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Classical Solutions for the Generalized Kawahara–KdV System

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Abstract: In this article, we investigate the generalized Kawahara–KdV system. A new topological approach is applied to prove the existence of at least one classical solution and at least two non-negative classical solutions. The arguments are based upon recent theoretical results.

Keywords: fractional derivatives; generalized Kawahara–KdV system; existence; classical solution

MSC: 37C25; 47H10

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1. Introduction

In the present paper, we investigate the Cauchy problem for the generalized Kawahara–KdV system:

$$\begin{aligned} \partial_t u + \sum_{k=0}^{N_1} \sum_{l=0}^{N_1-k} \partial_x \left\{ \sum_{m=0}^{N_1-k} \partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right) \right\} + \sum_{k=1}^{N_2} a_k(t,x) \partial_x^{2k+1} u &= 0 \\ \partial_t v + \sum_{k=0}^{N_3} \sum_{l=0}^{N_3-k} \partial_x^k \left\{ \sum_{m=0}^{N_3-k} \partial_x^m v^p Q_{k,l,m}\left(\partial_x^l u\right) \right\} + \sum_{k=1}^{N_4} b_k(t,x) \partial_x^{2k+1} v &= 0, \\ t \in [0,\infty), x \in \mathbb{R}, \quad u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}, & \end{aligned} \tag{1}$$

where

Hypothesis 1. *$u_0, v_0 \in \mathcal{C}^q(\mathbb{R})$, $0 \le u_0, v_0 \le \mathcal{B}$ on $\mathbb{R}$ for $\mathcal{B} > 1$, a_j, $b_k \in \mathcal{C}([0,\infty) \times \mathbb{R})$, $0 \le |a_j|, |b_k| \le \mathcal{B}$ on $[0,\infty) \times \mathbb{R}$, $j = 1, \dots, N_2$, $k = 1, \dots, N_4$,*

$$\begin{aligned} P_{k,l,m}(z) &= \sum_{r=0}^{N_5} c_{k,l,m,r}(t,x) z^r, \\ Q_{k,l,m}(z) &= \sum_{r=0}^{N_6} d_{k,l,m,r}(t,x) z^r, \quad t \in [0,\infty), x \in \mathbb{R}, \quad z \in \mathbb{R}, \end{aligned}$$

$c_{k,l,m,j}, d_{k,l,m,r} \in \mathcal{C}([0,\infty), \mathcal{C}^q(\mathbb{R})),$

$$0 \leq \left|\partial_x^{p_1} c_{k,l,m,j}\right|, \quad \left|\partial_x^{p_1} d_{k,l,m,r}\right| \leq \mathcal{B},$$

on $[0,\infty) \times \mathbb{R}$, $j = 1,\ldots,N_5$, $r = 1,\ldots,N_6$, $p_1 = 1,\ldots,N_1$, $p, N_1, N_2, N_3, N_4, N_5, N_6 \in \mathbb{N}$,

$$q = \max\{N_1, 2N_2+1, N_3, 2N_4+1\}.$$

Kondo and Pes [1] proved the local well-posedness of this system in analytic Gevrey spaces $G^{\sigma,s}(\mathbb{R})$ with $s \geq 2N + 1/2$, $N = \max\{N_2, N_4\}$.

The range of the type of equations that this model encompasses is obviously broad and can represent many physical phenomena. As examples, we can consider the nonlinear case

$$\sum_{k=0}^{N_1} \sum_{\ell=0}^{N_1-k} \partial_x^k \left\{ \sum_{m=0}^{N_1-k} \partial_x^m w^p P_{k,\ell,m}\left(\partial_x^\ell z\right) \right\}.$$

When $N_1 = 1$, we have

$$\begin{aligned} &w^p P_{0,0,0}(z) + \partial_x w^p P_{0,0,1}(z) + w^p P_{0,1,0}(\partial_x z) + \partial_x w^p P_{0,1,1}(\partial_x z) \\ &\quad + \partial_x [w^p P_{1,0,0}(z) + \partial_x w^p P_{1,0,1}(z)]. \end{aligned} \tag{2}$$

When $N_1 = N_2 = 1$ and $p = 1$, taking $P_{0,0,0} = P_{0,1,0} = P_{0,0,1} = P_{0,1,1} = P_{1,0,1} \equiv 0$, $P_{1,0,0}(x) = x^2$, $a_1 = 1$ and taking the same choices to $Q_{k,\ell,m}$ with $N_3 = N_4 = 1$ and $b_1 = 1$, we have a coupled system of modified KdV equations (see [2,3]):

$$\begin{cases} \partial_t w + \partial_x^3 w + \partial_x\left(wz^2\right) = 0, \\ \partial_t z + \partial_x^3 z + \partial_x\left(zw^2\right) = 0, \\ w(x,0) = w_0(x), z(x,0) = z_0(x). \end{cases}$$

Considering in (2) the case when $p = q \in \mathbb{N}$ and $P_{1,0,0}(x) = x^{q+1}$, since $N_1 = N_2 = N_3 = N_4 = 1$, we obtain a more general system, treated in [4] as

$$\begin{cases} \partial_t w + \partial_x^3 w + \partial_x\left(w^q z^{q+1}\right) = 0, \\ \partial_t z + \partial_x^3 z + \partial_x\left(z^q w^{q+1}\right) = 0, \\ w(x,0) = w_0(x), z(x,0) = z_0(x). \end{cases}$$

In order to find a more general and more complicated systems, we can consider $N_1 = 2$ and $p = 1$; then, we notice that the term nonlinear is more general:

$$\begin{aligned} &wP_{0,0,0}(z) + \partial_x wP_{0,0,1}(z) + \partial_x^2 wP_{0,0,2}(z) \\ &\quad + wP_{0,1,0}(\partial_x z) + \partial_x wP_{0,1,1}(\partial_x z) + \partial_x^2 wP_{0,1,2}(\partial_x z) \\ &\quad + wP_{0,2,0}\left(\partial_x^2 z\right) + \partial_x wP_{0,2,1}\left(\partial_x^2 z\right) + \partial_x^2 wP_{0,2,2}\left(\partial_x^2 z\right) \\ &\quad + \partial_x \Big[wP_{1,0,0}(z) + \partial_x wP_{1,0,1}(z) + \partial_x^2 wP_{1,0,2}(z) \\ &\quad + wP_{1,1,0}(\partial_x z) + \partial_x wP_{1,1,1}(\partial_x z) + \partial_x^2 wP_{1,1,2}(\partial_x z)\Big] \\ &\quad + \partial_x^2 \Big[wP_{2,0,0}(z) + \partial_x wP_{2,0,1}(z) + \partial_x^2 wP_{2,0,2}(z)\Big]. \end{aligned}$$

