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On Several Results Associated with the Apéry-like Series

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Abstract: In 1979, Apéry proved the irrationality of $\zeta(2)$ and $\zeta(3)$. Since then, there has been much research interest in investigating the Apéry-like series for values of Riemann zeta function, Ramanujan-like series for π and other infinite series involving central binomial coefficients. The purpose of this work is to present the first 20 results related to the Apéry-like series in the form of 4 lemmas, each containing 5 results. The Sherman's results are applied to attain this. Thereafter, these 20 results are further used to establish up to 104 results pertaining to the Apéry-like series in the form of 4 theorems, with 26 results each. These findings are finally been described in terms of the generalized hypergeometric functions. Symmetry occurs naturally in the generalized hypergeometric functions.

Keywords: central binomail coefficients; reciprocals; Apéry-like series; combinatorics; pochhammer symbol; gamma function; generalized hypergeometyric function

MSC: 33B15; 11B65; 33C20; 33C05

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1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator q denominator parameters is defined by [1–5]

$$ {}_pF_q\left[\begin{array}{cccc} \alpha_1, & \dots, & \alpha_r & \\ & & & ;\ z \\ \beta_1, & \dots, & \beta_s & \end{array}\right] = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \dots (\alpha_r)_m}{(\beta_1)_m \dots (\beta_s)_m} \frac{z^m}{m!} $$

where $(\alpha)_m$ is well-known Pochhammer's symbol defined by

$$ (\alpha)_m = \begin{cases} \alpha(\alpha+1)\dots(\alpha+m-1) & ; m \in \mathbb{N} \\ 1 & ; m = 0 \end{cases} $$

In terms of gamma function, we have

$$ (\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} $$

As usual, r and s are non-negative integers; $\alpha_j (j = 1, 2, \dots, r)$ and $\beta_j (j = 1, 2, \dots, s)$ can take arbitrary complex values with one exception: $\beta_j (j = 1, 2, \dots, s)$ should not be zero or a negative integer. The series converges for all $|z| < \infty$ whenever $r \le s$. It also converges when $r = s + 1$, provided that $|z| < 1$ and $|z| = 1$ when $r = s + 1$ and $Re(\delta) > 0$, where

$$ \delta = \sum_{j=1}^{s} \beta_j - \sum_{j=1}^{r} \alpha_j $$

It is appropriate to note that the generalized hypergeometric function and its many special and limiting cases, including Gauss's hypergeometric function ${}_2F_1$ and confluent hypergeometric function ${}_1F_1$, occur in a variety of theoretical and real-world circumstances in applied mathematics, theoretical physics, engineering, and statistics.

One might refer to [1–5] for more information on the generalized hypergeometric function ${}_pF_q$.

However, for non-negative integers m and k, the binomial coefficient is defined as follows:

$$\binom{m}{k} = \begin{cases} \dfrac{m!}{k!(m-k)!}; & m \geq k \\ 0; & m < k \end{cases}$$

The central binomial coefficient is defined by

$$\binom{m}{2m} = \frac{(2m)!}{(m!)^2}, \qquad (m = 0, 1, 2, \ldots).$$

It is interesting to note that the central binomial coefficients and reciprocal of the central binomial coefficients, which have been extensively researched, play a key role in numerous branches of mathematics, including number theory, combinatorics, probability theory, and statistics. The books by Koshy [6] and Riordan [7] are also good resources and contain many details regarding these coefficients. Gould [8] contains many identities concerning central binomial coefficients and their reciprocals. Many intriguing and practical identities can occasionally be seen in the research papers published by notable researchers, such as Lehmer [9] and Leshchiner [10], Mansour [11], Pla [12], Sprugnoli [13,14], Sury [15], Sury et al. [16], Trif [17], Wheelon [18], Zhao and Wang [19], Kumar et al. [20], Zhang and Ji [21], and references therein. In addition to this, for a new asymptotic series and estimates related to the well-known Euler–Mascheroni constant, we also refer to an interesting research paper by Cristea [22].

In a very interesting and useful research paper, Apéry [23], in 1979, proved the irrationality of $\zeta(3)$ by making use of the identity viz.

$$\zeta(3) = \frac{5}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}(m!)^2}{(2m)!\, m^3},$$

and the irrationality $\zeta(2)$ by making use of the identity viz.

$$\zeta(2) = 3 \sum_{m=1}^{\infty} \frac{(m!)^2}{(2m)!\, m^2}.$$

In addition to this, following Apéry proof, a large number of a similar series

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} f(m) = \sum_{m=0}^{\infty} \frac{f(m)}{\binom{2m}{m}},$$

which were frequently referred to in the literature as Apéry-like series have been examined by eminent scholars including Van der Poorten [24], and Borwein et al. [25]. Berndt and Joshi [26] and Zucker [27] have also documented a large number of simila formulas in their analysis of Chapter 9 of Ramanujan's second notebook.

In 2000, Sherman [28] established a large number of Apéry-like series, the following of which will be used in our investigations:

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}2^m = \frac{\pi}{2}+2. \tag{1}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(m+1)} = \pi-\frac{\pi^2}{8}. \tag{2}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(m+2)} = \frac{5\pi}{2}-\frac{3\pi^2}{8}-3. \tag{3}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(m+3)} = 6\pi-\frac{15\pi^2}{16}-\frac{35}{4}. \tag{4}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(m+4)} = \frac{83\pi}{6}-\frac{35\pi^2}{16}-\frac{763}{36}. \tag{5}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(m+5)} = 31\pi-\frac{315\pi^2}{64}-\frac{193}{4}. \tag{6}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m(m!)^2}{(2m)!2^m} = \frac{8}{9}-\frac{4\ln 2}{27}. \tag{7}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(m+1)} = \frac{8}{3}(\ln 2)-2(\ln 2)^2. \tag{8}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(m+2)} = -\frac{100}{3}(\ln 2)+24(\ln 2)^2+12. \tag{9}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(m+3)} = \frac{998}{3}(\ln 2)-240(\ln 2)^2-115. \tag{10}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(m+4)} = -\frac{9316}{3}(\ln 2)+2240(\ln 2)^2+\frac{9688}{9}. \tag{11}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(m+5)} = \frac{83843}{3}(\ln 2)-20160(\ln 2)^2-\frac{38743}{4}. \tag{12}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(2m+1)} = \frac{\pi}{2}. \tag{13}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(2m+3)} = \frac{3\pi}{2}-4. \tag{14}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(2m+5)} = \frac{23\pi}{6}-\frac{104}{9}. \tag{15}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(2m+7)} = \frac{91\pi}{10}-\frac{2116}{75}. \tag{16}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{2^m}{(2m+9)} = \frac{1451\pi}{70}-\frac{238192}{3675}. \tag{17}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(2m+1)} = \frac{4}{3}(\ln 2). \tag{18}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(2m+3)} = -\frac{68}{3}(\ln 2)+16. \tag{19}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(2m+5)} = \frac{724}{3}(\ln 2)-\frac{1504}{9}. \tag{20}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(2m+7)}=-\frac{34756}{15}(\ln 2)+\frac{120464}{75}. \tag{21}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m}{2^m(2m+9)}=\frac{2224364}{105}(\ln 2)-\frac{53963072}{3675}. \tag{22}$$

