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Fermatean Fuzzy CODAS Approach with Topology and Its Application to Sustainable Supplier Selection

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Abstract: A Fermatean fuzzy set (FFS) is a reliable method for representing uncertainty in "multi-criteria decision-making" (MCDM). This research seeks to examine the topological properties of FFSs and to establish the notion of "Fermatean fuzzy topology" (FFT). An FFT is the generalisation of existing fuzzy topologies. Several aspects of FFT are examined and various novel concepts are proposed, which include Fermatean fuzzy α-continuity between FFTSs and Fermatean fuzzy connectedness. To deal multiple challenges in sustainable supply chain management, a Fermatean fuzzy "combinative distance-based assessment" (CODAS) method was developed. The proposed FF CODAS technique involves various key features for MCDM. Firstly, a known reputation vector or equal expert weights is determined based on the reputation, experience and qualifications of the experts. Secondly, the Fermatean fuzzy direct rating approach is used to establish the relative relevance of criteria based on the expert group's evaluation preferences. Thirdly, the Fermatean fuzzy CODAS approach is used to construct alternative orderings based on their assessment scores. Finally, an application is developed to show the benefit of the suggested supplier selection approach. Additionally, the symmetry of an optimal decision in application is carried out by a comparison analysis of the suggested models with some existing models.

Keywords: Fermatean fuzzy topology; Fermatean fuzzy α-continuity; topological data analysis; CODAS; supplier selection

MSC: 03E72; 94D05; 90B50

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Citation: Farid, H.M.A.; Bouye, M.; Riaz, M.; Jamil, N. Fermatean Fuzzy CODAS Approach with Topology and Its Application to Sustainable Supplier Selection. *Symmetry* **2023**, *15*, 433. https://doi.org/10.3390/sym15020433

Academic Editor: Hsien-Chung Wu

Received: 2 January 2023
Revised: 30 January 2023
Accepted: 1 February 2023
Published: 6 February 2023

1. Introduction

In computational intelligence challenges, inferring topological and geometrical information from data can provide a novel perspective in mathematical modelling. Methods for "topological data analysis" (TDA) are rapidly expanding methodologies for inferring the permanent and essential characteristics for potentially complicated data. TDA can be utilised autonomously or in tandem with other information-processing and analytical instructional methods. In modern data science, topological methods have been utilised to evaluate the structural aspects of big data, leading to additional information analysis. Utilising a variety of mathematical techniques, topology and geometry are natural instruments for evaluating big datasets. Traditional and basic analysis in a range of computer intelligence domains are inspired by topology and big data. In addition, topology is the link between geographical structures and characteristics, and it may be utilised to explain certain spatial functions and create datasets with a greater level of dependability and integrity. Topological notions such as continuity, convergence and homeomorphisms have a solid geometrical interpretation.

Researchers have investigated a lot of key features in classical set theory and classical topology. Nevertheless, traditional approaches cannot handle ambiguous and unclear information. To solve these concerns, Zadeh [1] developed the concept of "fuzzy set" (FS) theory and "membership function" (MSF), Pawlak [2] proposed rough set theory and approximation spaces, and Molodtsov [3] developed soft set theory and parameterisation. Chang [4] expanded the concept of FSs to fuzzy topological space (TS) and studied several essential features, including open set, closed set, compactness, and continuity in terms of FS theory. Lowen presented a different notion of fuzzy TSs [5,6]. Atanassov [7] further revealed the notion of an "intuitionistic fuzzy set" (IFS) using MSF and "non-membership function" (NMSF). Later, Coker established the concept of intuitionistic fuzzy TS and researched the equivalent versions of traditional topology concepts such as continuity and compactness [8,9]. Additional results on intuitionistic fuzzy TSs are defined in [10,11]. Fuzzy metric spaces provided solutions to the topics concerning distance-like functions, etc. [12]. Thus, instead of particular fixed components, the idea of topology incorporates the MSF structure of fuzziness. The concepts of open and alpha open maps in terms of fuzzy and continuous functions were introduced by Singal and Rajvanshi [13]. Ajmal and Kohli [14] developed the idea of "connectedness in fuzzy TSs" and Chaudhuri and Das [15] initiated the notion of "some fuzzy connected sets in fuzzy TSs". Olgun et al. [16] introduced the "Pythagorean fuzzy TSs", Turkarslan et al. [17] proposed some sort of "q-rung orthopair fuzzy TSs", Joseline and Ajay [18] gave the idea of "Pythagorean fuzzy α-continuity". Haydar defined connectedness for the fuzzy Pythagorean TS [19].

In the modern scientific era, modelling uncertainty as part of MCDM approaches is essential for addressing problems that occur in the actual world. In order to determine how reliable human judgments are, many MCDM techniques have been devised. These procedures include comparing a group of potential outcomes to a set of criteria and rating each possibility accordingly. The gathering and synthesis of information is essential to the operation of a wide variety of technological processes, including machine learning, pattern classification, photogrammetry, and selection. In a general sense, the process of aggregation involves the combination of a great deal of data in order to obtain a conclusion. It was also shown that fundamental data processing algorithms that are based on crisp integers cannot be used to accurately reflect the operating circumstances of human cognitive processes. Because of these strategies, "decision makers" (DMs) are left with confusing facts and judgements that are difficult to understand. As a consequence of this, in order to cope with the ambiguous and fuzzy situations that are present in the world, DMs look for new philosophies that will enable them to understand confusing data values and preserve their judgement demands in response to a variety of settings.