If we change z by w, and consider again all identical null polynomials, except $P_{0,0,1}(x) = x^k$, we obtain the Kawahara system [5]

$$\partial_t u + \partial_x^3 u + \partial_x^5 u + u^k \partial_x u = 0.$$

The study of nonlinear partial differential equations (PDEs) has garnered significant attention in recent years due to their wide-ranging applications in various fields such as fluid dynamics, plasma physics, and optical communications [6–8]. In particular, fractional-order PDEs, which generalize classical PDEs by incorporating nonlocal effects, have been the subject of extensive research, including the analysis of the Kaup–Kupershmidt equation and Korteweg–De Vries (KdV)-type equations within different operators [6,7]. Additionally, the investigation of nonlinear wave phenomena in plasma and fluid systems has led to the development of analytical solutions for various nonlinear PDEs, such as the nonlinear Schrodinger equation with a detuning term [8].

Shah et al. [6] conducted a comparative analysis of the fractional-order Kaup–Kupershmidt equation using different operators, offering valuable insights into the behavior of the equation and its solutions. Similarly, Shah et al. [7] explored the analytical investigation of fractional-order KdV-type equations under the Atangana–Baleanu–Caputo operator, focusing on the modeling of nonlinear waves in plasma and fluid systems. Furthermore, Shah et al. [8] analyzed optical solitons for the nonlinear Schrodinger equation with a detuning term using the iterative transform method, which has important implications for the understanding and control of optical communication systems.

Building on these foundational studies, our research aims to further advance the understanding of nonlinear PDEs by applying a novel topological approach to the generalized Kawahara–KdV system. We seek to demonstrate the existence of classical and non-negative solutions, thus contributing to the broader knowledge of nonlinear PDEs and their applications in various scientific and engineering contexts.

Theorem 1. *We suppose that Hypothesis 1 holds. Then, the initial value problem* (1) *has at least one solution*

$$(u, v) \in \left(\mathcal{C}^1([0,\infty), \mathcal{C}^q(\mathbb{R}))\right)^2.$$

Theorem 2. *We suppose that Hypothesis 1 holds. Then, the initial value problem* (1) *has at least two non-negative solutions*

$$(u_1, v_1), (u_2, v_2) \in \left(\mathcal{C}^1([0,\infty), \mathcal{C}^q(\mathbb{R}))\right)^2.$$

We organized the paper as follows. In the second section, we introduce and state some auxiliary results related the to our system and its symmetrical problem. In the next Section 3, we prove Theorem 1 for the existence of at least one solution. In Section 4, we show the existence of at least two non-negative solutions in in Theorem 2. In Section 5, we introduce an example illustrating the main results.

2. Preliminary Results

In order to prove the existence of the solution, we shall use the following fixed-point Theorem.

Theorem 3. *Let* $0 < \epsilon > 0$, $\mathcal{B} > 0$, $\mathcal{E}$ *be a Banach space and*

$$\mathcal{W} = \{x \in \mathcal{E} : \|x\| \leq \mathcal{B}\}.$$

Let also $\mathcal{T}x = -\epsilon x$, $x \in \mathcal{W}$, $S : \mathcal{W} \to \mathcal{E}$ *be a continuous function,* $(I - S)(\mathcal{W})$ *reside in a compact subset of* $\mathcal{E}$, *and*

$$\{x \in \mathcal{E} : x = \lambda(I - S)x, \quad \|x\| = \mathcal{B}\} = \emptyset, \forall \lambda \in \left(0, \frac{1}{\epsilon}\right). \tag{3}$$

Then, there exists $x^* \in \mathcal{W}$ *such that*

$$\mathcal{T}x^* + Sx^* = 0.$$

Proof. Define

$$r\left(-\frac{1}{\epsilon}x\right) = \begin{cases} -\frac{1}{\epsilon}x, & \text{if} \quad \mathcal{B}\epsilon \geq \|x\| \\ \frac{\mathcal{B}x}{\|x\|}, & \text{if} \quad \mathcal{B}\epsilon < \|x\|. \end{cases}$$

Then,

$$r\left(-\frac{1}{\epsilon}(I-S)\right) : \mathcal{W} \to \mathcal{W}$$

is compact and continuous. Thus, owing to the Schauder fixed-point theorem, it follows that there exists $x^* \in \mathcal{W}$ such that

$$r\left(-\frac{1}{\epsilon}(I-S)x^*\right) = x^*.$$

Assume that $-\frac{1}{\epsilon}(I-S)x^* \notin \mathcal{W}$. Thus,

$$\mathcal{B}\epsilon < \left\|(I-S)x^*\right\|, \quad \mathcal{B}\|(I-S)x^*\|^{-1} < \frac{1}{\epsilon},$$

and

$$\begin{aligned} x^* &= \mathcal{B}\|(I-S)x^*\|^{-1}(I-S)x^* \\ &= r\left(-\frac{1}{\epsilon}(I-S)x^*\right). \end{aligned}$$

Then, $\|x^*\| = \mathcal{B}$ contradicts (3). Thus, $-\frac{1}{\epsilon}(I-S)x^* \in \mathcal{W}$ and

$$x^* = r\left(-\frac{1}{\epsilon}(I-S)x^*\right) = -\frac{1}{\epsilon}(I-S)x^*,$$

or

$$-\epsilon x^* + Sx^* = x^*,$$

or

$$\mathcal{T}x^* + Sx^* = x^*,$$

which completes our proof. □

Let $\mathcal{W}$ be a real Banach space.

Definition 1. *A mapping $\mathcal{K} : \mathcal{W} \to \mathcal{W}$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The definition of l-set contraction is related to the Kuratowski measure of noncompactness, which we recall for completeness.

Definition 2. *Let $\Gamma_{\mathcal{W}}$ be the class of all bounded sets of $\mathcal{W}$. The Kuratowski measure of noncompactness*

$$\alpha : \Gamma_{\mathcal{W}} \to [0, \infty)$$

is defined by

$$\alpha(\varrho) = \inf\left\{ \delta > 0 : \varrho = \bigcup_{j=1}^{m} \varrho_j \quad \textit{and} \quad \textit{diam}(\varrho_j) \leq \delta, \quad j = 1, \dots, m \right\},$$

where

$$\textit{diam}(\varrho_j) = \sup\{\|x-y\|_{\mathcal{W}} : x, y \in \varrho_j\},$$

is the diameter of ϱ_j, $j = 1, \dots, m$.