The remaining sections of the paper are arranged as follows. Twenty results will be established in Section 2 as four lemmas, with five results each. This is accomplished by the findings provided by Sherman [28], as well as 104 results connected to the Apéry-like series in the form of 4 theorems, each comprising 26 results. These findings are presented in Section 3 as generalized hypergeometric functions that could be beneficial for applications. The generalized hypergeometric function ${}_pF_q$ exhibits symmetry in the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$, as well as in the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$. This means that every arrangement of the numerator parameters for the generalized hypergeometric function, $\alpha_1, \alpha_2, \ldots, \alpha_p$, and every arrangement of the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$, produces the same function. The paper ends with a concluding remark.

2. Main Results

In this section, we first establish 20 results related to the Apéry-like series in the form of 4 lemmas containing 5 results each. These are:

Lemma 1. *For $m \in \mathbb{N}$, the following results are true.*

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+1)}=\frac{\pi^2}{8}-\frac{\pi}{2}+2. \tag{23}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+2)}=\frac{3\pi^2}{4}-\frac{9\pi}{2}+8. \tag{24}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+3)}=\frac{45\pi^2}{16}-\frac{35\pi}{2}+\frac{113}{4}. \tag{25}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+4)}=\frac{35\pi^2}{4}-\frac{329\pi}{6}+\frac{781}{9}. \tag{26}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+5)}=\frac{1575\pi^2}{64}-\frac{309\pi}{2}+\frac{973}{4}. \tag{27}$$

Lemma 2. *For $m \in \mathbb{N}$, the following results are true.*

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)}=2(ln2)^2-\frac{76}{27}ln2+\frac{8}{9}. \tag{28}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+2)}=-48(ln2)^2+\frac{1796}{27}ln2-\frac{208}{9}. \tag{29}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+3)}=720(ln2)^2-\frac{26950}{27}ln2+\frac{3113}{9}. \tag{30}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+4)}=-8960(ln2)^2+\frac{335372}{27}ln2-\frac{38744}{9}. \tag{31}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+5)}=100800(ln2)^2-\frac{3772939}{27}ln2+\frac{1743443}{36}. \tag{32}$$

Lemma 3. *For $m \in \mathbb{N}$, the following results are true.*

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(2m+1)} = 1. \tag{33}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(2m+3)} = -2\pi + 7. \tag{34}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(2m+5)} = \frac{-28\pi}{3} + \frac{269}{9}. \tag{35}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(2m+7)} = \frac{-158\pi}{5} + \frac{7481}{75}. \tag{36}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(2m+9)} = \frac{-3256\pi}{35} + \frac{358513}{1225}. \tag{37}$$

Lemma 4. *For $m \in \mathbb{N}$, the following results are true.*

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{(-1)^m m}{2^m(2m+1)} = \frac{-20}{27} ln2 + \frac{4}{9}. \tag{38}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{(-1)^m m}{2^m(2m+3)} = \frac{916}{27} ln2 - \frac{212}{9}. \tag{39}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{(-1)^m m}{2^m(2m+5)} = \frac{-16292}{27} ln2 + \frac{3764}{9}. \tag{40}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{(-1)^m m}{2^m(2m+7)} = \frac{1094804}{135} ln2 - \frac{1264772}{225}. \tag{41}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{(-1)^m m}{2^m(2m+9)} = \frac{-90086812}{945} ln2 + \frac{728506372}{11025}. \tag{42}$$

Proof. In the above 4 lemmas, the 20 Apéry-like series were derived in a rather simple manner. Therefore, we chose to establish just one result, for example, (33), and the other results can be established in a similar manner. To establish result (33), we proceeded as follows. Denoting the left-hand side of (33) by S, we have

$$\begin{aligned} S &= \sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(2m+1)} = \sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} 2^m \left\{ \frac{1}{2} - \frac{1}{2(2m+1)} \right\} \\ &= \frac{1}{2} \sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} 2^m - \frac{1}{2} \sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{2^m}{(2m+1)} \end{aligned}$$

Evaluating the first and second sums with the help of results (1) and (2), we could easily arrive at the right hand side of (33). This completes the proof of the result (33) asserted in Lemma 3. □

Now, we establish as many as 104 results related to the Apéry-like series in the form of 4 theorems containing 26 results each, with the help of the 4 Lemmas presented above. These are as follows.

Theorem 1. *The following 26 Apéry-like series hold true:*

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+2)} = -\frac{5\pi^2}{8} + 4\pi - 6. \tag{43}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+3)} = -\frac{43\pi^2}{32} - \frac{17\pi}{2} - \frac{105}{8}. \tag{44}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+4)} = -\frac{23\pi^2}{8} + \frac{163\pi}{9} - \frac{763}{27}. \tag{45}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+5)} = -\frac{1567\pi^2}{256} + \frac{77\pi}{2} - \frac{965}{16}. \tag{46}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+2)(m+3)} = -\frac{33\pi^2}{16} + 13\pi - \frac{81}{4}. \tag{47}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+2)(m+4)} = -4\pi^2 + \frac{151\pi}{16} - \frac{709}{18}. \tag{48}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+2)(m+5)} = -\frac{509\pi^2}{64} + 50\pi - \frac{941}{12}. \tag{49}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+3)(m+4)} = -\frac{95\pi^2}{16} + \frac{112\pi}{3} - \frac{2107}{36}. \tag{50}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+3)(m+5)} = -\frac{1395\pi^2}{128} + \frac{137\pi}{2} - \frac{215}{2}. \tag{51}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+4)(m+5)} = -\frac{1015\pi^2}{64} + \frac{299\pi}{3} - \frac{5633}{36}. \tag{52}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+2)(m+3)} = \frac{23\pi^2}{32} - \frac{9\pi}{2} + \frac{57}{8}. \tag{53}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+2)(m+4)} = \frac{9\pi^2}{8} - \frac{127\pi}{18} + \frac{601}{54}. \tag{54}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+2)(m+5)} = \frac{469\pi^2}{256} - \frac{23\pi}{2} + \frac{869}{48}. \tag{55}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+3)(m+4)} = \frac{49\pi^2}{32} - \frac{173\pi}{18} + \frac{3269}{216}. \tag{56}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+3)(m+5)} = -\frac{14617\pi^2}{512} - 15\pi + \frac{755}{32}. \tag{57}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+1)(m+4)(m+5)} = \frac{831\pi^2}{256} - \frac{367\pi}{18} + \frac{13847}{432}. \tag{58}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+2)(m+3)(m+4)} = \frac{31\pi^2}{16} - \frac{73\pi}{6} + \frac{689}{36}. \tag{59}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+2)(m+3)(m+5)} = \frac{377\pi^2}{128} - \frac{37\pi}{2} + \frac{349}{12}. \tag{60}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+2)(m+4)(m+5)} = \frac{253\pi^2}{64} - \frac{149\pi}{6} + \frac{1405}{36}. \tag{61}$$