Fuzzy logic is a sort of multi-valued logic in which variable values might vary between 0 and 1. The premise of fuzzy logic is that individuals make judgments based on confusing and non-numerical information. Fuzzy sets are mathematical representations of ambiguity and inaccurate information. They are commonly known as fuzzy models. It is a word used to explain the concept of partial truth, according to which the truth value might vary between false and true. Zadeh proposed a formal framework for making decisions based on imprecise data representations. This concept is based on the fuzzy set, which is a set with no obvious bounds and can only contain things to a given degree; in other words, elements can only be members to a certain degree. Researchers have noticed that the structure of the membership grades is a big concern when utilising fuzzy sets as a result of this. The uncertainty associated with giving an exact numerical membership value to each element in the supplied fuzzy collection is the source of the difficulty. Yager introduced the "Pythagorean fuzzy set" (PFS) [20–22] as well as the "q-rung orthopair fuzzy set" [23]. Senapati and Yager [24] developed the concept of FFSs.

Data processing is crucial for decision making in business, government, sociology, science, intellectual, cognitive, and autonomous systems. In general, awareness of the alternatives has been regarded as a numerical or verbal quantity. Unfortunately, the data cannot easily be amalgamated with regard to ambiguity. Xu et al. [25,26] established geometric and averaging AOs for IFS. Akram et al. [27] proposed the idea of complex Fermatean fuzzy N-soft sets as a hybrid model of N-soft set, FFS, and complex FS. They developed a new hybrid method for decision making with regard to the terrific capability problem based on the Fermatean fuzzy TOPSIS technique approach. A robust work related to the proposed work can be seen in [28].

Riaz et al. [29] developed "linear Diophantine fuzzy prioritised AOs" and Iampan et al. [30] proposed Einstein AOs for LDFSs. Ashraf and Abdullah [31] proposed some mathematical modelling for COVID-19 under a spherical fuzzy set. Some extensive work related to AOs can be seen in [32–37]. Peng et al. [38] introduced some AOs for a "single-valued neutrosophic number" (SVNN). Liu et al. [39] developed some AOs for SVNNs based on "Hamacher operations". Farid and Riaz [40] proposed Einstein interactive AOs for SVNNs. Some extensive work related to operational science in a fuzzy framework was given in [41–44]. The main goals of this study are given as follows:

1. To define the topological structure of FSSs and to propose the concept of fuzzy Fermatean topology;
2. To address the characterisation of Fermatean fuzzy TSs, such as interior, closure, and boundary, etc.;
3. Examining noteworthy results regarding images and inverse images of FFSs under Fermatean fuzzy mapping;
4. To define Fermatean fuzzy α-continuity between FFTSs and Fermatean fuzzy connectedness;
5. Modelling uncertain information in MCDM with Fermatean fuzzy CODAS methods.
6. Data analysis with Fermatean fuzzy CODAS approach for supplier selection and supply chain. A numerical example is illustrated to explain FF CODAS for supplier selection.

The remaining sections of the paper are structured as follows. Section 2 covers the fundamental FFS ideas. In Section 3, the concept of FFT is defined and associated findings are examined. In Section 4, the idea of FF α-continuity is introduced. Section 5 provides basic ideas pertaining to FF connectivity, whereas Section 6 suggests a CODAS structure under the FFSs. Section 7 provides an application regarding the selection of suppliers. Section 8 provides a summary of the planned outcomes, techniques, findings, and their benefits. This section also includes future directions.

2. Fundamental Concepts

In this section of the manuscript, we will review some of the fundamental notions.

Definition 1 ([24])**.** *Assume FFS $\hbar$ in $\overline{\Im}$ is defined as*

$$\hbar = \{\langle \rho, {\mu^\eta}_\hbar(\rho), {\nu^\eta}_\hbar(\rho)\rangle : \rho \in \overline{\Im}\}$$

where ${\mu^\eta}_\hbar, {\nu^\eta}_\hbar : \overline{\Im} \to [0,1]$ denotes the MSF and NMSF of the alternative $\rho \in \overline{\Im}$ and $\forall\rho$, so we have

$$0 \leq {\mu^\eta}^3_\hbar(\rho) + {\nu^\eta}^3_\hbar(\rho) \leq 1.$$

Furthermore, $\pi_\hbar(\rho) = \sqrt[3]{1 - {\mu^\eta}^3_\hbar(\rho) - {\nu^\eta}^3_\hbar(\rho)}$ is called the "indeterminacy degree" of ρ to $\hbar$.

Definition 2 ([24])**.** *Let $\tilde{\delta}^\curlywedge_1 = \langle {\mu^\eta}_1, {\nu^\eta}_1\rangle$ and $\tilde{\delta}^\curlywedge_2 = \langle {\mu^\eta}_2, {\nu^\eta}_2\rangle$ be FFNs. Since $\sigma > 0$, then*

(1) $(\tilde{\delta}^\curlywedge_1)^c = \langle {\nu^\eta}_1, {\mu^\eta}_1\rangle$

(2) $\tilde{\delta}^\curlywedge_1 \vee \tilde{\delta}^\curlywedge_2 = \langle \max\{{\mu^\eta}_1, {\nu^\eta}_1\}, \min\{{\mu^\eta}_2, {\nu^\eta}_2\}\rangle$

(3) $\tilde{\delta}^\curlywedge_1 \wedge \tilde{\delta}^\curlywedge_2 = \langle \min\{{\mu^\eta}_1, {\nu^\eta}_1\}, \max\{{\mu^\eta}_2, {\nu^\eta}_2\}\rangle$

(4) $\tilde{\delta}^\curlywedge_1 \oplus \tilde{\delta}^\curlywedge_2 = \langle \sqrt[3]{{\mu^\eta}^3_1 + {\mu^\eta}^3_2 - {\mu^\eta}^3_1{\mu^\eta}^3_2}, {\nu^\eta}_1{\nu^\eta}_2\rangle$

(5) $\tilde{\delta}^\curlywedge_1 \otimes \tilde{\delta}^\curlywedge_2 = \langle {\mu^\eta}_1{\mu^\eta}_2, \sqrt[3]{{\nu^\eta}^3_1 + {\nu^\eta}^3_2 - {\nu^\eta}^3_1{\nu^\eta}^3_2}\rangle$

