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Some Identities with Multi-Generalized q-Hyperharmonic Numbers of Order r

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Abstract: The main purpose of this paper is to define multiple alternative q-harmonic numbers, $H_n(k;q)$ and multi-generalized q-hyperharmonic numbers of order r, $H_n^r(k;q)$ by using q-multiple zeta star values (q-MZSVs). We obtain some finite sum identities and give some applications of them for certain combinations of q-multiple polylogarithms $Li_{q;k_1,k_2,\ldots,k_d}(t_1,t_2,\ldots,t_d)$ with the help of generating functions. Additionally, one of the applications is the sum involving q-Stirling numbers and q-Bernoulli numbers.

Keywords: q-multiple zeta values; q-analogue of multiple polylogaritms; multi-generalized q-hyperharmonic numbers of order r; q-series

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1. Introduction

The Riemann zeta function is defined for a complex variable $k \in \mathbb{C}$ with $Re(k) > 1$ by

$$\varsigma(k) = \sum_{n=0}^{\infty} \frac{1}{n^k}.$$

This function appears in mathematics and physics in many different contexts. It plays an important role in analytic number theory and has applications in physics, engineering and applied statistics.

For $k = 2$, the well-known constant [1] is

$$\varsigma(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6},$$

and for $k = 3$, Apéry's constant is

$$\varsigma(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202056903\cdots.$$

Generally, more is known for even zeta values $\varsigma(2m)$. For example, the closed representation is given by

$$\varsigma(2m) = \frac{(-1)^{m+1} B_{2m} (2\pi)^{2m}}{2(2m)!},$$

where the Bernoulli numbers B_n are defined as the coefficients of the generating function of $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$.

In the 1990s, Zagier [2] and Hoffman [3] introduced the multi-variable variant of the Riemann zeta function. By virtue of appearing in the study of many branches of mathematics and theoretical physics, these special values have attracted a lot of attention and interest [4–10].

The Euler–Zagier sums or multiple zeta values are given by for the multi-index $k = (k_1, k_2, \dots, k_d) \in (\mathbb{Z}^+)^d$ with $k_1 > 1$ and $k_j \geq 1$ for $2 \leq j \leq d$,

$$\varsigma(k) := \varsigma(k_1, k_2, \dots, k_d) = \sum_{0<n_1<\cdots<n_d} \prod_{i=1}^{d} \frac{1}{n_i^{k_i}}.$$

The multiple zeta function is a generalization of the Riemann zeta function to a k-tuple of arguments, but for number theoretic reasons, we are mainly interested in the case where the arguments are positive integers. For $d = 1$, they are the values of the Riemann zeta function at positive integers.

In [11,12], the authors defined generalized of $Li_k(t)$ that for $(k_1, k_2, \dots, k_d) \in (\mathbb{Z}^+)^d$,

$$Li_{k_1,k_2,\dots,k_d}(t) = \sum_{0<n_1<n_2<\cdots<n_d} \frac{t^{n_d}}{n_1^{k_1} n_2^{k_2} \dots n_d^{k_d}}, \quad |t| < 1,$$

where d is an integer with a value greater than one. In particular,

$$Li_{1,1,\dots,1}(t) = \frac{1}{r!}(-\log(1-t))^r = \sum_{n=r}^{\infty} (-1)^{n-r} S_1(n,r) \frac{t^n}{n!},$$

where $S_1(n,r)$ stands for the Stirling numbers of the first kind.

Throughout this paper, we assume that q is a real number with $0 < q < 1$.

The q-Pochhammer symbol is given by

$$(x;q)_0 = 1 \text{ and } (x;q)_n = \prod_{k=0}^{n-1} \left(1 - xq^k\right).$$

For any $m, n \in \mathbb{N}$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}},$$

if $n \geq m$, and if $n < m$, then $\begin{bmatrix} n \\ m \end{bmatrix}_q = 0$. It is clear that

$$\lim_{q \to 1} \begin{bmatrix} n \\ m \end{bmatrix}_q = \binom{n}{m},$$

where $\binom{n}{m}$ is the usual binomial coefficient.

