symmetry

MDPI

Article

An Intuitionistic Fuzzy Version of Hellinger Distance Measure and Its Application to Decision-Making Process

Xiang Li [1], Zhe Liu [2,*], Xue Han [3], Nan Liu [1] and Weihua Yuan [1]

[1] School of Computer Science and Technology, Shandong Jianzhu University, Jinan 250101, China
[2] School of Computer Science and Technology, Hainan University, Haikou 570228, China
[3] School of Management, Xi'an University of Architecture and Technology, Xi'an 710055, China
* Correspondence: liuzhe921@gmail.com or zheliu@ieee.org

Abstract: Intuitionistic fuzzy sets (IFSs), as a representative variant of fuzzy sets, has substantial advantages in managing and modeling uncertain information, so it has been widely studied and applied. Nevertheless, how to perfectly measure the similarities or differences between IFSs is still an open question. The distance metric offers an elegant and desirable solution to such a question. Hence, in this paper, we propose a new distance measure, named $\mathbb{D}_{IFS}$, inspired by the Hellinger distance in probability distribution space. First, we provide the formal definition of the new distance measure of IFSs, and analyze the outstanding properties and axioms satisfied by $\mathbb{D}_{IFS}$, which means it can measure the difference between IFSs well. Besides, on the basis of $\mathbb{D}_{IFS}$, we further present a normalized distance measure of IFSs, denoted $\mathbb{D}_{\widetilde{IFS}}$. Moreover, numerical examples verify that $\mathbb{D}_{\widetilde{IFS}}$ can obtain more reasonable and superior results. Finally, we further develop a new decision-making method on top of $\mathbb{D}_{\widetilde{IFS}}$ and evaluate its performance in two applications.

Keywords: intuitionistic fuzzy sets; Hellinger distance; decision making; uncertain information; pattern classification

check for updates

Citation: Li, X.; Liu, Z.; Han, X.; Liu, N.; Yuan, W. An Intuitionistic Fuzzy Version of Hellinger Distance Measure and Its Application to Decision-Making Process. *Symmetry* **2023**, *15*, 500. https://doi.org/10.3390/sym15020500

Academic Editor: Cengız Kahraman

Received: 31 January 2023
Revised: 8 February 2023
Accepted: 10 February 2023
Published: 14 February 2023

1. Introduction

Decision making is ubiquitous and surrounds our daily lives. Ideally, all information and knowledge are represented by a crisp and certain value. Unfortunately, because of the uncertainty and complexity of real applications, the information we collect is usually imperfect, i.e., information with uncertainty [1–3]. A major challenge, therefore, is how to process uncertain information to improve decision-making efficiency effectively [4–7]. At present, many methodologies have been proposed to solve this problem, such as fuzzy sets [8], intuitionistic fuzzy sets [9–11], rough sets [12–14], evidence theory [15–17] and R-number [18–20]. Among them, intuitionistic fuzzy sets (IFSs), an extension of fuzzy sets (FSs), has a unique advantage in processing uncertain information. The most important advantage of IFSs is that each element is represented by membership degree, as well as nonmembership degree and hesitancy degree, which makes it more flexible and effective in representing uncertain information. As a result, IFSs have gained great popularity and have been devoted to pattern classification [21–24], medical diagnosis [25–27], information fusion [28–30] and other fields [31–35].

The distance and divergence measures of IFSs, one of the essential branches of fuzzy sets theory, have attracted continuous attention. Up to now, many distance measures of IFSs [36–41] have been developed, which can be roughly classified into two types:

- *Two-dimensional (2D) distance measures of IFSs (2D distance measures of IFSs is to consider membership degree and nonmembership degree).* Atanassov [42] defined four 2*D* distance measures of IFSs according to Hamming and Euclidean distances. Later, Glazoczewski [43] proposed a Hausdorff distance measure to distinguish the difference of IFSs. Wang and Xin [44] defined two new distance measures of IFSs and proposed the axiom definition of distance measure. Recently, Mahanta and Panda [32] proposed a nonlinear distance measure to account for the difference between IFSs with high hesitation degrees. Gohain et al. [45] introduced the difference between the minimum and maximum cross-evaluation factor to measure IFSs. However, this distance does not satisfy the property of symmetry.

- *Three-dimensional (3D) distance measures of IFSs* (*3D distance measures of IFSs is to consider membership degree, nonmembership degree and hesitation degree*). Szmidt and Kacprzyk [46] developed $3D$ Hamming and Euclidean distances and corresponding normalized versions, respectively. Based on Hausdorff distance [43], Yang and Chiclana [47] proposed the corresponding $3D$ distance measure of IFSs and discussed its compatibility with $2D$ distance measure. Park et al. [48] presented a distance measure of IFSs by combining the distance measures of Yang and Chiclana [47] and Szmidt and Kacprzyk [46]. Xiao [22] suggested a new Jensen–Shannon divergence-based distance measure of IFSs. There are other ways to measure the difference of IFSs, see [49–53].

Although the above methods have achieved good results in certain scenarios, there are still some shortcomings:

- Several existing distance measures of IFSs do not fully satisfy the axiomatic definition.
- Most distance measures of IFSs tend to produce counterintuitive results when computing the difference between IFSs.
- In pattern classification problems, some distance measures cannot effectively predict the class of query pattern.

Hence, how distinguishing the difference between IFSs is an open and hot topic. In this paper, inspired by the Hellinger distance [54], we propose a new distance measure between IFSs, named $\mathbb{D}_{IFS}$, and prove the several properties of $\mathbb{D}_{IFS}$. Furthermore, we present a normalized version of $\mathbb{D}_{IFS}$, called $\mathbb{D}_{\widetilde{IFS}}$, and verify its effectiveness on some numerical examples. Finally, we propose a decision-making method based on $\mathbb{D}_{\widetilde{IFS}}$ and verify its performance in three applications.

1.1. Contribution

- A new distance measure ($\mathbb{D}_{IFS}$) is proposed, which makes full use of the membership degree, nonmembership degree, and hesitancy degree to consider the difference between IFSs.
- $\mathbb{D}_{IFS}$ meets the properties of the axiomatic definition of the distance measure, which reflects its validity in measuring the difference between IFSs.
- Based on $\mathbb{D}_{\widetilde{IFS}}$, a decision-making method is devised. The applications in pattern recognition verify that $\mathbb{D}_{\widetilde{IFS}}$ has good performance.

1.2. Organization

In Section 2, we review the basic definitions of IFSs and Hellinger distance. Section 3 shows the proposed distance measure and proves some of its properties. Numerical examples verify the effectiveness of $\mathbb{D}_{IFS}$ in Section 4. In Section 5, we propose a new decision-making method based on $\mathbb{D}_{IFS}$, and two applications are employed to verify the performance of the proposed method in Section 6. Finally, we make a conclusion in Section 7.

2. Preliminaries

In this section, we shortly introduce the related definitions of IFSs and Hellinger distance.

2.1. Basics of Intuitionistic Fuzzy Sets

Definition 1 (Fuzzy sets)**.** *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe of discourse (UOD). A fuzzy set* [8] $\bar{\mathcal{I}}$ in *X is defined as follows*:

$$\bar{\mathcal{I}} = \{\langle x, \xi_{\bar{\mathcal{I}}}(x)\rangle | x \in X\} \tag{1}$$

where $\xi_{\bar{\mathcal{I}}}(x) : X \to [0,1]$ *denotes the membership degree of* $x \in X$.

Definition 2 (Intuitionistic fuzzy sets)**.** *Let* $X = \{x_1, x_2, \cdots, x_n\}$ *be a UOD. An intuitionistic fuzzy set* [9] $\mathcal{I}$ in X *is defined as follows*:

$$\mathcal{I} = \{\langle x, \xi_{\mathcal{I}}(x), \zeta_{\mathcal{I}}(x)\rangle | x \in X\} \tag{2}$$

where $\xi_{\mathcal{I}}(x), \zeta_{\mathcal{I}}(x) : X \to [0,1]$ *represent the membership degree and non-membership degree of* $x \in X$*, respectively. For each* $x \in X$*, we have*:

$$0 \leq \xi_{\mathcal{I}}(x) + \zeta_{\mathcal{I}}(x) \leq 1 \tag{3}$$

and

$$\vartheta_{\mathcal{I}}(x) = 1 - \xi_{\mathcal{I}}(x) - \zeta_{\mathcal{I}}(x) \tag{4}$$

where $\vartheta_{\mathcal{I}}(x) : X \to [0,1]$ *denote the hesitancy degree or uncertainty degree of* $x \in X$.

Distance measures of IFSs is an essential task in decision-making. Wang and Xin [44] first proposed the properties of distance measure.

Definition 3 (The properties of IFSs)**.** *Let* $\mathcal{I}$*,* $\mathcal{J}$ *and* $\mathcal{K}$ *be three IFSs in* X*, and the mapping* $\mathbb{D} : IFS(X) \times IFS(X) \to [0,1]$ *is a distance measure of IFSs if* $\mathbb{D}(\mathcal{I}, \mathcal{J})$ *satisfies:*

1. $0 \leq \mathbb{D}(\mathcal{I}, \mathcal{J}) \leq 1$;
2. $\mathbb{D}(\mathcal{I}, \mathcal{J}) = 0$ *if and only if* $\mathcal{I} = \mathcal{J}$;
3. $\mathbb{D}(\mathcal{I}, \mathcal{J}) = \mathbb{D}(\mathcal{J}, \mathcal{I})$;
4. *If* $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{K}$*, then* $\mathbb{D}(\mathcal{I}, \mathcal{J}) \leq \mathbb{D}(\mathcal{I}, \mathcal{K})$ *and* $\mathbb{D}(\mathcal{J}, \mathcal{K}) \leq \mathbb{D}(\mathcal{I}, \mathcal{K})$.

Table 1 displays some classical distance measures between IFSs. For more concepts about IFSs, see [9,42].

Table 1. Existing distance measures of IFSs.