(6) $\sigma\tilde{\delta}^\curlywedge_1 = \langle \sqrt[3]{1 - (1 - {\mu^\eta}^3_1)^\sigma}, {\nu^\eta}^\sigma_1\rangle$

(7) $(\tilde{\delta}^\curlywedge_1)^\sigma = \langle {\mu^\eta}^\sigma_1, \sqrt[3]{1 - (1 - {\nu^\eta}^3_1)^\sigma}\rangle$

Definition 3 ([24]). *Let $\tilde{\delta}^{\curlywedge} = \langle \mu^{\eta}, \nu^{\eta} \rangle$ be the FFN, and the "score function" (SF) S of $\tilde{\delta}^{\curlywedge}$ is defined as*

$$S(\tilde{\delta}^{\curlywedge}) = \mu^{\eta 3} - \nu^{\eta 3}$$

$S(\tilde{\delta}^{\curlywedge}) \in [-1, 1]$. The FFN score will determine its ranking, with the greatest score dictating the FFN's top priority. In other instances, however, the SF is not favourable for FFN. Consequently, using the SF to examine FFNs is inadequate. We are introducing a new function, namely the "accuracy function" (AF).

Definition 4 ([24]). *Let $\tilde{\delta}^{\curlywedge} = \langle \mu^{\eta}, \nu^{\eta} \rangle$ be the FFN, then an AF $\check{L}$ of $\tilde{\delta}^{\curlywedge}$ is defined as*

$$\check{L}(\tilde{\delta}^{\curlywedge}) = \mu^{\eta 3} + \nu^{\eta 3}$$

$\check{L}(\tilde{\delta}^{\curlywedge}) \in [0, 1]$.

Definition 5. *Consider $\tilde{\delta}^{\curlywedge} = \langle \mu^{\eta}{}_{\tilde{\delta}^{\curlywedge}}, \nu^{\eta}{}_{\tilde{\delta}^{\curlywedge}} \rangle$ and $\breve{\mho} = \langle \mu^{\eta}{}_{\breve{\mho}}, \nu^{\eta}{}_{\breve{\mho}} \rangle$ as two FFN, and $S(\tilde{\delta}^{\curlywedge}), S(\breve{\mho})$ are the SF of $\tilde{\delta}^{\curlywedge}$ and $\breve{\mho}$, and $\check{L}(\tilde{\delta}^{\curlywedge}), \check{L}(\breve{\mho})$ are the AFs of $\tilde{\delta}^{\curlywedge}$ and $\breve{\mho}$, respectively, then:*
(a) If $S(\tilde{\delta}^{\curlywedge}) > S(\breve{\mho})$, then $\tilde{\delta}^{\curlywedge} > \breve{\mho}$;
(b) If $S(\tilde{\delta}^{\curlywedge}) = S(\breve{\mho})$ then;

If $\check{L}(\tilde{\delta}^{\curlywedge}) > \check{L}(\breve{\mho})$ then $\tilde{\delta}^{\curlywedge} > \breve{\mho}$;
If $\check{L}(\tilde{\delta}^{\curlywedge}) = \check{L}(\breve{\mho})$, then $\tilde{\delta}^{\curlywedge} = \breve{\mho}$.

Definition 6 ([45]). *Considering $\tilde{\delta}_1^{\curlywedge} = \langle \mu^{\eta}{}_1, \nu^{\eta}{}_1 \rangle$ and $\tilde{\delta}_2^{\curlywedge} = \langle \mu^{\eta}{}_2, \nu^{\eta}{}_2 \rangle$ as FFNs, we define the subtraction and division of FFNs as*

- $\tilde{\delta}_1^{\curlywedge} \ominus \tilde{\delta}_2^{\curlywedge} = \left(\sqrt[3]{\frac{\mu^{\eta 3}_1 - \mu^{\eta 3}_2}{1 - \mu^{\eta 3}_2}}, \frac{\nu^{\eta}1}{\nu^{\eta}2} \right)$, *if* $\mu^{\eta}{}_1 \geq \mu^{\eta}{}_2, \nu^{\eta}{}_1 \leq \min\left\{ \nu^{\eta}{}_2, \frac{\nu^{\eta}2\pi_1}{\pi_2} \right\}$
- $\tilde{\delta}_1^{\curlywedge} \oslash \tilde{\delta}_2^{\curlywedge} = \left(\frac{\mu^{\eta}{}_1}{\mu^{\eta}{}_2}, \sqrt[3]{\frac{\nu^{\eta}1^3 - \nu^{\eta}2^3}{1 - \nu^{\eta}2^3}} \right)$ *if* $\nu^{\eta}{}_1 \geq \nu^{\eta}{}_2, \mu^{\eta}{}_1 \leq \min\left\{ \mu^{\eta}{}_2, \frac{\mu^{\eta}{}_2\pi_1}{\pi_2} \right\}$

3. Main Results

In this section, the concept of Fermatean fuzzy topology (FFT) and numerous related results are proposed.

Definition 7. *Let $\overline{\Im} \neq \emptyset$ be the universe and $\hbar^{\daleth}$ be the assemblage of FF subsets of $\overline{\Im}$. If $\hbar^{\daleth}$ satisfies the axioms:*

T1 $0_{\overline{\Im}}, 1_{\overline{\Im}} \in \hbar^{\daleth}$;
T2 *For any $\chi^{\zeta}{}_1, \chi^{\zeta}{}_2 \in \hbar^{\daleth}$, we have $\chi^{\zeta}{}_1 \cap \chi^{\zeta}{}_2 \in \hbar^{\daleth}$;*
T3 *For any $\{\chi^{\zeta}{}_i\}_{i \in I} \subseteq \hbar^{\daleth}$, we have $\cup_{i \in I} \chi^{\zeta}{}_i \in \hbar^{\daleth}$.*

Then, $\hbar^{\daleth}$ is called an FFT on $\overline{\Im}$ and the pair $(\overline{\Im}, \hbar^{\daleth})$ is said to be an FFTS. Each member of $\hbar^{\daleth}$ is called an FF open set (FFOS). The complement of an FF open set is called an FF closed set (FFCS).

