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Regular and Intra-Regular Ternary Semirings in Terms of *m*-Polar Fuzzy Ideals

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Abstract: In practical applications, the basic fuzzy set is used via symmetric uncertainty variables. In the research field, it is comparatively rare to discuss two-fold uncertainty due to its complication. To deal with the multi-polar uncertainty in real life problems, m-polar (multi-polar) fuzzy (m-PF) sets are put forward. The main objective of this paper is to explore the idea of m-PF sets, which is a generalization of bipolar fuzzy (BPF) sets, in ternary semirings. The major aspects and novel distinctions of this work are that it builds any multi-person, multi-period, multi-criteria, and complex hierarchical problems. The main focus of this study is to confine generalization of some important results of BPF sets to the results of m-PF sets. In this research, the notions of m-polar fuzzy ternary subsemiring (m-PFSS), m-polar fuzzy ideal (m-PFI), m-polar fuzzy generalized bi-ideal (m-PFGBI), m-polar fuzzy bi-ideal (m-PFBI), and m-polar fuzzy quasi-ideal (m-PFQI) in ternary semirings are introduced. Moreover, this paper deals with several important properties of m-PFIs and characterizes regular and intra-regular ternary semiring in terms of these ideals.

Keywords: m-polar fuzzy ternary subsemiring; m-polar fuzzy ideals; regular ternary semiring; intra-regular ternary semiring

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1. Introduction

High-ranking organization makes native decisions in every field with multiple aspects of a situation. This concept can also be applied to create a formalized model system for conducting fair elections. For the growth and development of democratic countries, free and fair elections are required. Since Pakistan is a multi-party democratic country where, in the national and provincial assemblies, political parties compete for seats. Specifically, in district Gujrat, the political parties are PMLQ, PPP, PMLN, PTI, and any independent candidates. There is always a fierce competition between them. It a big challenge in Gujrat as well as in Pakistan. To handle the hard problems occurring in elections, we can use the model based on the m-polar fuzzy set to select the appropriate candidate conveniently. This would increase the reliability of that selected party between its followers and the voters. If we choose a leader without any rigging, then we can build a peaceful society based on justice. In the same way, all illegal activities can be stopped and soon the country can move forward. It is a lasting solution for the government and citizens to conflict and insecurity. With regards to this, the idea of m-polar (multi-polar) fuzzy set, which is the generalization of bi-polar fuzzy set, is explored. In the study of m-polar fuzzy set, we need to evaluate the existence of multi-polar information about the given set. When have assigned the membership degree to many objects according to multi-polar information, then the m-polar fuzzy set will work successfully. The m-PF set is an extension of the BPF set. Fuzzy set is an appropriate theory for handling the uncertainties of world problems. The theory of the fuzzy set is well established, it has a wide range of applications in various disciplines containing medical diagnosis, computer networks, artificial intelligence, social sciences, management sciences, decision making problems, and many more. A membership function with a unit interval $[0,\ 1]$ range demonstrates a fuzzy set. The degree of belongingness of entries to a set is known as the membership degree. In 1965, Zadeh proposed the idea of fuzzy set for the first time [1]. The structure of the fuzzy group was first introduced by Rosenfeld in 1971 [2]. There have been several different types of fuzzy set extensions over the years, such as interval-valued fuzzy sets, intuitionistic fuzzy sets, and so on. It is difficult to express the distinction between irrelevant and contrary elements in a fuzzy set by using membership degrees ranging from 0 to 1 based on these studies. Zhang [3] introduced the concept of BPF set. The BPF set is in fact an extension of the fuzzy set with membership degree $[-1,1]$. The membership degree 0 indicates that element does not belong to the corresponding property, the membership degree $(0,1]$ shows that elements fulfil the property to a certain extent, and the membership degree $[-1,0]$ shows that elements fulfil the implicit counter property to a certain extent in a BPF set. For the importance of BPF set, see [4–8].

The BPF set and 2-polar fuzzy set are cryptomorphic mathematical concepts that can be obtained briefly from one another. Actually, a natural one-to-one correspondence exists between BPF set and 2-polar fuzzy set. By using this idea of one-to-one correspondence, the BPF set is extended to m-PF set. Sometimes, different objects have been monitored in different ways. This led to the study of m-PF set. The concept that lies behind such description is linked with the existence of multi-polar information about the given set. The m-PF set works successfully to assign the membership degrees to several objects regarding multi-polar information. Here, no membership degree will be assumed as negative as m-PF set provides only a positive degree of memberships of each element [9].

An m-PF set can be considered as m different fuzzy sets, similar to the case of BPF set. Therefore, in this condition every input is represented by an m-dimentional vector of numbers from the unit interval $[0,1]$, all indicate a confidence degree. Assume that the set $K = \{1,2,\ldots,m\}$ is the set of context. Then, for each $\kappa \in K$, an m-PF set will represent the element's satisfaction degree regarding to kth context [10]. For example, if we take a fuzzy set "good person" then there are different interpretations among the people of a particular area.

We have noticed that we face many problems in real world having multipolar information from multi-agents. For instance, $[0,\ 1]^n$ ($n \approx 7 \times 109$) contains the exact degree of mankind's telecommunication safety due to monitoring of different person in different times. There are many other instances, such as degree of inclusion (rough measures, fuzziness measures, accuracy measures, decision performance evaluations, and approximation qualities), ordering results of universities, and ordering results of magazines. The objective of this study is to make extension of bipolar fuzzy ideals into m-PFIs, because we have uncertainty and vagueness of data involving multi-attributes and multi-agents in real life problems.

1.1. Innovative Contribution

A semiring is an algebraic structure that is very useful in mathematics and used in engineering, physics, computer, coding, topological space, automata theory, formal languages, modelling, and graph theory. There are numerous structures that are not handled by using binary multiplication of semiring, which is a reason for the existence of ternary framework. Lehmer [11] was the first to propose the idea of a ternary algebraic structure to deal with such types of problems. For example, $\mathbb{Z}$ is a ring that is essential in ring theory and its subset $\mathbb{Z}^+$ is a subsemiring of $\mathbb{Z}$, but $\mathbb{Z}^-$ is not closed under binary product, therefore, it is not a subsemiring of $\mathbb{Z}$. While $\mathbb{Z}^-$ is closed under ternary multiplication. In this paper, we initiate the study of m-PFIs in ternary semirings.

1.2. Related Works

Kar and Dutta created the idea of ternary semiring, and they studied the characteristics of ternary semirings in [12,13]. Kavikumar and Khamis studied fuzzy ideals and fuzzy quasi-ideals in ternary semirings [14] and Kavikumar et al. studied fuzzy bi-ideals in ternary semirings [15]. The study of m-PF algebraic structure began with the concept of m-PF Lie subalgebras [16]. Later on, the theory of m-PF Lie ideals was studied in Lie algebras [17]. The idea of the m-PF graphs was given in [18,19]. In 2019, Ahmad and Al-Masarwah introduced the concept of m-PF (commutative) ideals and m-polar (α, β)-fuzzy ideals [20,21] in BCK/BCI-algebras. By continuing this work, they introduced a new form of generalized m-PFIs in [22] and studied normalization of m-PF subalgebras in [23]. In 2021, Shabir et al. [24] studied m-PFIs in terms of LA-semigroups. In 2021, m-PFIs of semigroup introduced by Bashir [25]. Recently, Bashir et al. [26] have worked on multi-polar fuzzy ideals of ternary semigroup.

1.3. Organization of the Paper

This paper is organized as follows: in Section 2, we give the basic concept of m-PF ternary semiring. Section 3 is the main section of this paper in which m-PF subsets, m-PFIs, m-PFGBIs, m-PFBIs, and m-PFQIs of ternary semirings are discussed. In Section 4, we have characterized regular ternary semiring by the properties of m-PFIs. In Section 5, a comparison of this paper to past work is provided. In the last, we provide the conclusions of our research as well as our future plans. The list of acronyms is given in Table 1.

Table 1. List of acronyms.