In [10,13,14], the q-analogue of the multiple polylogarithms are defined by

$$Li_{q;k_1,k_2,\dots,k_d}(t_1, t_2, \dots, t_d) = \sum_{0<n_1<n_2<\cdots<n_d} \frac{t_1^{n_1} \dots t_d^{n_d}}{[n_1]_q^{k_1} [n_2]_q^{k_2} \dots [n_d]_q^{k_d}}, \; |t_i| < 1, \; i = 1, 2, \dots, d, \tag{1}$$

where $[n]_q = (1 - q^n)/(1 - q)$ is the q-analogue of the positive integer n. In a special case, for $d = 1$, q-polylogarithms are given by

$$Li_{q;k}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}.$$

There are various different versions of q-analogues of multiple zeta values in the literature. We consider the most common version that was first independently studied by Bradley and Zhao [4,10].

For the multi-index $k = (k_1, k_2, \ldots, k_d) \in (\mathbb{Z}^+)^d$ with $k_1 > 1$ and $k_j \geq 1$ for $2 \leq j \leq d$, the multiple q-zeta function [4,7] is the nested infinite series

$$\varsigma(q\,;k) := \varsigma(q\,;k_1, k_2, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \frac{q^{(k_1-1)n_1} \ldots q^{(k_{d-1}-1)n_d} q^{(k_d-1)n_d}}{[n_1]_q^{k_1} \ldots [n_{d-1}]_q^{k_{d-1}} [n_d]_q^{k_d}}, \tag{2}$$

where the sum is over all positive integers n_j satisfying the indicated inequalities, When $d = 0$, the argument list in (2) is empty, and shows $\varsigma[.] := 1$. If the arguments in (2) are positive integers (with $k_1 > 1$ for convergence), it is refered to (2) as a multiple q-zeta value (q-MZV). Note that $\lim_{q \to 1^-} \varsigma(q\,;k) = \varsigma(k)$. For MZVs, there are many linear relations and algebraic ones over $\mathbb{Q}$. For example, these relations are the cyclic sum formula, the Ohno relation and the Ohno–Zagier relation [15–17]. Okuda et al. [7] gave the q-analogue of the Ohno–Zagier relation for the multiple zeta values (MZV's).

Clearly,

$$\varsigma(q\,;k) = Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1}, q^{k_2-1}, \ldots, q^{k_d-1}\right).$$

Kentaro et al. [18] defined the q-multiple zeta star value (q-MZSV) given by

$$\varsigma_n(q\,;k) := \varsigma_n(q\,;k_1, k_2, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d \leq n} \frac{q^{(k_1-1)n_1} \ldots q^{(k_{d-1}-1)n_d} q^{(k_d-1)n_d}}{[n_1]_q^{k_1} \ldots [n_{d-1}]_q^{k_{d-1}} [n_d]_q^{k_d}}. \tag{3}$$

It is clearly seen that

$$\lim_{n \to \infty} \varsigma_n(q\,;k) = \varsigma(q\,;k).$$

Using Rothe's formula [19], it is obtained that

$$\sum_{i=0}^{n} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q t^i = (t;q)_n, \tag{4}$$

and also it is known that

$$\frac{1}{(t;q)_n} = \sum_{i=0}^{\infty} \begin{bmatrix} n+i-1 \\ i \end{bmatrix}_q t^i. \tag{5}$$

The q-extension of the exponential function [20,21] is defined as follows

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}, \; t \in \mathbb{C} \text{ with } |t| < 1. \tag{6}$$

In [22], the q-Stirling numbers of the second kind are given by

$$\frac{1}{[l]_q!} \left(e_q(t) - 1\right)^l = \sum_{n=l}^{\infty} S_{2,q}(n,l) \frac{t^n}{[n]_q!}. \tag{7}$$

In [23], Koparal et al. showed that for $n \geq 0$,

$$\sum_{i=1}^{n+1} q^{(i-1)(m-t)} \begin{bmatrix} t+i-m-2 \\ i-1 \end{bmatrix}_q = q^{n(m-t)} \begin{bmatrix} n+t-m \\ n \end{bmatrix}_q, \tag{8}$$

where m and t are positive integers such that $0 < m \leq t-1$.