Ref.	Distance Measures
Atanassov [42]	$\mathbb{D}_{Ham-2}(\mathcal{I}, \mathcal{J}) = \frac{1}{2n}\sum_{i=1}^{n}(\lvert\xi_{\mathcal{I}}(x_i) - \xi_{\mathcal{J}}(x_i)\rvert + \lvert\zeta_{\mathcal{I}}(x_i) - \zeta_{\mathcal{J}}(x_i)\rvert)$
Atanassov [42]	$\mathbb{D}_{E-2}(\mathcal{I}, \mathcal{J}) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left((\xi_{\mathcal{I}}(x_i) - \xi_{\mathcal{J}}(x_i))^2 + (\zeta_{\mathcal{I}}(x_i) - \zeta_{\mathcal{J}}(x_i))^2\right)}$
Szmidt and Kacprzyk [46]	$\mathbb{D}_{Ham-3}(\mathcal{I}, \mathcal{J}) = \frac{1}{2n}\sum_{i=1}^{n}(\lvert\xi_{\mathcal{I}}(x_i) - \xi_{\mathcal{J}}(x_i)\rvert + \lvert\zeta_{\mathcal{I}}(x_i) - \zeta_{\mathcal{J}}(x_i)\rvert + \lvert\vartheta_{\mathcal{I}}(x_i) - \vartheta_{\mathcal{J}}(x_i)\rvert)$
Szmidt and Kacprzyk [46]	$\mathbb{D}_{E-3}(\mathcal{I}, \mathcal{J}) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left((\xi_{\mathcal{I}}(x_i) - \xi_{\mathcal{J}}(x_i))^2 + (\zeta_{\mathcal{I}}(x_i) - \zeta_{\mathcal{J}}(x_i))^2 + (\vartheta_{\mathcal{I}}(x_i) - \vartheta_{\mathcal{J}}(x_i))^2\right)}$
Glazoczewski [43]	$\mathbb{D}_{Hau}(\mathcal{I}, \mathcal{J}) = \frac{1}{n}\sum_{i=1}^{n}\max(\lvert\xi_{\mathcal{I}}(x_i) - \xi_{\mathcal{J}}(x_i)\rvert, \lvert\zeta_{\mathcal{I}}(x_i) - \zeta_{\mathcal{J}}(x_i)\rvert)$
Wang and Xin [44]	$\mathbb{D}^1_{WX}(\mathcal{I}, \mathcal{J}) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\lvert\xi_{\mathcal{I}}(x_i) - \xi_{\mathcal{J}}(x_i)\rvert + \lvert\zeta_{\mathcal{I}}(x_i) - \zeta_{\mathcal{J}}(x_i)\rvert}{4} + \frac{max(\lvert\xi_{\mathcal{I}}(x_i) - \xi_{\mathcal{J}}(x_i)\rvert, \lvert\zeta_{\mathcal{I}}(x_i) - \zeta_{\mathcal{J}}(x_i)\rvert)}{2}\right)$
Wang and xin [44]	$\mathbb{D}^2_{WX}(\mathcal{I}, \mathcal{J}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\lvert\xi_{\mathcal{I}}(x_i) - \xi_{\mathcal{J}}(x_i)\rvert + \lvert\zeta_{\mathcal{I}}(x_i) - \zeta_{\mathcal{J}}(x_i)\rvert}{2}$

Table 1. *Cont.*

Ref.	Distance Measures
Park et al. [48]	$\mathbb{D}_P(\mathcal{I},\mathcal{J}) = \frac{1}{4n}\sum_{i=1}^{n}[\lvert\xi_{\mathcal{I}}(x_i)-\xi_{\mathcal{J}}(x_i)\rvert + \lvert\zeta_{\mathcal{I}}(x_i)-\zeta_{\mathcal{J}}(x_i)\rvert + \lvert\vartheta_{\mathcal{I}}(x_i)-\vartheta_{\mathcal{J}}(x_i)\rvert$ $+2\max(\lvert\xi_{\mathcal{I}}(x_i)-\xi_{\mathcal{J}}(x_i)\rvert, \lvert\zeta_{\mathcal{I}}(x_i)-\zeta_{\mathcal{J}}(x_i)\rvert, \lvert\vartheta_{\mathcal{I}}(x_i)-\vartheta_{\mathcal{J}}(x_i)\rvert)]$
Yang and Chiclana [47]	$\mathbb{D}_Y(\mathcal{I},\mathcal{J}) = \frac{1}{n}\sum_{i=1}^{n}\max(\lvert\xi_{\mathcal{I}}(x_i)-\xi_{\mathcal{J}}(x_i)\rvert, \lvert\zeta_{\mathcal{I}}(x_i)-\zeta_{\mathcal{J}}(x_i)\rvert, \lvert\vartheta_{\mathcal{I}}(x_i)-\vartheta_{\mathcal{J}}(x_i)\rvert)$
Shen et al. [51]	$\mathbb{D}_{SH}(\mathcal{I},\mathcal{J}) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{(\tilde{\xi}_{\mathcal{I}}(x_i)-\tilde{\xi}_{\mathcal{J}}(x_i))^2+(\tilde{\zeta}_{\mathcal{I}}(x_i)-\tilde{\zeta}_{\mathcal{J}}(x_i))^2}{2}\right)^{\frac{1}{2}}$
Song et al. [50]	$\mathbb{D}_S(\mathcal{I},\mathcal{J}) = 1-\frac{1}{3n}\sum_{i=1}^{n}(2\sqrt{\xi_{\mathcal{I}}(x_i)\xi_{\mathcal{J}}(x_i)} + 2\sqrt{\zeta_{\mathcal{I}}(x_i)\zeta_{\mathcal{J}}(x_i)} + \sqrt{\vartheta_{\mathcal{I}}(x_i)\vartheta_{\mathcal{J}}(x_i)}$ $+\sqrt{(1-\xi_{\mathcal{I}}(x_i))(1-\xi_{\mathcal{J}}(x_i))} + \sqrt{(1-\zeta_{\mathcal{I}}(x_i))(1-\zeta_{\mathcal{J}}(x_i))})$
Chen and Deng [55]	$\mathbb{D}_{Ham-h}(\mathcal{I},\mathcal{J}) = \frac{1}{2n}\sum_{i=1}^{n}(\lvert\xi_{\mathcal{I}}(x_i)-\xi_{\mathcal{J}}(x_i)\rvert + \lvert\zeta_{\mathcal{I}}(x_i)-\zeta_{\mathcal{J}}(x_i)\rvert) \times \left(1-\frac{1}{2}\lvert\vartheta_{\mathcal{I}}(x_i)-\vartheta_{\mathcal{J}}(x_i)\rvert\right)$
Chen and Deng [55]	$\mathbb{D}_{Ham-c}(\mathcal{I},\mathcal{J}) = \frac{1}{2n}\sum_{i=1}^{n}(\lvert\xi_{\mathcal{I}}(x_i)-\xi_{\mathcal{J}}(x_i)\rvert + \lvert\zeta_{\mathcal{I}}(x_i)-\zeta_{\mathcal{J}}(x_i)\rvert) \times \left(\cos\frac{\vartheta}{6}\lvert\pi_{\mathcal{I}}(x_i)-\vartheta_{\mathcal{J}}(x_i)\rvert\right)$
Xiao [22]	$\mathbb{D}_X(\mathcal{I},\mathcal{J}) = \frac{1}{2}\Big[\xi_{\mathcal{I}}(x_i)\log\frac{2\xi_{\mathcal{I}}(x_i)}{\xi_{\mathcal{I}}(x_i)+\xi_{\mathcal{J}}(x_i)} + \xi_{\mathcal{J}}(x_i)\log\frac{2\xi_{\mathcal{J}}(x_i)}{\xi_{\mathcal{I}}(x_i)+\xi_{\mathcal{J}}(x_i)} + \zeta_{\mathcal{I}}(x_i)\log\frac{2\zeta_{\mathcal{I}}(x_i)}{\zeta_{\mathcal{I}}(x_i)+\zeta_{\mathcal{J}}(x_i)}$ $+\zeta_{\mathcal{J}}(x_i)\log\frac{2\zeta_{\mathcal{J}}(x_i)}{\zeta_{\mathcal{I}}(x_i)+\zeta_{\mathcal{J}}(x_i)} + \vartheta_{\mathcal{I}}(x_i)\log\frac{2\vartheta_{\mathcal{I}}(x_i)}{\vartheta_{\mathcal{I}}(x_i)+\vartheta_{\mathcal{J}}(x_i)} + \vartheta_{\mathcal{J}}(x_i)\log\frac{2\vartheta_{\mathcal{J}}(x_i)}{\vartheta_{\mathcal{I}}(x_i)+\vartheta_{\mathcal{J}}(x_i)}\Big]^{\frac{1}{2}}$
Gohain et al. [45]	$\mathbb{D}_G(\mathcal{I},\mathcal{J}) = \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{2}(\frac{\lvert\xi_{\mathcal{I}}(x_i)-\xi_{\mathcal{J}}(x_i)\rvert+\lvert\zeta_{\mathcal{I}}(x_i)-\zeta_{\mathcal{J}}(x_i)\rvert}{(1-\xi_{\mathcal{I}}(x_i))(1-\xi_{\mathcal{J}}(x_i))+(1+\zeta_{\mathcal{I}}(x_i))(1+\zeta_{\mathcal{J}}(x_i))})$ $+\frac{1}{4}(\lvert\min(\xi_{\mathcal{I}}(x_i),\zeta_{\mathcal{J}}(x_i))-\min(\xi_{\mathcal{J}}(x_i),\zeta_{\mathcal{I}}(x_i))\rvert$ $+\lvert\max(\xi_{\mathcal{I}}(x_i),\zeta_{\mathcal{J}}(x_i))-\max(\xi_{\mathcal{J}}(x_i),\zeta_{\mathcal{I}}(x_i))\rvert)]$

2.2. Hellinger Distance

To qualify the difference between two probability distributions, a new distance measure named Hellinger distance was presented [54].

Definition 4 (Hellinger distance)**.** *Consider two probability distributions $P = \{p_1, p_2, \cdots, p_n\}$ and $Q = \{q_1, q_2, \cdots, q_n\}$, the Hellinger distance between P and Q is defined as follows:*

$$D_H(P,Q) = \frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{n}(\sqrt{p_i}-\sqrt{q_i})^2} \tag{5}$$

Remark 1. *A larger value of $D_H(P,Q)$ corresponds to a larger difference between P and Q, and conversely, a smaller value of $D_H(P,Q)$ corresponds to a smaller difference between P and Q.*

3. An Intuitionistic Fuzzy Version of Hellinger Distance Measure

In this section, we present an intuitionistic fuzzy version of Hellinger distance measure, called $\mathbb{D}_{IFS}$. In parallel, we demonstrate that $\mathbb{D}_{IFS}$ satisfies some outstanding properties.