Acronyms	Representation
m-PF	m-Polar fuzzy
BFS	Bipolar fuzzy set
RI (resp, LI, MI)	Right ideal (resp. Left ideal, Lateral ideal)
GBI (resp. BI, QI)	Generalized bi-ideal (resp. Bi-ideal, Quasi ideal)
m-PFTS	m-Polar fuzzy ternary subsemiring
m-PFI	m-Polar fuzzy ideal
m-PFLI	m-Polar fuzzy left ideal
m-PFRI	m-Polar fuzzy right ideal
m-PFMI	m-Polar fuzzy lateral ideal
m-PFGBI	m-Polar fuzzy generalized bi-ideal
m-PFBI	m-Polar fuzzy bi-ideal
m-PFQI	m-Polar fuzzy quasi-ideal

2. Preliminaries

This section includes some basic and necessary definitions, preliminaries, and results based on ternary semirings that are important in their own right. These are prerequisites for later sections. A non-empty set E with the dual operations '$+$' (usual addition) and '$*$' (ternary product) is stated as ternary semiring if E is abelian monoid under '$+$' and semigroup under ternary multiplication and distributive laws of ternary multiplication over addition holds. Identity of the ternary semiring is an element 'e' such that $eet = tee = ete = t$ for all $t \in E$ [27,28]. Throughout this paper, E will represent a ternary semiring except otherwise specified. An additive subsemigroup Y of E is called a ternary subsemiring of E if $Y_1Y_2Y_3 \subseteq Y$ for all $Y_1, Y_2, Y_3 \subseteq Y$. If Y is a non-empty subset of E, then it is called LI (resp. MI, RI) of E if Y is closed under binary addition and $EEY \subseteq Y$ (resp. $EYE \subseteq Y, YEE \subseteq Y$). In this paper, subset means non-empty subset. A ternary subsemiring Y of E is called BI of E if $YEYEY \subseteq Y$ [15]. Let $(Y, +)$ be an additive subsemigroup of E then Y is called QI of E if $YEE \cap (EYE + EEYEE) \cap EEY \subseteq Y$ [14].

An m-PF subset (or a $[0,1]^m$-set) on E is a mapping $\psi : E \to [0,1]^m$. An m-PF set is the m-tuple of membership degree function of E that is $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ where $\psi_\kappa : E \to [0,1]$ is a mapping for all $\kappa \in \{1, 2, \ldots, m\}$. Now, we express operations $\psi \wedge \mu \wedge \gamma$ and $\psi \vee \mu \vee \gamma$ on three m-PF subsets $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ of E.$(\psi \wedge \mu \wedge \gamma)(s) = \psi(s) \wedge \mu(s) \wedge \gamma(s)$ and $(\psi \vee \mu \vee \gamma)(s) = \psi(s) \vee \mu(s) \vee \gamma(s)$, that is $(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(s) = \psi_\kappa(s) \wedge \mu_\kappa(s) \wedge \gamma_\kappa(s)$ and $(\psi_\kappa \vee \mu_\kappa \vee \gamma_\kappa)(s) = \psi_\kappa(s) \vee \mu_\kappa(s) \vee \gamma_\kappa(s)$, for all $s \in E$ and $\kappa \in \{1, 2, \ldots, m\}$. Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$,

$\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ be m-PF subsets of E. Then product $\psi \circ \mu \circ \gamma = (\psi_1 \circ \mu_1 \circ \gamma_1, \psi_2 \circ \mu_2 \circ \gamma_2, \ldots, \psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa)$ and addition $\psi + \mu = (\psi_1 + \mu_1, \psi_2 + \mu_2, \ldots, \psi_\kappa + \mu_\kappa)$ is defined by

$$(\psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa)(x) = \begin{cases} \bigvee_{x=rst} \{\psi_\kappa(r) \wedge \mu_\kappa(s) \wedge \gamma_\kappa(t)\}, & \text{if } x = rst; \\ 0 & \text{otherwise}; \end{cases}$$

for some $r, s, t \in E$, and for all $\kappa \in \{1, 2, \ldots, m\}$.

$$(\psi_\kappa + \mu_\kappa)(x) = \begin{cases} \bigvee_{x=r+s} \{\psi_\kappa(r) \wedge \mu_\kappa(s)\}, & \text{if } x = r + s; \\ 0 & \text{otherwise}; \end{cases}$$

for some $r, s \in E$, and for all $\kappa \in \{1, 2, \ldots, m\}$.

The next Example shows the product and addition of m-PF subsets ψ, μ and γ of E for $m = 3$.

Example 1. *Let $E = \{0, v, w\}$ be a ternary semiring under the operations given in Tables 2–5.*

Table 2. Addition.

+	**0**	v	w
0	0	v	w
v	v	v	w
w	w	v	w

Table 3. Multiplication under **0**.

0	**0**	v	w
0	0	0	0
v	0	0	0
w	0	0	0

Table 4. Multiplication under v.

v	**0**	v	w
0	0	0	0
v	0	w	v
w	0	v	w

Table 5. Multiplication under w.

w	**0**	v	w
0	0	0	0
v	0	v	w
w	0	w	v

We define 3-PF subset $\psi = (\psi_1, \psi_2, \psi_3)$, $\mu = (\mu_1, \mu_2, \mu_3)$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ as follows:

$$\psi(0) = (0.1, 0.9, 0.7),\ \psi(v) = (0.5, 0.2, 0.8), \psi(w) = (0.2, 0.4, 0.6);$$

$$\mu(0) = (0.3, 0.5, 0.4), \mu(v) = (0.3, 0.4, 0.8), \mu(w) = (0.6, 0.8, 0.4)$$

$$\gamma(0) = (0.3, 0.4, 0.1), \gamma(v) = (0.6, 0.9, 0.7), \gamma(w) = (0.7, 0.2, 0.5).$$

By definition we have

$$(\psi_1 \circ \mu_1 \circ \gamma_1)(0) = (0.3), (\psi_1 \circ \mu_1 \circ \gamma_1)(v) = (0.5), (\psi_1 \circ \mu_1 \circ \gamma_1)(w) = (0.5);$$

$$(\psi_2 \circ \mu_2 \circ \gamma_2)(0) = (0.8), (\psi_2 \circ \mu_2 \circ \gamma_2)(v) = (0.4), (\psi_2 \circ \mu_2 \circ \gamma_2)(w) = (0.4);$$

$$(\psi_3 \circ \mu_3 \circ \gamma_3)(0) = (0.7), (\psi_3 \circ \mu_3 \circ \gamma_3)(v) = (0.6), (\psi_3 \circ \mu_3 \circ \gamma_3)(w) = (0.7).$$

Hence, the product of $\psi = (\psi_1, \psi_2, \psi_3)$, $\mu = (\mu_1, \mu_2, \mu_3)$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is defined by $(\psi \circ \mu \circ \gamma)(0) = (0.3, 0.4, 0.7), (\psi \circ \mu \circ \gamma)(v) = (0.5, 0.4, 0.6), (\psi \circ \mu \circ \gamma)(w) = (0.5, 0.4, 0.7)$.

Also, $(\psi_1 + \mu_1)(0) = (0.1), (\psi_1 + \mu_1)(v) = (0.3), (\psi_1 + \mu_1)(w) = (0.5)$; $(\psi_2 + \mu_2)(0) = (0.5), (\psi_2 + \mu_2)(v) = (0.4), (\psi_2 + \mu_2)(w) = (0.8)$; $(\psi_3 + \mu_3)(0) = (0.4), (\psi_3 + \mu_3)(v) = (0.8), (\psi_3 + \mu_3)(w) = (0.4)$;

Hence, addition of $\psi = (\psi_1, \psi_2, \psi_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ is defined by

$$(\psi + \mu)(0) = (0.1, 0.5, 0.4), (\psi + \mu)(v) = (0.3, 0.4, 0.8), (\psi + \mu)(w) = (0.5, 0.8, 0.4).$$

3. Characterization of Ternary Semirings by m-Polar Fuzzy Sets

This is the most essential portion, because here we make our major contributions with the help of several lemmas, theorems, and examples, the notions of m-PFTSs and m-PFIs of ternary semirings are explained in this section. For semigroups and ternary semigroups, Bashir et al. [24–26] have proven the results we have generalized the results in ternary semirings. Throughout the paper, δ is the m-PF subset of E mapping every element of E on $(1, 1, \ldots, 1)$.

Definition 1. *Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ be an m-PF subset of E.*

(1) *Then for all $t = (t_1, t_2, \ldots, t_m) \in (0, 1]^m$, the set $\psi_t = \{s \in E | \psi(s) \geq t\}$ that is $\psi_\kappa \geq t_k$ for all $\kappa \in \{1, 2, \ldots, m\}$, is called a t-cut or a level set.*

(2) *The support of $\psi : E \to [0, 1]^m$ is defined to be the set $Supp(\psi) = \{s \in E | \psi(s) \geq (0, 0, \ldots, 0)\ m-\text{tuple}\}$ that is $\psi_\kappa(s) \geq 0$ for all $\kappa \in \{1, 2, \ldots, m\}$.*

3.1. m-Polar Fuzzy Ternary Subsemirings and Ideals in Ternary Semirings

Here, we define the m-PFTS and m-PFIs of a ternary semirings with examples and explain the related lemmas.