Recently, there have been works involving some identities of symmetry for special numbers [24–29]. Bernoulli numbers and polynomials have received much considerable attention throughout mathematical literature [27,30–32]. These numbers are rational numbers. Carlitz first studied q-analogues of Bernoulli numbers [33,34].

In [35], the q-analogues of Bernoulli numbers are defined by the generating function to be

$$\frac{t}{e_q(t)-1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{[n]_q!}. \tag{9}$$

The first few terms of them are $1, -1/[2]_q, q^2/[2]_q[3]_q, \cdots$.

2. Some Sums Involving Multi-Generalized q-Hyperharmonic Numbers of Order r

In this section, first, we will define multiple alternative q-harmonic numbers, $H_n(k;q)$ and multi-generalized q-hyperharmonic numbers of the order r, $H_n^r(k;q)$ and then give applications of them.

Definition 1. *For $n \in \mathbb{Z}^+$ and the multi-index $k = (k_1, k_2, \ldots, k_d) \in (\mathbb{Z}^+)^d$, multiple alternative q-harmonic numbers are defined by*

$$H_0(k;q) := 0 \text{ and } H_n(k;q) := \sum_{j=1}^{n} q^j \varsigma_j(q\,;k) = \sum_{j=1}^{n} q^j \sum_{0<j_1<\cdots<j_d\leq j} \prod_{i=1}^{d} \frac{q^{(k_i-1)j_i}}{[j_i]_q^{k_i}}.$$

Now, we will define multi-generalized q-hyperharmonic numbers of order r using $H_n(k;q)$ as follows

Definition 2. *For the multi-index $k = (k_1, k_2, \ldots, k_d) \in (\mathbb{Z}^+)^d$, when $r < 0$ or $n \leq 0$, $H_n^r(k;q) = 0$, and when $n \geq 1$, multi-generalized q-hyperharmonic numbers of order r, $H_n^r(k;q)$ are defined by*

$$H_n^r(k;q) = H_n^r(k_1, k_2, \ldots, k_d; q) = \sum_{i=1}^{n} q^i H_i^{r-1}(k;q), \quad r \geq 1,$$

where $H_n^0(k;q) = \varsigma_n(q\,;k)$.

When $r = 1$, $H_n^1(k;q) = H_n(k;q)$. It is clearly seen that

$$H_n^r(k;q) = q^n H_n^{r-1}(k;q) + H_{n-1}^r(k;q). \tag{10}$$

Lemma 1. *For the multi-index $k = (k_1, k_2, \ldots, k_d) \in (\mathbb{Z}^+)^d$ and $n, r \in \mathbb{Z}^+$, we have*

$$\sum_{i=1}^{r} q^{n(r-i)} H_{n-1}^i(k;q) = H_n^r(k;q) - q^{nr}\varsigma_n(q\,;k).$$

Proof. For the proof of this sum, we will use double induction on n and r. For $n = r = 1$, by Definition 1, it is clear. Suppose that for $n \geq 1$, the claim is true. In that

$$H_n(k;q) = H_{n-1}(k;q) + q^n \varsigma_n(q\,;k).$$

For $n+1$, using induction hypothesis and Definition 1, we have

$$\begin{aligned}
& H_n(k;q) + q^{n+1}\varsigma_{n+1}(q\,;k) \\
= \; & H_{n-1}(k;q) + q^n \varsigma_n(q\,;k) + q^{n+1}\varsigma_{n+1}(q\,;k) \\
= \; & H_{n-2}(k;q) + q^{n-1}\varsigma_{n-1}(q\,;k) + q^n \varsigma_n(q\,;k) + q^{n+1}\varsigma_{n+1}(q\,;k) \\
= \; & \ldots
\end{aligned}$$

$$
\begin{aligned}
&= H_0(k;q) + q\varsigma_1(q\,;k) + q^2\varsigma_2(q\,;k) + \ldots + q^n\varsigma_n(q\,;k) + q^{n+1}\varsigma_{n+1}(q\,;k) \\
&= \sum_{j=1}^{n+1} q^j\varsigma_j(q\,;k) = H_{n+1}(k;q).
\end{aligned}
$$

Finally, let $r \geq 1$, and suppose that the claim is true for $r-1$. In that for all $n \geq 1$,

$$
\sum_{i=1}^{r-1} q^{n(r-i-1)} H_{n-1}^{i}(k;q) = H_n^{r-1}(k;q) - q^{n(r-1)}\varsigma_n(q\,;k).
$$