We refer the reader to [9] for the main symmetrical properties of the measure of noncompactness.

Definition 3. *A mapping $\mathcal{A} : \mathcal{W} \to \mathcal{W}$ is said to be an l-set contraction if it is continuous, bounded, and there exists a constant $0 \leq l$ such that*

$$\beta(\mathcal{A}(\mathcal{Z})) \leq l\beta(\mathcal{Z})$$

for any bounded set $\mathcal{Z} \subset \mathcal{W}$. The mapping $\mathcal{A}$ is said to be a strict set contraction if $1 > l$.

If $\mathcal{A} : \mathcal{W} \to \mathcal{W}$ is a completely continuous mapping, then $\mathcal{A}$ is 0-set contraction (see [10] (p. 264)).

Definition 4. *Let $\mathcal{W}$ and $\mathcal{Z}$ be real Banach spaces. A mapping $\mathcal{A} : \mathcal{W} \to \mathcal{Z}$ is said to be expansive if there exists a constant $\alpha > 1$ such that*

$$\|\mathcal{A}x - \mathcal{A}z\|_{\mathcal{Z}} \geq \alpha\|x - z\|_{\mathcal{W}}, \forall x, z \in \mathcal{W}.$$

Definition 5. *A closed, convex set ϖ in ϱ is said to be cone if*

1. *$\beta y \in \varpi$ for any $\beta \geq 0$ and for any $y \in \varpi$;*
2. *$y, -y \in \varpi$ implies $y = 0$.*

Let us denote $\varpi^* = \varpi \backslash \{0\}$.

Lemma 1. *Let ϱ be a convex closed subset of a Banach space $\mathcal{E}$ and $\mathcal{X} \subset \varrho$ be a bounded open subset where $0 \in \mathcal{X}$. For small enough values of $\varepsilon > 0$, let $\mathcal{A} : \overline{\mathcal{X}} \to \varrho$ be a strict k-set contraction that satisfies*

$$\mathcal{A}y \notin \{y,\ \lambda y\}, \forall y \in \partial\mathcal{X}, \lambda \geq 1 + \varepsilon.$$

Thus, $i\,(\mathcal{A}, \mathcal{X}, \varrho) = 1$.

Proof. Let the homotopic deformation be

$$\mathcal{H} : [0, 1] \times \overline{\mathcal{X}} \to \varrho,$$

defined by

$$\mathcal{H}(t, y) = \frac{1}{\varepsilon + 1}\, t\mathcal{A}y.$$

For each y, the operator $\mathcal{H}$ is continuous and uniformly continuous in t, where $\mathcal{H}(t, .)$ is a strict set contraction for each $t \in [0, 1]$. Notice that $\mathcal{H}(t, .)$ has no fixed point on $\partial\mathcal{X}$. On the contrary,

- If $t = 0$, $\exists y_0 \in \partial\mathcal{X}$ such that $y_0 = 0$, contradicting $y_0 \in \mathcal{X}$.
- If $t \in (0, 1]$, $\exists y_0 \in \mathcal{P} \cap \partial\mathcal{X}$ such that $\frac{1}{\varepsilon+1}\, t\mathcal{A}y_0 = y_0$; then, $\mathcal{A}y_0 = \frac{1+\varepsilon}{t} y_0$ with $\frac{1+\varepsilon}{t} \geq 1 + \varepsilon$, contradicting the assumption. From the invariance under homotopy and the normalization symmetrical properties of the index, we deduce

$$i\,(\frac{1}{\varepsilon + 1}\,\mathcal{A}, \mathcal{X}, \varrho) = i\,(0, \mathcal{X}, \varrho) = 1.$$

We show that

$$i\,(\mathcal{A}, \mathcal{X}, \varrho) = i\,(\frac{1}{\varepsilon + 1}\,\mathcal{A}, \mathcal{X}, \varrho).$$

Then,

$$\frac{1}{\varepsilon + 1}\mathcal{A}y \neq y,\ \forall\, y \in \partial\mathcal{X}. \tag{4}$$

Thus, $\exists\gamma > 0$ so that

$$\|y - \frac{1}{\varepsilon+1}\,\mathcal{A}y\| \geq \gamma,\ \forall\, y \in \partial\mathcal{X}.$$

We have $\frac{1}{\varepsilon+1}\mathcal{A}y \to \mathcal{A}y$ as $\epsilon \to 0$, for $x \in \overline{\mathcal{X}}$.
So, for small enough ε,

$$\|\mathcal{A}y - \frac{1}{\varepsilon+1}\,\mathcal{A}y\| < \frac{\gamma}{2},\ \forall\, y \in \partial\mathcal{X}.$$

Let us define the convex deformation $F : [0,1] \times \overline{\mathcal{X}} \to \varrho$ by

$$F(t,y) = t\mathcal{A}y + (1-t)\frac{1}{\varepsilon+1}\mathcal{A}y.$$

For all x, F is continuous, and uniformly continuous in t. The mapping $F(t,.)$ is a strict set contraction $\forall t \in [0,1]$. We mention that $F(t,.)$ has no fixed point on $\partial\mathcal{X}$. We have $\forall x \in \partial\mathcal{X}$, and thus we have

$$\begin{array}{rcl} \|y - F(t,y)\| & = & \|y - t\mathcal{A}y - (1-t)\frac{1}{\varepsilon+1}\mathcal{A}y\| \\ & \geq & \|y - \frac{1}{\varepsilon+1}\mathcal{A}y\| - t\|\mathcal{A}y - \frac{1}{\varepsilon+1}\mathcal{A}y\| \\ & > & \gamma - \frac{\gamma}{2} > \frac{\gamma}{2}, \end{array}$$

According to the invariance properties, the homotopy of the index ensures the claim. □

3. Proof of Theorem 1

Let $\mathcal{W}_1 = \mathcal{C}^1([0,\infty), \mathcal{C}^q(\mathbb{R}))$ be a space endowed with

$$\begin{array}{rcl} \|u\|_2 & = & \max\{ \sup\limits_{t\in[0,\infty),x\in\mathbb{R}} |u|, \quad \sup\limits_{t\in[0,\infty),x\in\mathbb{R}} |\partial_t u|, \\ & & \sup\limits_{t\in[0,\infty),x\in\mathbb{R}} |\partial_x^j u|, \quad j \in \{1,\ldots,q\}\}, \end{array}$$

provided it exists. Define $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_1$ with

$$\|(u,v)\| = \max\{\|u\|_2, \quad \|v\|_2\}.$$

We define for $(u,v) \in \mathcal{W}$

$$\begin{array}{rcl} Q_1(u,v) & = & \sum\limits_{k=0}^{N_1}\sum\limits_{l=0}^{N_1-k} \partial_x \left\{ \sum\limits_{m=0}^{N_1-k} \partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right) \right\}, \\ Q_2(u,v) & = & \sum\limits_{k=0}^{N_3}\sum\limits_{l=0}^{N_3-k} \partial_x^k \left\{ \sum\limits_{m=0}^{N_3-k} \partial_x^m v^p Q_{k,l,m}\left(\partial_x^l u\right) \right\}, \quad t \in [0,\infty), x \in \mathbb{R}. \end{array}$$