Definition 5 (Hellinger distance of IFSs)**.** *Let X be a UOD, $\mathcal{I}$ and $\mathcal{J}$ be two IFSs on X, where $\mathcal{I} = \{\langle x, \xi_{\mathcal{I}}(x), \zeta_{\mathcal{I}}(x)\rangle \mid x \in X\}$ and $\mathcal{J} = \{\langle x, \xi_{\mathcal{J}}(x), \zeta_{\mathcal{J}}(x)\rangle \mid x \in X\}$. The distance measure, named $\mathbb{D}_{IFS}(\mathcal{I},\mathcal{J})$, between IFSs $\mathcal{I}$ and $\mathcal{J}$ is defined as follows:*

$$\mathbb{D}_{IFS}(\mathcal{I},\mathcal{J}) = \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{\xi_{\mathcal{I}}(x)}-\sqrt{\xi_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{I}}(x)}-\sqrt{\zeta_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{I}}(x)}-\sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2} \tag{6}$$

Remark 2. *A larger value of $\mathbb{D}_{IFS}(\mathcal{I},\mathcal{J})$ corresponds to a larger difference between the two IFSs $\mathcal{I}$ and $\mathcal{J}$; similarly, a smaller value of $\mathbb{D}_{IFS}(\mathcal{I},\mathcal{J})$ corresponds to a smaller difference between the two IFSs $\mathcal{I}$ and $\mathcal{J}$.*

The properties of $\mathbb{D}_{IFS}$ are deduced as follows:

Property 1. $0 \leq \mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) \leq 1$.

Proof. Given two IFSs $\mathcal{I} = \{\langle x, \xi_{\mathcal{I}}(x), \zeta_{\mathcal{I}}(x) \rangle | x \in X\}$ and $\mathcal{J} = \{\langle x, \xi_{\mathcal{J}}(x), \zeta_{\mathcal{J}}(x) \rangle | x \in X\}$ on UOD X, and we have:

$$\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) = \frac{1}{\sqrt{2}} \sqrt{\left(\sqrt{\xi_{\mathcal{I}}(x)} - \sqrt{\xi_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{I}}(x)} - \sqrt{\zeta_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{I}}(x)} - \sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2}$$

We can easily obtain $\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) \geq 0$. What is more, $\xi_{\mathcal{I}}(x) + \zeta_{\mathcal{I}}(x) + \vartheta_{\mathcal{I}}(x) = 1$ and $\xi_{\mathcal{J}}(x) + \zeta_{\mathcal{J}}(x) + \vartheta_{\mathcal{J}}(x) = 1$, we can thus infer:

$$\begin{aligned}
\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) &= \frac{1}{\sqrt{2}} \left[\left(\sqrt{\xi_{\mathcal{A}}(x)} - \sqrt{\xi_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{I}}(x)} - \sqrt{\zeta_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{I}}(x)} - \sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2\right]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \Big[\xi_{\mathcal{I}}(x) - 2\sqrt{\xi_{\mathcal{I}}(x)\xi_{\mathcal{J}}(x)} + \xi_{\mathcal{J}}(x) + \zeta_{\mathcal{I}}(x) - 2\sqrt{\zeta_{\mathcal{I}}(x)\zeta_{\mathcal{J}}(x)} + \zeta_{\mathcal{J}}(x) + \vartheta_{\mathcal{I}}(x) \\
&\quad -2\sqrt{\vartheta_{\mathcal{I}}(x)\vartheta_{\mathcal{J}}(x)} + \vartheta_{\mathcal{J}}(x)\Big]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \left[2 - 2\left(\sqrt{\xi_{\mathcal{I}}(x)\xi_{\mathcal{J}}(x)} + \sqrt{\zeta_{\mathcal{I}}(x)\zeta_{\mathcal{J}}(x)} + \sqrt{\vartheta_{\mathcal{I}}(x)\vartheta_{\mathcal{J}}(x)}\right)\right]^{\frac{1}{2}} \\
&= 1 - \left[\sqrt{\xi_{\mathcal{I}}(x)\xi_{\mathcal{J}}(x)} + \sqrt{\zeta_{\mathcal{I}}(x)\zeta_{\mathcal{J}}(x)} + \sqrt{\vartheta_{\mathcal{I}}(x)\vartheta_{\mathcal{J}}(x)}\right]^{\frac{1}{2}} \\
&\leq 1
\end{aligned}$$

Therefore, we can prove that $0 \leq \mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) \leq 1$. □

Property 2. $\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) = 0$ *if and only if* $\mathcal{I} = \mathcal{J}$.

Proof. Given two same IFSs $\mathcal{I} = \{\langle x, \xi_{\mathcal{I}}(x), \zeta_{\mathcal{I}}(x) \rangle | x \in X\}$ and $\mathcal{J} = \{\langle x, \xi_{\mathcal{J}}(x), \zeta_{\mathcal{J}}(x) \rangle | x \in X\}$ on UOD X, we have:

$$\xi_{\mathcal{I}}(x) = \xi_{\mathcal{J}}(x),\ \zeta_{\mathcal{I}}(x) = \zeta_{\mathcal{J}}(x),\ \vartheta_{\mathcal{I}}(x) = \vartheta_{\mathcal{J}}(x)$$

Then,

$$\begin{aligned}
\left(\sqrt{\xi_{\mathcal{I}}(x)} - \sqrt{\xi_{\mathcal{J}}(x)}\right)^2 &= 0 \\
\left(\sqrt{\zeta_{\mathcal{I}}(x)} - \sqrt{\zeta_{\mathcal{J}}(x)}\right)^2 &= 0 \\
\left(\sqrt{\vartheta_{\mathcal{I}}(x)} - \sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2 &= 0
\end{aligned}$$

Therefore, we can obtain:

$$\begin{aligned}
\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) &= \frac{1}{\sqrt{2}} \sqrt{\left(\sqrt{\xi_{\mathcal{I}}(x)} - \sqrt{\xi_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{I}}(x)} - \sqrt{\zeta_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{I}}(x)} - \sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2} \\
&= 0
\end{aligned}$$

For any $x \in X$, if $\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) = 0$, we have:

$$\begin{aligned}
\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) &= \frac{1}{\sqrt{2}} \sqrt{\left(\sqrt{\xi_{\mathcal{I}}(x)} - \sqrt{\xi_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{I}}(x)} - \sqrt{\zeta_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{I}}(x)} - \sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2} \\
&= 0
\end{aligned}$$

Hence, we can obtain:

$$\left(\sqrt{\xi_{\mathcal{I}}(x)} - \sqrt{\xi_{\mathcal{J}}(x)}\right)^2 = 0$$
$$\left(\sqrt{\zeta_{\mathcal{I}}(x)} - \sqrt{\zeta_{\mathcal{J}}(x)}\right)^2 = 0$$
$$\left(\sqrt{\vartheta_{\mathcal{I}}(x)} - \sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2 = 0$$

Since $0 \leq \xi_{\mathcal{I}}(x) \leq 1, 0 \leq \zeta_{\mathcal{I}}(x) \leq 1, 0 \leq \vartheta_{\mathcal{I}}(x) \leq 1$, $0 \leq \xi_{\mathcal{J}}(x) \leq 1, 0 \leq \zeta_{\mathcal{J}}(x) \leq 1$ and $0 \leq \vartheta_{\mathcal{J}}(x) \leq 1$.

Thus, we have:

$$\xi_{\mathcal{I}}(x) = \xi_{\mathcal{J}}(x),\ \zeta_{\mathcal{I}}(x) = \zeta_{\mathcal{J}}(x),\ \vartheta_{\mathcal{I}}(x) = \vartheta_{\mathcal{J}}(x)$$

Thereby, we can prove that $\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) = 0$ if and only if $\mathcal{I} = \mathcal{J}$. □

Property 3. $\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) = \mathbb{D}_{IFS}(\mathcal{J}, \mathcal{I})$.

Proof. Given two IFSs $\mathcal{I} = \{\langle x, \xi_{\mathcal{I}}(x), \zeta_{\mathcal{I}}(x)\rangle | x \in X\}$ and $\mathcal{J} = \{\langle x, \xi_{\mathcal{J}}(x), \zeta_{\mathcal{J}}(x)\rangle | x \in X\}$ on UOD X, we have:

$$\begin{aligned}
\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) &= \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{\xi_{\mathcal{I}}(x)} - \sqrt{\xi_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{I}}(x)} - \sqrt{\zeta_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{I}}(x)} - \sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{\xi_{\mathcal{J}}(x)} - \sqrt{\xi_{\mathcal{I}}(x)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{J}}(x)} - \sqrt{\zeta_{\mathcal{I}}(x)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{J}}(x)} - \sqrt{\vartheta_{\mathcal{I}}(x)}\right)^2} \\
&= \mathbb{D}_{IFS}(\mathcal{J}, \mathcal{I})
\end{aligned}$$

Hence, we can prove that $\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) = \mathbb{D}_{IFS}(\mathcal{J}, \mathcal{I})$. □

Property 4. *If $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{K}$, then $\mathbb{D}(\mathcal{I}, \mathcal{J}) \leq \mathbb{D}(\mathcal{I}, \mathcal{K})$ and $\mathbb{D}(\mathcal{J}, \mathcal{K}) \leq \mathbb{D}(\mathcal{I}, \mathcal{K})$.*

Proof. Given three IFSs $\mathcal{I} = \{\langle x, \xi_{\mathcal{I}}(x), \zeta_{\mathcal{I}}(x)\rangle | x \in X\}$, $\mathcal{J} = \{\langle x, \xi_{\mathcal{J}}(x), \zeta_{\mathcal{J}}(x)\rangle | x \in X\}$ and $\mathcal{K} = \{\langle x, \xi_{\mathcal{K}}(x), \zeta_{\mathcal{K}}(x)\rangle | x \in X\}$ on UOD X. If $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{K}$, we have:

$$\xi_{\mathcal{I}}(x) \leq \xi_{\mathcal{J}}(x) \leq \xi_{\mathcal{K}}(x),\ \zeta_{\mathcal{I}}(x) \leq \zeta_{\mathcal{J}} \leq \zeta_{\mathcal{K}}(x),\ \vartheta_{\mathcal{I}}(x) \leq \vartheta_{\mathcal{J}}(x) \leq \vartheta_{\mathcal{K}}(x)$$

Hence, we obtain:

$$\begin{aligned}
\mathbb{D}_{IFS}(\mathcal{I}, \mathcal{J}) &= \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{\xi_{\mathcal{I}}(x)} - \sqrt{\xi_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{I}}(x)} - \sqrt{\zeta_{\mathcal{J}}(x)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{I}}(x)} - \sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2} \\
&\leq \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{\xi_{\mathcal{I}}(x)} - \sqrt{\xi_{\mathcal{K}}(x)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{I}}(x)} - \sqrt{\zeta_{\mathcal{K}}(x)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{I}}(x)} - \sqrt{\vartheta_{\mathcal{J}}(x)}\right)^2} \\
&= \mathbb{D}_{IFS}(\mathcal{I}, \mathcal{K})
\end{aligned}$$

Similarly, we can obtain $\mathbb{D}_{IFS}(\mathcal{J}, \mathcal{K}) \leq \mathbb{D}_{IFS}(\mathcal{I}, \mathcal{K})$.