We will show that the claim is true for r. Using the induction hypothesis and (10), we write

$$
\begin{aligned}
\sum_{i=1}^{r} q^{n(r-i)} H_{n-1}^{i}(k;q) &= H_{n-1}^{r}(k;q) + q^n \sum_{i=1}^{r-1} q^{n(r-1-i)} H_{n-1}^{i}(k;q) \\
&= H_{n-1}^{r}(k;q) + q^n \left(H_n^{r-1}(k;q) - q^{n(r-1)}\varsigma_n(q\,;k) \right) \\
&= H_{n-1}^{r}(k;q) + q^n H_n^{r-1}(k;q) - q^{nr}\varsigma_n(q\,;k) \\
&= H_n^{r}(k;q) - q^{nr}\varsigma_n(q\,;k).
\end{aligned}
$$

By the principle of double induction, for all $n \geq 1$ and $r \geq 1$, the desired result is true. This completes the proof. □

Theorem 1. *For the multi-index $k = (k_1, k_2, \ldots, k_d) \in (\mathbb{Z}^+)^d$ and $n, r \in \mathbb{Z}^+$, we have*

$$
H_n^{r}(k;q) = \sum_{i=1}^{n} q^{ri} \begin{bmatrix} n-i+r-1 \\ r-1 \end{bmatrix}_q \varsigma_i(q\,;k).
$$

Proof. By double induction on n and r, we will start the proof. For $n = r = 1$ and for $n = 1$ and $r > 1$, by Definition 2, it is clear. Let $n \geq 1$, and suppose that the claim is true for $n-1$. In that, for all $r \geq 1$,

$$
H_{n-1}^{r}(k;q) = \sum_{i=1}^{n-1} q^{ri} \begin{bmatrix} n-i+r-2 \\ r-1 \end{bmatrix}_q \varsigma_i(q\,;k).
$$

We will show that the claim is true for n. Using Lemma 1 and the induction hypothesis, we write

$$
\begin{aligned}
H_n^{r}(k;q) &= q^{nr}\varsigma_n(q\,;k) + \sum_{j=1}^{r} q^{n(r-j)} H_{n-1}^{j}(k;q) \\
&= q^{nr}\varsigma_n(q\,;k) + \sum_{j=1}^{r} q^{n(r-j)} \sum_{i=1}^{n-1} q^{ij} \begin{bmatrix} n+j-i-2 \\ j-1 \end{bmatrix}_q \varsigma_i(q\,;k) \\
&= q^{nr}\varsigma_n(q\,;k) + q^{nr} \sum_{i=1}^{n-1} \varsigma_i(q\,;k) \sum_{j=1}^{r} q^{j(i-n)} \begin{bmatrix} n+j-i-2 \\ j-1 \end{bmatrix}_q.
\end{aligned}
$$

From (8), we have

$$
\begin{aligned}
H_n^{r}(k;q) &= q^{nr}\varsigma_n(q;k) + \sum_{i=1}^{n-1} q^{ri} \begin{bmatrix} n-i+r-1 \\ r-1 \end{bmatrix}_q \varsigma_i(q\,;k) \\
&= \sum_{i=1}^{n} q^{ri} \begin{bmatrix} n-i+r-1 \\ r-1 \end{bmatrix}_q \varsigma_i(q\,;k).
\end{aligned}
$$

By the principle of double induction, for all $n \geq 1$ and $r \geq 1$, the desired result is true. This completes the proof. □

Lemma 2. *For the multi-index $k = (k_1, k_2, \ldots, k_d) \in (\mathbb{Z}^+)^d$ and $r \in \mathbb{Z}^+$, we have*

$$\sum_{n=1}^{\infty} q^{rn} \varsigma_n(q\,; k) t^n = \frac{1}{1-q^r t} Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1}, q^{k_2-1}, \ldots, q^{k_d+r-1} t\right). \tag{11}$$

