L^p-Type Solution Pertinent to the Fredholm Integral Equation

Let $\mathcal{A} = L^p(0, +\infty), (1 < p < +\infty)$. Define $\varepsilon : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ by $\varepsilon(\jmath(\rho), \varrho(\rho)) = \parallel \jmath(\rho) - \varrho(\rho) \parallel_{L^p}$. So, it is easy to check that $\mathcal{A}$ is an α_Γ-F orthogonal fuzzy contractive mapping.

We consider the L^p solutions $(1 < p < +\infty)$ of the linear Fredholm integral equation of the second kind.

$$\jmath(\rho) = \psi(\rho) + \propto \int_0^{+\infty} G(p, \rho)\jmath(p)dp; \tag{20}$$

where $\jmath, \psi \in L^p(0, +\infty)$.

For $1 < p < +\infty, \frac{1}{p} + \frac{1}{p'} = 1$. The integral Kernal $G(p, \rho) = G(\rho, p)$ is symmetric and non-negative almost on $(0, +\infty) \times (0, +\infty)$.

Given $Z \in L^p(0, +\infty)$ and $H \in L^{p'}(0, +\infty)$, one obtains

$$\kappa Z(\rho) = \int_0^{+\infty} G(p, \rho)Z(p)dp; \rho \in (0, +\infty),$$

and

$$\kappa H(p) = \int_0^{+\infty} G(p, \rho)H(p)dp; p \in (0, +\infty),$$

for any $\varepsilon > 0$ and $p > 0$; we define

$$G_\varepsilon(r, p) = \int_0^{+\infty} G(p, \rho) \left(\frac{p}{\rho} \right)^{\frac{1+\varepsilon}{r}} d\rho; r = p \text{ or } p'$$

and

$$G_o(r, p) = \int_0^{+\infty} G(p, \rho) \left(\frac{p}{\rho} \right)^{\frac{1}{r}} d\rho; r = p \text{ or } p'$$

(c_1): If $G_0(p) = \kappa_o(r, p), \left(r = p \text{ or } p' \right)$ is independent of $p > 0$, then $\kappa : L^p(0, +\infty) \to L^p(0, +\infty)$ is a continuous linear operator and $\parallel \kappa \parallel_{L^p \to L^p} \leq \kappa_o(p)$.

(c_2): If $G_\varepsilon(p) = \kappa_\varepsilon(r, \jmath)$, $\left(r = p \text{ or } p'\right)$ is independent of $\jmath > 0$, then $\kappa_\varepsilon(p) = \kappa_o(p) + o(1)$, $(\varepsilon \to 0^+)$ is a continuous linear operator and $\| \kappa \|_{L^p \to L^p} = \kappa_o(p)$.

(c_3): Moreover, if the conditions in (c_2) are fulfilled, for any $Z \in L^p(0, +\infty)$ and $\| Z \|_{L^p} > 0$, then $\| \kappa \|_{L^p} < \| \kappa \|_{L^p \to L^p} \| Z \|_{L^p}$ holds.

Define an operator $\propto \in L^p(0, +\infty)$ as,

$$\propto \jmath(\rho) = \psi(\rho) + \propto \int_0^{+\infty} G(p, \rho)\jmath(p)dp.$$

Consider,

$$\begin{aligned}
&\| \propto \jmath(\rho) - \propto \varrho(\rho) \|_{L^p} \\
&= \| \psi(\rho) + \propto \int_0^{+\infty} G(p, \rho)\jmath(p)dp - \psi(\rho) - \propto \int_0^{+\infty} G(p, \rho)\varrho(p)dp \|_{L^p}. \\
&= \| \propto \int_0^{+\infty} G(p, \rho)(\jmath(p) - \varrho(p))dp \|_{L^p}. \\
&\leq |\propto| \, \kappa_o(p) \| \jmath(p) - \varrho(p) \|_{L^p}. \\
&= |\propto| \, \kappa_o(p)\zeta(\jmath, \varrho).
\end{aligned}$$

By assuming $|\propto| \, \kappa_o(p) < \frac{1}{\tau}$, we obtain

$$\zeta(\propto \jmath, \propto \varrho) \leq |\propto| \, \kappa_o(p)\zeta(\jmath, \varrho).$$