Then, the IVP (1) can be rewritten as

$$\begin{array}{rcl} \partial_t u + Q_1(u,v) + \sum\limits_{k=1}^{N_2} a_k(t,x) + \partial_x^{2k+1} u & = & 0, \\ \partial_t v + Q_2(u,v) + \sum\limits_{k=1}^{N_4} b_k(t,x)\partial_x^{2k+1} v & = & 0, \quad t \in [0,\infty), x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) & = & v_0(x), \quad x \in \mathbb{R}. \end{array} \tag{5}$$

Let

$$C_1 = \Bigg\{ \sum_{k=0}^{N_1} \sum_{l=0}^{N_1-k} \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} ((k-r+m)!)^2 p! \mathcal{B}^p \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} (i!)^2 j! \mathcal{B}^{j+1},$$

$$\sum_{k=0}^{N_3} \sum_{l=0}^{N_3-k} \sum_{m=0}^{N_3-k} \sum_{r=0}^{k} \binom{k}{r} ((k-r+m)!)^2 p! \mathcal{B}^p \sum_{j=0}^{N_6} \sum_{i=0}^{r} \binom{r}{i} (i!)^2 j! \mathcal{B}^{j+1} \Bigg\}.$$

Lemma 2. *Suppose* $((Hyp1)$. *If* $(u,v) \in \mathcal{W}$ *and* $\|(u,v)\| \le \mathcal{B}$, *then*

$$|Q_1(u,v)|, \quad |Q_2(u,v)| \le C_1, \quad t \in [0,\infty), x \in \mathbb{R}.$$

Proof. We have

$$\begin{aligned}
&\partial_x \left\{ \sum_{m=0}^{N_1-k} \partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right) \right\} \\
&= \sum_{m=0}^{N_1-k} \partial_x^k \left(\partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right)\right) \\
&= \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} \partial_x^{k-r+m} u^p \partial_x^r P_{k,l,m}\left(\partial_x^l v\right) \\
&= \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} \partial_x^{k-r+m} u^p \partial_x^r \sum_{j=0}^{N_5} c_{k,l,m,j}\left(\partial_x^l v\right)^j \\
&= \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} \partial_x^{k-r+m} u^p \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} \partial_x^{r-i} c_{k,l,m,j} \partial_x \left(\partial_x^l v\right)^j,
\end{aligned}$$

$k \in \mathbb{N}$, $0 \le k \le N_1$. Since $\mathcal{B} > 1$, we have

$$\left|\partial_x^{r_1} u^{r_2}\right| \le (r_1!)^2 r_2! \mathcal{B}^{r_2},$$

for any $r_1, r_2 \in \mathbb{N}$, $r_1 \le q$. Then,

$$\begin{aligned}
&\left| \partial_x \left\{ \sum_{m=0}^{N_1-k} \partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right) \right\} \right| \\
&\le \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} \left|\partial_x^{k-r+m} u^p\right| \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} \left|\partial_x^{r-i} c_{k,l,m,j}\right| \left|\partial_x^i \left(\partial_x^l v\right)^j\right| \\
&\le \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} ((k-r+m)!)^2 p! \mathcal{B}^p \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} (i!)^2 j! \mathcal{B}^{j+1},
\end{aligned}$$

on $[0,\infty) \times \mathbb{R}$, $0 \le k \le N_1$, and then

$$\begin{aligned}
|Q_1(u,v)| &\le \sum_{k=0}^{N_1} \sum_{l=0}^{N_1-k} \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} ((k-r+m)!)^2 p! B^p \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} (i!)^2 j! \mathcal{B}^{j+1} \\
&\le C_1,
\end{aligned}$$

on $[0,\infty) \times \mathbb{R}$. As above,

$$|Q_2(u,v)| \le C_1,$$

on $[0,\infty) \times \mathbb{R}$. The proof is now completed. □

For $(u,v) \in \mathcal{W}$, we define the operators

$$\begin{aligned} S_1^1(u,v) &= u - u_0(x) \\ &\quad + \int_0^t \left(Q_1(u,v)(s,x) + \sum_{k=1}^{N_2} a_k(s,x)\partial_x^{2k+1} u(s,x) \right) ds, \\ S_1^2(u,v) &= v - v_0(x) \\ &\quad + \int_0^t \left(Q_2(u,v)(s,x) + \sum_{k=1}^{N_4} b_k(s,x)\partial_x^{2k+1} v(s,x) \right) ds, \\ S(u,v) &= \left(S_1^1(u,v), S_1^2(u,v) \right), \end{aligned}$$

$t \in [0,\infty)$, $x \in \mathbb{R}$.

Lemma 3. *Suppose ((Hyp1). If $(u,v) \in \mathcal{W}$ satisfies*

$$S_1(u,v) = 0, \quad t \in [0,\infty), x \in \mathbb{R},$$

then (u,v) is a solution to (1).

Proof. We have

$$\begin{aligned} 0 &= u - u_0(x) \\ &\quad + \int_0^t \left(Q_1(u,v)(s,x) + \sum_{k=1}^{N_2} a_k(s,x)\partial_x^{2k+1} u(s,x) \right) ds, \\ 0 &= v - v_0(x) \\ &\quad + \int_0^t \left(Q_2(u,v)(s,x) + \sum_{k=1}^{N_4} b_k(s,x)\partial_x^{2k+1} v(s,x) \right) ds, \end{aligned} \tag{6}$$

$t \in [0,\infty)$, $x \in \mathbb{R}$, where we differentiate with respect to t to have (5). Let $t = 0$ in (6). We thus obtain

$$\begin{aligned} 0 &= u(0,x) - u_0(x) \\ 0 &= v(0,x) - v_0(x), \quad x \in \mathbb{R}. \end{aligned}$$

Thus, (u,v) is a solution to (1). The proof is now completed. □

Let

$$\mathcal{B}_1 = \max\{2\mathcal{B}, C_1 + N_2\mathcal{B}^2, C_1 + N_4\mathcal{B}^2\}.$$