Proof. By (3), consider that

$$\begin{aligned}
\sum_{n=1}^{\infty} q^{rn} \varsigma_n(q\,; k) t^n &= \sum_{n=1}^{\infty} \sum_{n_d=1}^{n} \sum_{0<n_1<\cdots<n_d} q^{rn} \frac{q^{(k_1-1)n_1} \cdots q^{(k_{d-1}-1)n_{d-1}} q^{(k_d-1)n_d}}{[n_1]_q^{k_1} \cdots [n_{d-1}]_q^{k_{d-1}} [n_d]_q^{k_d}} t^n \\
&= \sum_{n_d=1}^{\infty} \sum_{0<n_1<\cdots<n_d} \frac{q^{(k_1-1)n_1} \cdots q^{(k_{d-1}-1)n_{d-1}} q^{(k_d-1)n_d}}{[n_1]_q^{k_1} \cdots [n_{d-1}]_q^{k_{d-1}} [n_d]_q^{k_d}} \sum_{n=n_d}^{\infty} q^{rn} t^n \\
&= \frac{1}{1-q^r t} Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1}, q^{k_2-1}, \ldots, q^{k_d+r-1} t\right),
\end{aligned}$$

as claimed. □

Theorem 2. *For the multi-index $k = (k_1, k_2, \ldots, k_d) \in (\mathbb{Z}^+)^d$ and $r \in \mathbb{Z}^+$, we have*

$$\sum_{n=1}^{\infty} H_n^r(k; q) t^n = \frac{1}{(t; q)_{r+1}} Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1}, q^{k_2-1}, \ldots, q^{k_d+r-1} t\right). \tag{12}$$

Proof. From (5) and (11), using Theorem 1, we have

$$\begin{aligned}
&\frac{1}{(t; q)_{r+1}} Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1}, q^{k_2-1}, \ldots, q^{k_d+r-1} t\right) \\
= &\frac{1}{(t; q)_r} \frac{1}{1-q^r t} Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1}, q^{k_2-1}, \ldots, q^{k_d+r-1} t\right) \\
= &\sum_{i=0}^{\infty} \begin{bmatrix} i+r-1 \\ r-1 \end{bmatrix}_q t^i \sum_{n=1}^{\infty} q^{rn} \varsigma_n(q\,; k) t^n \\
= &\sum_{n=1}^{\infty} \sum_{i=1}^{n} q^{ri} \begin{bmatrix} n-i+r-1 \\ r-1 \end{bmatrix}_q \varsigma_i(q\,; k) t^n = \sum_{n=1}^{\infty} H_n^r(k; q) t^n,
\end{aligned}$$

as claimed. □

Theorem 3. *For the multi-index $k = (k_1, k_2, \ldots, k_d) \in (\mathbb{Z}^+)^d$ and $n, r \in \mathbb{Z}^+$, we have*

$$\sum_{i=1}^{n} (-1)^{n-i} q^{\binom{n-i}{2} - rn} \begin{bmatrix} r \\ n-i \end{bmatrix}_q H_i^r(k; q) = \varsigma_n(q\,; k).$$

Proof. From (4), (11) and (12), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} q^{rn} \varsigma_n(q\,; k) t^n &= \frac{1}{1-q^r t} Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1}, q^{k_2-1}, \ldots, q^{k_d+r-1} t\right) \\
&= (t; q)_r \sum_{i=1}^{\infty} H_i^r(k; q) t^i \\
&= \sum_{i=0}^{\infty} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}_q t^i \sum_{i=1}^{\infty} H_i^r(k; q) t^i \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} (-1)^{n-i} q^{\binom{n-i}{2}} \begin{bmatrix} r \\ n-i \end{bmatrix}_q H_i^r(k; q) t^n.
\end{aligned}$$

This completes the proof. □

Theorem 4. *For the multi-index $k=(k_1,k_2,\ldots,k_d)\in(\mathbb{Z}^+)^d$ and $n,r\in\mathbb{Z}^+$, we have*

$$\sum_{i=1}^{n}(-1)^i[i]_q!H_i^r(k;q)S_{2,q}(n,i) = \sum_{m=1}^{n}\sum_{j=0}^{n-m}\sum_{i=1}^{m}q^{ri}(-1)^{i+j}[i]_q![j]_q!\begin{bmatrix}j+r-1\\ j\end{bmatrix}_q\begin{bmatrix}n\\ m\end{bmatrix}_q \times S_{2,q}(n-m,j)S_{2,q}(m,i)\varsigma_i(q;k).$$