$$\sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+3)(m+4)(m+5)} = \frac{635\pi^2}{128} - \frac{187\pi}{6} + \frac{1763}{36}. \tag{62}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+1)(m+2)(m+3)(m+4)}=-\frac{13\pi^2}{32}+\frac{23\pi}{9}-\frac{865}{216}. \tag{63}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+1)(m+2)(m+3)(m+5)}=-\frac{285\pi^2}{512}+\frac{7\pi}{2}-\frac{527}{96}. \tag{64}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+1)(m+2)(m+4)(m+5)}=-\frac{181\pi^2}{256}+\frac{40\pi}{9}-\frac{3013}{432}. \tag{65}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+1)(m+3)(m+4)(m+5)}=-\frac{439\pi^2}{512}+\frac{97\pi}{18}-\frac{7309}{864}. \tag{66}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+2)(m+3)(m+4)(m+5)}=-\frac{129\pi^2}{128}+\frac{331\pi}{48}-\frac{179}{18}. \tag{67}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(m+1)(m+2)(m+3)(m+4)(m+5)}=\frac{77\pi^2}{512}-\frac{217\pi}{144}+\frac{1283}{864}. \tag{68}$$

Theorem 2. *The following 26 Apéry-like series hold true:*

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+2)}=50(\ln 2)^2-\frac{208}{3}(\ln 2)+24. \tag{69}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(1)^m m}{2^m(m+1)(m+3)}=-359(\ln 2)^2+\frac{1493}{3}(\ln 2)-\frac{345}{2}. \tag{70}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+4)}=\frac{8962}{3}(\ln 2)^2-\frac{12424}{3}(\ln 2)+\frac{38752}{27}. \tag{71}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+5)}=-\frac{100798}{4}(\ln 2)^2+\frac{419207}{12}(\ln 2)+\frac{581137}{48}. \tag{72}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+2)(m+3)}=-768(\ln 2)^2+\frac{3194}{3}(\ln 2)-\frac{2905}{9}. \tag{73}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+2)(m+4)}=4456(\ln 2)^2-\frac{18532}{3}(\ln 2)+2164. \tag{74}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+2)(m+5)}=-33616(\ln 2)^2+\frac{139805}{3}(\ln 2)-\frac{436069}{27}. \tag{75}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+3)(m+4)}=9680(\ln 2)^2-\frac{40258}{3}(\ln 2)+\frac{41857}{9}. \tag{76}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+3)(m+5)}=-50040(\ln 2)^2+\frac{416221}{6}(\ln 2)-\frac{576997}{24}. \tag{77}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+4)(m+5)}=-109760(\ln 2)^2+\frac{456479}{3}(\ln 2)-\frac{1898419}{36}. \tag{78}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+2)(m+3)}=-507(\ln 2)^2+409(\ln 2)+\frac{2705}{18}. \tag{79}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+2)(m+4)}=-\frac{4454}{3}(\ln 2)^2-\frac{308}{9}(\ln 2)-\frac{25684}{9}. \tag{80}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+2)(m+5)}=\frac{16833}{2}(\ln 2)^2-\frac{46671}{4}(\ln 2)+\frac{582289}{144}. \tag{81}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+3)(m+4)}=-\frac{10016}{3}(\ln 2)^2+4639(\ln 2)-\frac{86819}{54}. \tag{82}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+3)(m+5)}=\frac{49681}{4}(\ln 2)^2-\frac{137745}{8}(\ln 2)+\frac{572857}{96}. \tag{83}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+4)(m+5)}=\frac{169121}{6}(\ln 2)^2-\frac{156301}{4}(\ln 2)+\frac{575617}{96}. \tag{84}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+2)(m+3)(m+4)}=-5224(\ln 2)^2+7242(\ln 2)-\frac{22589}{9}. \tag{85}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+2)(m+3)(m+5)}=16424(\ln 2)^2-\frac{45537}{2}(\ln 2)+\frac{1716875}{216}. \tag{86}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+2)(m+4)(m+5)}=38072(\ln 2)^2-522779(\ln 2)+\frac{1510763}{108}. \tag{87}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+3)(m+4)(m+5)}=59720(\ln 2)^2-\frac{165579}{2}(\ln 2)+\frac{2065847}{72}. \tag{88}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+2)(m+3)(m+4)}=\frac{5633}{3}(\ln 2)^2-2603(\ln 2)+\frac{48715}{54}. \tag{89}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+2)(m+3)(m+5)}=-\frac{17455}{4}(\ln 2)^2+\frac{1306681}{216}(\ln 2)-\frac{553993}{288}. \tag{90}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+2)(m+4)(m+5)}=-\frac{59311}{6}(\ln 2)^2+\frac{54815}{4}(\ln 2)-\frac{2051699}{432}. \tag{91}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+3)(m+4)(m+5)}=-\frac{189199}{12}(\ln 2)^2+\frac{174857}{8}(\ln 2)-\frac{164311}{108}. \tag{92}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+2)(m+3)(m+4)(m+5)}=-21648(\ln 2)^2+\frac{60021}{2}(\ln 2)-\frac{748853}{72}. \tag{93}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(m+1)(m+2)(m+3)(m+4)(m+5)}=\frac{70577}{12}(\ln 2)^2-\frac{65227}{8}(\ln 2)+\frac{2441419}{864}. \tag{94}$$