Lemma 4. *Suppose ((Hyp1). If $(u,v) \in \mathcal{W}$ and $\|(u,v)\| \leq \mathcal{B}$; then,*

$$\begin{aligned} |S_1^1(u,v)| &\leq \mathcal{B}_1(1+t), \\ |S_1^2(u,v)| &\leq \mathcal{B}_1(1+t), \quad t \in [0,\infty), x \in \mathbb{R}. \end{aligned}$$

Proof. We have

$$
\begin{aligned}
|S_1^1(u,v)| &= \Bigg| u - u_0(x) \\
&\quad + \int_0^t \left(Q_1(u,v)(s,x) + \sum_{k=1}^{N_2} a_k(s,x)\partial_x^{2k+1} u(s,x) \right) ds \Bigg| \\
&\le |u| + |u_0(x)| \\
&\quad + \int_0^t \left(|Q_1(u,v)(s,x)| + \sum_{k=1}^{N_2} |a_k(s,x)||\partial_x^{2k+1} u(s,x)| \right) ds \\
&\le 2\mathcal{B} + \int_0^t (C_1 + N_2\mathcal{B}^2) ds \\
&\le \mathcal{B}_1(1+t), \quad t \in [0,\infty), x \in \mathbb{R}.
\end{aligned}
$$

As above,

$$
|S_1^2(u,v)| \le \mathcal{B}_1(1+t), \quad t \in [0,\infty), x \in \mathbb{R},
$$

which completes the proof. □

Let

Hypothesis 2. *There exists a function $g \in \mathcal{C}([0,\infty) \times \mathbb{R})$, $g > 0$ on $(0,\infty) \times (\mathbb{R} \backslash \{0\})$, $g(0,x) = g(t,0) = 0$, $t \in [0,\infty), x \in \mathbb{R}$, and $\mathcal{A} > 0$ such that*

$$
q! \cdot 2^{q+1}(1+t+t^2)(1+|x|+\cdots+|x|^q) \int_0^t \left| \int_0^x g(t_2,x_2) dx_2 \right| dt_2 \le \mathcal{A},
$$

$t \in [0,\infty), x \in \mathbb{R}$.

We will give some examples for g and $\mathcal{A}$ that satisfy Hypothesis 2. For $(u,v) \in \mathcal{W}$, define the operators

$$
\begin{aligned}
S_2^1(u,v) &= \int_0^t \int_0^x (t-t_2)(x-x_2)^q g(t_2,x_2) S_1^1(u,v)(t_2,x_2) dx_2 dt_2, \\
S_2^2(u,v) &= \int_0^t \int_0^x (t-t_2)(x-x_2)^q g(t_2,x_2) S_1^2(u,v)(t_2,x_2) dx_2 dt_2, \\
S_2(u,v) &= (S_2^1(u,v), S_2^2(u,v)), \quad t \in [0,\infty), x \in \mathbb{R}.
\end{aligned}
$$

Lemma 5. *Suppose Hypothesis 1 and Hypothesis 2. If $(u,v) \in \mathcal{W}$ satisfies*

$$
S_2(u,v) = 0, \quad t \in [0,\infty), x \in \mathbb{R},
$$

then (u,v) is a solution to (1).

Proof. Differentiating the Equation (5) two times in t and $q+1$ times in x, we have

$$
g(t,x) S_1^1(u,v) = g(t,x) S_1^2(u,v) = 0, \quad t \in [0,\infty), x \in \mathbb{R}.
$$

Hence,

$$
S_1^1(u,v) = S_1^2(u,v) = 0, \quad t, \in (0,\infty), x \in (\mathbb{R} \backslash \{0\}).
$$

Since $S_1^1(u,v)(\cdot,\cdot)$ and $S_1^2(u,v)(\cdot,\cdot)$ are continuous functions on $[0,\infty)\times\mathbb{R}$, we have

$$\begin{aligned}
0 &= S_1^1(u,v)(0,x) = S_1^2(u,v)(0,x) \\
&= \lim_{t\to 0} S_1^1(u,v) = \lim_{t\to 0} S_1^2(u,v) \\
&= \lim_{x\to 0} S_1^1(u,v) = \lim_{x\to 0} S_1^2(u,v) \\
&= S_1^1(u,v)(t,0) = S_1^2(u,v)(t,0), \quad t\in[0,\infty), x\in\mathbb{R}.
\end{aligned}$$

Therefore,

$$S_1^1(u,v) = S_1^2(u,v) = 0, \quad t\in[0,\infty), x\in\mathbb{R}.$$

Using Lemma 3, we obtain the main result. □

Lemma 6. *Suppose Hypothesis 1 and Hypothesis 2. If $(u,v)\in\mathcal{W}$, $\|(u,v)\|\le\mathcal{B}$, then*

$$\|S_2(u,v)\| \le \mathcal{A}\mathcal{B}_1.$$

Proof. The inequality $(z+w)^r \le 2^r(z^r+w^r)$, $w,z,q\ge 0$ will be used. We have

$$\begin{aligned}
|S_2^1(u,v)| &= \left|\int_0^t\int_0^x (t-t_2)(x-x_2)^q g(t_2,x_2) S_1^1(u,v)(t_2,x_2)dx_2dt_2\right| \\
&\le \int_0^t\left|\int_0^x (t-t_2)|x-x_2|^q g(t_2,x_2)|S_1^1(u,v)(t_2,x_2)|dx_2\right|dt_2 \\
&\le \mathcal{B}_1\int_0^t\left|\int_0^x (t-t_2)(1+t_2)|x-x_2|^q g(t_2,x_2)dx_2\right|dt_2 \\
&\le \mathcal{B}_1 t(1+t)2^{q+1}|x|^q\int_0^t\left|\int_0^x g(t_2,x_2)dx_2\right|dt_2 \\
&\le \mathcal{A}\mathcal{B}_1, \quad t\in[0,\infty), x\in\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
|\partial_t S_2^1(u,v)| &= \left|\int_0^t\int_0^x (x-x_2)^q g(t_2,x_2) S_1^1(u,v)(t_2,x_2)dx_2dt_2\right| \\
&\le \int_0^t\left|\int_0^x |x-x_2|^q g(t_2,x_2)|S_1^1(u,v)(t_2,x_2)|dx_2\right|dt_2 \\
&\le \mathcal{B}_1\int_0^t\left|\int_0^x (1+t_2)|x-x_2|^q g(t_2,x_2)dx_2\right|dt_2 \\
&\le \mathcal{B}_1(1+t)2^{q+1}|x|^q\int_0^t\left|\int_0^x g(t_2,x_2)dx_2\right|dt_2 \\
&\le \mathcal{A}\mathcal{B}_1, \quad t\in[0,\infty), x\in\mathbb{R},
\end{aligned}$$