Definition 2. *An m-PF subset $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ of E is called an m-PFTS of E if for all $r, s, t \in E$, it satisfies the following conditions:*

(1) *$\psi(r + s) \geq \psi(r) \wedge \psi(s)$ that is $\psi_\kappa(r + s) \geq \psi_\kappa(r) \wedge \psi_\kappa(s)$;*

(2) *$\psi(rst) \geq \psi(r) \wedge \psi(s) \wedge \psi(t)$ that is $\psi_\kappa(rst) \geq \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$ for all $\kappa \in \{1, 2, \ldots, m\}$.*

Definition 3. *An m-PF subset $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ of E is called an m-PFRI(resp. m-PFLI, m-PFMI) of E if for all $r, s, t \in E$:*

(1) *$\psi_\kappa(r + s) \geq \psi_\kappa(r) \wedge \psi_\kappa(s)$;*

(2) *$\psi_\kappa(rst) \geq \psi_\kappa(r)$ (resp. $\psi_\kappa(rst) \geq \psi_\kappa(t)$, $\psi_\kappa(rst) \geq \psi_\kappa(s)$) for all $\kappa \in \{1, 2, \ldots, m\}$.*

If an m-PF subset ψ is m-PFRI, m-PFLI and m-PFMI of E, then ψ is called m-PFI.

The following Example is of 3-PFI.

Example 2. *Let $E = \{0, v, w\}$ be a ternary semiring given as in Tables 2–5 of Example 1. We define a 3-PF subset $\psi = (\psi_1, \psi_2, \psi_3)$ of E as follows:*

$\psi(0) = (0.7, 0.7, 0.6), \psi(v) = \psi(w) = (0.6, 0.5, 0.4)$. Clearly, $\psi = (\psi_1, \psi_2, \psi_3)$ is a 3-PFRI, 3-PFLI and 3-PFMI of E. Hence, ψ is 3-PFI of E.

Definition 4. *Let Y be a subset of E. Then the m-polar characteristic function $C_Y : Y \to [0,1]^m$ of Y is defined as*

$$C_Y(s) = \begin{cases} (1,1,\ldots,1),\ m\text{-tuple if } s \in Y; \\ (0,0,\ldots,0),\ m\text{-tuple if } s \notin Y. \end{cases}$$

Lemma 1. *Let X, Y and Z be subsets of E. Then the following hold.*

(1) $C_X \wedge C_Y \wedge C_Z = C_{X\cap Y\cap Z}$;
(2) $C_X \circ C_Y \circ C_Z = C_{XYZ}$;
(3) $C_X + C_Y = C_{X+Y}$.

Proof. Obviously hold. □

Lemma 2. *Let Y be subset of E. Then the following hold.*

(1) *Y is a ternary subsemiring of E if and only if C_Y is an m-*PFTS *of E;*
(2) *Y is a LI (resp. MI, RI) of E if and only i C_Y is an m-*PFLI *(resp. m-*PFMI*, m-*PFRI*) of E.*

Proof. (1) : Let Y be a teranry subsemiring of E. We have to show that $C_Y(r+s) \geq \min\{C_Y(r), C_Y(s)\}$ and $C_Y(rst) \geq \min\{C_Y(r), C_Y(s), C_Y(t)\}$ for all $r, s, t \in E$. We consider the following cases:

Case 1: Let $r, s \in Y$. Then $C_Y(r) = (1,1,\ldots,1) = C_Y(s)$. Since Y is ternary subsemiring of E. So $r+s \in Y$ implies that $C_Y(r+s) = (1,1,\ldots,1)$. Hence, $C_Y(r+s) \geq C_Y(r) \wedge C_Y(s)$.

Case 2: Let $r \in Y, s \notin Y$. Then $C_Y(r) = (1,1,\ldots,1)$ and $C_Y(s) = (0,0,\ldots,0)$. Clearly, $C_Y(r+s) \geq (0,0,\ldots,0) = C_Y(r) \wedge C_Y(s)$. Hence, $C_Y(r+s) \geq C_Y(r) \wedge C_Y(s)$.

Case 3: Let $r, s \notin Y$. This implies that $C_Y(r) = C_Y(s) = (0,0,\ldots,0)$. Clearly, $C_Y(r+s) \geq (0,0,\ldots,0) = C_Y(r) \wedge C_Y(s)$. Hence, $C_Y(r+s) \geq C_Y(r) \wedge C_Y(s)$.

Also, Case 1: Let $r, s, t \in Y$. Then $C_Y(r) = C_Y(s) = C_Y(t) = (1,1,\ldots,1)$. Since Y is ternary subsemiring of E. So $rst \in Y$ implies that $C_Y(rst) = (1,1,\ldots,1)$. Hence, $C_Y(rst) \geq C_Y(r) \wedge C_Y(s) \wedge C_Y(t)$.

Case 2: Let $r \in Y$ and $s, t \notin Y$. Then $C_Y(r) = (1,1,\ldots,1)$. $C_Y(s) = C_Y(t) = (0,0,\ldots,0)$. Clearly, $C_Y(rst) \geq (0,0,\ldots,0) = C_Y(r) \wedge C_Y(s) \wedge C_Y(t)$. Hence, $C_Y(rst) \geq C_Y(r) \wedge C_Y(s) \wedge C_Y(t)$.

Case 3: Let $r \notin Y$, and $s, t \in Y$. Then, $C_Y(r) = (0,0,\ldots,0)$ and $C_Y(s) = C_Y(t) = (1,\ldots\ldots,1)$. Clearly, $C_Y(rst) \geq (\ldots 0,\ldots,0) = C_Y(r) \wedge C_Y(s) \wedge C_Y(t)$. Hence, $C_Y(rst) \geq C_Y(r) \wedge C_Y(s) \wedge C_Y(t)$.

Case 4: Let $r, s, t \notin Y$. Then $C_Y(r) = C_Y(s) = C_Y(t \ldots 0,0,\ldots,0)$. Clearly,$C_Y(rs\ldots) \geq (0,0,\ldots,0) = C_Y(r) \wedge C_Y(s) \wedge C_Y(t)$. Hence, $C_Y(rst) \geq C_Y(r) \wedge C_Y(s) \wedge C_Y(t)$.

Conversely, assume that C_Y is an m-PF ternary subsemiring of E. Let $r, s, t \in Y$. Then, $C_Y(r) = C_Y(s) = \ldots(t) = (1,1,\ldots,1)$. By definition, $C_Y(rst) \geq C_Y(r) \wedge C_Y(s) \wedge \ldots(t) = (\ldots 1,\ldots,1\ldots 1,1,\ldots\ldots) \wedge (1,1,\ldots,1) = (1,1,\ldots,1)$, we have $C_Y(rst) = (1,1,\ldots,1)$. This implies that $rst \in Y$. Also, $C_Y(r+s) \geq \ldots(r) \wedge \ldots(s) = (\ldots 1,\ldots,1) \wedge (1,1,\ldots,1) = (1,1,\ldots,1)$, we have $C_Y(r+s) = (1,1,\ldots,1)$.

This implies that $r+s \in Y$. So, Y is ternary subsemiring of E.

(2) : Suppose Y is LI of E. We show that C_Y is an m-PFLI of E i.e., $(C_Y(r+s) \geq \min\{C_Y(r), C_Y(s)\}$ and $C_Y(rst) \geq C_Y(t)$.

Case 1: Let $t\ldots$ and $r, s \in E$. Then $C_Y(t) = (1,1,\ldots,1)$. Since Y is LI of E, So $r\ldots \in Y$. Implies that $C_Y(rst) = (1,1,\ldots,1)$. Hence, $C_Y(rst) \geq C_Y(t)$.

Case 2: Let $t \notin Y$, and $r, s \in E$. Then $C_Y(t) = (0, 0, \ldots, 0)$. Clearly $C_Y(rst) \geq (0, 0, \ldots, 0) = C_Y(t)$. Hence, $C_Y(rst) \geq C_Y(t)$. Now we show that $C_Y(r+s) \geq \min\{C_Y(r), C_Y(s)\}$.

Conversely, assume that C_Y is an m-PFLI of E. Let $t \in Y$ and $r, s \in E$. Then $C_Y(t) = (1, 1, \ldots, 1)$. By definition $C_Y(rst) \geq C_Y(t) = (1, 1, \ldots, 1)$. We have $C_Y(rst) = (1, 1, \ldots, 1)$. This implies that $rst \in Y$. Also, Let $r, s \in Y$. Then $C_Y(r) = (1, 1, \ldots, 1)$ and $C_Y(s) = (1, 1, \ldots, 1)$. By definition, $C_Y(r+s) \geq C_Y(r) \wedge C_Y(s) = (1, 1, \ldots, 1)$. This implies that $r + s \in Y$. So Y is LI of E.