Theorem 3. *The following 26 Apéry-like series hold true:*

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+3)}=\frac{15\pi^2}{4}+\frac{\pi}{2}-1. \tag{95}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+5)}=\frac{7\pi}{3}-\frac{65}{9}. \tag{96}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+7)}=\frac{79\pi}{15}-\frac{3703}{225}. \tag{97}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+9)}=\frac{407\pi}{35}-\frac{44661}{1225}. \tag{98}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+3)(2m+5)}=\frac{11\pi}{3}-\frac{103}{9}. \tag{99}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+3)(2m+7)}=\frac{37\pi}{5}-\frac{1739}{75}. \tag{100}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+3)(2m+9)}=\frac{531\pi}{35}-\frac{58323}{1225}. \tag{101}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+5)(2m+7)}=\frac{167\pi}{15}-\frac{7859}{225}. \tag{102}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+5)(2m+9)}=\frac{2197\pi}{105}-\frac{724273}{11025}. \tag{103}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+7)(2m+9)}=\frac{215\pi}{7}-\frac{70897}{735}. \tag{104}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+3)(2m+5)}=-\frac{2\pi}{3}+\frac{19}{9}. \tag{105}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+3)(2m+7)}=-\frac{16\pi}{15}+\frac{757}{225}. \tag{106}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+3)(2m+9)}=-\frac{62\pi}{35}+\frac{6831}{1225}. \tag{107}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+5)(2m+7)}=-\frac{22\pi}{15}+\frac{1039}{225}. \tag{108}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+5)(2m+9)}=-\frac{244\pi}{105}+\frac{80581}{11025}. \tag{109}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+7)(2m+9)}=-\frac{334\pi}{105}+\frac{110251}{11025}. \tag{110}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+3)(2m+5)(2m+7)}=-\frac{28\pi}{15}+\frac{1321}{225}. \tag{111}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+3)(2m+5)(2m+9)}=-\frac{302\pi}{105}+\frac{99683}{11025}. \tag{112}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+3)(2m+7)(2m+9)}=-\frac{136\pi}{35}+\frac{44879}{3675}. \tag{113}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+5)(2m+7)(2m+9)}=-\frac{514\pi}{105}+\frac{169591}{11025}. \tag{114}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+3)(2m+5)(2m+7)}=\frac{\pi}{5}-\frac{47}{75}. \tag{115}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+3)(2m+5)(2m+9)}=\frac{29\pi}{105}-\frac{9551}{11025}. \tag{116}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+3)(2m+7)(2m+9)}=\frac{37\pi}{105}-\frac{12193}{11025}. \tag{117}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+5)(2m+7)(2m+9)}=\frac{3\pi}{7}-\frac{989}{735}. \tag{118}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+3)(2m+5)(2m+7)(2m+9)}=\frac{53\pi}{105}-\frac{17477}{11025}. \tag{119}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{m2^m}{(2m+1)(2m+3)(2m+5)(2m+7)(2m+9)}=-\frac{4\pi}{105}+\frac{1321}{11025}. \tag{120}$$

Theorem 4. *The following 26 Apéry-like series hold true:*

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+3)}=-\frac{52}{3}\ln 2+12. \tag{121}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(1)^m m}{2^m(2m+1)(2m+5)}=\frac{452}{3}\ln 2-\frac{940}{9}. \tag{122}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+7)}=\frac{210812}{225}\ln 2-\frac{20276}{15}. \tag{123}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+9)}=-\frac{5}{54}\ln 2+\frac{241883501}{66150}. \tag{124}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+3)(2m+5)}=\frac{956}{3}\ln 2-\frac{1988}{9}. \tag{125}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+3)(2m+7)}=-\frac{30284}{15}\ln 2+\frac{104956}{75}. \tag{126}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+3)(2m+9)}=\frac{1668868}{105}\ln 2-\frac{13495668}{1225}. \tag{127}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+5)(2m+7)}=-\frac{65348}{15}\ln 2+\frac{679436}{225}. \tag{128}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+5)(2m+9)}=\frac{2486572}{105}\ln 2-\frac{180973868}{11025}. \tag{129}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+7)(2m+9)}=\frac{1086116}{21}\ln 2-\frac{5269868}{147}. \tag{130}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+3)(2m+5)}=-84\ln 2+\frac{524}{9}. \tag{131}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+3)(2m+7)}=\frac{1668}{5}\ln 2-\frac{52028}{225}. \tag{132}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+3)(2m+9)}=-\frac{69612}{35}\ln 2+\frac{1688796}{1225}. \tag{133}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+5)(2m+7)}=\frac{3756}{5}\ln 2-\frac{39052}{75}. \tag{134}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+5)(2m+9)}=-\frac{102948}{35}\ln 2+\frac{22477796}{11025}. \tag{135}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+7)(2m+9)}=-\frac{232188}{35}\ln 2+\frac{50696236}{11025}. \tag{136}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+3)(2m+5)(2m+7)}=\frac{5844}{5}\ln 2-\frac{182284}{225}. \tag{137}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+3)(2m+5)(2m+9)}=-\frac{136284}{35}\ln 2+\frac{29756428}{11025}. \tag{138}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+3)(2m+7)(2m+9)}=-\frac{313476}{35}\ln 2+\frac{22814924}{3675}. \tag{139}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+5)(2m+7)(2m+9)}=-\frac{490668}{35}\ln 2+\frac{107133116}{11025}. \tag{140}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+3)(2m+5)(2m+7)}=-\frac{1044}{5}\ln 2+\frac{32564}{225}. \tag{141}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+3)(2m+5)(2m+9)}=\frac{16668}{35}\ln 2-\frac{3639316}{11025}. \tag{142}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+3)(2m+7)(2m+9)}=\frac{40644}{35}\ln 2-\frac{8874268}{11025}. \tag{143}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+5)(2m+7)(2m+9)}=\frac{12924}{7}\ln 2-\frac{2821844}{2205}. \tag{144}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+3)(2m+5)(2m+7)(2m+9)}=\frac{88596}{35}\ln 2-\frac{19344172}{11025}. \tag{145}$$

$$\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+3)(2m+5)(2m+7)(2m+9)}=-\frac{11988}{35}\ln 2+\frac{872492}{3675}. \tag{146}$$

Proof. In the above 4 theorems, the 104 Apéry-like series were derived in a rather simple manner. Because of this, we chose to establish just one result, say (146), and the other results can be established in a similar manner. Thus, in order to establish result (146), we proceeded as follows. Denoting the left-hand side of (146) by S, we have

$$S=\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!}\frac{(-1)^m m}{2^m(2m+1)(2m+3)(2m+5)(2m+7)(2m+9)}.$$