Therefore, we can prove that when $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{K}$, $\mathbb{D}(\mathcal{I}, \mathcal{J}) \leq \mathbb{D}(\mathcal{I}, \mathcal{K})$ and $\mathbb{D}(\mathcal{J}, \mathcal{K}) \leq \mathbb{D}(\mathcal{I}, \mathcal{K})$. □

Definition 6 (Normalized Hellinger distance of IFSs)**.** *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a UOD, $\mathcal{I}$ and $\mathcal{J}$ be two IFSs on X, where $\mathcal{I} = \{\langle x_i, \xi_{\mathcal{I}}(x_i), \zeta_{\mathcal{I}}(x_i)\rangle | x_i \in X\}$ and $\mathcal{J} = \{\langle x_i, \xi_{\mathcal{J}}(x_i), \zeta_{\mathcal{J}}(x_i)\rangle | x_i \in X\}$, the normalized distance measure, named $\mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{J})$, between IFSs $\mathcal{I}$ and $\mathcal{J}$ is defined as follows:*

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{J}) = \frac{1}{\sqrt{2n}}\sqrt{\sum_{i=1}^{n}\left[\left(\sqrt{\xi_{\mathcal{I}}(x_i)} - \sqrt{\xi_{\mathcal{J}}(x_i)}\right)^2 + \left(\sqrt{\zeta_{\mathcal{I}}(x_i)} - \sqrt{\zeta_{\mathcal{J}}(x_i)}\right)^2 + \left(\sqrt{\vartheta_{\mathcal{I}}(x_i)} - \sqrt{\vartheta_{\mathcal{J}}(x_i)}\right)^2\right]} \quad (7)$$

4. Numerical Examples

This section verifies the validity and properties of the proposed distance measure with several numerical examples. In the paper, we execute all the distance measures and generate the corresponding figures with MATLAB and run them on a computer with Intel Core i7 2.4 GHz CPU with 16 GB RAM.

Example 1. *Given three IFSs $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ in UOD $X = \{x_1, x_2\}$. These IFSs are expressed as follows:*

$$\mathcal{I} = \{\langle x_1, 0.20, 0.40\rangle, \langle x_2, 0.50, 0.10\rangle\}$$
$$\mathcal{J} = \{\langle x_1, 0.20, 0.40\rangle, \langle x_2, 0.50, 0.10\rangle\}$$
$$\mathcal{K} = \{\langle x_1, 0.60, 0.25\rangle, \langle x_2, 0.15, 0.30\rangle\}$$

The $\mathbb{D}_{\widetilde{IFS}}$ between IFSs $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ can be measured as:

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{J}) = 0, \mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{K}) = 0.2969, \mathbb{D}_{\widetilde{IFS}}(\mathcal{J}, \mathcal{K}) = 0.2969$$
$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{J}, \mathcal{I}) = 0, \mathbb{D}_{\widetilde{IFS}}(\mathcal{K}, \mathcal{I}) = 0.2969, \mathbb{D}_{\widetilde{IFS}}(\mathcal{K}, \mathcal{J}) = 0.2969$$

Based on the above results, we can easily verify the Property 2 and Property 3 of the proposed distance measure.

Example 2. *Given three IFSs $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ in UOD $X = \{x_1, x_2\}$. These IFSs are expressed as follows:*

$$\mathcal{I} = \{\langle x_1, 0.30, 0.60\rangle, \langle x_2, 0.40, 0.50\rangle\}$$
$$\mathcal{J} = \{\langle x_1, 0.50, 0.40\rangle, \langle x_2, 0.60, 0.20\rangle\}$$
$$\mathcal{K} = \{\langle x_1, 0.70, 0.10\rangle, \langle x_2, 0.80, 0.10\rangle\}$$

Clearly, $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{K}$. Thereby, we have:

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{J}) = 0.1940, \mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{K}) = 0.3647, \mathbb{D}_{\widetilde{IFS}}(\mathcal{J}, \mathcal{K}) = 0.2137$$

and

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{J}) \leq \mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{K})$$
$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{J}, \mathcal{K}) \leq \mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{K})$$

Therefore, we verify Property 4.

Example 3. *Given two IFSs $\mathcal{I}$ and $\mathcal{J}$ in UOD $X = \{x\}$, these IFSs are expressed as follows:*

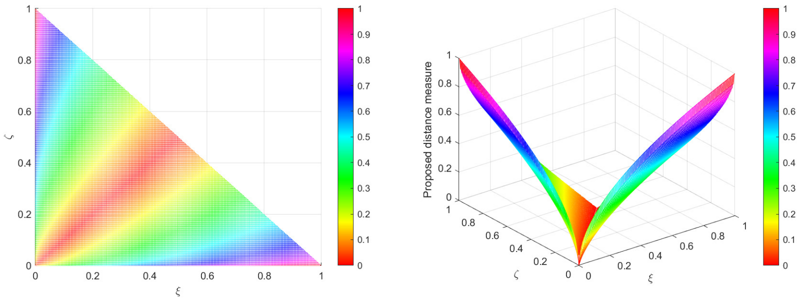
$$\mathcal{I} = \{\langle x, \xi, \zeta\rangle\}, \mathcal{J} = \{\langle x, \zeta, \xi\rangle\}$$

The parameters ξ and ζ are in the range of $[0, 1]$ and satisfy $\xi + \zeta \leq 1$, as shown in Figure 1a. The $\mathbb{D}_{\widetilde{IFS}}$ between IFSs $\mathcal{I}$ and $\mathcal{J}$ can be measured as shown in Figure 1b. From Figure 1b, we can observe that as ξ and ζ change, $\mathbb{D}_{\widetilde{IFS}}$ always changes between 0 and 1. Therefore, we can verify Property 1 of $\mathbb{D}_{\widetilde{IFS}}$.

Figures 2–4 show the influence of ξ and ζ as well as parameters α and β on $\mathbb{D}_{\widetilde{IFS}}$ under different cases, respectively. It can be intuitively observed from Figures 2–4 that the relationship between $\mathbb{D}_{\widetilde{IFS}}$ and ξ and ζ of IFSs is nonlinear, which reflects the nonlinear characteristics of $\mathbb{D}_{\widetilde{IFS}}$. In addition, the maximum value of $\mathbb{D}_{\widetilde{IFS}}$ provided by ξ and ζ in different cases is as follows:

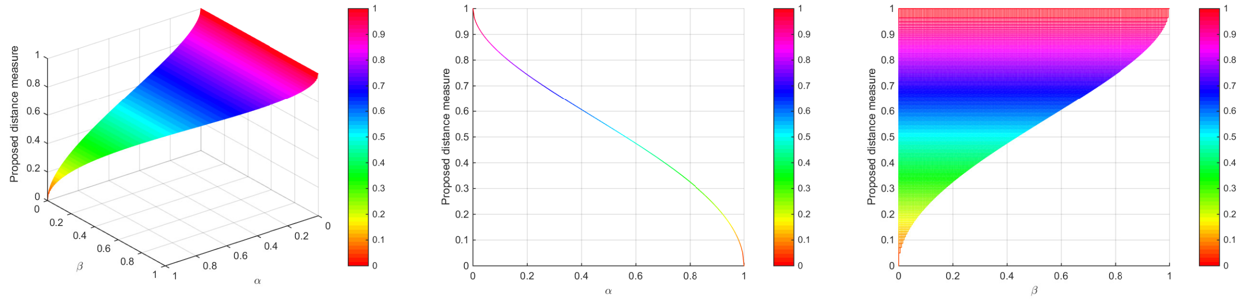
- $\mathcal{J} = \{\langle x, 1, 0\rangle\} \to \mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{J}) = 1$, for $\mathcal{I} = \{\langle x, 0, \beta\rangle\}$, $\beta \in [0, 1]$.
- $\mathcal{J} = \{\langle x, 0, 1\rangle\} \to \mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{J}) = 1$, for $\mathcal{I} = \{\langle x, \alpha, 0\rangle\}$, $\alpha \in [0, 1]$.

- $\mathcal{J} = \{\langle x, 0.5, 0.5 \rangle\} \rightarrow \mathbb{D}_{\widetilde{IFS}}(\mathcal{I}, \mathcal{J}) = 1$, for $\mathcal{I} = \{\langle x, 0, 0 \rangle\}$.



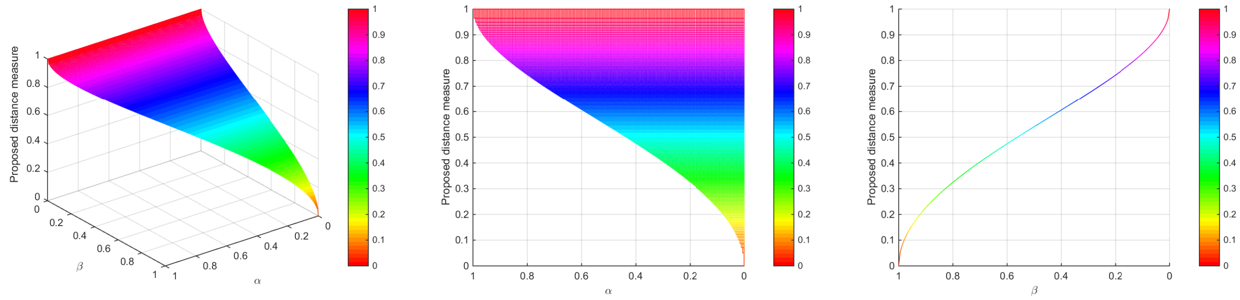
(**a**) (**b**)

Figure 1. The values of $\mathbb{D}_{\widetilde{IFS}}$ varying with ξ and ζ in Example 3. (**a**) Variations of ξ and ζ; (**b**) $\mathbb{D}_{\widetilde{IFS}}$.