In the similar way we can prove for RI and MI. □

Lemma 3. *Let $\psi = (\psi_1, \psi_2, \ldots \psi_m)$ be an m-PF subset of E. Then the following properties hold.*

(1) *ψ is an m-PFTS of E if and only if $\psi \circ \psi \circ \psi \leq \psi$ and $\psi + \psi \leq \psi$;*
(2) *ψ is an m-PFLI of E if and only if $\delta \circ \delta \circ \psi \leq \psi$ and $\psi + \psi \leq \psi$;*
(3) *ψ is an m-PFRI of E if and only if $\psi \circ \delta \circ \delta \leq \psi$ and $\psi + \psi \leq \psi$;*
(4) *ψ is an m-PFMI of E if and only if $\delta \circ \psi \circ \delta \leq \psi, \delta \circ \delta \circ \psi \circ \delta \circ \delta \leq \psi$ and $\psi + \psi \leq \psi$.*

Proof. (1) : Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ is an m-PFTS of E that is $\psi_\kappa(r+s) \geq \min\{\psi_\kappa(r), \psi_\kappa(s)\}$ and $\psi_\kappa(rst) \geq \min\{\psi_\kappa(r), \psi_\kappa(s), \psi_\kappa(t)\}$ for all $r, s, t \in E$ and $\kappa \in \{1, 2, \ldots, m\}$. Let $a \in E$ if a is not expressible as $a = ghi$ for some $g, h, i \in E$. Then $(\psi \circ \psi \circ \psi)(a) = 0$. Hence, $(\psi \circ \psi \circ \psi)(a) \leq \psi(a)$. But if a is expressible as $a = rst$ for some $r, s, t \in E$, then

$$(\psi_\kappa \circ \psi_\kappa \circ \psi_\kappa)(a) = \bigvee_{a=rst}\{\psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)\}$$

$$\leq \bigvee_{a=rst}\{\psi_\kappa(rst)\} = \psi_\kappa(a) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.$$

Hence, $\psi \circ \psi \circ \psi \leq \psi$. Also, let $a \in E$, if a is not expressible as $a = r + s$ for some $r, s \in E$. Then $(\psi + \psi)(u) = 0$. Hence, $(\psi + \psi)(a) \leq \psi(a)$. But if u is expressible as $a = r + s$ for some $r, s \in E$ then

$$(\psi + \psi)(a) = \bigvee_{a=r+s}\{\psi_\kappa(r) \wedge \psi_\kappa(s)\}$$

$$\leq \bigvee_{a=r+s}\{\psi_\kappa(r+s)\} = \psi_\kappa(a) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.$$

Hence, $\psi + \psi \leq \psi$. Conversely, let $\psi \circ \psi \circ \psi \leq \psi$ and $\psi + \psi \leq \psi$. We show that ψ is an m-PF ternary subsemiring of E. Let $r, s, t \in E$ then,

$$\psi_\kappa(rst) \geq (\psi \circ \psi \circ \psi)(rst)$$

$$= \bigvee_{rst=uvw}\{\psi_\kappa(u) \wedge \psi_\kappa(v) \wedge \psi_\kappa(w)\}$$

$$\geq \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$$

So, $\psi_\kappa(rst) \geq \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$.
Also, $\psi_k(r+s) \geq (\psi_k + \psi_k)(r+s)$

$$= \bigvee_{r+s=u+v}\{\psi_k(u) \wedge \psi_k(v)\} \geq \{\psi_k(r) \wedge \psi_k(s)\} \text{ for all } \kappa \in \{1, 2, \ldots, m\}.$$

So, $\psi_\kappa(r+s) \geq \psi_\kappa(r) \wedge \psi_\kappa(s)$. Hence proved.

(2) Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ be an m-PFLI of E that is $\psi_\kappa(r+s) \geq \psi_\kappa(r) \wedge \psi_\kappa(s)$ and $\psi_\kappa(rst) \geq \psi_\kappa(t)$ for all $\kappa \in \{1, 2, \ldots, m\}$ and $r, s, t \in E$. Let $a \in E$. If a is not expressible

as $a = lmr$ for some $l, m, r \in E$. Then $(\delta \circ \delta \circ \psi)(a) = 0$. Hence, $(\delta \circ \delta \circ \psi)(a) \leq \psi(a)$. Although if a is expressible as $a = rst$ for some $r, s, t \in E$, then

$$(\delta_\kappa \circ \delta_\kappa \circ \psi_\kappa)(a) = \bigvee_{a=rst}\{\delta_\kappa(r) \wedge \delta_\kappa(s) \wedge \psi_\kappa(t)\}$$

$$= \bigvee_{a=rst}\{1 \wedge 1 \wedge \psi_\kappa(t)\}$$

$$\leq \bigvee_{a=rst}\{\psi_\kappa(rst)\} = \psi_\kappa(a) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.$$

Hence, $\delta \circ \delta \circ \psi \leq \psi$. Additionally,

Let $a \in E$, if a is not expressible as $a = u + w$ for some $u, w \in E$. Then $(\psi + \psi)(a) = 0$. Hence $(\psi + \psi)(a) \leq \psi(a)$. Though if a is expressible as $a = r + s$ for some $r, s \in E$ then

$$(\psi_\kappa + \psi_\kappa)(a) = \bigvee_{a=r+s}\{\psi_\kappa(r) \wedge \psi_\kappa(s)\} \leq \bigvee_{a=r+s}\{\psi_\kappa(r+s)\} = \psi_\kappa(a).$$

Hence, $\psi + \psi \leq \psi$. Conversely, Let $\delta \circ \delta \circ \psi \leq \psi$ and $\psi + \psi \leq \psi$. We show that ψ is m-PFLI of E. Let $r, s, t \in E$. Then, $\psi_\kappa(rst) \geq (\delta \circ \delta \circ \psi)(rst) = \bigvee_{rst=uvw}\{\delta_\kappa(u) \wedge \delta_\kappa(v) \wedge \psi_\kappa(w)\}$ $\geq\ \delta_\kappa(r) \wedge \delta_\kappa(s) \wedge \psi_\kappa(t) = \psi_\kappa(t)$ for all $\kappa \in \{1, 2, \ldots, m\}$.

So, $\psi_k(rst) \geq \psi_k(t)$.

Additionally, $\psi_\kappa(r+s) \geq (\psi_\kappa + \psi_\kappa)(r+s) = \vee_{r+s=u+v}\{\psi_\kappa(u) \wedge \psi_\kappa(v)\} \geq \{\psi_\kappa(r) \wedge \psi_\kappa(s)\}$.

So, $\psi_\kappa(r+s) \geq \psi_\kappa(r) \wedge \psi_\kappa(s)$ for all $\kappa \in \{1, 2, \ldots, m\}$.

Similarly, we can prove the parts (3) and (4). □

Lemma 4. *The following assertions are true in E.*

(1) *Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ be three m-PF ternary subsemirings of E. Then $\psi \wedge \mu \wedge \gamma$ is also an m-PF ternary subsemiring of E;*

(2) *Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ and be three m-PFRI (resp. m-PFMI, m-PFLI) of E. Then $\psi \wedge \mu \wedge \gamma$ is also an m-PFLI (resp. m-PFMI, m-PFRI) of E.*

Proof. Straightforward. □

Proposition 1. *Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ be an m-PF subset of ternary semiring E. Then ψ is an m-PFTS (resp. m-PFLI, m-PFMI, m-PFRI) of E if and only if $\psi_t = \{s \in E \mid \psi(s) \geq t\} \neq \varphi$ is a ternary subsemiring (resp. LI, MI, RI) of E for all $t = (t_1, t_2, \ldots, t_m) \in (0, 1]^m$.*

Proof. Let ψ be an m-PFTS of E and $r, s, t \in \psi_t$, then $\psi_\kappa(r) \geq t_k$, $\psi_\kappa(s) \geq t_k$ and $\psi_\kappa(t) \geq t_k$ for all $\kappa \in \{1, 2, \ldots, m\}$. Since ψ is an m-PFTS of E. We have $\psi_\kappa(r+s) \geq \psi_\kappa(r) \wedge \psi_\kappa(s) \geq t_k \wedge t_k \geq t_k$ for all $k \in \{1, 2, \ldots, m\}$. Also, $\psi_\kappa(rst) \geq \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t) \geq t_k \wedge t_k \wedge t_k \geq t_k$ for all $k \in \{1, 2, \ldots, m\}$. Hence, ψ_t is a ternary subsemiring of E.

Conversely, suppose that ψ_t is a ternary subsemiring of E. Let $r, s, t \in E$. Suppose on contrary that ψ_t is an m-PFTS of E, such that $\psi_\kappa(r+s) < \psi_\kappa(r) \wedge \psi_\kappa(s)$ and take $t_k = \psi_\kappa(r) \wedge \psi_\kappa(s)$. Then $r, s \in \psi_t$ but $r + s \notin \psi_t$. Hence $\psi_\kappa(r+s) \geq \psi_\kappa(r) \wedge \psi_\kappa(s)$. Also for $\psi_\kappa(rst) \geq \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$ suppose on contrary that $\psi_\kappa(rst) < \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$ and take $t_k \in [0, 1]^m$ such that $t_k = \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$, then $r, s, t \in \psi_t$. But $rst \notin \psi_t$. So $\psi_\kappa(rst) \geq \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$. Thus ψ is an m-PFTS of E. □

3.2. m-Polar Fuzzy Generalized Bi-Ideals in Ternary Semirings

Here, we define m-PFGBI of ternary semirings.