Remark 1. *Each IFS and PFS may be considered as an FFS, leading to the conclusion that any IFS topology and PFS topology is an FFT. However, the converse fails to hold.*

Example 1. *Let $\overline{\Im} = \{\breve{\alpha}_1^{\gamma}, \breve{\alpha}_2^{\gamma}, \breve{\alpha}_3^{\gamma}\}$. Consider the following family of FF subsets $\hbar^{\daleth} = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_1, \ldots, \chi^{\zeta}{}_4\}$, where*

$$\chi^{\zeta}{}_1 = \{\langle \breve{\alpha}_1^{\gamma}, 0.58, 0.78 \rangle, \langle \breve{\alpha}_2^{\gamma}, 0.67, 0.58 \rangle, \langle \breve{\alpha}_3^{\gamma}, 0.28, 0.18 \rangle\},$$
$$\chi^{\zeta}{}_2 = \{\langle \breve{\alpha}_1^{\gamma}, 0.60, 0.76 \rangle, \langle \breve{\alpha}_2^{\gamma}, 0.72, 0.53 \rangle, \langle \breve{\alpha}_3^{\gamma}, 0.30, 0.16 \rangle\},$$
$$\chi^{\zeta}{}_3 = \{\langle \breve{\alpha}_1^{\gamma}, 0.64, 0.72 \rangle, \langle \breve{\alpha}_2^{\gamma}, 0.74, 0.48 \rangle, \langle \breve{\alpha}_3^{\gamma}, 0.34, 0.14 \rangle\},$$
$$\chi^{\zeta}{}_4 = \{\langle \breve{\alpha}_1^{\gamma}, 0.71, 0.67 \rangle, \langle \breve{\alpha}_2^{\gamma}, 0.80, 0.43 \rangle, \langle \breve{\alpha}_3^{\gamma}, 0.44, 0.10 \rangle\}.$$

One can see that $(\overline{\Im}, \hbar^{\daleth})$ is an FFTS.

Moreover, $\hbar^{\daleth}{}_1 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_1\}$, $\hbar^{\daleth}{}_2 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_2\}$, $\hbar^{\daleth}{}_3 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_3\}$, $\hbar^{\daleth}{}_4 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_4\}$, $\hbar^{\daleth}{}_5 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_2\}$, $\hbar^{\daleth}{}_6 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_3\}$, $\hbar^{\daleth}{}_7 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_4\}$, $\hbar^{\daleth}{}_8 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_2, \chi^{\zeta}{}_3\}$, $\hbar^{\daleth}{}_9 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_2, \chi^{\zeta}{}_4\}$, $\hbar^{\daleth}{}_{10} = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_3, \chi^{\zeta}{}_4\}$, $\hbar^{\daleth}{}_{11} = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_2, \chi^{\zeta}{}_3\}$ $\hbar^{\daleth}{}_{12} = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_3, \chi^{\zeta}{}_4\}$ are also FFTSs.

Definition 8. *Let $\overline{\Im}$ and $\breve{\amalg}$ be two non-empty sets, let $\sqsupset : \overline{\Im} \to \breve{\amalg}$ be a mapping, and let $\check{D}$ and $\tilde{F}$ be FF subsets of $\overline{\Im}$ and $\breve{\amalg}$, respectively. The image of $\check{D}$ under mapping $\sqsupset$ is denoted by $\sqsupset[\check{D}]$. Then, the MSF and NMSF of the image set $\sqsupset[\check{D}]$ are defined by*

$$\mu^{\eta}{}_{\sqsupset[\check{D}]}(y) = \begin{cases} \sup_{z \in \sqsupset^{-1}(y)} \mu^{\eta}{}_{\check{D}}(z), & \text{if } \sqsupset^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu^{\eta}{}_{\sqsupset[\check{D}]}(y) = \begin{cases} \inf_{z \in \sqsupset^{-1}(y)} \nu^{\eta}{}_{\check{D}}(z), & \text{if } \sqsupset^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

respectively. The MSF and NMSF of the pre-image of $\tilde{F}$ with respect to $\sqsupset$ that is denoted by $\sqsupset^{-1}[\tilde{F}]$ are defined by

$$\mu^{\eta}{}_{\sqsupset^{-1}[\tilde{F}]}(x) = \mu^{\eta}{}_{\tilde{F}}(\sqsupset(x)) \text{ and } \nu^{\eta}{}_{\sqsupset^{-1}[\tilde{F}]}(x) = \nu^{\eta}{}_{\tilde{F}}(\sqsupset(x)) \text{ respectively.}$$

which show that the $\mu^{\eta}{}^{3}_{\sqsupset[\check{D}]} + \nu^{\eta}{}^{3}_{\sqsupset[\check{D}]} \leq 1$ FF membership condition is provided for the FF image and pre-image.

Proposition 1. *Let $\overline{\Im}$ and $\breve{\amalg}$ be two non-empty sets and $\sqsupset : \overline{\Im} \to \breve{\amalg}$ be an FF mapping. Then, we have*

1. *$\sqsupset^{-1}[\tilde{F}^c] = (\sqsupset^{-1}[\tilde{F}])^c$ for any FF subset $\tilde{F}$ of $\breve{\amalg}$;*
2. *$(\sqsupset[\check{D}])^c \subseteq \sqsupset[\check{D}^c]$ for any FF subset $\check{D}$ of $\overline{\Im}$;*
3. *If $\tilde{F}_1 \subseteq \tilde{F}_2$, then $\sqsupset^{-1}[\tilde{F}_1] \subseteq \sqsupset^{-1}[\tilde{F}_2]$ where $\tilde{F}_1$ and $\tilde{F}_2$ are FF subsets of $\breve{\amalg}$;*
4. *If $\check{D}_1 \subseteq \check{D}_2$ then $\sqsupset[\check{D}_1] \subseteq \sqsupset[\check{D}_2]$ where $\check{D}_1$ and $\check{D}_2$ are FF subsets of $\overline{\Im}$;*
5. *$\sqsupset[\sqsupset^{-1}[\tilde{F}]] \subseteq \tilde{F}$ for any FF subset $\tilde{F}$ of $\breve{\amalg}$;*
6. *$\check{D} \subseteq \sqsupset^{-1}[\sqsupset[\check{D}]]$ for any FF subset $\check{D}$ of $\overline{\Im}$.*