Proof. By (7) and (12), we have

$$\begin{aligned}
&\frac{Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1},q^{k_2-1},\ldots,q^{k_d+r-1}(1-e_q(-t))\right)}{(1-e_q(-t);q)_{r+1}}\\
&= \sum_{i=1}^{\infty}H_i^r(k;q)\frac{(1-e_q(-t))^i}{[i]_q!}[i]_q! \\
&= \sum_{i=1}^{\infty}(-1)^i[i]_q!H_i^r(k;q)\sum_{n=i}^{\infty}(-1)^nS_{2,q}(n,i)\frac{t^n}{[n]_q!}\\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n}(-1)^{n+i}H_i^r(k;q)S_{2,q}(n,i)\frac{[i]_q!}{[n]_q!}t^n,
\end{aligned} \tag{13}$$

and by (5), (7) and (11), then

$$\begin{aligned}
&\frac{Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1},q^{k_2-1},\ldots,q^{k_d+r-1}(1-e_q(-t))\right)}{(1-e_q(-t);q)_{r+1}}\\
&= \sum_{j=0}^{\infty}\begin{bmatrix}j+r-1\\ r-1\end{bmatrix}_q(1-e_q(-t))^j\sum_{i=1}^{\infty}\varsigma_i(q;k)q^{ri}(1-e_q(-t))^i\\
&= \sum_{j=0}^{\infty}(-1)^j\begin{bmatrix}j+r-1\\ r-1\end{bmatrix}_q[j]_q!\sum_{n=j}^{\infty}(-1)^nS_{2,q}(n,j)\frac{t^n}{[n]_q!}\\
&\quad\times\sum_{i=1}^{\infty}(-1)^iq^{ri}[i]_q!\varsigma_i(q;k)\sum_{n=i}^{\infty}(-1)^nS_{2,q}(n,i)\frac{t^n}{[n]_q!}\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^{n+j}\begin{bmatrix}j+r-1\\ r-1\end{bmatrix}_q\frac{[j]_q!}{[n]_q!}S_{2,q}(n,j)t^n\\
&\quad\times\sum_{n=1}^{\infty}\sum_{i=1}^{n}(-1)^{n+i}q^{ri}\frac{[i]_q!}{[n]_q!}\varsigma_i(q;k)S_{2,q}(n,i)t^n\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{j=0}^{n-m}\sum_{i=1}^{m}(-1)^{n+i+j}\begin{bmatrix}j+r-1\\ r-1\end{bmatrix}_q\begin{bmatrix}n\\ m\end{bmatrix}_q\frac{[i]_q![j]_q!}{[n]_q!}\\
&\quad\times q^{ri}\varsigma_i(q;k)S_{2,q}(n-m,j)S_{2,q}(m,i)t^n.
\end{aligned} \tag{14}$$

Thus, by comparing the coefficients on the right sides of (13) and (14), this completes the proof. □

For example, for $d=1$, we write $k_1=l$ and $q\to 1^-$; then

$$\sum_{i=1}^{n}(-1)^i i!H_i^r(l)S_2(n,i) = \sum_{m=1}^{n}\sum_{j=0}^{n-m}\sum_{i=1}^{m}(-1)^{i+j}i!j!\binom{j+r-1}{j}\binom{n}{m}S_2(n-m,j)S_2(m,i)H_{i,l}\,,$$

where $H_{n,l}=\sum\limits_{j=1}^{n}\frac{1}{j^l}$ is harmonic number of order l.