Definition 5. *An m-PF subset $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ of E is called an m-PFGBI of E if for all $r, s, t, e_1, e_2 \in E$ it satisfies the following conditions $\psi E \psi E \psi \leq \psi$ that is $\psi_\kappa(re_1se_2t) \geq \min\{\psi_\kappa(r), \psi_\kappa(s), \psi_\kappa(t)\}$ for all $r, s, t, e_1, e_2 \in E$ and $\kappa \in \{1, 2, \ldots, m\}$.*

Lemma 5. *A subset Y of E is a GBI of E if and only if C_Y is an m-PFGBI of E.*

Proof. Similar to Lemma 2. □

Lemma 6. *An m-PFTS of E is an m-PFGBI of E if and only if $\psi \circ \delta \circ \psi \circ \delta \circ \psi \leq \psi$.*

Proof. Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ be an m-PFGBI of E. Let $a \in E$. If a is not expressible as $a = lmn$ for some $l, m, n \in E$, then $\psi \circ \delta \circ \psi \circ \delta \circ \psi \leq \psi$. But if a is expressible as $a = rst$ for some $r, s, t \in E$. Then for all $\kappa \in \{1, 2, \ldots, m\}$. We have $((\psi_\kappa \circ \delta_\kappa \circ \psi_\kappa) \circ \delta_\kappa \circ \psi_\kappa)(a) = \vee_{a=rst}\{(\psi_\kappa \circ \delta_\kappa \circ \psi_\kappa)(r) \wedge \delta_\kappa(s) \wedge \psi_\kappa(t)\} = \vee_{a=rst}\{\vee_{r=uvw}\{\psi_\kappa(u) \wedge \delta_\kappa(v) \wedge \psi_\kappa(w)\} \wedge \delta_\kappa(s) \wedge \psi_\kappa(t)\} = \vee_{a=rst}\{\vee_{r=uvw}\{\psi_\kappa(u) \wedge \psi_\kappa(v)\} \wedge \psi_\kappa(w)\} \leq \vee_{a=rst} \vee_{r=uvw} \{\psi_\kappa((uvw)st)\} = \vee_{a=rst}\{\psi_\kappa(rst)\} = \psi_\kappa(a)$ for all $\kappa \in \{1, 2, \ldots, m\}$.

Conversely, let $\psi \circ \delta \circ \psi \circ \delta \circ \psi \leq \psi$ and $re_1se_2t \in E$. So

$$\begin{aligned}
\psi_\kappa(re_1se_2t) &\geq ((\psi_\kappa \circ \delta_\kappa \circ \psi_\kappa) \circ \delta_\kappa \circ \psi_\kappa)(re_1se_2t) \\
&= \vee_{re_1se_2t=uvw}\{(\psi_\kappa \circ \delta_\kappa \circ \psi_\kappa)(u) \wedge \delta_\kappa(v) \wedge \psi_\kappa(w)\} \\
&\geq \{(\psi_\kappa \circ \delta_\kappa \circ \psi_\kappa)(re_1s) \wedge \delta_\kappa(e_2) \wedge \psi_\kappa(t)\} \\
&= \vee_{re_1s=abc}\{\psi_\kappa(a) \wedge \delta_\kappa(b) \wedge \psi_\kappa(c)\} \wedge \delta_\kappa(e_2) \wedge \psi_\kappa(t) \\
&\geq \{\psi_\kappa(r) \wedge \delta_\kappa(e_1) \wedge \psi_\kappa(s)\} \wedge \delta_\kappa(e_2) \wedge \psi_\kappa(t) \\
&= \psi_\kappa(r) \wedge \delta_\kappa(e_1) \wedge \psi_\kappa(s)\} \wedge \delta_\kappa(e_2) \wedge \psi_\kappa(t) \\
&= \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}$$

Hence, $\psi(re_1se_2t) \geq \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$. □

Proposition 2. *Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ be an m-PFTS of E. Then ψ is an m-PFGBI of E if and only if $\psi_t = \{s \in E | \psi(s) \geq t\}$ is a GBI of E for all $t = (t_1, t_2, \ldots, t_m) \in (0, 1]^m$.*

Proof. Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ be an m-PFTS of E and ψ is an m-PFGBI of E. Let $r, s, t \in \psi_t$. Then $\psi_\kappa(r) \geq t_\kappa, \psi_\kappa(s) \geq t_\kappa$ and $\psi_\kappa(t) \geq t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m)$. Since ψ is m-PFGBI of E. So we have $r, s, t \in \psi_t$ and $e_1, e_2 \in E$ such that $\psi_\kappa(re_1se_2t) \geq \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t) \geq t_\kappa \wedge t_\kappa \wedge t_\kappa \geq t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m\}$. Hence ψ_t is GBI of E. Conversely, suppose that ψ_t is a GBI of E. On contrary suppose that ψ is not an m-PFGBI of E. Let $r, s, t, e_1, e_2 \in E$ such that $\psi_\kappa(re_1se_2t) < \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$. Take $t_\kappa = \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$. Then $r, e_1, s, e_2, t \in \psi_t$ but $re_1se_2t \notin \psi_t$ which is contradiction. So, $\psi_\kappa(re_1se_2t) \geq \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$. So ψ is an m-PFGBI of E. □

3.3. m-Polar Fuzzy Bi-Ideals in Ternary Semirings

Here, we define m-PFBI of ternary semirings and explain some lemmas based on this definition.

Definition 6. *An m-PFTS $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ of E is called an m-PFBI of E if for all $r, s, t, e_1, e_2 \in E$ it satisfies the following conditions $\psi E \psi E \psi \leq \psi$ that is $\psi_\kappa(re_1se_2t) \geq \min\{\psi_\kappa(r), \psi_\kappa(s), \psi_\kappa(t)\}$ for all $r, s, t, e_1, e_2 \in E$ and $\kappa \in \{1, 2, \ldots, m\}$.*

Lemma 7. *A subset Y of E is a BI of E if and only if C_Y is an m-PFBI of E.*

Proof. Similar to Lemma 2 and Lemma 5. □

Lemma 8. *An m-PFTS of E is an m-PFBI of E if and only if*

(1) $\psi + \psi \leq \psi$;
(2) $\psi \circ \psi \circ \psi \leq \psi$;
(3) $\psi \circ \delta \circ \psi \circ \delta \circ \psi \leq \psi$.

Proof. Proof of (1) and (2) are follows from Lemma 3 and Proof of (3) follows from Lemma 6. □

Proposition 3. *Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ be an m-PFTS of E. Then ψ is an m-PFBI of E if and only if $\psi_t = \{s \in E | \psi(s) \geq t\}$ is a BI of E for all $t = (t_1, t_2, \ldots, t_m) \in (0,1]^m$.*

Proof. Follows from Proposition 2. □

3.4. m-Polar Fuzzy Quasi-Ideals in Ternary Semirings

Now, we define m-PFQI of ternary semirings and some its characteristics.

Definition 7. *An m-PF additive subsemigroup $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ of E is called an m-PFQI of E if $\psi EE \cap (E\psi E + EE\psi EE) \cap EE\psi \leq \psi$* i.e., $\psi(x) \geq \min\{(\psi EE)(x), (E\psi E + EE\psi EE)(x), EE\psi(x)\}$, *that is $\psi_\kappa E_\kappa E_\kappa \cap (E_\kappa \psi_\kappa E_\kappa + E_\kappa E_\kappa \psi_\kappa E_\kappa E_\kappa) \cap E_\kappa E_\kappa \psi_\kappa \leq \psi$* i.e., $\psi(x) \geq \min\{(\psi_\kappa E_\kappa E_\kappa)(x), (E_\kappa \psi_\kappa E_\kappa + E_\kappa E_\kappa \psi_\kappa E_\kappa E_\kappa)(x), E_\kappa E_\kappa \psi_\kappa(x)\}$ *for all $\kappa \in \{1, 2, \ldots, m\}$.*

Lemma 9. *Let Y be an additive subsemigroup of E. Then Y is QI of E if and only if C_Y is an m-PFQI of E.*

Proof. Let Y is a quasi-ideal. Then obviously, C_Y is fuzzy subsemigroup of E.

$$\begin{aligned}
C_Y E_\kappa E_\kappa \cap E_\kappa C_Y E_\kappa + E_\kappa E_\kappa C_Y E_\kappa E_\kappa \cap E_\kappa E_\kappa C_Y \\
&= C_Y C_E C_E \cap C_E C_Y C_E + C_E C_E C_Y C_E C_E \cap C_E C_E C_Y \\
&= C_{YEE} \cap C_{(EYE+EEYEE)} \cap C_{EEY} \\
&= C_{YEE \cap (EYE+EEYEE) \cap EEY} \\
&\subseteq C_Y.
\end{aligned}$$

This means that C_Y is m-PFQI. Conversely, let s is any element of $YEE \cap (EYE + EEYEE) \cap EEY$. Then we have

$$C_Y(s) \geq \{C_Y EE \cap EC_Y E + EEC_Y EE \cap EEC_Y\}(s)$$
$$= min\{C_Y EE(s), (EC_Y E + EEC_Y EE)(s), EEC_Y(s)\}$$
$$= min\Big\{C_{YEE}(s), C_{(EYE+EEYEE)}(s), C_{EEY}(s)\Big\}$$
$$= C_{YEE \cap (EYE+EEYEE) \cap EEY}(s) = (1, 1 \ldots, 1).$$