and

$$|\partial_x S_2^1(u,v)| = q\left|\int_0^t\int_0^x (t-t_2)(x-x_2)^{q-1} g(t_2,x_2) S_1^1(u,v)(t_2,x_2)dx_2dt_2\right|$$

$$
\begin{aligned}
&\le q\int_0^t \left|\int_0^x (t-t_2)|x-x_2|^{q-1}g(t_2,x_2)|S_1^1(u,v)(t_2,x_2)|dx_2\right|dt_2 \\
&\le q\mathcal{B}_1\int_0^t \left|\int_0^x (t-t_2)(1+t_2)|x-x_2|^{q-1}g(t_2,x_2)dx_2\right|dt_2 \\
&\le q\mathcal{B}_1 t(1+t)2^q|x|^{q-1}\int_0^t \left|\int_0^x g(t_2,x_2)dx_2\right|dt_2 \\
&\le \mathcal{A}\mathcal{B}_1, \quad t\in[0,\infty), x\in\mathbb{R},
\end{aligned}
$$

and so on. As above,

$$|S_2^2(u,v)| \le \mathcal{A}\mathcal{B}_1, \quad |\partial_t S_2^2(u,v)| \le \mathcal{A}\mathcal{B}_1,$$

$$|\partial_x^j S_2^2(u,v)| \le \mathcal{A}\mathcal{B}_1, \quad j\in\{1,\ldots,q\}.$$

$t\in[0,\infty), x\in\mathbb{R}$. Thus,

$$\|S_2(u,v)\| \le \mathcal{A}\mathcal{B}_1,$$

which completes our proof. □

Suppose

Hypothesis 3. *Let $\epsilon\in(0,1)$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}_1$ satisfy $\epsilon\mathcal{B}_1(1+\mathcal{A})<1$ and $\mathcal{B}>\mathcal{A}\mathcal{B}_1$.*

Let $\tilde{\tilde{\tilde{\varrho}}}$ denote the set of all equi-continuous in $\mathcal{W}$ with respect to the norm $\|\cdot\|$. Also let $\tilde{\tilde{\varrho}} = \overline{\tilde{\tilde{\tilde{\varrho}}}}$ be the closure of $\tilde{\tilde{\varrho}}$, where

$$\tilde{\varrho} = \tilde{\tilde{\varrho}} \cup \{(u_0,v_0)\},$$

and

$$\varrho = \{(u,v)\in\tilde{\varrho} : (u,v)\ge 0, \quad \|(u,v)\|\le\mathcal{B}\}.$$

Note that ϱ is a compact set in $\mathcal{W}$. For $(u,v)\in\mathcal{W}$, we define

$$
\begin{aligned}
\mathcal{T}(u,v) &= -\epsilon(u,v), \\
S(u,v) &= (u,v)+\epsilon(u,v)+\epsilon S_2(u,v), \quad t\in[0,\infty), x\in\mathbb{R}.
\end{aligned}
$$

Owing to the Lemma 5, we have f $(u,v)\in\varrho$

$$
\begin{aligned}
\|(I-S)(u,v)\| &= \|\epsilon(u,v)-\epsilon S_2(u,v)\| \\
&\le \epsilon\|(u,v)\|+\epsilon\|S_2(u,v)\| \\
&\le \epsilon\mathcal{B}_1+\epsilon\mathcal{A}\mathcal{B}_1 \\
&= \epsilon\mathcal{B}_1(1+\mathcal{A}) \\
&< \mathcal{B}.
\end{aligned}
$$

Thus, $S : \varrho \to \mathcal{W}$ is continuous, and $(I - S)(\varrho)$ resides in a compact subset of $\mathcal{W}$. One can suppose that $\exists (u, v) \in \mathcal{W}$ such that $\|(u, v)\| = \mathcal{B}$ and

$$(u, v) = \lambda(I - S)(u, v),$$

or

$$\frac{1}{\lambda}(u, v) = (I - S)(u, v) = -\epsilon(u, v) - \epsilon S_2(u, v),$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right)(u, v) = -\epsilon S_2(u, v),$$

for $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, $\|S_2(u, v)\| \leq \mathcal{A}\mathcal{B}_1 < \mathcal{B}$,

$$\epsilon\mathcal{B} < \left(\frac{1}{\lambda} + \epsilon\right)\mathcal{B} = \left(\frac{1}{\lambda} + \epsilon\right)\|(u, v)\| = \epsilon\|S_2(u, v)\| < \epsilon\mathcal{B}.$$

This is a contradiction. By Theorem 3, we see that $\mathcal{T} + S$ has a fixed point $(u^*, v^*) \in \varrho$. Then,

$$\begin{aligned} (u^*, v^*) &= \mathcal{T}(u^*, v^*) + S(u^*, v^*) \\ &= -\epsilon(u^*, v^*) + (u^*, v^*) + \epsilon(u^*, v^*) + \epsilon S_2(u^*, v^*), \end{aligned}$$

$t \in [0, \infty), x \in \mathbb{R}$, whereupon

$$0 = S_2(u^*, v^*), \quad t \in [0, \infty), x \in \mathbb{R}.$$

Owing to the Lemma 5, we have (u^*, v^*) as a solution to (1), which completes the proof.

4. Proof of Theorem 2

Let $\mathcal{W}$ be the space used in the previous section (see [11]).

Hypothesis 4. *Let* $0 < m$ *be large enough and* $r, \mathcal{A}, \mathcal{B}, L, R_1 > 0$ *satisfy*

$$\mathcal{B} \geq R_1 > r, \quad 0 < \epsilon, \quad R_1 > \left(\frac{2}{5m} + 1\right)L,$$

$$\mathcal{A}\mathcal{B}_1 < \frac{L}{5}.$$

Define

$$\widetilde{P} = \{(u, v) \in \mathcal{W} : 0 \leq (u, v) \quad \text{on} \quad [0, \infty) \times \mathbb{R}\}.$$

We denote by ϖ the set of all equi-continuous families in $\widetilde{P}$. For $(u, v) \in \mathcal{W}$, define

$$\begin{aligned} \mathcal{T}_1(u, v) &= (1 + m\epsilon)(u, v) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \\ S_3(u, v) &= -\epsilon S_2(u, v) - m\epsilon(u, v) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \end{aligned}$$