Definition 9. *Let $\left\{\chi^{\zeta}{}_i = \left\{\left\langle \breve{\aleph}, \mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}), \nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}) \right\rangle : \breve{\aleph} \in \overline{\Im}\right\}\right\}_{i \in I}$ be the assemblage of FF sets over $\overline{\Im}$. Then,*

$$\bigcap_{i \in I} \chi^{\zeta}{}_i = \left\{\left\langle \breve{\aleph}, \in \sqsupset\left\{\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}, \sup\left\{\nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\} \right\rangle : \breve{\aleph} \in \overline{\Im}\right\},$$

$$\bigcup_{i \in I} \chi^{\zeta}{}_i = \left\{\left\langle \breve{\aleph}, \sup\left\{\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}, \in \sqsupset\left\{\nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\} \right\rangle : \breve{\aleph} \in \overline{\Im}\right\}.$$

Note that $\bigcap_{i \in I} \chi^{\zeta}{}_i$ and $\cup_{i \in I} \chi^{\zeta}{}_i$ are FF sets over $\overline{\Im}$. We shall $\bigcap_{i \in I} \chi^{\zeta}{}_i$ define $\bigcap_{i \in I} \chi^{\zeta}{}_i = \left\{\left\langle \breve{\aleph}, \alpha_{\cap} \cap_i, \beta_{i \in I} \chi^{\zeta}{}_i \right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$ such that $\alpha_{n \in \chi^{\zeta}{}_i} = \in \sqsupset\left\{\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}$ and $\beta_{i \in I} \chi^{\zeta}{}_i = \sup\left\{\nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}$. In order to for $\bigcap_{i \in I} \chi^{\zeta}{}_i$ to be an FF set, we must have that $\alpha^3_{i \in I} \chi^{\zeta}{}_i(\breve{\aleph}) + \beta^3_{i \in I} \chi^{\zeta}{}_i(\breve{\aleph}) \leq 1$. We see since $\beta^3_{i \in I} \chi^{\zeta}{}_i(\breve{\aleph}) = \sup\left\{\nu^{\eta}{}^{3}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}$, then

$$\begin{aligned}\beta^3_{\cap\in I}\chi^{\zeta}{}_i(\breve{\aleph}) &= \sup\left\{\nu^{\eta 3}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\} = \sup\left\{r_i^3 - \mu^{\eta 3}{}_{\chi^{\zeta}{}_i}, r_i^3 - \nu^{\eta 3}{}_{\chi^{\zeta}{}_i}\right\} \\ &\le \sup\left\{r_i^3 - \in \beth\left\{\mu^{\eta 3}{}_{\chi^{\zeta}{}_i}, \nu^{\eta 3}{}_{\chi^{\zeta}{}_i}\right\}, r_i^3 - \in \beth\left\{\mu^{\eta 3}{}_{\chi^{\zeta}{}_i}, \nu^{\eta 3}{}_{\chi^{\zeta}{}_i}\right\}\right\} \\ \beta^3_{i\in I}(\breve{\aleph}) &\le \sup\left\{1 - \in \beth\left\{\mu^{\eta 3}{}_{\chi^{\zeta}{}_i}, \nu^{\eta 3}{}_{\chi^{\zeta}{}_i}\right\}, 1 - \in \beth\left\{\mu^{\eta 3}{}_{\chi^{\zeta}{}_i}, \nu^{\eta 3}{}_{\chi^{\zeta}{}_i}\right\}\right\} \\ &\le 1 - \in \beth\left\{\mu^{\eta 3}{}_{\chi^{\zeta}{}_i}, \nu^{\eta 3}{}_{\chi^{\zeta}{}_i}\right\}\end{aligned}$$

where $\mu^{\eta 3}{}_{\chi^{\zeta}{}_i} + \nu^{\eta 3}{}_{\chi^{\zeta}{}_i} = r_i^3$ *for every* $i \in I$. *From this, we see that* $\alpha^3_{\cap\in I}\chi^{\zeta}{}_i(\breve{\aleph}) + \beta^3_{\cap\in I}\chi^{\zeta}{}_i(\breve{\aleph}) \le \in \beth\left\{\mu^{\eta 3}{}_{\chi^{\zeta}{}_i}, \nu^{\eta 3}{}_{\chi^{\zeta}{}_i}\right\} + 1 - \in \beth\left\{\mu^{\eta 3}{}_{\chi^{\zeta}{}_i}, \nu^{\eta 3}{}_{\chi^{\zeta}{}_i}\right\} \le 1$. *Thus,* $\bigcap_{i\in I}\chi^{\zeta}{}_i$ *is an FF set.*
The proof is trivial for $\cup_{i\in I}\chi^{\zeta}{}_i$.

Theorem 1. *Let* $\left\{\chi^{\zeta}{}_i = \left\{\left\langle\breve{\aleph}, \mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}), \nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\rangle : \breve{\aleph} \in \overline{\Im}\right\}\right\}_{i\in I}$ *be the assemblage of FF sets over* $\overline{\Im}$. *Then,*

1. $\overline{\bigcap_{i\in I}\chi^{\zeta}{}_i} = \cup_{i\in I}\overline{\chi^{\zeta}{}_i}$;
2. $\overline{\bigcup_{i\in I}\chi^{\zeta}{}_i} = \bigcap_{i\in I}\overline{\chi^{\zeta}{}_i}$.