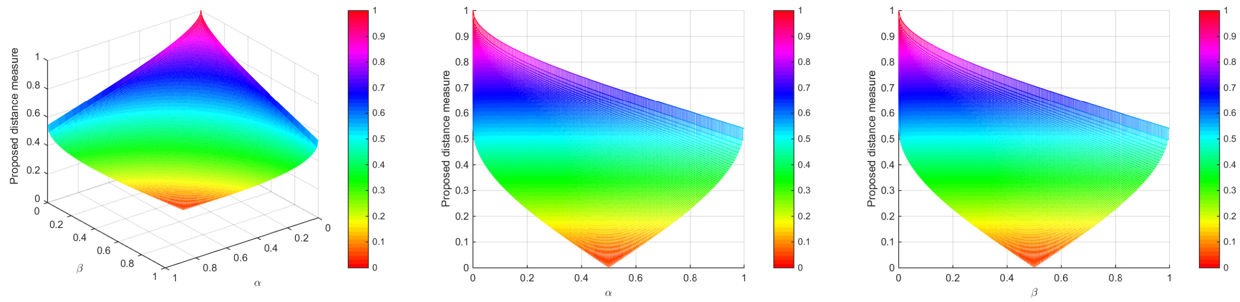
(**a**) (**b**) (**c**)

Figure 2. $\xi = 1, \zeta = 0$ in Example 4. (**a**) $\mathbb{D}_{\widetilde{IFS}}$; (**b**) $\mathbb{D}_{\widetilde{IFS}}$ with different α; (**c**) $\mathbb{D}_{\widetilde{IFS}}$ with different β .



(**a**) (**b**) (**c**)

Figure 3. $\xi = 0, \zeta = 1$ in Example 4. (**a**) $\mathbb{D}_{\widetilde{IFS}}$; (**b**) $\mathbb{D}_{\widetilde{IFS}}$ with different α; (**c**) $\mathbb{D}_{IFS}$ with different β.



(**a**) (**b**) (**c**)

Figure 4. $\xi = 0.5, \zeta = 0.5$ in Example 4. (**a**) $\mathbb{D}_{\widetilde{IFS}}$; (**b**) $\mathbb{D}_{\widetilde{IFS}}$ with different α; (**c**) $\mathbb{D}_{\widetilde{IFS}}$ with different β.

Example 4. *Given two IFSs $\mathcal{I}$ and $\mathcal{J}$ in UOD $X = \{x\}$. These IFSs are expressed as follows:*

$$\mathcal{I} = \{\langle x, \alpha, \beta \rangle\}, \mathcal{J} = \{\langle x, \zeta, \xi \rangle\}$$

Example 5. *Given the IFSs $\mathcal{I}_i$ and $\mathcal{J}_i$ in UOD $X = \{x_1, x_2\}$ under Case $i(i = 1, \cdots, 6)$, these IFSs are expressed as shown in Table 2.*

Table 2. Two IFSs $\mathcal{I}_i$ and $\mathcal{J}_i$ under different cases in Example 5.

IFSs	*Case* 1	*Case* 2
$\mathcal{I}_i$	$\{\langle x_1, 0.35, 0.25\rangle, \langle x_2, 0.45, 0.35\rangle\}$	$\{\langle x_1, 0.35, 0.25\rangle, \langle x_2, 0.45, 0.35\rangle\}$
$\mathcal{J}_i$	$\{\langle x_1, 0.15, 0.20\rangle, \langle x_2, 0.25, 0.30\rangle\}$	$\{\langle x_1, 0.13, 0.22\rangle, \langle x_2, 0.23, 0.32\rangle\}$
IFSs	*Case* 3	*Case* 4
$\mathcal{I}_i$	$\{\langle x_1, 0.30, 0.20\rangle, \langle x_2, 0.40, 0.30\rangle\}$	$\{\langle x_1, 0.50, 0.40\rangle, \langle x_2, 0.40, 0.30\rangle\}$
$\mathcal{J}_i$	$\{\langle x_1, 0.25, 0.25\rangle, \langle x_2, 0.50, 0.40\rangle\}$	$\{\langle x_1, 0.55, 0.35\rangle, \langle x_2, 0.30, 0.20\rangle\}$
IFSs	*Case* 5	*Case* 6
$\mathcal{I}_i$	$\{\langle x_1, 0.50, 0.40\rangle, \langle x_2, 0.40, 0.30\rangle\}$	$\{\langle x_1, 0.50, 0.50\rangle, \langle x_2, 1.00, 0.00\rangle\}$
$\mathcal{J}_i$	$\{\langle x_1, 0.50, 0.50\rangle, \langle x_2, 0.45, 0.35\rangle\}$	$\{\langle x_1, 0.00, 0.00\rangle, \langle x_2, 0.00, 1.00\rangle\}$

Table 3 shows the results of different distance measures. We can see that most distance measures can sometimes produce counterintuitive results. Specifically, $\mathbb{D}_{Ham-2}$, $\mathbb{D}_{Ham-3}$, $\mathbb{D}^2_{WX}$, $\mathbb{D}_P$ and $\mathbb{D}_{YC}$ generate counterintuitive results in Case $1-$ Case 4, $\mathbb{D}_{E-2}$, $\mathbb{D}_{E-3}$, $\mathbb{D}_{Hau}$, $\mathbb{D}^1_{WX}$, $\mathbb{D}_{Ham-h}$ and $\mathbb{D}_{Ham-c}$ generate counterintuitive results in Case $3-$ Case 4, and $\mathbb{D}_X$ cannot output results in Case $5-$ Case 6. However, $\mathbb{D}_{SH}$, $\mathbb{D}_S$, $\mathbb{D}_G$ and the proposed distance measure $\mathbb{D}_{\widetilde{IFS}}$ can well distinguish the differences between IFSs in different cases, which also shows their potential in practical application.

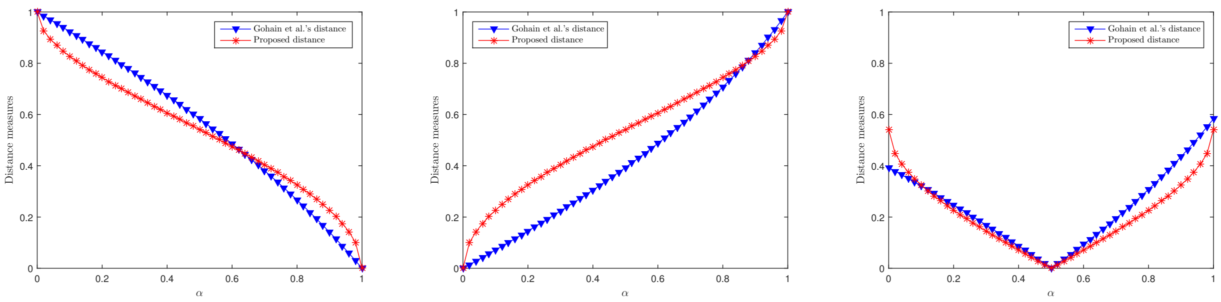
Table 3. Comparisons of various distance measures in Example 5.

Methods	***Case* 1**	***Case* 2**	***Case* 3**	***Case* 4**	***Case* 5**	***Case* 6**
$\mathbb{D}_{Ham-2}$[42]	0.1250	0.1250	0.0750	0.0750	0.0500	0.7500
$\mathbb{D}_{E-2}$ [42]	0.1458	0.1570	0.0791	0.0791	0.0612	0.7906
$\mathbb{D}_{Ham-3}$ [46]	0.2500	0.2500	0.1250	0.1250	0.1000	1.0000
$\mathbb{D}_{E-3}$ [46]	0.2291	0.2364	0.1275	0.1275	0.0935	0.9354
$\mathbb{D}_{Hau}$ [43]	0.2000	0.2200	0.0750	0.0750	0.0750	0.7500
$\mathbb{D}^1_{WX}$ [44]	0.1625	0.1725	0.0750	0.0750	0.0625	0.7500
$\mathbb{D}^2_{WX}$ [44]	0.1250	0.1250	0.0750	0.0750	0.0500	0.7500
$\mathbb{D}_P$ [48]	0.2500	0.2500	0.1250	0.1250	0.1000	1.0000
$\mathbb{D}_{YC}$ [47]	0.2500	0.2500	0.0625	0.0625	0.1000	1.0000
$\mathbb{D}_{SH}$ [51]	0.1374	0.1638	0.0590	0.0604	0.0399	0.7500
$\mathbb{D}_S$ [50]	0.0218	0.0249	0.0083	0.0063	0.0112	0.7643
$\mathbb{D}_{Ham-h}$ [55]	0.0812	0.0806	0.0483	0.0483	0.3208	0.4583
$\mathbb{D}_{Ham-c}$ [55]	0.1248	0.1224	0.0749	0.0749	0.0500	0.7333
$\mathbb{D}_X$ [22]	0.2358	0.2473	0.1385	0.1103	NAN	NAN
$\mathbb{D}_G$ [45]	0.0968	0.1057	0.0484	0.0621	0.0351	0.6250
$\mathbb{D}_{\widetilde{IFS}}$	0.1974	0.2072	0.1336	0.1067	0.1725	1.0000

Example 6. *Given two IFSs $\mathcal{I}$ and $\mathcal{J}$ in UOD $X = \{x\}$. These IFSs are expressed as follows:*

$$\mathcal{I} = \{\langle x, \alpha, 1-\alpha \rangle\}, \mathcal{J} = \{\langle x, \zeta, \xi \rangle\}$$

Figure 5 shows the results of $\mathbb{D}_G$ and $\mathbb{D}_{\widetilde{IFS}}$ in three cases. Under different α values, we can see that the changes of $\mathbb{D}_G$, $\mathbb{D}_{\widetilde{IFS}}$ and IFSs are nonlinear. Furthermore, in Figure 5c, we can see that $\mathbb{D}_G$ does not satisfy the symmetry of the axiomatic definition of the distance measure, whereas $\mathbb{D}_{\widetilde{IFS}}$ is a better measure of the difference between IFSs.