This implies that $s \in Y$, and so $YEE \cap (EYE + EEYEE) \cap EEY \subseteq Y$. □

Proposition 4. *Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m$ be an m-PFTS of E. Then ψ is an m-PFQI of E if and only if $\psi_t = \{s \in E | \psi(s) \geq t\} \neq \varphi$ is a QI of E for all $t = (t_1, t_2, \ldots, t_m) \in (0,1]^m$.*

Proof. Let ψ is *m-PFQI of E*. Suppose that $a \in \psi_t EE \cap E\psi_t E + EE\psi_t EE \cap EE\psi_t$, then $a \in \psi_t EE$ an $a \in E\psi_t E + EE\psi_t EE$ and $a \in EE\psi_t$. So $a = re_1e_2$ and $a = e_1s_1e_2 + e_1e_2s_2e_1e_2$ and $a = e_1e_2t$ for some $r, s_1, s_2, t \in \psi_t$ and $e_1, e_2 \in E$. Thus $\psi_\kappa(r) \geq t_\kappa$, $\psi_\kappa(s_1) \geq t_\kappa$, $\psi_\kappa(s_2) \geq t_\kappa$ and $\psi_\kappa(t) \geq t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m\}$. Now $(\psi_\kappa E_\kappa E_\kappa)(a) = \vee_{a=re_1e_2}\{\psi_\kappa(r)\}(E_\kappa \psi_\kappa E_\kappa + E_\kappa E_\kappa \psi_\kappa E_\kappa E_\kappa)(a) = \vee_{a=e_1s_1e_2+e_1e_2s_2e_1e_2}\{\psi_\kappa(s_1) \wedge \psi_\kappa(s_2)\}(E_\kappa E_\kappa \psi_\kappa)(a) = \vee_{a=e_1e_2t}\{\psi_\kappa(t)\}$.

Thus, $(\psi_\kappa E_\kappa E_\kappa \cap (E_\kappa \psi_\kappa E_\kappa + E_\kappa E_\kappa \psi_\kappa E_\kappa E_\kappa) \cap E_\kappa E_\kappa \psi_\kappa)(a) = (\psi_\kappa E_\kappa E_\kappa)(a) \wedge (E_\kappa \psi_\kappa E_\kappa + E_\kappa E_\kappa \psi_\kappa E_\kappa E_\kappa)(a) \wedge (E_\kappa E_\kappa \psi_\kappa)(a) = t_\kappa \wedge t_\kappa \wedge t_\kappa = t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m\}$.

So $(\psi EE \cap E\psi E + EE\psi EE \cap EE\psi) \geq t$. Since $\psi(a) \geq (\psi EE \cap E\psi E + EE\psi EE \cap EE\psi)(a) \geq t$. This means that $a \in \psi_t$. So ψ_t is a QI of E. Additionally, as ψ is a m-PFQI of E that is $\psi_\kappa(r+s) \geq \psi_\kappa(r) \wedge \psi_\kappa(s)$. Let $r, s \in \psi_t$, then $\psi_\kappa(r) \geq t_\kappa$, $\psi_\kappa(s) \geq t_\kappa$. So $\psi_\kappa(r+s) \geq t_\kappa + t_\kappa$ and thus $r, s \in \psi_t$. Conversely, assume that ψ is not an m-PFQI of E. Let $a \in E$ such that $\psi_\kappa(a) < (\psi_\kappa E_\kappa E_\kappa)(a) \wedge (E_\kappa \psi_\kappa E_\kappa + E_\kappa E_\kappa \psi_\kappa E_\kappa E_\kappa)(a) \wedge (E_\kappa E_\kappa \psi_\kappa)(a)$ for all $\kappa \in \{1, 2, \ldots, m\}$. Choose $t_\kappa \in (0, 1]$ such that $t_\kappa = (\psi_\kappa E_\kappa E_\kappa)(a) \wedge (E_\kappa \psi_\kappa E_\kappa + E_\kappa E_\kappa \psi_\kappa E_\kappa E_\kappa)(a) \wedge (E_\kappa E_\kappa \psi_\kappa)(a)$ for all $\kappa \in \{1, 2, \ldots, m\}$. This implies that $a \in (\psi_\kappa E_\kappa E_\kappa)_{t_\kappa}$, $a \in (E_\kappa \psi_\kappa E_\kappa + E_\kappa E_\kappa \psi_\kappa E_\kappa E_\kappa)_{t_\kappa}$ and $a \in (E_\kappa E_\kappa \psi_\kappa)_{t_\kappa}$ but $a \notin (\psi_\kappa)t_\kappa$ for some κ. Hence $a \in (\psi EE)_t$, $a \in (E\psi E + EE\psi EE)_t$ and $a \in (EE\psi)_t$ but $a \notin \psi_t$ which is the contradiction. Hence $\psi \geq \psi EE \cap E\psi E + EE\psi EE \cap EE\psi$. □

Lemma 10. *Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ be m-PFRI, m-PFMI and m-PFLI of E, respectively. Then $\psi \wedge \mu \wedge \gamma$ is an m-PFQI of E.*

Proof. Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be m-PFRI, m-PFMI and m-PFLI of E, respectively. Let $x \neq ae_1e_2 = e_1(b + e_1ce_2)e_2 = e_1e_2d$, where $a, b, c, d, e_1, e_2 \in E$. Then $(\psi \wedge \mu \wedge \gamma)EE \cap E(\psi \wedge \mu \wedge \gamma)E + EE(\psi \wedge \mu \wedge \gamma)EE \cap EE(\psi \wedge \mu \wedge \gamma) \leq \psi \wedge \mu \wedge \gamma$. If $x = re_1e_2 = e_1(s_1 + e_1s_2e_2)e_2 = e_1e_2t$, where $r, s_1, s_2, t, e_1, e_2 \in E$. Then $\{(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)E_\kappa E_\kappa \cap E_\kappa(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)E_\kappa + E_\kappa E_\kappa(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)E_\kappa E_\kappa \cap E_\kappa E_\kappa(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)\}(x) = \min\{((\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)E_\kappa E_\kappa)(x), (E_\kappa(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)E_\kappa)(x) + (E_\kappa E_\kappa(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)E_\kappa E_\kappa)(x), E_\kappa E_\kappa(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)\} = \min\{\vee_{x=re_1e_2}(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(r), \vee_{x=e_1(s_1+e_1s_2e_2)e_2}\{(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(s_1) \wedge (\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(s_2)\}, \vee_{x=e_1e_2t}(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(t)\} \leq \min\{1, \{(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(x), 1\} = (\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(x)$. □

Hence, $(\psi \wedge \mu \wedge \gamma)$ is an m-PFQI of E.

Lemma 11. *Every m-PFLI (resp. m-PFMI, m-PFRI) of E is an m-PFQI of E.*

Proof. The Proof follows from Lemma 3. □

The next example shows that the converse of Lemma 11 may not be true.

Example 3. *Consider a ternary semiring $E = \{p, q, r, s, t\}$ under operations as given in Tables 6 and 7.*

Table 6. Addition '+'.

+	*p*	*q*	*r*	*s*	*t*
p	*p*	*q*	*r*	*s*	*t*
q	*q*	*p*	*q*	*r*	*q*
r	*r*	*q*	*p*	*q*	*r*
s	*s*	*r*	*q*	*p*	*t*
t	*t*	*q*	*r*	*t*	*p*

Table 7. Ternary Multiplication '*'.

*	*p*	*q*	*r*	*s*	*t*
p	*p*	*p*	*p*	*p*	*p*
q	*q*	*p*	*p*	*p*	*p*
r	*p*	*p*	*p*	*p*	*p*
s	*p*	*p*	*p*	*p*	*p*
t	*p*	*q*	*r*	*s*	*t*

Define a 3-PF subset $\psi = (\psi_1, \psi_2, \psi_3)$ of E as follows:

$\psi(p) = \psi(q) = (0.7, 0.5, 0.6), \psi(r) = \psi(s) = (0, 0, 0)$ and $\psi(t) = (0, 4, 0.3, 0.2)$. Then simple calculation shows that ψ_t is a QI of E. Therefore, by using Proposition 3, ψ is a 3-PFQI of E. Now, $\psi(r) = \psi(ttr) \ngeq \psi(t) = (0.4, 0.3, 0.2)$. So ψ is not a 3-PFRI of E.