Proof. (1) We have $\bigcap_{i\in I}\chi^{\zeta}{}_i = \left\{\left\langle\breve{\aleph}, \in \beth\left\{\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}, \sup\left\{\nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}\right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$. Then,

$$\overline{\bigcap_{i\in I}\chi^{\zeta}{}_i} = \left\{\left\langle\breve{\aleph}, \sup\left\{\nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}), \in \beth\left\{\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}\right\}\right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$$

and $\overline{\chi^{\zeta}{}_i} = \left\{\left\langle\breve{\aleph}, \nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}), \mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$ and so
$\cup\bar{u}_{i\in I}\overline{\chi^{\zeta}{}_i} = \left\{\left\langle\breve{\aleph}, \sup\left\{\nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}), \in \beth\left\{\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}\right\}\right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$. That is, $\overline{\bigcap_{i\in I}\chi^{\zeta}{}_i} = \cup_{i\in I}\overline{\chi^{\zeta}{}_i}$.

(2) It is proven similar to (1). □

Definition 10. *Let* $(\overline{\Im}, \hbar^{\daleth})$ *be an FFTS and* $\chi^{\zeta} = \left\{\left\langle\breve{\aleph}, \mu^{\eta}{}_{\chi^{\zeta}}(\breve{\aleph}), \nu^{\eta}{}_{\chi^{\zeta}}(\breve{\aleph})\right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$ *be an FFS over* $\overline{\Im}$. *Then, the FF interior, FF closure, and FF boundary of* χ^{ζ} *are defined by:*

1. $Int(\chi^{\zeta}) = \cup\{\mathfrak{G} : \mathfrak{G}$ *is an FFOS in* $\overline{\Im}$ *and* $\mathfrak{G} \subseteq \chi^{\zeta}\}$;
2. $Cl(\chi^{\zeta}) = \cap\{\mathfrak{K} : \mathfrak{K}$ *is an FFCS in* $\overline{\Im}$ *and* $\chi^{\zeta} \subseteq \mathfrak{K}\}$;
3. $Fr(\chi^{\zeta}) = Cl(\chi^{\zeta}) \cap Cl\left(\chi^{\zeta^c}\right)$;
4. $Ext(\chi^{\zeta}) = Int(\chi^{\zeta^c})$.

It is clear that:
(a) Int (χ^{ζ}) *is the largest FFOS containing* χ^{ζ};
(b) $Cl(\chi^{\zeta})$ *is the smallest FFCS containing* χ^{ζ}.

Example 2. *Assume* $\overline{\Im} = \{\breve{\alpha}_1^{\gamma}, \breve{\alpha}_2^{\gamma}, \breve{\alpha}_3^{\gamma}\}$. *Consider the family of FFSs*

$$\hbar^{\daleth} = \left\{1_{\overline{\Im}}, 0_{\overline{\Im}}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_2, \chi^{\zeta}{}_3, \chi^{\zeta}{}_4,\right\},$$

where

$$\begin{aligned}\chi^{\zeta}{}_1 &= \{\langle\breve{\alpha}_1^{\gamma}, 0.58, 0.78\rangle, \langle\breve{\alpha}_2^{\gamma}, 0.67, 0.58\rangle, \langle\breve{\alpha}_3^{\gamma}, 0.28, 0.18\rangle\}, \\ \chi^{\zeta}{}_2 &= \{\langle\breve{\alpha}_1^{\gamma}, 0.60, 0.76\rangle, \langle\breve{\alpha}_2^{\gamma}, 0.72, 0.53\rangle, \langle\breve{\alpha}_3^{\gamma}, 0.30, 0.16\rangle\}, \\ \chi^{\zeta}{}_3 &= \{\langle\breve{\alpha}_1^{\gamma}, 0.64, 0.72\rangle, \langle\breve{\alpha}_2^{\gamma}, 0.74, 0.48\rangle, \langle\breve{\alpha}_3^{\gamma}, 0.34, 0.14\rangle\}, \\ \chi^{\zeta}{}_4 &= \{\langle\breve{\alpha}_1^{\gamma}, 0.71, 0.67\rangle, \langle\breve{\alpha}_2^{\gamma}, 0.80, 0.43\rangle, \langle\breve{\alpha}_3^{\gamma}, 0.44, 0.10\rangle\}.\end{aligned}$$

It is clear that $(\overline{\Im}, \hbar^{\daleth})$ *is an FF TS. Now, assume that*

$$\urcorner = \{\langle \breve{\alpha}_1^{\gamma}, 0.78, 0.48\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.88, 0.28\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.57, 0.08\rangle\}$$

is an FF subset over $\overline{\Im}$*. Then,*

$$\begin{aligned} Int(\urcorner) &= 0_{\overline{\Im}} \cup \chi^{\zeta}{}_1 \cup \chi^{\zeta}{}_2 \cup \chi^{\zeta}{}_3 \cup \chi^{\zeta}{}_4 \\ &= \chi^{\zeta}{}_4 = \{\langle \breve{\alpha}_1^{\gamma}, 0.71, 0.67\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.80, 0.43\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.44, 0.10\rangle\} \end{aligned}$$

In fact, $Int(\urcorner)$ *is the largest FFOS contained in FFS* $\urcorner$*.*
On the other hand, in order to find the FF closure of χ^{ζ}*, it necessary to determine the FFCSs over* $\overline{\Im}$*. Then,*

$$\begin{aligned} \chi^{\zeta c}{}_1 &= \{\langle \breve{\alpha}_1^{\gamma}, 0.78, 0.58\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.58, 0.67\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.18, 0.28\rangle\}, \\ \chi^{\zeta c}{}_2 &= \{\langle \breve{\alpha}_1^{\gamma}, 0.76, 0.60\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.53, 0.72\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.16, 0.30\rangle\}, \\ \chi^{\zeta c}{}_3 &= \{\langle \breve{\alpha}_1^{\gamma}, 0.72, 0.64\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.48, 0.74\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.14, 0.34\rangle\}, \\ \chi^{\zeta c}{}_4 &= \{\langle \breve{\alpha}_1^{\gamma}, 0.67, 0.71\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.43, 0.80\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.10, 0.44\rangle\}. \end{aligned}$$