(**a**) (**b**) (**c**)

Figure 5. Comparisons of $\mathbb{D}_{\mathbb{G}}$ and $\mathbb{D}_{\widetilde{IFS}}$ in Example 6. (**a**) $\xi = 1, \zeta = 0$; (**b**) $\xi = 0, \zeta = 1$; (**c**) $\xi = 0.5, \zeta = 0.5$.

5. A New Decision-Making Method Based on $\mathbb{D}_{\widetilde{IFS}}$

In this section, a new decision-making method based on $\mathbb{D}_{\widetilde{IFS}}$ is designed for the pattern recognition problems.

Problem statement: Suppose there exist a finite UOD $X = \{x_1, x_2, \cdots, x_n\}$, and t patterns $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_t\}$, which can be expressed in IFSs: $\mathcal{P}_j = \{\langle x_i, \xi_{\mathcal{P}_j}(x_i), \zeta_{\mathcal{P}_j}(x_i)\rangle | x_i \in X\}, (j = 1, 2, \cdots, t)$. For m query patterns $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_m\}$, expressed in IFSs as: $\mathcal{S}_k = \{\langle x_i, \xi_{\mathcal{S}_k}(x_i), \zeta_{\mathcal{S}_k}(x_i)\rangle | x_i \in X\}, (k = 1, 2, \cdots, m)$. We aim to classify the query pattern in accordance with the known pattern.

Step 1: The distance measure $\mathbb{D}_{\widetilde{IFS}}$ is adopted to compute the distance between the query pattern $\mathcal{S}_k, (k = 1, 2, \cdots, m)$ and known pattern $\mathcal{P}_j, (j = 1, 2, \cdots, t)$.

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_j, \mathcal{S}_k) = \frac{1}{\sqrt{2n}} \sqrt{\sum_{i=1}^{n} \left[\left(\sqrt{\xi_{\mathcal{P}_j}(x_i)} - \sqrt{\xi_{\mathcal{S}_k}(x_i)} \right)^2 + \left(\sqrt{\zeta_{\mathcal{P}_j}(x_i)} - \sqrt{\zeta_{\mathcal{S}_k}(x_i)} \right)^2 + \left(\sqrt{\vartheta_{\mathcal{P}_j}(x_i)} - \sqrt{\vartheta_{\mathcal{S}_k}(x_i)} \right)^2 \right]} \tag{8}$$

Step 2: The minimum distance $\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_g, \mathcal{S}_k)$ between the query pattern $\mathcal{S}_k, (k = 1, 2, \cdots, m)$ and known pattern $\mathcal{P}_j, (j = 1, 2, \cdots, t)$ will be selected as follows:

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_g, \mathcal{S}_k) = \min_{1 \le j \le t} \mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_j, \mathcal{S}_k) \tag{9}$$

Step 3: The classification result of the query pattern $\mathcal{S}_k$ is described as follows:

$$g = \arg \min_{1 \le j \le t} \mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_j, \mathcal{S}_k), \ \mathcal{P}_g \to \mathcal{S}_k \tag{10}$$

The pseudocode of the proposed decision-making method is shown in Algorithm 1.

Algorithm 1 The proposed decision-making method.

Require: $\mathcal{P}_j, (j = 1, 2, \cdots, t), \mathcal{S}_k, (k = 1, 2, \cdots, m)$
Ensure: Classification result of $\mathcal{S}_k$
for $j = 1, j \le t$ **do**
  for $k = 1, k \le m$ **do**
    Calculate the distance measure $\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_j, \mathcal{S}_k)$ using (8);
  end for
end for
for $k = 1, k \le n$ **do**
  Obtain the minimum distance $\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_g, \mathcal{S}_k)$ using (9);
end for
for $k = 1, k \le n$ **do**
  Obtain the classification result of $\mathcal{S}_k$ using (10);
end for

6. Application in Pattern Recognition

In this section, we apply the new decision-making method to pattern recognition scenarios. By comparing fifteen existing methods, we verify that the proposed method has advantages over the existing methods.

Application 1 ([22,56]). *Let us consider a pattern classification scenario, where IFSs represent all data. Table 4 shows three known patterns $\mathcal{P}_i(i = 1, 2, 3.)$ and a query pattern $\mathcal{S}$. Our goal is to obtain the class of the query pattern $\mathcal{S}$ depending on known patterns.*

Table 4. IFSs of pattern classification problem in **Application 1**.

Pattern	Feature		
	x_1	x_2	x_3
$\mathcal{P}_1$	$\langle x_1, 0.15, 0.25\rangle$	$\langle x_2, 0.25, 0.35\rangle$	$\langle x_3, 0.35, 0.45\rangle$
$\mathcal{P}_2$	$\langle x_1, 0.05, 0.15\rangle$	$\langle x_2, 0.15, 0.25\rangle$	$\langle x_3, 0.25, 0.35\rangle$
$\mathcal{P}_3$	$\langle x_1, 0.16, 0.26\rangle$	$\langle x_2, 0.26, 0.36\rangle$	$\langle x_3, 0.36, 0.46\rangle$
$\mathcal{S}$	$\langle x_1, 0.30, 0.20\rangle$	$\langle x_2, 0.40, 0.30\rangle$	$\langle x_3, 0.50, 0.40\rangle$

The implementation of the decision-making method is expressed below:

Step 1: *The distance measure $\mathbb{D}_{\widetilde{IFS}}$ is adopted to compute the distances between $\mathcal{S}$ and $\mathcal{P}_i(i = 1, 2, 3.)$, the results are depicted as:*

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_1, \mathcal{S}) = 0.1236$$
$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_2, \mathcal{S}) = 0.2589$$
$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_3, \mathcal{S}) = 0.1129$$

Step 2: *The minimum distance between $\mathcal{S}$ and $\mathcal{P}_i$ is expressed as follows:*

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_3, \mathcal{S}) = 0.1129$$

Step 3: *The classification result of $\mathcal{S}$ is described as follows:*

$$\mathcal{P}_3 \to \mathcal{S}$$

Table 5 and Figure 6 show the results of distance measures by different methods. From Table 5 and Figure 6, we can see that $\mathbb{D}_{Ham-2}$, $\mathbb{D}^2_{WX}$, $\mathbb{D}_{Ham-h}$ and $\mathbb{D}_{Ham-c}$ all produce counterintuitive results, where $\mathbb{D}(\mathcal{P}_1, \mathcal{S}) = \mathbb{D}(\mathcal{P}_3, \mathcal{S})$. Therefore, it is difficult for these methods to obtain satisfactory decision results in practical application. Other comparison methods and the proposed method can all obtain reasonable classification results, whereas $\mathbb{D}_{E-2}$, $\mathbb{D}^1_{WX}$, $\mathbb{D}_{YC}$, $\mathbb{D}_S$ and $\mathbb{D}_G$ are very close to the distance from $\mathcal{S}$ to $\mathcal{P}_1$ and $\mathcal{P}_3$, which may limit their application in some complex cases.

Table 5. Pattern classification results of various methods in **Application 1**.

Method	Distance Measures			Classification
	$(\mathcal{P}_1, \mathcal{S})$	$(\mathcal{P}_2, \mathcal{S})$	$(\mathcal{P}_3, \mathcal{S})$	
$\mathbb{D}_{Ham-2}$ [42]	0.1000	0.1500	0.1000	Uncertainty
$\mathbb{D}_{E-2}$ [42]	0.1118	0.1803	0.1077	$\mathcal{P}_3$
$\mathbb{D}_{Ham-3}$ [46]	0.1500	0.3000	0.1400	$\mathcal{P}_3$
$\mathbb{D}_{E-3}$ [46]	0.1323	0.2784	0.1217	$\mathcal{P}_3$
$\mathbb{D}_{Hau}$ [43]	0.1500	0.2500	0.1400	$\mathcal{P}_3$
$\mathbb{D}^1_{WX}$ [44]	0.1250	0.2000	0.1200	$\mathcal{P}_3$
$\mathbb{D}^2_{WX}$ [44]	0.1000	0.1500	0.1000	Uncertainty

Table 5. *Cont.*

Method	Distance Measures			Classification
	$(\mathcal{P}_1, \mathcal{S})$	$(\mathcal{P}_2, \mathcal{S})$	$(\mathcal{P}_3, \mathcal{S})$	
$\mathbb{D}_P$ [48]	0.1500	0.3000	0.1400	$\mathcal{P}_3$
$\mathbb{D}_{YC}$ [47]	0.1500	0.3000	0.1400	$\mathcal{P}_3$
$\mathbb{D}_{SH}$ [51]	0.1373	0.1914	0.1347	$\mathcal{P}_3$
$\mathbb{D}_S$ [50]	0.0102	0.0381	0.0089	$\mathcal{P}_3$
$\mathbb{D}_{Ham-h}$ [55]	0.0988	0.1463	0.0988	Uncertainty
$\mathbb{D}_{Ham-c}$ [55]	0.0100	0.1497	0.1000	Uncertainty
$\mathbb{D}_X$ [22]	0.1479	0.3060	0.1351	$\mathcal{P}_3$
$\mathbb{D}_G$ [57]	0.0452	0.0699	0.0450	$\mathcal{P}_3$
$\mathbb{D}_{\widetilde{IFS}}$	0.1236	0.2589	0.1129	$\mathcal{P}_3$

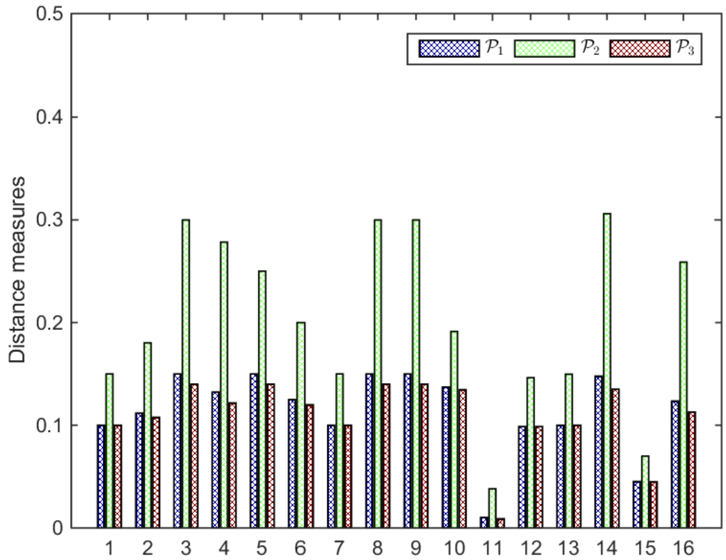


Figure 6. Comparisons of various distance measures in **Application 1**.