Using partial fraction technique, we have

$$\begin{aligned}S=&\sum_{m=0}^{\infty}\frac{(m!)^2(-1)^m m}{(2m)!2^m}\left\{\frac{1}{384(2m+1)}-\frac{1}{96(2m+3)}+\frac{1}{64(2m+5)}\right.\\&\left.-\frac{1}{96(2m+7)}+\frac{1}{384(2m+9)}\right\}.\end{aligned}$$

Now, separating this into five series, we have

$$\begin{aligned}S=&\frac{1}{384}\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!2^m}\frac{(-1)^m m}{(2m+1)}-\frac{1}{96}\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!2^m}\frac{(-1)^m m}{(2m+3)}+\frac{1}{64}\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!2^m}\frac{(-1)^m m}{(2m+5)}\\&-\frac{1}{96}\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!2^m}\frac{(-1)^m m}{(2m+7)}+\frac{1}{384}\sum_{m=0}^{\infty}\frac{(m!)^2}{(2m)!2^m}\frac{(-1)^m m}{(2m+9)}.\end{aligned}$$

Finally, using the results (38)–(42), we easily arrive, after some simplification, at the desired right-hand side of (146). This completes the derivation of result (146). We conclude this section by noting that the derivations of rest of the results are fairly straight-forward and are left as an exercise for interested readers. □

3. Results (43)–(146) in Terms of the Generalized Hypergeometric Functions

In this section, we shall express results (43)–(146) in terms of the generalized hypergeometric functions, which may be useful from an applications perspective. These are as follows:

$${}_3F_2\left[\begin{matrix}2, & 2, & 2\\ \frac{3}{2}, & & 4\end{matrix};\frac{1}{2}\right]=-\frac{15}{4}\pi^2+24\pi-36. \tag{147}$$

$${}_4F_3\left[\begin{matrix}2, & 2, & 2, & 4\\ \frac{3}{2}, & 3, & 5 & \end{matrix};\frac{1}{2}\right]=-\frac{43}{4}\pi^2+68\pi-105. \tag{148}$$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 5 \\ \frac{3}{2}, & 3, & 6 & \end{matrix};\frac{1}{2}\right] = -\frac{115}{4}\pi^2 + \frac{1630}{9}\pi - \frac{7630}{27}. \tag{149} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 6 \\ \frac{3}{2}, & 3, & 7 & \end{matrix};\frac{1}{2}\right] = -\frac{4701}{64}\pi^2 + 462\pi - \frac{2895}{4}. \tag{150} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 3 \\ \frac{3}{2}, & 5 & \end{matrix};\frac{1}{2}\right] = -\frac{99}{4}\pi^2 + 156\pi - 243. \tag{151} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 3, & 5 \\ \frac{3}{2}, & 4, & 6 & \end{matrix};\frac{1}{2}\right] = -60\pi^2 + \frac{2265}{16}\pi - \frac{3545}{6}. \tag{152} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 3, & 6 \\ \frac{3}{2}, & 4, & 7 & \end{matrix};\frac{1}{2}\right] = -\frac{4581}{32}\pi^2 + 900\pi - \frac{2823}{2}. \tag{153} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 4 \\ \frac{3}{2}, & 6 & \end{matrix};\frac{1}{2}\right] = -\frac{475}{4}\pi^2 + \frac{2240}{3}\pi - \frac{10535}{9}. \tag{154} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 4, & 6 \\ \frac{3}{2}, & 5, & 7 & \end{matrix};\frac{1}{2}\right] = -\frac{4185}{16}\pi^2 + 1644\pi - 2580. \tag{155} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 5 \\ \frac{3}{2}, & 7 & \end{matrix};\frac{1}{2}\right] = -\frac{15225}{32}\pi^2 + 2990\pi - \frac{28165}{6}. \tag{156} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 2 \\ \frac{3}{2}, & 5 & \end{matrix};\frac{1}{2}\right] = \frac{69}{4}\pi^2 - 216\pi + 342. \tag{157} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 5 \\ \frac{3}{2}, & 4, & 6 & \end{matrix};\frac{1}{2}\right] = \frac{135}{4}\pi^2 - \frac{635}{3}\pi + \frac{3005}{9}. \tag{158} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 6 \\ \frac{3}{2}, & 4, & 7 & \end{matrix};\frac{1}{2}\right] = \frac{4221}{64}\pi^2 - 414\pi + \frac{2607}{4}. \tag{159} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 4 \\ \frac{3}{2}, & 3, & 6 & \end{matrix};\frac{1}{2}\right] = \frac{245}{4}\pi^2 - \frac{3460}{9}\pi + \frac{16345}{27}. \tag{160} $$