Now, we obtain

$$\urcorner^c = \{\langle \breve{\alpha}_1^{\gamma}, 0.48, 0.78\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.28, 0.88\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.08, 0.57\rangle\}$$

The computations for $Cl(\urcorner)$*,* $Fr(\urcorner)$*, and* $Ext(\urcorner)$ *are as follows,*

$$\begin{aligned} Cl(\urcorner) =& 1_{\overline{\Im}} \qquad (1_{\overline{\Im}} \textit{ is only closed superset of } \urcorner) \\ Cl(\urcorner^c) =& 1_{\overline{\Im}} \cap \chi^{\zeta c}{}_1 \cap \chi^{\zeta c}{}_2 \cap \chi^{\zeta c}{}_3 \cap \chi^{\zeta c}{}_4 \\ =& \chi^{\zeta c}{}_4 \qquad (\chi^{\zeta c}{}_4 \textit{ is the smallest closed superset of } \urcorner) \\ Fr(\urcorner) =& Cl(\urcorner) \cap Cl(\urcorner^c) \\ =& 1_{\overline{\Im}} \cap \chi^{\zeta c}{}_4 \\ =& \chi^{\zeta c}{}_4 \qquad (\chi^{\zeta c}{}_4 \subseteq 1_{\overline{\Im}}) \\ Ext(\urcorner) =& Int(\urcorner^c) \\ =& 0_{\overline{\Im}} \end{aligned}$$

Remark 2. *In addition, we analyse why certain findings that hold in crisp topology fail in FFT. The results of crisp topology and FFT are shown in Table 1.*

Table 1. Comparison of some results of crisp topology and FFT.

Crisp Topology	Fermatean Fuzzy Topology
$Int(\urcorner) \bigcup Ext(\urcorner) \bigcup Fr(\urcorner) = \overline{\Im}$	$Int(\urcorner) \bigcup Ext(\urcorner) \bigcup Fr(\urcorner) \neq 1_{\overline{\Im}}$
$Int(\urcorner) \bigcap Ext(\urcorner) = \varnothing$	$Int(\urcorner) \bigcap Ext(\urcorner) \neq 0_{\overline{\Im}}$
$Ext(\urcorner) \bigcap Fr(\urcorner) = \varnothing$	$Ext(\urcorner) \bigcap Fr(\urcorner) \neq 0_{\overline{\Im}}$
$Int(\urcorner) \bigcap Fr(\urcorner) = \varnothing$	$Int(\urcorner) \bigcap Fr(\urcorner) \neq 0_{\overline{\Im}}$

Proposition 2. *Let* $(\overline{\Im}, \hbar^{\daleth})$ *be an FFTS and* $\chi^{\zeta}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_2$ *be FFSs over* $\overline{\Im}$*. Then, the following properties hold.*

1. $Int(\chi^{\zeta}) \subseteq \chi^{\zeta}$;
2. $Int(Int(\chi^{\zeta})) = Int(\chi^{\zeta})$;
3. $\chi^{\zeta}{}_1 \subseteq \chi^{\zeta}{}_2 \Rightarrow Int(\chi^{\zeta}{}_1) \subseteq Int(\chi^{\zeta}{}_2)$;
4. $Int(\chi^{\zeta}{}_1 \cap \chi^{\zeta}{}_2) = Int(\chi^{\zeta}{}_1) \cap Int(\chi^{\zeta}{}_2)$;
5. $Int(1_{\overline{\Im}}) = 1_{\overline{\Im}}, Int(0_{\overline{\Im}}) = 0_{\overline{\Im}}$;
6. $Int(\chi^{\zeta}{}_1 \cup \chi^{\zeta}{}_2) \subseteq Int(\chi^{\zeta}{}_1) \cap Int(\chi^{\zeta}{}_2)$.

Proof. We can see that (1), (2), (3), and (5) are readily available from the FF interior description.
For (4), we obtain
$Int(\chi^{\zeta}{}_1 \cap \chi^{\zeta}{}_2) \subseteq Int(\chi^{\zeta}{}_1) \cap$ Int $(\chi^{\zeta}{}_2)$ from $Int(\chi^{\zeta}{}_1 \cap \chi^{\zeta}{}_2) \subseteq Int(\chi^{\zeta}{}_1)$ and $Int(\chi^{\zeta}{}_1 \cap \chi^{\zeta}{}_2) \subseteq Int(\chi^{\zeta}{}_2)$. On the other hand, from the facts $Int(\chi^{\zeta}{}_1) \subseteq \chi^{\zeta}{}_1$ and $Int(\chi^{\zeta}{}_2) \subseteq \chi^{\zeta}{}_2 \Rightarrow Int(\chi^{\zeta}{}_1) \cap Int(\chi^{\zeta}{}_2) \subseteq \chi^{\zeta}{}_1 \cap \chi^{\zeta}{}_2$ and $Int(\chi^{\zeta}{}_1) \cap Int(\chi^{\zeta}{}_2) \in \hbar^{\daleth}$, we have $Int(\chi^{\zeta}{}_1) \cap Int(\chi^{\zeta}{}_2) \subseteq Int(\chi^{\zeta}{}_1 \cap \chi^{\zeta}{}_2)$. Thus, the proof of the axioms (4) is obtained from these two inequalities. □