Application 2. *Suppose there has a pattern classification scenario where IFSs represent all data. Table 6 displays three known patterns $\mathcal{P}_i(i = 1, 2, 3.)$ and a query pattern $\mathcal{S}$. We intend to predict the class of the query pattern $\mathcal{S}$ based on known patterns $\mathcal{P}_i(i = 1, 2, 3.)$.*

Table 6. IFSs of pattern classification problem in **Application 2**.

Pattern	Feature		
	x_1	x_2	x_3
$\mathcal{P}_1$	$\langle x_1, 0.34, 0.34\rangle$	$\langle x_2, 0.19, 0.48\rangle$	$\langle x_3, 0.02, 0.12\rangle$
$\mathcal{P}_2$	$\langle x_1, 0.35, 0.33\rangle$	$\langle x_2, 0.20, 0.47\rangle$	$\langle x_3, 0.00, 0.14\rangle$
$\mathcal{P}_3$	$\langle x_1, 0.33, 0.35\rangle$	$\langle x_2, 0.21, 0.46\rangle$	$\langle x_3, 0.01, 0.13\rangle$
$\mathcal{S}$	$\langle x_1, 0.37, 0.31\rangle$	$\langle x_2, 0.23, 0.44\rangle$	$\langle x_3, 0.04, 0.10\rangle$

Step 1: *$\mathbb{D}_{\widetilde{IFS}}$ is adopted to compute the distances between $\mathcal{S}$ and $\mathcal{P}_i(i = 1, 2, 3.)$. The results are depicted as:*

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_1, \mathcal{S}) = 0.0375$$
$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_2, \mathcal{S}) = 0.0871$$
$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_3, \mathcal{S}) = 0.0500$$

Step 2: *The minimum distance between $\mathcal{S}$ and $\mathcal{P}_i$ is described as follows:*

$$\mathbb{D}_{\widetilde{IFS}}(\mathcal{P}_1, \mathcal{S}) = 0.0375$$

Step 3: *The classification result of $\mathcal{S}$ is described as follows:*

$$\mathcal{P}_1 \rightarrow \mathcal{S}$$

Table 7 and Figure 7 show the results of distance measures by different methods. From Table 7 and Figure 7, we can find that $\mathbb{D}_{Ham-2}$, $\mathbb{D}_{E-2}$, $\mathbb{D}_{Ham-3}$, $\mathbb{D}_{E-3}$, $\mathbb{D}_{Hau}$, $\mathbb{D}^1_{WX}$, $\mathbb{D}^2_{WX}$, $\mathbb{D}_P$, $\mathbb{D}_{YC}$, $\mathbb{D}_{Ham-h}$ and $\mathbb{D}_{Ham-c}$ fail to classify the query pattern $\mathcal{S}$ reasonably because they are not effective in distinguishing the distances between $\mathcal{S}$ and $\mathcal{P}_i (i = 1, 2, 3)$. Specifically, these methods obtain the same distance on different patterns, namely $\mathbb{D}(\mathcal{P}_1, \mathcal{S}) = \mathbb{D}(\mathcal{P}_2, \mathcal{S}) = \mathbb{D}(\mathcal{P}_3, \mathcal{S})$. In addition, $\mathbb{D}_X$ can give unreasonable results in calculating the distances between $\mathcal{S}$ to $\mathcal{P}_i (i = 1, 2, 3)$. In contrast, $\mathbb{D}_{SH}$, $\mathbb{D}_S$, $\mathbb{D}_G$ and $\mathbb{D}_{\widetilde{IFS}}$ can all effectively identify the class of $\mathcal{S}$. It is worth noting that $\mathbb{D}_{\widetilde{IFS}}$ has a relatively large difference in calculating the distances between $\mathcal{S}$ to $\mathcal{P}_i (i = 1, 2, 3)$, which also reflects that $\mathbb{D}_{\widetilde{IFS}}$ can more easily make the correct decision results.

Table 7. Pattern classification results of various methods in **Application 2**.

Method	Distance Measures			Classification
	$(\mathcal{P}_1, \mathcal{S})$	**$(\mathcal{P}_2, \mathcal{S})$**	**$(\mathcal{P}_3, \mathcal{S})$**	
$\mathbb{D}_{Ham-2}$ [42]	0.0300	0.0300	0.0300	Uncertainty
$\mathbb{D}_{E-2}$ [42]	0.0311	0.0311	0.0311	Uncertainty
$\mathbb{D}_{Ham-3}$ [46]	0.0300	0.0300	0.0300	Uncertainty
$\mathbb{D}_{E-3}$ [46]	0.0311	0.0311	0.0311	Uncertainty
$\mathbb{D}_{Hau}$ [43]	0.0300	0.0300	0.0300	Uncertainty
$\mathbb{D}^1_{WX}$ [44]	0.0300	0.0300	0.0300	Uncertainty
$\mathbb{D}^2_{WX}$ [44]	0.0300	0.0300	0.0300	Uncertainty
$\mathbb{D}_P$ [48]	0.0300	0.0300	0.0300	Uncertainty
$\mathbb{D}_{YC}$ [47]	0.0300	0.0300	0.0300	Uncertainty
$\mathbb{D}_{SH}$ [51]	0.0438	0.0490	0.0464	$\mathcal{P}_1$
$\mathbb{D}_S$ [50]	0.0011	0.0052	0.0018	$\mathcal{P}_1$
$\mathbb{D}_{Ham-h}$ [55]	0.0300	0.0300	0.0300	Uncertainty
$\mathbb{D}_{Ham-c}$ [55]	0.0300	0.0300	0.0300	Uncertainty
$\mathbb{D}_X$ [22]	0.0438	NAN	0.0516	Unreasonable
$\mathbb{D}_G$ [57]	0.0125	0.0128	0.0131	$\mathcal{P}_1$
$\mathbb{D}_{\widetilde{IFS}}$	0.0375	0.0871	0.0500	$\mathcal{P}_1$

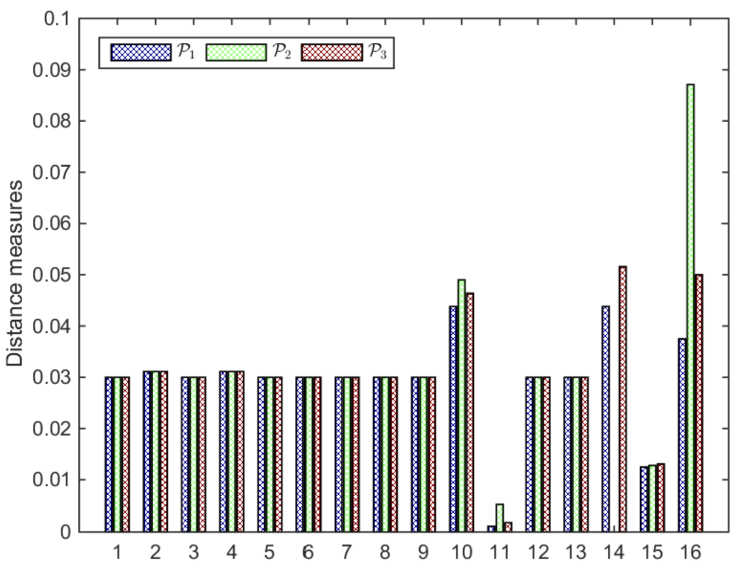


Figure 7. Comparisons of various distance measures in **Application 2**.

7. Conclusions

In this paper, we propose a new distance measure ($\mathbb{D}_{IFS}$) of IFSs based on Hellinger distance. $\mathbb{D}_{IFS}$ meets the axiomatic definition of distance measure well, showing that it can effectively measure the difference between IFSs. In addition, we also present a normalized distance measure ($\mathbb{D}_{\widetilde{IFS}}$), and numerical examples explain the potential of $\mathbb{D}_{\widetilde{IFS}}$. Finally, we design a $\mathbb{D}_{\widetilde{IFS}}$-based decision-making method and employ it to several applications. It is worth noting that the proposed method still faces several challenges. For example, in applications, different attributes often have different importance in decision making, while the importance of each attribute is treated equally in the proposed method. Therefore, in future work, we will further explore the weighted Hellinger distance to consider the difference between IFSs. It is necessary to apply $\mathbb{D}_{IFS}$ to medical diagnosis, image processing, and risk analysis.

Author Contributions: Conceptualization, Z.L.; methodology, Z.L.; software, X.L.; validation, X.L. and X.H.; formal analysis, Z.L.; investigation, Z.L.; resources, Z.L.; data curation, X.L.; writing—original draft preparation, X.L. and X.H.; writing—review and editing, Z.L.; visualization, X.H.; supervision, N.L. and W.Y.; funding acquisition, N.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work has been partially supported by the National Natural Science Foundation of China (61902221).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All data generated or analyzed during this work are included in this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Keith, A.J.; Ahner, D.K. A survey of decision making and optimization under uncertainty. *Ann. Oper. Res.* **2021**, *300*, 319–353. [CrossRef]
2. Hariri, R.H.; Fredericks, E.M.; Bowers, K.M. Uncertainty in big data analytics: Survey, opportunities, and challenges. *J. Big Data* **2019**, *6*, 44. [CrossRef]
3. Dutt, L.S.; Kurian, M. Handling of uncertainty—A survey. *Int. J. Sci. Res. Publ.* **2013**, *3*, 1–4.
4. Li, Y.; Chen, J.; Feng, L. Dealing with uncertainty: A survey of theories and practices. *IEEE Trans. Knowl. Data Eng.* **2013**, *25*, 2463–2482. [CrossRef]
5. Wang, X.; Song, Y. Uncertainty measure in evidence theory with its applications. *Appl. Intell.* **2018**, *48*, 1672–1688. [CrossRef]
6. Yager, R.R. On using the Shapley value to approximate the Choquet integral in cases of uncertain arguments. *IEEE Trans. Fuzzy Syst.* **2018**, *26*, 1303–1310. [CrossRef]
7. Zavadskas, E.K.; Turskis, Z.; Vilutienė, T.; Lepkova, N. Integrated group fuzzy multi-criteria model: Case of facilities management strategy selection. *Expert Syst. Appl.* **2017**, *82*, 317–331. [CrossRef]
8. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
9. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
10. Xu, Z.; Chen, J.; Wu, J. Clustering algorithm for intuitionistic fuzzy sets. *Inf. Sci.* **2008**, *178*, 3775–3790. [CrossRef]
11. Yager, R.R. Some aspects of intuitionistic fuzzy sets. *Fuzzy Optim. Decis. Mak.* **2009**, *8*, 67–90. [CrossRef]
12. Aggarwal, M. Rough information set and its applications in decision making. *IEEE Trans. Fuzzy Syst.* **2017**, *25*, 265–276. [CrossRef]
13. Ayub, S.; Shabir, M.; Riaz, M.; Mahmood, W.; Bozanic, D.; Marinkovic, D. Linear diophantine fuzzy rough sets: A new rough set approach with decision making. *Symmetry* **2022**, *14*, 525. [CrossRef]
14. Wei, W.; Liang, J. Information fusion in rough set theory: An overview. *Inf. Fusion* **2019**, *48*, 107–118. [CrossRef]
15. Yager, R.R. Generalized dempster–shafer structures. *IEEE Trans. Fuzzy Syst.* **2019**, *27*, 428–435. [CrossRef]
16. Deng, Y. Uncertainty measure in evidence theory. *Sci. China Inf. Sci.* **2020**, *63*, 1–19. [CrossRef]
17. Ma, Z.; Liu, Z.; Luo, C.; Song, L. Evidential classification of incomplete instance based on K-nearest centroid neighbor. *J. Intell. Fuzzy Syst.* **2021**, *41*, 7101–7115. [CrossRef]
18. Seiti, H.; Hafezalkotob, A.; Martínez, L. R-sets, comprehensive fuzzy sets risk modeling for risk-based information fusion and decision-making. *IEEE Trans. Fuzzy Syst.* **2021**, *29*, 385–399. [CrossRef]