Theorem 1. *Every m-PFQI of E is an m-PFBI of E.*

Proof. Suppose that $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ is an m-PFQI of E. Let $r, s, t \in E$. Then, $\psi_\kappa(rst) \geq ((\psi_\kappa \circ \delta_\kappa \circ \delta_\kappa) \wedge (\delta_\kappa \circ \psi_\kappa \circ \delta_\kappa) \wedge (\delta_\kappa \circ \delta_\kappa \circ \psi_\kappa))(rst) = (\psi_\kappa \circ \delta_\kappa \circ \delta_\kappa)(rst) \wedge (\delta_\kappa \circ \psi_\kappa \circ \delta_\kappa)(rst)$ $\wedge\, (\delta_\kappa \circ \delta_\kappa \circ \psi_\kappa)(rst) = \{\vee_{rst=abc}\{\psi_\kappa(a) \wedge \delta_\kappa(b) \wedge \delta_\kappa(c)\}\} \wedge \{\vee_{rst=def}\{\delta_\kappa(d) \wedge \psi_\kappa(e) \wedge \delta_\kappa(f)$ $\}\} \wedge \left\{\vee_{rst-ghi}\{\delta_\kappa(g) \wedge \delta_\kappa(h) \wedge \psi_\kappa(i)\}\right\} \geq \{\psi_\kappa(r) \wedge \delta_\kappa(s) \wedge \delta_\kappa(t),\} \wedge \{\delta_\kappa(r) \wedge \psi_\kappa(s) \wedge \delta_\kappa(t)\}$ $\wedge \{\delta_\kappa(r) \wedge \delta_\kappa(s) \wedge \psi_\kappa(t)\} = \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$.

Also, $\boldsymbol{\psi_\kappa}(re_1se_2t) \geq \{(\psi_\kappa \circ \delta_\kappa \circ \delta_\kappa) \wedge (\delta_\kappa \circ \delta_\kappa \circ \psi_\kappa \circ \delta_\kappa \circ \delta_\kappa) \wedge (\delta_\kappa \circ \delta_\kappa \circ \psi_\kappa)\}(re_1se_2t)$ $= (\psi_\kappa \circ \delta_\kappa \circ \delta_\kappa)(re_1se_2t) \wedge (\delta_\kappa \circ \delta_\kappa \circ \psi_\kappa \circ \delta_\kappa \circ \delta_\kappa)(re_1se_2t) \wedge (\delta_\kappa \circ \delta_\kappa \circ \psi_\kappa)(re_1se_2t) =$ $\{\vee_{re_1se_2t=abc}\{\psi_\kappa(a) \wedge \delta_\kappa(b) \wedge \delta_\kappa(c)\}\} \wedge \{\vee_{re_1se_2t=pqruv}\{\delta_\kappa(p) \wedge \delta_\kappa(q) \wedge \psi_\kappa(r)\} \wedge \delta_\kappa(u) \wedge$ $\delta_\kappa(v)\}\} \wedge \{\vee_{re_1se_2t=ghi}\{\delta_\kappa(g) \wedge \delta_\kappa(h) \wedge \psi_\kappa(i)\}\} \geq \{\psi_\kappa(r) \wedge \delta_\kappa(e_1se_2) \wedge \delta_\kappa(t)\} \wedge \{\delta_\kappa(r) \wedge$ $\delta_\kappa(e_1) \wedge \psi_\kappa(s) \wedge \delta_\kappa(e_2) \wedge \delta_\kappa(t)\} \wedge \{\delta_\kappa(r) \wedge \delta_\kappa(e_1se_2) \wedge \psi_\kappa(t)\} = \psi_\kappa(r) \wedge \psi_\kappa(s) \wedge \psi_\kappa(t)$.

So, $\psi(re_1se_2t) \geq \psi(r) \wedge \psi(s) \wedge \psi(t)$. Hence, ψ is an m-PFBI of E. □

3.5. Applications of Proposed Work

The m-PF set has a broad variety of applications in real-life challenges regarding multi-agent, multi-objects, multi-attributes, multi-index, and multi-polar information. The m-PF sets can also be applied in decision-making, cooperative games, and diagnosis data, among other applications. These sets may also be used to describe multi-relationships and as a model for clustering or grouping. We will give an example to demonstrate it.

Let $Z = \{r, s, t, u\}$ be the set of 4 persons. We characterized them as a good person according to seven qualities in the form of 7-PF subset given in Table 8:

Table 8. Table of qualities with their membership values.

	Honesty	Loyalty	Enthusiasm	Cooperative	Self-Control	Maturity	Courage
r	0.4	0.5	0	0.1	0.5	0.8	1
s	0	0.7	0.4	0.6	0.3	0.5	0.7
t	0.8	0.5	0.9	0.3	0.4	0.8	0.6
u	0.5	0.4	0.7	1	0.6	0.7	0.8

Thus, we get a 7-PF subset $\psi : Z \to [0, 1]^7$ such that

$$\psi(r) = (0.4, 0.5, 0, 0.1, 0.5, 0.8, 1)$$

$$\psi(s) = (0, 0.7, 0.4, 0.6, 0.3, 0.5, 0.7)$$

$$\psi(t) = (0.8, 0.5, 0.9, 0.3, 0.4, 0.8, 0.6)$$

$$\psi(u) = (0.5, 0.4, 0.7, 1, 0.6, 0.7, 0.8).$$

Here, is the graphical representation of a 7-PF subset shown in Figure 1.

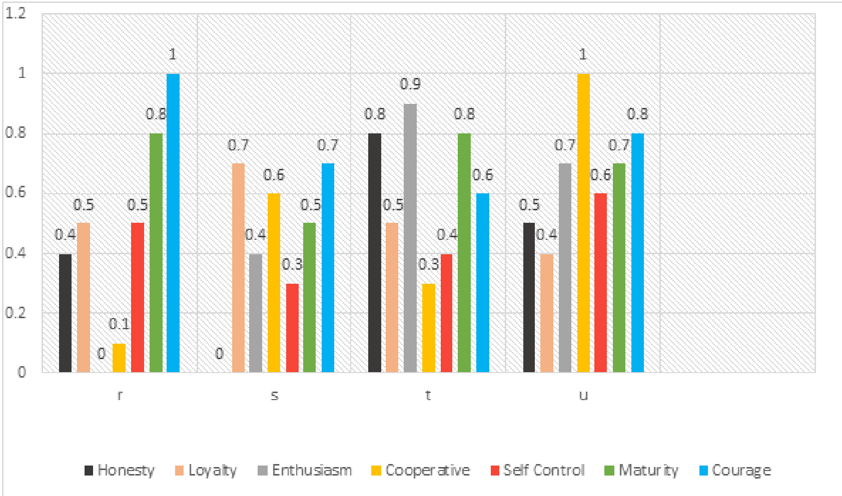

Figure 1. Graphical representation of a 7-PF subset.

Here, 1 represents good remarks, 0.5 represents average, and 0 represents bad remarks.

4. Characterization of Regular, and Intra-Regular Ternary Semirings by m-Polar Fuzzy Ideals

This section presents many important results on regular and intra-regular ternary semirings in terms of m-PFIs. Many results of Bashir et al. [25,26] were studied and generalized in the structure of ternary semirings by m-PFIs.

Definition 8. *An element 'a' of E is called regular if there exist elements $r, s \in E$ such that $a = arasa$ [5].*

Theorem 2. *[15] A ternary semiring E is regular if and only if $\psi * \mu * \gamma = \psi \cap \mu \cap \gamma$ for every RI ψ, MI μ and LI γ of E.*

Theorem 3. *For E the following conditions are equivalent.*

(1) *E is regular;*
(2) *$\psi \wedge \mu \wedge \gamma = \psi \circ \mu \circ \gamma$ for each m-PFRI ψ, m-PFMI μ and m-PFLI γ of E.*

Proof. $(1) \Rightarrow (2):$ Suppose that $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ be m-PFRI, m-PFMI, and m-PFLI of E, respectively. Let $a \in E$, we have $(\psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa)(a) = \vee_{a=rst}\{\psi_\kappa(r) \wedge \mu_\kappa(s) \wedge \gamma_\kappa(t)\} \leq \vee_{a=rst}\{\psi_\kappa(rst) \wedge \mu_\kappa(rst) \wedge \gamma_\kappa(rst) = \psi_\kappa(a) \wedge \mu_\kappa(a) \wedge \gamma_\kappa(a) = (\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(a)$.

Hence, $(\psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa)(a) \leq (\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(a)$ for all $\kappa \in \{1, 2, \ldots, m\}$. Since E is regular so for each $a \in E$ there exist an element $r, s \in E$ such that $a = arasa$. $(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(a) = \psi_\kappa(a) \wedge \mu_\kappa(a) \wedge \gamma_\kappa(a) \leq \psi_\kappa(a) \wedge \mu_\kappa(ras) \wedge \gamma_\kappa(a) = \vee_{a=rst}\{\psi_\kappa(r) \wedge \mu_\kappa(s) \wedge \gamma_\kappa(t)\} = (\psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa)(a)$ for all $\kappa \in \{1, 2, \ldots, m\}$. So, $(\psi_\kappa \wedge \gamma_\kappa \wedge \gamma_\kappa) \leq (\psi_\kappa \circ \gamma_\kappa \circ \gamma_\kappa)$. Therefore, $\psi \wedge \mu \wedge \gamma = \psi \circ \mu \circ \gamma$.