$t \in [0, \infty)$. We have any fixed point $(u, v) \in \mathcal{W}$ of the operator $\mathcal{T}_1 + S_3$ is a solution to (1). Define

$$\mathcal{X}_1 = \varpi_r = \{(u, v) \in \varpi : \|(u, v)\| < r\},$$

$$\begin{aligned}\mathcal{X}_2 &= \varpi_L = \{(u,v) \in \varpi : \|(u,v)\| < L\},\\ \mathcal{X}_3 &= \varpi_{R_1} = \{(u,v) \in \varpi : \|(u,v)\| < R_1\},\\ R_2 &= R_1 + \frac{\mathcal{A}}{m}\mathcal{B}_1 + \frac{L}{5m},\\ \Gamma &= \overline{\varpi_{R_2}} = \{(u,v) \in \varpi : \|(u,v)\| \le R_2\}.\end{aligned}$$

1. For $(u_1,v_1),(u_2,v_2) \in \Gamma$, we have

$$\|\mathcal{T}_1(u_1,v_1) - \mathcal{T}_1(u_2,v_2)\| = (1+m\varepsilon)\|(u_1,v_1) - (u_2,v_2)\|,$$

where $\mathcal{T}_1 : \Gamma \to \mathcal{W}$ is an expansive operator with a constant $1 < h = 1 + m\varepsilon$.

2. For $(u,v) \in \overline{\varpi}_{R_1}$, we have

$$\begin{aligned}\|S_3(u,v)\| &\le \varepsilon\|S_2(u,v)\| + m\varepsilon\|(u,v)\| + \varepsilon\frac{L}{10}\\ &\le \varepsilon\left(\mathcal{A}\mathcal{B}_1 + mR_1 + \frac{L}{10}\right).\end{aligned}$$

Then, $S_3(\overline{\varpi}_{R_1})$ is uniformly bounded. As $S_3 : \overline{\varpi}_{R_1} \to \mathcal{W}$ is continuous, we note that $S_3(\overline{\varpi}_{R_1})$ is equi-continuous. Then, $S_3 : \overline{\varpi}_{R_1} \to \mathcal{W}$ is a 0-set contraction.

3. Let $(u_1,v_1) \in \overline{\varpi}_{R_1}$. Set

$$(u_2,v_2) = (u_1,v_1) + \frac{1}{m}S_2(u_1,v_1) + \left(\frac{L}{5m}, \frac{L}{5m}\right).$$

We have $0 \le S_2u_1 + \frac{L}{5}$, $0 \le S_2v_1 + \frac{L}{5}$ on $[0,\infty) \times \mathbb{R}$. We have $0 \le u_2, v_2$ on $[0,\infty) \times \mathbb{R}$ and

$$\begin{aligned}\|(u_2,v_2)\| &\le \|(u_1,v_1)\| + \frac{1}{m}\|S_2(u_1,v_1)\| + \frac{L}{5m}\\ &\le R_1 + \frac{\mathcal{A}}{m}\mathcal{B}_1 + \frac{L}{5m}\\ &= R_2.\end{aligned}$$

Then, $(u_2,v_2) \in \Gamma$ and

$$-\varepsilon m(u_2,v_2) = -\varepsilon m(u_1,v_1) - \varepsilon S_2(u_1,v_1) - \varepsilon\left(\frac{L}{10}, \frac{L}{10}\right) - \varepsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$\begin{aligned}(I - \mathcal{T}_1)(u_2,v_2) &= -\varepsilon m(u_2,v_2) + \varepsilon\left(\frac{L}{10}, \frac{L}{10}\right)\\ &= S_3(u_1,v_1).\end{aligned}$$

Thus, $S_3(\overline{\varpi}_{R_1}) \subset (I - \mathcal{T}_1)(\Gamma)$.

4. $\forall (u_0,v_0) \in \varpi^*$, $\exists 0 \le \lambda$ and $(u,v) \in \partial\varpi_r \cap (\Gamma + \lambda(u_0,v_0))$ or $v \in \partial\varpi_{R_1} \cap (\Gamma + \lambda(u_0,v_0))$ so that

$$S_3(u,v) = (I - \mathcal{T}_1)((u,v) - \lambda(u_0,v_0)).$$

Thus,

$$-\epsilon S_2(u,v) - m\epsilon(u,v) - \epsilon\left(\frac{L}{10},\frac{L}{10}\right) = -m\epsilon((u,v) - \lambda(u_0,v_0)) + \epsilon\left(\frac{L}{10},\frac{L}{10}\right),$$

or

$$-S_2(u,v) = \lambda m(u_0,v_0) + \left(\frac{L}{5},\frac{L}{5}\right).$$

Hence,

$$\|S_2 v\| = \left\|\lambda m(u_0,v_0) + \left(\frac{L}{5},\frac{L}{5}\right)\right\| > \frac{L}{5}.$$

This contradicts our claim.

5. $\forall \epsilon_1 \geq 0$ small enough $\exists (u_1,v_1) \in \partial\varpi_L$ and $\lambda_1 \geq 1+\epsilon_1$ so that $\lambda_1(u_1,v_1) \in \overline{\varpi}_{R_1}$ and

$$S_3(u_1,v_1) = (I - \mathcal{T}_1)(\lambda_1(u_1,v_1)). \tag{7}$$

In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $(u_1,v_1) \in \partial\varpi_L$, $\lambda_1(u_1,v_1) \in \overline{\varpi}_{R_1}$, $\lambda_1 \geq 1+\epsilon_1$ and (7) holds. Since $(u_1,v_1) \in \partial\varpi_L$ and $\lambda_1(u_1,v_1) \in \overline{\varpi}_{R_1}$, then

$$\left(\frac{2}{5m}+1\right)L < \lambda_1 L = \lambda_1\|(u_1,v_1)\| \leq R_1.$$

Moreover,

$$-\epsilon S_2(u_1,v_1) - m\epsilon(u_1,v_1) - \epsilon\left(\frac{L}{10},\frac{L}{10}\right) = -\lambda_1 m\epsilon(u_1,v_1) + \epsilon\left(\frac{L}{10},\frac{L}{10}\right),$$

or

$$S_2(u_1,v_1) + \left(\frac{L}{5},\frac{L}{5}\right) = (\lambda_1 - 1)m(u_1,v_1).$$

Then,

$$2\frac{L}{5} \geq \left\|S_2(u_1,v_1) + \left(\frac{L}{5},\frac{L}{5}\right)\right\| = (\lambda_1-1)m\|(u_1,v_1)\| = (\lambda_1-1)mL,$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which contradicts out claim.