19. Seiti, H.; Hafezalkotob, A.; Martínez, L. R-numbers, a new risk modeling associated with fuzzy numbers and its application to decision making. *Inf. Sci.* **2019**, *483*, 206–231. [CrossRef]
20. Seiti, H.; Fathi, M.; Hafezalkotob, A.; Herrera-Viedma, E.; Hameed, I.A. Developing the modified R-numbers for risk-based fuzzy information fusion and its application to failure modes, effects, and system resilience analysis (FMESRA). *ISA Trans.* **2021**, *113*, 9–27. [CrossRef]
21. Luo, X.; Li, W.; Zhao, W. Intuitive distance for intuitionistic fuzzy sets with applications in pattern recognition. *Appl. Intell.* **2018**, *48*, 2792–2808. [CrossRef]
22. Xiao, F. A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 3980–3992. [CrossRef]
23. Ejegwa, P.A.; Ahemen, S. Enhanced intuitionistic fuzzy similarity operators with applications in emergency management and pattern recognition. *Granul. Comput.* **2022** , 1–12.
24. Zeng, W.; Cui, H.; Liu, Y.; Yin, Q.; Xu, Z. Novel distance measure between intuitionistic fuzzy sets and its application in pattern recognition. *Iran. J. Fuzzy Syst.* **2022**, *19*, 127–137.
25. Khatibi, V.; Montazer, G.A. Intuitionistic fuzzy set vs. fuzzy set application in medical pattern recognition. *Artif. Intell. Med.* **2009**, *47*, 43–52. [CrossRef]
26. Ejegwa, P.A.; Onyeke, I.C.; Adah, V. An algorithm for an improved intuitionistic fuzzy correlation measure with medical diagnostic application. *Ann. Optim. Theory Pract.* **2020**, *3*, 51–66.
27. Garg, H.; Kaur, G. Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis. *Granul. Comput.* **2020**, *5*, 169–184. [CrossRef]
28. Xu, Z.; Zhao, N. Information fusion for intuitionistic fuzzy decision making: An overview. *Inf. Fusion* **2016**, *28*, 10–23. [CrossRef]
29. Garg, H. Novel intuitionistic fuzzy decision making method based on an improved operation laws and its application. *Eng. Appl. Artif. Intell.* **2017**, *60*, 164–174. [CrossRef]
30. Yu, D. Prioritized information fusion method for triangular intuitionistic fuzzy set and its application to teaching quality evaluation. *Int. J. Intell. Syst.* **2013**, *28*, 411–435. [CrossRef]
31. Gao, Y.; Li, D.s.; Zhong, H. A novel target threat assessment method based on three-way decisions under intuitionistic fuzzy multi-attribute decision making environment. *Eng. Appl. Artif. Intell.* **2020**, *87*, 103276. [CrossRef]
32. Mahanta, J.; Panda, S. A novel distance measure for intuitionistic fuzzy sets with diverse applications. *Int. J. Intell. Syst.* **2021**, *36*, 615–627. [CrossRef]
33. Konwar, N.; Debnath, P. Some new contractive conditions and related fixed point theorems in intuitionistic fuzzy n-Banach spaces. *J. Intell. Fuzzy Syst.* **2018**, *34*, 361–372. [CrossRef]
34. Rahman, K.; Ayub, S.; Abdullah, S. Generalized intuitionistic fuzzy aggregation operators based on confidence levels for group decision making. *Granul. Comput.* **2021**, *6*, 867–886. [CrossRef]
35. Feng, F.; Fujita, H.; Ali, M.I.; Yager, R.R.; Liu, X. Another view on generalized intuitionistic fuzzy soft sets and related multiattribute decision making methods. *IEEE Trans. Fuzzy Syst.* **2019**, *27*, 474–488. [CrossRef]
36. Xu, Z.; Chen, J. An overview of distance and similarity measures of intuitionistic fuzzy sets. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2008**, *16*, 529–555. [CrossRef]
37. Baccour, L.; Alimi, A.M.; John, R.I. Similarity measures for intuitionistic fuzzy sets: State of the art. *J. Intell. Fuzzy Syst.* **2013**, *24*, 37–49. [CrossRef]
38. Papakostas, G.A.; Hatzimichailidis, A.G.; Kaburlasos, V.G. Distance and similarity measures between intuitionistic fuzzy sets: A comparative analysis from a pattern recognition point of view. *Pattern Recognit. Lett.* **2013**, *34*, 1609–1622. [CrossRef]
39. Szmidt, E. *Distances and Similarities in Intuitionistic Fuzzy Sets*; Springer: Berlin/Heidelberg, Germany, 2014; Volume 307.
40. He, X.; Li, Y.; Qin, K.; Meng, D. Distance measures on intuitionistic fuzzy sets based on intuitionistic fuzzy dissimilarity functions. *Soft Comput.* **2020**, *24*, 523–541. [CrossRef]
41. Wu, X.; Chen, G.; Wang, T.; Zhu, Z.; Liu, P. Strict intuitionistic fuzzy distance/similarity measures based on Jensen-Shannon divergence. *arXiv* **2022**, arXiv:2207.06980
42. Atanassov, K.T. Intuitionistic fuzzy sets. In *Intuitionistic Fuzzy Sets*; Springer: Berlin/Heidelberg, Germany, 1999; pp. 1–137.
43. Grzegorzewski, P. Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric. *Fuzzy Sets Syst.* **2004**, *148*, 319–328. [CrossRef]
44. Wang, W.; Xin, X. Distance measure between intuitionistic fuzzy sets. *Pattern Recognit. Lett.* **2005**, *26*, 2063–2069. [CrossRef]
45. Gohain, B.; Dutta, P.; Gogoi, S.; Chutia, R. Construction and generation of distance and similarity measures for intuitionistic fuzzy sets and various applications. *Int. J. Intell. Syst.* **2021**, *36*, 7805–7838. [CrossRef]
46. Szmidt, E.; Kacprzyk, J. Distances between intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **2000**, *114*, 505–518. [CrossRef]
47. Yang, Y.; Chiclana, F. Consistency of 2D and 3D distances of intuitionistic fuzzy sets. *Expert Syst. Appl.* **2012**, *39*, 8665–8670. [CrossRef]
48. Park, J.H.; Lim, K.M.; Kwun, Y.C. Distance measure between intuitionistic fuzzy sets and its application to pattern recognition. *J. Korean Inst. Intell. Syst.* **2009**, *19*, 556–561. [CrossRef]
49. Vlachos, I.K.; Sergiadis, G.D. Intuitionistic fuzzy information–applications to pattern recognition. *Pattern Recognit. Lett.* **2007**, *28*, 197–206. [CrossRef]

50. Song, Y.; Wang, X.; Quan, W.; Huang, W. A new approach to construct similarity measure for intuitionistic fuzzy sets. *Soft Comput.* **2019**, *23*, 1985–1998. [CrossRef]
51. Shen, F.; Ma, X.; Li, Z.; Xu, Z.; Cai, D. An extended intuitionistic fuzzy TOPSIS method based on a new distance measure with an application to credit risk evaluation. *Inf. Sci.* **2018**, *428*, 105–119. [CrossRef]
52. Jiang, Q.; Jin, X.; Lee, S.J.; Yao, S. A new similarity/distance measure between intuitionistic fuzzy sets based on the transformed isosceles triangles and its applications to pattern recognition. *Expert Syst. Appl.* **2019**, *116*, 439–453. [CrossRef]
53. Garg, H.; Rani, D. Novel distance measures for intuitionistic fuzzy sets based on various triangle centers of isosceles triangular fuzzy numbers and their applications. *Expert Syst. Appl.* **2022**, *191*, 116228. [CrossRef]
54. Hellinger, E. Neue begründung der theorie quadratischer formen von unendlichvielen veränderlichen. *J. FÜR Die Reine Und Angew. Math.* **1909**, *1909*, 210–271. [CrossRef]
55. Chen, C.; Deng, X. Several new results based on the study of distance measures of intuitionistic fuzzy sets. *Iran. J. Fuzzy Syst.* **2020**, *17*, 147–163.
56. Hatzimichailidis, A.G.; Papakostas, G.A.; Kaburlasos, V.G. A novel distance measure of intuitionistic fuzzy sets and its application to pattern recognition problems. *Int. J. Intell. Syst.* **2012**, *27*, 396–409. [CrossRef]
57. Garg, H.; Rani, D. Novel similarity measure based on the transformed right-angled triangles between intuitionistic fuzzy sets and its applications. *Cogn. Comput.* **2021**, *13*, 447–465. [CrossRef]