$(2) \Rightarrow (1):$ Let $a \in E$. Then $L = aEE$ is a LI of E, $R = EEa$ is a RI of E and $T = EaE$ is a lateral ideal of E. Then by using Lemma 2, C_L, C_R, C_T the m-polar characteristic functions of L, R, and T are m-PFLI, m-PFRI and m-PFMI of E, respectively. Then we have $C_{RTL} = C_R \circ C_T \circ C_L$ by Lemma 1 $= C_R \wedge C_T \wedge C_L = C_{R \cap T \cap L}$

Thus, $R \cap T \cap L = RTL$. Hence it follows from Theorem 1 that E is regular. □

Theorem 4. *The following conditions are equivalent in E.*

(1) *E is regular;*

(2) $\psi = \psi \circ \delta \circ \psi \circ \delta \circ \psi$ *for every* m-PFBI *of* E;
(3) $\psi = \psi \circ \delta \circ \psi \circ \delta \circ \psi$ *for every* m-PFQI *of* E.

Proof. $(1) \Rightarrow (2):$ Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ be m-PFBI of E. Let $a \in E$. As E is regular, so there exists elements $r, s \in E$ such that $a = arasa$. We have for some $r, s, t \in E$ $(\psi_\kappa \circ \delta_\kappa \circ \psi_\kappa \circ \delta_\kappa \circ \psi_\kappa)(a) = \vee_{a=rst}\{\psi_\kappa(r) \wedge (\delta_\kappa \circ \psi_\kappa \circ \delta_\kappa)(s) \wedge \psi_\kappa(t)\} \geq \psi_\kappa(a) \wedge (\delta_\kappa \circ \psi_\kappa \circ \delta_\kappa)(ras) \wedge \psi_\kappa(a) = \psi_\kappa(a) \wedge \vee_{ras=pqr}\{\psi_\kappa(p) \wedge \Lambda_\kappa(q) \wedge \psi_\kappa(r)\} \wedge \psi_\kappa(a) \geq \psi_\kappa(a) \wedge \{\delta_\kappa(r) \wedge \psi_\kappa(a) \wedge \delta_\kappa(s)\} \wedge \psi_\kappa(a) = \psi_\kappa(a)$ for all $\kappa \in \{1, 2, \ldots, m\}$.

So, $(\psi \circ \delta \circ \psi \circ \delta \circ \psi) \geq \psi$.

Since ψ is m-PFBI of E. So we have for some $r, s, t \in E$ $(\psi_\kappa \circ \delta_\kappa \circ \psi_\kappa \circ \delta_\kappa \circ \psi_\kappa)(a) = \vee_{a=rst}\{(\psi_\kappa \circ \delta_\kappa \circ \psi_\kappa(r)) \wedge \delta_\kappa(s) \wedge \psi_\kappa(t)\} = \vee_{a=rst}\{\vee_{r=uvw}\{\psi_\kappa(u) \wedge \delta_\kappa(v) \wedge \psi_\kappa(w)\} \wedge \delta_\kappa(s) \wedge \psi_\kappa(t)\} = \vee_{a=rst}\{\vee_{r=uvw}\{\psi_\kappa(u) \wedge \psi_\kappa(w)\} \wedge \psi_\kappa(t)\} \leq \vee_{a=rst}\{\vee_{r=uvw}\psi_\kappa(uvwst)\} = \vee_{a=rst}\{\psi_\kappa(rst)\} = \psi_\kappa(a)$ for all $\kappa \in$ for all $\kappa \in \{1, 2, \ldots, m\}$. So, $(\psi \circ \delta \circ \psi \circ \delta \circ \psi) \leq \psi$. Thus $\psi = (\psi \circ \delta \circ \psi \circ \delta \circ \psi)$.

$(2) \Rightarrow (3):$ It is obvious.

$(3) \Rightarrow (1):$ Let ψ, μ and γ be m-PFLI, m-PFMI and m-PFRI of E, respectively. Then $\psi \wedge \mu \wedge \gamma$ is m-PFQI of E. Hence by hypothesis $(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa) \leq (\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa) \circ \delta_\kappa \circ (\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa) \circ \delta_\kappa \circ (\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa) \leq \psi_\kappa \circ \delta_\kappa \circ \mu_\kappa \circ \delta_\kappa \circ \gamma_\kappa = \psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa$.

So, $\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa \leq \psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa$. But $\psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa \leq \psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa$ always hold. Hence $\psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa = \psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa$ that is $\psi \circ \mu \circ \gamma = \psi \wedge \mu \wedge \gamma$. Thus, by Theorem 3, E is regular. □

Definition 9. *[5] An element 'a' of* E *is stated as intra-regular if there exist* $r, s \in E$ *such that* $a = ra^5s$.

Theorem 5. *[5] A ternary semiring* E *is intra-regular if and only if* $X \cap Y \cap Z \subseteq XYZ$ *for every LI* X*, MI* Y *and every RI* Z *of* E.

Theorem 6. E *is intra-regular if and only if* $(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa) \leq \psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa$ *for every m-PFLI* ψ_κ*, m-PFMI* μ_κ *and m-PFRI ideal* γ_κ *of* E.

Proof. Let E be intra-regular ternary semiring. Let $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ be three m-PFRI ψ, m-PFMI μ and m-PFLI γ of E, respectively. Let $a \in E$ then there exist $r, s \in E$ such that $a = ra^5s$. $(\psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa)(a) = \vee_{a=ra^5s}\{\psi_\kappa(raa) \wedge \mu_\kappa(a) \wedge \gamma_\kappa(aas)\} \geq \vee_{a=rst}\{\psi_\kappa(a) \wedge \mu_\kappa(a) \wedge \gamma_\kappa(a)\} \geq \psi_\kappa(a) \wedge \mu_\kappa(a) \wedge \gamma_\kappa(a) = (\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa)(a)$ for all $\kappa \in \{1, 2, \ldots, m\}$. Hence, $(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa) \leq \psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa$.

Conversely, let $(\psi_\kappa \wedge \mu_\kappa \wedge \gamma_\kappa) \leq (\psi_\kappa \circ \mu_\kappa \circ \gamma_\kappa)$ for m-PFLI ψ_κ, m-PFMI μ_κ and m-PFRI ideal γ_κ of E. Let X, Y and Z be LI, MI and RI, respectively. Then C_X, C_Y and C_Z are m-PFLI, m-PFMI and m-PFRI of E, respectively. Now by our supposition $C_X \wedge C_Y \wedge C_Z \leq C_X \circ C_Y \circ C_Z$ implies $C_{X \cap Y \cap Z} \leq C_{XYZ}$, then $X \cap Y \cap Z \subseteq XYZ$. So E is intra-regular by Theorem 5. □

5. Comparative Study

In this section, we have described how this research work is better and related to previous work. Shabir and Bashir [24–26] used m-PF ideals for the characterization of the regular LA-semigroup, semigroups and ternary semigroups, respectively. Our work is superior to Shabir and Bashir [24–26], as there are numerous structures that cannot be handled by using binary operations, such as the fact that $\mathbb{Z}^-$ is not a semiring but it is a ternary semiring. Similarly, Q^- $(\textit{the set of negative rational numbers})$ and $\mathrm{R}^-(\textit{the set of negative real numbers})$ are ternary semirings under ternary multiplication. To get over this problem, we have applied the ternary operation and generalized entire results of Shabir & Bashir [24–26] in the ternary semirings. Several results are generalized, and new results are found. As a result, our methodology offers a broad variety of applications than Shabir & Bashir [24–26].

In existing techniques, the alternatives are used directly and they are time consuming and error-based.

In the result, the existing methods do not give precise outcomes. The bipolar fuzzy environment minimizes this problem due to pairwise comparison and m-polar fuzzy environment is much better environment to minimize this issue because of presence of multi-attributes based data on real world problems. Our technique may achieve efficiently the precise outcomes.

6. Conclusions

The m-PF set theory is a beneficial mathematical tool for resolving uncertainty. In this paper, the definition of m-PF set is applied on the structure of ternary semiring. We converted the fundamental algebraic structure of Shabir and Bashir [24–26] into a ternary semiring from the LA-semigroup, ternary semigroup, and semiring. Most significantly, we have proven some results related to ternary semirings in terms of m-PFIs, m-PFGBIs, m-PFBIs, and m-PFQIs. This paper has a vast range of applications of m-PF set theory. Additionally, we have studied the characterization of regular and intra-regular ternary semirings in terms of m-PFIs. In future, we may work on regular and intra-regular ternary semirings regarding intuitionistic fuzzy ideals, picture fuzzy ideals, interval-valued fuzzy ideals, cubic fuzzy ideals, and many other extensions of fuzzy ideals. We hope that this research work will be a basis for further study of the ternary semiring theory.

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