$$ {}_5F_4\left[\begin{matrix} 2, & 2, & 2, & 4, & 6 \\ \frac{3}{2}, & 3, & 5, & 7 & \end{matrix};\frac{1}{2}\right] = -\frac{43851}{32}\pi^2 - 720\pi + \frac{2265}{2}. \tag{161} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 5 \\ \frac{3}{2}, & 3, & 7 & \end{matrix};\frac{1}{2}\right] = \frac{12465}{64}\pi^2 - \frac{3670}{3}\pi + \frac{69235}{36}. \tag{162} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 3 \\ \frac{3}{2}, & 6 & \end{matrix};\frac{1}{2}\right] = \frac{465}{4}\pi^2 - 730\pi + \frac{3445}{3}. \tag{163} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 3, & 6 \\ \frac{3}{2}, & 5, & 7 & \end{matrix};\frac{1}{2}\right] = \frac{3393}{16}\pi^2 - 1332\pi + 2094. \tag{164} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 3, & 5 \\ \frac{3}{2}, & 4, & 7 & \end{matrix};\frac{1}{2}\right] = \frac{11385}{32}\pi^2 - 2235\pi + \frac{7025}{2}. \tag{165} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 4 \\ \frac{3}{2}, & 7 & \end{matrix};\frac{1}{2}\right] = \frac{9525}{16}\pi^2 - 3740\pi + \frac{17630}{3}. \tag{166} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 2 \\ \frac{3}{2}, & 6 & \end{matrix};\frac{1}{2}\right] = -\frac{195}{4}\pi^2 + \frac{920}{3}\pi - \frac{4325}{9}. \tag{167} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 6 \\ \frac{3}{2}, & 5, & 7 & \end{matrix};\frac{1}{2}\right] = -\frac{2565}{32}\pi^2 + 504\pi - \frac{1581}{2}. \tag{168} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 5 \\ \frac{3}{2}, & 4, & 7 & \end{matrix};\frac{1}{2}\right] = -\frac{8145}{64}\pi^2 + 800\pi - \frac{15065}{12}. \tag{169} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 4 \\ \frac{3}{2}, & 3, & 7 & \end{matrix};\frac{1}{2}\right] = -\frac{6585}{32}\pi^2 + \frac{3880}{3}\pi - \frac{36545}{18}. \tag{170} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 3 \\ \frac{3}{2}, & 7 & \end{matrix};\frac{1}{2}\right] = -\frac{5805}{16}\pi^2 + \frac{4965}{2}\pi - 3580. \tag{171} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 2 \\ \frac{3}{2}, & 7 & \end{matrix};\frac{1}{2}\right] = \frac{3465}{32}\pi^2 - 1085\pi\frac{6415}{6}. \tag{172} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 2 \\ \frac{3}{2}, & 4 & \end{matrix};-\frac{1}{8}\right] = -1200(\ln 2)^2 + 1664(\ln 2) - 576. \tag{173} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 4 \\ \frac{3}{2}, & 3, & 5 & \end{matrix};-\frac{1}{8}\right] = 11488(\ln 2)^2 - \frac{47776}{3}(\ln 2) + 5520. \tag{174} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 5 \\ \frac{3}{2}, & 3, & 6 & \end{matrix};-\frac{1}{8}\right] = -\frac{358480}{3}(\ln 2)^2 + \frac{496960}{3}(\ln 2) - \frac{1550080}{27}. \tag{175} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 6 \\ \frac{3}{2}, & 3, & 7 & \end{matrix};-\frac{1}{8}\right] = 49209576(\ln 2)^2 - 1676828(\ln 2) + 581137. \tag{176} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 3 \\ \frac{3}{2}, & 5 & \end{matrix};-\frac{1}{8}\right] = 36864(\ln 2)^2 - 51104(\ln 2) + \frac{46480}{3}. \tag{177} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 3, & 5 \\ \frac{3}{2}, & 4, & 6 & \end{matrix};-\frac{1}{8}\right] = -267360(\ln 2)^2 + 370640(\ln 2) - 129840. \tag{178} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 3, & 6 \\ \frac{3}{2}, & 4, & 7 & \end{matrix};-\frac{1}{8}\right] = 2420352(\ln 2)^2 - 3355320(\ln 2) + \frac{3488552}{3}. \tag{179} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 4 \\ \frac{3}{2}, & 6 & \end{matrix};-\frac{1}{8}\right] = -774400(\ln 2)^2 + \frac{3220640}{3}(\ln 2) - \frac{3348560}{9}. \tag{180} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 4, & 6 \\ \frac{3}{2}, & 5, & 7 & \end{matrix};-\frac{1}{8}\right] = 4803840(\ln 2)^2 - 6659536(\ln 2) + 2307988. \tag{181} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 5 \\ \frac{3}{2}, & 7 & \end{matrix};-\frac{1}{8}\right] = 13171200(\ln 2)^2 - 18259160(\ln 2) + \frac{18984190}{3}. \tag{182} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 2 \\ \frac{3}{2}, & 5 & \end{matrix};-\frac{1}{8}\right] = 48672(\ln 2)^2 - 39264(\ln 2) - \frac{43280}{3}. \tag{183} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 5 \\ \frac{3}{2}, & 4, & 6 & \end{matrix};-\frac{1}{8}\right] = 176240(\ln 2)^2 - 244320(\ln 2) + \frac{762080}{9}. \tag{184} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 6 \\ \frac{3}{2}, & 4, & 7 & \end{matrix};-\frac{1}{8}\right] = -1211976(\ln 2)^2 + 1680156(\ln 2) - 582289. \tag{185} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 4 \\ \frac{3}{2}, & 3, & 6 & \end{matrix};-\frac{1}{8}\right] = \frac{16022560}{3}(\ln 2)^2 - 742240(\ln 2) + \frac{6945520}{27}. \tag{186} $$