Then, conditions of Theorem 2 hold, and (1) has at least two solutions (u_1,v_1) and (u_2,v_2) so that

$$\|(u_1,v_1)\| = L < \|(u_2,v_2)\| < R_1,$$

or

$$r < \|(u_1,v_1)\| < L < \|(u_2,v_2)\| < R_1.$$

5. Example

Let $\mathcal{B} = 1$ and

$$R_1 = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad \mathcal{A} = \frac{1}{10\mathcal{B}_1}, \quad \epsilon = \frac{1}{5\mathcal{B}_1(1+\mathcal{A})},$$

$N_j = 5$, $j \in \{1,\ldots,4\}$, $p = 10$. Then,

$$\mathcal{A}\mathcal{B}_1 = \frac{1}{10} < \mathcal{B}, \quad \epsilon\mathcal{B}_1(1+\mathcal{A}) < 1,$$

i.e., (Hypothesis 3) holds. Next,

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right)L, \quad \mathcal{AB}_1 < \frac{L}{5},$$

i.e., (Hypothesis 4) holds. Take

$$h(s) = \log \frac{1 + s^{q+1}\sqrt{2} + s^{2q+2}}{1 - s^{q+1}\sqrt{2} + s^{2q+2}}, \quad l(s) = \arctan \frac{s^{q+1}\sqrt{2}}{1 - s^{2q+2}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then,

$$\begin{aligned} h'(s) &= \frac{2\sqrt{2}(q+1)s^q(1 - s^{2q+2})}{(1 - s^{q+1}\sqrt{2} + s^{2q+2})(1 - s^{q+1}\sqrt{2} + s^{2q+2})}, \\ l'(s) &= \frac{(q+1)\sqrt{2}s^q(1 + s^{2q+2})}{1 + s^{4q+4}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{s\to\pm\infty} \sum_{r=0}^{q+1} s^r h(s) = \lim_{s\to\pm\infty} \frac{h(s)}{\frac{1}{\sum_{r=0}^{l+1} s^r}} \\ &= \lim_{s\to\pm\infty} \frac{h'(s)}{-\frac{\sum_{r=0}^{q}(r+1)s^r}{\left(\sum_{r=0}^{q+1} s^r\right)^2}} \\ &= -\lim_{s\to\pm\infty} \frac{2\sqrt{2}(q+1)s^q(1 - s^{2q+2})\left(\sum_{r=0}^{q+1} s^r\right)^2}{\left(\sum_{r=0}^{q}(r+1)s^r\right)(1 - s^{q+1}\sqrt{2} + s^{2q+2})(1 - s^{q+1}\sqrt{2} + s^{2q+2})} \\ &\neq \pm\infty, \end{aligned}$$

and

$$\begin{aligned} &\lim_{s\to\pm\infty} \sum_{r=0}^{q+1} s^r l(s) = \lim_{s\to\pm\infty} \frac{l(s)}{\frac{1}{\sum_{r=0}^{q+1} s^r}} \\ &= \lim_{s\to\pm\infty} \frac{l'(s)}{-\frac{\sum_{r=0}^{q}(r+1)s^r}{\left(\sum_{r=0}^{q+1} s^r\right)^2}} \\ &= -\lim_{s\to\pm\infty} \frac{(q+1)\sqrt{2}s^q(1 + s^{2q+2})\left(\sum_{r=0}^{q+1} s^r\right)^2}{\left(1 + s^{4q+4}\right)\left(\sum_{r=0}^{q}(r+1)s^r\right)} \\ &\neq \pm\infty. \end{aligned}$$

Consequently,

$$\begin{aligned} -\infty &< \lim_{s\to\pm\infty} \left(\sum_{r=0}^{q+1} s^r\right) h(s) < \infty, \\ -\infty &< \lim_{s\to\pm\infty} \left(\sum_{r=0}^{q+1} s^r\right) l(s) < \infty. \end{aligned}$$

Hence, there exists $C_2 > 0$ such that

$$\sum_{r=0}^{q+1} |s|^r \left(\frac{1}{(4q+4)\sqrt{2}} \log \frac{1+s^{q+1}\sqrt{2}+s^{2q+2}}{1-s^{q+1}\sqrt{2}+s^{2q+2}} + \frac{1}{(2q+2)\sqrt{2}} \arctan \frac{s^{q+1}\sqrt{2}}{1-s^{2q+2}} \right) \quad \leq \quad C_2,$$

$s \in \mathbb{R}$. Note that acccording to $\lim_{s\to\pm1} l(s) = \frac{\pi}{2}$ and [12] (p. 707, Integral 79), we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^q}{(1+s^{4q+4})}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x,y) = Q(t)Q(x), \quad t \in [0,\infty), \quad x \in \mathbb{R}.$$

Then, $\exists C > 0$ such that

$$2^{q+1}(q+1)!(1+t+t^2)\left(\sum_{r=0}^{q} |x|^r\right) \int_0^t \left| \int_0^x g_1(t_2,x_2)dx_2 \right| dt_2 \leq C, \quad t \in [0,\infty), x \in \mathbb{R}.$$

Let

$$g(t,x) = \frac{\mathcal{A}}{C} g_1(t,x), \quad t \in [0,\infty), x \in \mathbb{R}.$$

Then,

$$2^{q+1}q!(1+t+t^2)\left(\sum_{r=0}^{q} |x|^r\right) \int_0^t \left| \int_0^x g(t_2,x_2)dx_2 \right| dt_2 \leq \mathcal{A}, \quad t \in [0,\infty), x \in \mathbb{R},$$

i.e., (Hypothesis 3) holds. Therefore, for the IVP

$$\begin{aligned} \partial_t u + \sum_{k=0}^{5}\sum_{l=0}^{5-k} \partial_x \left\{ \sum_{m=0}^{5-k} \partial_x^m u^{10} \partial_x^l v \right\} + \sum_{k=1}^{5} \frac{1}{(1+t^{2k})(1+x^{2k})} \partial_x^{2k+1} u &= 0 \\ \partial_t v + \sum_{k=0}^{5}\sum_{l=0}^{5-k} \partial_x^k \left\{ \sum_{m=0}^{5-k} \partial_x^m v^{10} \partial_x^l u \right\} + \sum_{k=1}^{5} \frac{1}{(2+t^{4k})(3+x^{6k})} \partial_x^{2k+1} v &= 0, \\ t \in [0,\infty), x \in \mathbb{R}, \quad u(0,x) = \frac{1}{1+x^4}, \quad v(0,x) = \frac{1}{3+4x^8}, \quad x \in \mathbb{R}, \end{aligned}$$

all conditions of Theorems 1 and 2 are fulfilled.

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