Theorem 5. *For the multi-index $k=(k_1,k_2,\ldots,k_d)\in(\mathbb{Z}^+)^d$ and $n\in\mathbb{Z}^+$, we have*

$$\begin{aligned}&\sum_{i=0}^{n}(-1)^i q^{r(i+1)}\varsigma_{i+1}(q\,;k)S_{2,q}(n,i)[i+1]_q!\\ =\ &\sum_{m=0}^{n}\sum_{i=1}^{m+1}(-1)^i q^{ri}\frac{[i]_q[i]_q!}{[m+1]_q}\begin{bmatrix}n\\ m\end{bmatrix}_q B_{n-m,q}\varsigma_i(q\,;k)S_{2,q}(m+1,i).\end{aligned}$$

Proof. By (1), we have

$$\begin{aligned}&\frac{1}{(1-e_q(-t))}\sum_{i=1}^{\infty}[i]_q\varsigma_i(q\,;k)q^{ri}(1-e_q(-t))^i\\ =\ &\sum_{i=1}^{\infty}\varsigma_i(q\,;k)q^{ri}[i]_q!\frac{(1-e_q(-t))^{i-1}}{[i-1]_q!}\\ =\ &\sum_{i=0}^{\infty}\varsigma_{i+1}(q\,;k)q^{r(i+1)}[i+1]_q!\frac{(1-e_q(-t))^{i}}{[i]_q!}\\ =\ &\sum_{i=0}^{\infty}(-1)^i\varsigma_{i+1}(q\,;k)q^{r(i+1)}[i+1]_q!\sum_{n=i}^{\infty}(-1)^n S_{2,q}(n,i)\frac{t^n}{[n]_q!}\\ =\ &\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^{n+i}q^{r(i+1)}\varsigma_{i+1}(q\,;k)S_{2,q}(n,i)\frac{[i+1]_q!}{[n]_q!}t^n,\end{aligned}\tag{15}$$

and by (9),

$$\begin{aligned}&\frac{1}{(1-e_q(-t))}\sum_{i=1}^{\infty}[i]_q\varsigma_i(q\,;k)q^{ri}(1-e_q(-t))^i\\ =\ &\frac{1}{t}\sum_{n=0}^{\infty}B_{n,q}\frac{(-1)^{n+1}t^n}{[n]_q!}\sum_{i=1}^{\infty}(-1)^i[i]_q[i]_q!\varsigma_i(q\,;k)q^{ri}\frac{(e_q(-t)-1)^i}{[i]_q!}\\ =\ &\frac{1}{t}\sum_{n=0}^{\infty}B_{n,q}\frac{(-1)^{n+1}t^n}{[n]_q!}\sum_{i=1}^{\infty}(-1)^i[i]_q[i]_q!\varsigma_i(q\,;k)q^{ri}\sum_{n=i}^{\infty}(-1)^n S_{2,q}(n,i)\frac{t^n}{[n]_q!}\\ =\ &\sum_{n=0}^{\infty}B_{n,q}\frac{(-1)^{n+1}t^n}{[n]_q!}\sum_{n=0}^{\infty}\sum_{i=1}^{n+1}(-1)^{n+i+1}q^{ri}\varsigma_i(q\,;k)S_{2,q}(n+1,i)\frac{[i]_q[i]_q!}{[n+1]_q!}t^n\\ =\ &\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{i=1}^{m+1}(-1)^{n+i}q^{ri}\frac{[i]_q[i]_q!}{[m+1]_q!}\frac{B_{n-m,q}}{[n-m]_q!}\varsigma_i(q\,;k)S_{2,q}(m+1,i)t^n.\end{aligned}\tag{16}$$

Thus, by combining the coefficients on the right sides of (15) and (16), this finishes the proof. □

Theorem 6. *For the multi-index $k=(k_1,k_2,\ldots,k_d)\in(\mathbb{Z}^+)^d$ and $n,r\in\mathbb{Z}^+$, we have*

$$\begin{aligned}&\sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{i+j}[i]_q!\begin{bmatrix}n\\ j\end{bmatrix}_q H_i^r(k;q)S_{2,q}(j,i)\\ =\ &\sum_{t=1}^{n}\sum_{i=0}^{n-t}\sum_{m=1}^{t}\sum_{j=1}^{m}(-1)^{n+m+j+t+i}q^{rj}[i]_q![j]_q!\\ &\quad\times\begin{bmatrix}n\\ t\end{bmatrix}_q\begin{bmatrix}t\\ m\end{bmatrix}_q\begin{bmatrix}i+r-1\\ r-1\end{bmatrix}_q\varsigma_j(q\,;k)S_{2,q}(m,j)S_{2,q}(n-t,i).\end{aligned}$$