$$ {}_5F_4\left[\begin{matrix} 2, & 2, & 2, & 4, & 6 \\ \frac{3}{2}, & 3, & 5, & 7 & \end{matrix};-\frac{1}{8}\right] = -2384688(\ln 2)^2 + 3305880(\ln 2) - 1145714. \tag{187} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 5 \\ \frac{3}{2}, & 3, & 7 & \end{matrix};-\frac{1}{8}\right] = -6764840(\ln 2)^2 + 9378060(\ln 2) - 2878085. \tag{188} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 3 \\ \frac{3}{2}, & 6 & \end{matrix};-\frac{1}{8}\right] = 1253760(\ln 2)^2 - 1738080\ (\ln 2) + \frac{1807120}{3}. \tag{189} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 3, & 6 \\ \frac{3}{2}, & 5, & 7 & \end{matrix};-\frac{1}{8}\right] = -4730112(\ln 2)^2 + 6557328(\ln 2) - \frac{6867500}{3}. \tag{190} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 3, & 5 \\ \frac{3}{2}, & 4, & 7 & \end{matrix};-\frac{1}{8}\right] = -13705920(\ln 2)^2 + 19000440(\ln 2) - \frac{15107630}{3}. \tag{191} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 4 \\ \frac{3}{2}, & 7 & \end{matrix};-\frac{1}{8}\right] = -28665600(\ln 2)^2 + 39738960(\ln 2) - \frac{41316940}{3}. \tag{192} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 5 \\ \frac{3}{2}, & 5, & 6 & \end{matrix};-\frac{1}{8}\right] = -901280(\ln 2)^2 + 1249440(\ln 2) - \frac{3897200}{9}. \tag{193} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 6 \\ & \frac{3}{2}, & 5, & 7 \end{matrix};-\frac{1}{8}\right]=2513520(\ln 2)^2-\frac{10453448}{3}(\ln 2)+1107986. \tag{194} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 5 \\ & \frac{3}{2}, & 4, & 7 \end{matrix};-\frac{1}{8}\right]=7117320(\ln 2)^2-9866700(\ln 2)+\frac{10258495}{3}. \tag{195} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & 2, & 4 \\ & \frac{3}{2}, & 3, & 7 \end{matrix};-\frac{1}{8}\right]=15135920(\ln 2)^2-20982840(\ln 2)+\frac{13144880}{9}. \tag{196} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 3 \\ & \frac{3}{2}, & 7 \end{matrix};-\frac{1}{8}\right]=31173120(\ln 2)^2-43215120(\ln 2)+14977060. \tag{197} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & 2 \\ & \frac{3}{2}, & 7 \end{matrix};-\frac{1}{8}\right]=-16938480(\ln 2)^2+23481720(\ln 2)-\frac{24414190}{3}. \tag{198} $$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ & \frac{7}{2} \end{matrix};\frac{1}{2}\right]=15\pi-45. \tag{199} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ & \frac{5}{2}, & \frac{9}{2} \end{matrix};\frac{1}{2}\right]=49\pi-\frac{455}{3}. \tag{200} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{9}{2} \\ & \frac{5}{2}, & \frac{11}{2} \end{matrix};\frac{1}{2}\right]=\frac{711}{5}\pi-\frac{11109}{25}. \tag{201} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{11}{2} \\ & \frac{5}{2}, & \frac{13}{2} \end{matrix};\frac{1}{2}\right]=\frac{13431}{35}\pi-\frac{1473813}{1225}. \tag{202} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{5}{2} \\ & \frac{3}{2}, & \frac{9}{2} \end{matrix};\frac{1}{2}\right]=\frac{385}{3}\pi-\frac{3605}{9}. \tag{203} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2}, & \frac{9}{2} \\ & \frac{3}{2}, & \frac{7}{2}, & \frac{11}{2} \end{matrix};\frac{1}{2}\right]=333\pi-\frac{5217}{5}. \tag{204} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2}, & \frac{11}{2} \\ & \frac{3}{2}, & \frac{7}{2}, & \frac{13}{2} \end{matrix};\frac{1}{2}\right]=\frac{5841}{7}\pi-\frac{641553}{245}. \tag{205} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ & \frac{3}{2}, & \frac{11}{2} \end{matrix};\frac{1}{2}\right]=\frac{3507}{5}\pi-\frac{55013}{25}. \tag{206} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{7}{2}, & \frac{11}{2} \\ & \frac{3}{2}, & \frac{9}{2}, & \frac{13}{2} \end{matrix};\frac{1}{2}\right]=\frac{24167}{15}\pi-\frac{7967003}{1575}. \tag{207} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{9}{2} \\ & \frac{3}{2}, & \frac{13}{2} \end{matrix};\frac{1}{2}\right]=\frac{21285}{7}\pi-\frac{2339601}{245}. \tag{208} $$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ & \frac{9}{2} \end{matrix};\frac{1}{2}\right]=-70\pi+\frac{665}{3}. \tag{209} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{9}{2} \\ & \frac{7}{2}, & \frac{11}{2} \end{matrix};\frac{1}{2}\right]=-144\pi+\frac{2271}{5}. \tag{210} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{11}{2} \\ & \frac{7}{2}, & \frac{13}{2} \end{matrix};\frac{1}{2}\right]=-\frac{2046}{7}\pi+\frac{225423}{245}. \tag{211} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ & \frac{5}{2}, & \frac{11}{2} \end{matrix};\frac{1}{2}\right]=-\frac{1386}{5}\pi+\frac{21819}{25}. \tag{212} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{7}{2}, & \frac{11}{2} \\ & \frac{5}{2}, & \frac{9}{2}, & \frac{13}{2} \end{matrix};\frac{1}{2}\right]=-\frac{2684}{5}\pi+\frac{886391}{525}. \tag{213} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{9}{2} \\ & \frac{5}{2}, & \frac{13}{2} \end{matrix};\frac{1}{2}\right]=-\frac{33066}{35}\pi+\frac{3638283}{1225}. \tag{214} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{5}{2} \\ & \frac{3}{2}, & \frac{11}{2} \end{matrix};\frac{1}{2}\right]=-588\pi+\frac{9247}{5}. \tag{215} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2}, & \frac{11}{2} \\ \frac{3}{2}, & \frac{9}{2}, & \frac{13}{2}. & \end{matrix};\frac{1}{2}\right] = -\frac{3322}{3}\pi + \frac{1096513}{315}. \tag{216} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2}, & \frac{9}{2} \\ \frac{3}{2}, & \frac{7}{2}, & \frac{13}{2} & \end{matrix};\frac{1}{2}\right] = -\frac{13464}{7}\pi + \frac{1481007}{245}. \tag{217} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ \frac{3}{2}, & \frac{13}{2} & \end{matrix};\frac{1}{2}\right] = -\frac{16962}{5}\pi + \frac{1865501}{175}. \tag{218} $$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ \frac{11}{2} & \end{matrix};\frac{1}{2}\right] = -189\pi + \frac{2961}{5}. \tag{219} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{11}{2} \\ \frac{9}{2}, & \frac{13}{2} & \end{matrix};\frac{1}{2}\right] = 319\pi - \frac{105061}{105}. \tag{220} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{9}{2}, & \frac{11}{2} \\ \frac{7}{2}, & \frac{11}{2}, & \frac{13}{2} & \end{matrix};\frac{1}{2}\right] = \frac{3663}{7}\pi - \frac{402369}{245}. \tag{221} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ \frac{3}{2}, & \frac{13}{2} & \end{matrix};\frac{1}{2}\right] = 297\pi - \frac{32637}{35}. \tag{222} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2} \\ \frac{3}{2}, & \frac{13}{2} & \end{matrix};\frac{1}{2}\right] = 1749\pi - \frac{192247}{35}. \tag{223} $$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ \frac{13}{2} & \end{matrix};\frac{1}{2}\right] = \frac{43593}{35}\pi - 396. \tag{224} $$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ \frac{7}{2} & \end{matrix};-\frac{1}{8}\right] = 1040(\ln 2) - 720. \tag{225} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ \frac{5}{2}, & \frac{9}{2} & \end{matrix};-\frac{1}{8}\right] = -12656(\ln 2) - \frac{26320}{3}. \tag{226} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{9}{2} \\ \frac{5}{2}, & \frac{11}{2} & \end{matrix};-\frac{1}{8}\right] = -\frac{2529744}{25}(\ln 2) + \frac{729936}{5}. \tag{227} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{11}{2} \\ \frac{5}{2}, & \frac{13}{2} & \end{matrix};-\frac{1}{8}\right] = \frac{110}{9}(\ln 2) - \frac{5321437022}{11025}. \tag{228} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{5}{2} \\ \frac{3}{2}, & \frac{9}{2} & \end{matrix} \quad -\frac{1}{8}\right] = -\frac{133840}{3}(\ln 2) + \frac{278320}{9}. \tag{229} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2}, & \frac{9}{2} \\ \frac{3}{2}, & \frac{7}{2}, & \frac{11}{2} & \end{matrix};-\frac{1}{8}\right] = 363408(\ln 2) - \frac{1259472}{5}. \tag{230} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2}, & \frac{11}{2} \\ \frac{3}{2}, & \frac{7}{2}, & \frac{13}{2} & \end{matrix};-\frac{1}{8}\right] = -\frac{73430192}{21}(\ln 2) + \frac{593809392}{245}. \tag{231} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ \frac{3}{2}, & \frac{11}{2} & \end{matrix};-\frac{1}{8}\right] = \frac{5489232}{5}(\ln 2) - \frac{19024208}{25}. \tag{232} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{7}{2}, & \frac{11}{2} \\ \frac{3}{2}, & \frac{9}{2}, & \frac{13}{2} & \end{matrix};-\frac{1}{8}\right] = -\frac{109409168}{15}(\ln 2) - \frac{7962850192}{1575}. \tag{233} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{9}{2} \\ \frac{3}{2}, & \frac{13}{2} & \end{matrix};-\frac{1}{8}\right] = -\frac{143367312}{7}(\ln 2) + \frac{695622576}{49}. \tag{234} $$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ \frac{9}{2} & \end{matrix};-\frac{1}{8}\right] = 35280(\ln 2) - \frac{73360}{3}. \tag{235} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{9}{2} \\ \frac{7}{2}, & \frac{11}{2} & \end{matrix};-\frac{1}{8}\right] = -180144(\ln 2) + \frac{624336}{5}. \tag{236} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{11}{2} \\ \frac{7}{2}, & \frac{13}{2} & \end{matrix};-\frac{1}{8}\right] = \frac{9188784}{7}(\ln 2) - \frac{222921072}{245}. \tag{237} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ \frac{5}{2}, & \frac{11}{2} & \end{matrix};-\frac{1}{8}\right] = -\frac{2839536}{5}(\ln 2) + \frac{9841104}{25}. \tag{238} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{7}{2}, & \frac{11}{2} \\ \frac{5}{2}, & \frac{9}{2}, & \frac{13}{2} & \end{matrix} ; -\frac{1}{8}\right] = \frac{13589136}{5}(\ln 2) - \frac{989023024}{525}. \tag{239} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{9}{2} \\ \frac{5}{2}, & \frac{13}{2} & \end{matrix} ; -\frac{1}{8}\right] = \frac{275839344}{35}(\ln 2) - \frac{6691903152}{1225}. \tag{240} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{5}{2} \\ \frac{3}{2}, & \frac{11}{2} & \end{matrix} ; -\frac{1}{8}\right] = -1472688(\ln 2) + \frac{5103952}{5}. \tag{241} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2}, & \frac{11}{2} \\ \frac{3}{2}, & \frac{9}{2}, & \frac{13}{2} & \end{matrix} ; -\frac{1}{8}\right] = 5996496(\ln 2) - \frac{1309282832}{315}. \tag{242} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2}, & \frac{9}{2} \\ \frac{3}{2}, & \frac{7}{2}, & \frac{13}{2} & \end{matrix} \; -\frac{1}{8}\right] = \frac{124136496}{7}(\ln 2) - \frac{3011569968}{245}. \tag{243} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ \frac{3}{2}, & \frac{13}{2} & \end{matrix} ; -\frac{1}{8}\right] = \frac{194304528}{5}(\ln 2) - \frac{4713857104}{175}. \tag{244} $$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ \frac{11}{2} & \end{matrix} ; -\frac{1}{8}\right] = 789264(\ln 2) - \frac{2735376}{5}. \tag{245} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{11}{2} \\ \frac{9}{2}, & \frac{13}{2} & \end{matrix} ; -\frac{1}{8}\right] = -2200176(\ln 2) + \frac{160129904}{105}. \tag{246} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{9}{2}, & \frac{11}{2} \\ \frac{7}{2}, & \frac{11}{2}, & \frac{13}{2} & \end{matrix} ; -\frac{1}{8}\right] = -\frac{48285072}{7}(\ln 2) + \frac{1171403376}{245}. \tag{247} $$