For the linear Fredholm integral Equation (20) if the Kernal of $G(p, \rho)$ is symmetric and almost every time $(0, +\infty) \times (0, +\infty)$ and it fulfils the conditions (c_1), (c_2) and (c_3), then for the linear Fredholm integral Equation (20) there exists a solution as long as $|\propto| \, \kappa_o(p) < \frac{1}{\tau}$

$$\begin{aligned}
\zeta(\propto \jmath, \propto \varrho) &\leq \frac{1}{\tau}\zeta(\jmath, \varrho) \\
\frac{\zeta(\propto \jmath, \propto \varrho)}{\rho} &\leq \frac{1}{\tau.\rho}\zeta(\jmath, \varrho) \\
\tau.\frac{\zeta(\propto \jmath, \propto \varrho)}{\rho} &\leq \frac{1}{\rho}\zeta(\jmath, \varrho) \\
-\frac{1}{\rho}\zeta(\jmath, \varrho) &\leq -\tau.\frac{\zeta(\propto \jmath, \propto \varrho)}{\rho} \\
\log e^{-\frac{1}{\rho}\zeta(\jmath, \varrho)} &\leq \log e.
\end{aligned}$$

Hence, the remaining proof is similar to the above application. Therefore, Λ is an α_Γ-F orthogonal fuzzy contractive mapping.

5. Conclusions

We introduced a new contractive condition called α_Γ-F orthogonal fuzzy contractive mapping on $\mathcal{A}$. We validated some fixed point results and a corollary and proved the existence and uniqueness of our results. As an example, we validated all the conditions for our main result and showed the existence and uniqueness of fixed points. We presented an application of the Atangana–Baleanu fractional integral operator and the Fredholm integral equations, in which we proved the uniqueness and existence of a solution. This work is extendable with respect to the existing literature [15–17].

Author Contributions: S.S., F.J., D.A.K. and U.I. contributed equally in this manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data will be available on demand from corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Information and control. *Fuzzy Sets* **1965**, *8*, 338–353.
2. Schweizer, B.; Sklar, A. Statistical metric spaces. *Pac. J. Math* **1960**, *10*, 313–334. [CrossRef]
3. Kramosil, I.; Michálek, J. Fuzzy metrics and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344.
4. Gregori, V.; Sapena, A. On fixed-point theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2002**, *125*, 245–252. [CrossRef]
5. Pedro, T. Contraction mappings in fuzzy quasi-metric spaces and [0, 1]-fuzzy posets. *Fixed Point Theory* **2012**, *13*, 273–283.
6. Miheţ, D. Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces. *Fuzzy Sets Syst.* **2008**, *159*, 739–744. [CrossRef]
7. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 1–6. [CrossRef]
8. Zhou, M.; Saleem, N.; Liu, X.; Fulga, A.; de Hierro, A.F.R.L. A new approach to Proinov-type fixed-point results in non-Archimedean fuzzy metric spaces. *Mathematics* **2021**, *9*, 3001. [CrossRef]
9. Hezarjaribi, M. Fixed point result in orthogonal fuzzy metric space. *Jordan J. Math. Stat.* **2018**, *11*, 295–308.
10. Panda, S.K.; Abdeljawad, T.; Ravichandran, C. Novel fixed point approach to Atangana–Baleanu fractional and Lp-Fredholm integral equations. *Alex. Eng. J.* **2020**, *59*, 1959–1970. [CrossRef]
11. Patel, U.D.; Radenović, S. An Application to Nonlinear Fractional Differential Equation via $\alpha - \mathbf{\Gamma F}$-Fuzzy Contractive Mappings in a Fuzzy Metric Space. *Mathematics* **2022**, *10*, 2831. [CrossRef]
12. Nazam, M.; Aydi, H.; Hussain, A. Existence theorems for (Ψ, Φ)-orthogonal interpolative contractions and an application to fractional differential equations. *Optimization* **2022**, *71* , 1–31. [CrossRef]
13. Ishtiaq, U.; Saleem, N.; Uddin, F.; Sessa, S.; Ahmad, K.; di Martino, F. Graphical Views of Intuitionistic Fuzzy Double-Controlled Metric-Like Spaces and Certain Fixed-Point Results with Application. *Symmetry* **2022**, *14*, 2364. [CrossRef]
14. Saleem, N.; Javed, K.; Uddin, F.; Ishtiaq, U.; Ahmed, K.; Abdeljawad, T.; Alqudah, M.A. Unique solution of integral equations via intuitionistic extended fuzzy b-metric-like spaces. *Comp. Model. Eng. Sci.* **2022**, *135*, 23. [CrossRef]
15. Humaira, H.H.A.; Sarwar, M.; De la Sen, M. Existence theorem for a unique solution to a coupled system of impulsive fractional differential equations in complex-valued fuzzy metric spaces. *Adv. Differ. Equ.* **2021**, *1*, 242. [CrossRef]
16. Hammad, H.A.; De la Sen, M. Analytical solution of Urysohn integral equations by fixed point technique in complex valued metric spaces. *Mathematics* **2019**, *7*, 852. [CrossRef]
17. Hammad, H.A.; Agarwal, P.; Momani, S.; Alsharari, F. Solving a fractional-order differential equation using rational symmetric contraction mappings. *Fractal Fract.* **2021**, *5*, 159. [CrossRef]