Proof. From (6) and (12), we have

$$
\begin{aligned}
&\frac{Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1},q^{k_2-1},\ldots,q^{k_d+r-1}(1-e_q(-t))\right)}{(1-e_q(-t);q)_{r+1}}e_q(t)\\
&= \sum_{i=1}^{\infty}(-1)^i[i]_q!H_i^r(k;q)\frac{(e_q(-t)-1)^i}{[i]_q!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\\
&= \sum_{i=1}^{\infty}(-1)^i[i]_q!H_i^r(k;q)\sum_{n=i}^{\infty}(-1)^n S_{2,q}(n,i)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n}(-1)^{n+i}H_i^r(k;q)S_{2,q}(n,i)\frac{[i]_q!}{[n]_q!}t^n\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{i+j}\begin{bmatrix} n\\ j\end{bmatrix}_q H_i^r(k;q)S_{2,q}(j,i)\frac{[i]_q!}{[n]_q!}t^n,
\end{aligned}
\tag{17}
$$

and by (1), (5) and (7), then

$$
\begin{aligned}
&\frac{Li_{q;k_1,k_2,\ldots,k_d}\left(q^{k_1-1},q^{k_2-1},\ldots,q^{k_d+r-1}(1-e_q(-t))\right)}{(1-e_q(-t);q)_{r+1}}e_q(t)\\
&= \sum_{i=0}^{\infty}\begin{bmatrix} r-1+i\\ r-1\end{bmatrix}_q(1-e_q(-t))^i\sum_{j=1}^{\infty}\varsigma_j(q;k)q^{rj}(1-e_q(-t))^j\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\\
&= \sum_{i=0}^{\infty}(-1)^i\begin{bmatrix} r-1+i\\ r-1\end{bmatrix}_q[i]_q!\sum_{n=i}^{\infty}(-1)^n S_{2,q}(n,i)\frac{t^n}{[n]_q!}\\
&\quad\times\sum_{j=1}^{\infty}(-1)^j\varsigma_j(q;k)q^{rj}[j]_q!\sum_{n=j}^{\infty}(-1)^n S_{2,q}(n,j)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^{n+i}\frac{[i]_q!}{[n]_q!}\begin{bmatrix} r-1+i\\ r-1\end{bmatrix}_q S_{2,q}(n,i)t^n\\
&\quad\times\sum_{n=1}^{\infty}\sum_{j=1}^{n}(-1)^{n+j}\varsigma_j(q;k)q^{rj}\frac{[j]_q!}{[n]_q!}S_{2,q}(n,j)t^n\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^{n+i}\frac{[i]_q!}{[n]_q!}\begin{bmatrix} r-1+i\\ r-1\end{bmatrix}_q S_{2,q}(n,i)t^n\\
&\quad\times\sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{j=1}^{m}(-1)^{m+j}q^{rj}\varsigma_j(q;k)\frac{[j]_q!}{[m]_q![n-m]_q!}S_{2,q}(m,j)t^n\\
&= \sum_{n=1}^{\infty}\sum_{t=1}^{n}\sum_{i=0}^{n-t}\sum_{m=1}^{t}\sum_{j=1}^{m}(-1)^{n+m+j+t+i}q^{rj}\frac{[j]_q![i]_q!}{[n]_q!}\\
&\quad\times\begin{bmatrix} n\\ t\end{bmatrix}_q\begin{bmatrix} t\\ m\end{bmatrix}_q\begin{bmatrix} i+r-1\\ r-1\end{bmatrix}_q\varsigma_j(q;k)S_{2,q}(m,j)S_{2,q}(n-t,i)t^n.
\end{aligned}
\tag{18}
$$

Thus, by comparing the coefficients on the right sides of (17) and (18), this completes the proof. □

3. Conclusions

In this paper, we defined multiple alternative q-harmonic numbers and multi-generalized q-hyperharmonic numbers of order r. We derived the generating functions of q-hyperharmonic numbers of order r. We gave the closed form of q-hyperharmonic numbers of order r in Theorem 1 and some sums involving q-hyperharmonic numbers of order r and the q-Stirling numbers of the second kind in Theorems 4–6. As one of our next thoughts, we would like to examine some applications of matrices with entries made up of these numbers. For example, we can derive explicit formulae for their $LU-$decompositions and inverses.

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