$$ {}_3F_2\left[\begin{matrix} 2, & 2, & \frac{7}{2} \\ \frac{3}{2}, & \frac{13}{2} & \end{matrix} ; -\frac{1}{8}\right] = -15353712(\ln 2) + \frac{372483408}{35}. \tag{248} $$

$$ {}_4F_3\left[\begin{matrix} 2, & 2, & \frac{5}{2} \\ \frac{3}{2}, & \frac{13}{2} & \end{matrix} ; -\frac{1}{8}\right] = -35084016(\ln 2) + \frac{851143568}{35}. \tag{249} $$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ \frac{13}{2} & \end{matrix} ; -\frac{1}{8}\right] = 14241744(\ln 2) - \frac{345506832}{35}. \tag{250} $$

Proof. The derivation of these mentioned results is quite straightforward, so we chose to establish only one of the results, say (156). To establish result (156), we proceeded as follows. We start with result (52), viz.

$$ \sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+4)(m+5)} = \frac{-1015\pi^2}{64} + \frac{299\pi}{3} - \frac{5633}{36}. \tag{251} $$

Denoting the left-hand side of (251) by S, we obtain

$$ S = \sum_{m=0}^{\infty} \frac{(m!)^2}{(2m)!} \frac{m2^m}{(m+4)(m+5)}. $$

Now, using $m = N+1$, convert all expressions in terms of the Pochhammer's symbol; after some simplification, we obtained

$$ \begin{aligned} S &= \frac{1}{30} \sum_{N=0}^{\infty} \frac{(2)_N\,'(2)_N\,(5)_N}{2^N (1)_N (\frac{3}{2})_N (7)_N} \\ &= \frac{1}{30}\, {}_3F_2\left[\begin{matrix} 2, & 2, & 5 \\ \frac{3}{2}, & 7 & \end{matrix} ; \frac{1}{2}\right] \end{aligned} \tag{252} $$

Finally, equating (251) and (252), we obtained result (156). Other results can be established in a similar way. Therefore, we prefer to omit the details. □

4. Concluding Remark

In this paper, we first developed 20 results in the form of 4 lemmas with 5 results each. This was accomplished using the findings provided by Sherman. By applying the results given in the 4 lemmas, we obtained 104 results related to the Apéry-like series in the form of 4 theorems containing 26 results each. These findings are finally described in terms of generalized hypergeometric functions and we believe that this study is novel to the literature and could make significant contributions to the theories of generalized hypergeometric functions and combinatorics. The findings presented in the research are straightforward, intriguing, and simple to verify.

We conclude this paper by noting that all the results given in this paper have been verified numerically by Mathematica.

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