Theorem 2. *Let $\mathcal{P}^{\xi} : FFS(\overline{\Im}) \to FFS(\overline{\Im})$ be a mapping. The family $\hbar^{\daleth} = \{\chi^{\zeta} \in FFS(\overline{\Im}) : \mathcal{P}^{\xi}(\chi^{\zeta}) = \chi^{\zeta}\}$ is an FF topology over $\overline{\Im}$, if the mapping $\mathcal{P}^{\xi}$ satisfies the following conditions:*
(i) $\mathcal{P}^{\xi}(\chi^{\zeta}) \subseteq \chi^{\zeta}$;
(ii) $\mathcal{P}^{\xi}(1_{\overline{\Im}}) = 1_{\overline{\Im}}$;
(iii) $\mathcal{P}^{\xi}(\mathcal{P}^{\xi}(\chi^{\zeta})) = \mathcal{P}^{\xi}(\chi^{\zeta})$;
(iv) $\mathcal{P}^{\xi}(\chi^{\zeta}{}_1 \cap \chi^{\zeta}{}_2) = \mathcal{P}^{\xi}(\chi^{\zeta}{}_1) \cap \mathcal{P}^{\xi}(\chi^{\zeta}{}_2)$. $\mathcal{P}^{\xi}(\chi^{\zeta}) = Int(\chi^{\zeta})$ for each FF set χ^{ζ} in this FF TS.

Proposition 3. *Let $(\overline{\Im}, \hbar^{\daleth})$ be an FFTS and $\chi^{\zeta}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_2$ be FFSs over $\overline{\Im}$. Then, the following properties hold.*

1. $\chi^{\zeta} \subseteq Cl(\chi^{\zeta})$;
2. $Cl(Cl(\chi^{\zeta})) = Cl(\chi^{\zeta})$;
3. $\chi^{\zeta}{}_1 \subseteq \chi^{\zeta}{}_2 \Rightarrow Cl(\chi^{\zeta}{}_1) \subseteq Cl(\chi^{\zeta}{}_2)$;
4. $Cl(\chi^{\zeta}{}_1 \cup \chi^{\zeta}{}_2) = Cl(\chi^{\zeta}{}_1) \cup Cl(\chi^{\zeta}{}_2)$;
5. $Cl(1_{\overline{\Im}}) = 1_{\overline{\Im}}, Cl(0_{\overline{\Im}}) = 0_{\overline{\Im}}$.

Proof. Here, (1), (2), (3) and (5) can be easily obtained from the definition of the FF closure. For (4), we obtain $Cl(\chi^{\zeta}{}_1) \cup Cl(\chi^{\zeta}{}_2) \subseteq Cl(\chi^{\zeta}{}_1 \cup \chi^{\zeta}{}_2)$ from $Cl(\chi^{\zeta}{}_1) \subseteq Cl(\chi^{\zeta}{}_1 \cup \chi^{\zeta}{}_2)$ and $Cl(\chi^{\zeta}{}_2) \subseteq Cl(\chi^{\zeta}{}_1 \cup \chi^{\zeta}{}_2)$. On the other hand, from the facts $\chi^{\zeta}{}_1 \subseteq Cl(\chi^{\zeta}{}_1)$ and $\chi^{\zeta}{}_2 \subseteq Cl(\chi^{\zeta}{}_2) \Rightarrow \chi^{\zeta}{}_1 \cup \chi^{\zeta}{}_2 \subseteq Cl(\chi^{\zeta}{}_1) \cup Cl(\chi^{\zeta}{}_2)$ and $Cl(\chi^{\zeta}{}_1) \cup Cl(\chi^{\zeta}{}_2) \in \chi^{\zeta}$, we have $Cl(\chi^{\zeta}{}_1 \cup \chi^{\zeta}{}_2) \subseteq Cl(\chi^{\zeta}{}_1) \cup Cl(\chi^{\zeta}{}_2)$. Thus, the proof of the axioms (4) is obtained from these two inequalities. □

Theorem 3. *Let $\mathcal{C} : FFS(\overline{\Im}) \to FFS(\overline{\Im})$ be a mapping. The family*
$\hbar^{\daleth} = \left\{\chi^{\zeta} \in FFS(\overline{\Im}) : \mathcal{C}\left(\chi^{\zeta^c}\right) = \chi^{\zeta^c}\right\}$ is an FF topology over $\overline{\Im}$ if the mapping $\mathcal{C}$ satisfies the following conditions:

1. $\chi^{\zeta} \subseteq \mathcal{C}(\chi^{\zeta})$;
2. $\mathcal{C}(0_{\overline{\Im}}) = 0_{\overline{\Im}}$;
3. $\mathcal{C}(\mathcal{C}(\chi^{\zeta})) = \mathcal{C}(\chi^{\zeta})$;
4. $\mathcal{C}(\chi^{\zeta}{}_1 \cup \chi^{\zeta}{}_2) = \mathcal{C}(\chi^{\zeta}{}_1) \cup \mathcal{C}(\chi^{\zeta}{}_2)$.

Furthermore, $\mathcal{C}(\chi^{\zeta}) = Cl(\chi^{\zeta})$ for each FF set χ^{ζ} in this FF TS.

Theorem 4. *Let $(\overline{\Im}, \hbar^{\daleth})$ be an FFTS and χ^{ξ} be an FFS over $\overline{\Im}$. Then,*

(a) $Cl\left(\chi^{\zeta^c}\right) = (Int(\chi^{\xi}))^c$;

(b) $Int\left(\chi^{\zeta^c}\right) = (Cl(\chi^{\zeta}))^c$.

Proof. **(a)** Let $\chi^{\xi} = \left\{\left\langle \breve{\aleph}, \mu^{\eta}{}_{\chi^{\zeta}}(\breve{\aleph}), \nu^{\eta}{}_{\chi^{\zeta}}(\breve{\aleph}) \right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$ and assume that the family of FFSs contained in χ^{ξ} are indexed by the family $\left\{\chi^{\zeta}{}_i = \left\{\left\langle \breve{\aleph}, \mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}), \nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}) \right\rangle : \breve{\aleph} \in \overline{\Im}\right\}\right\}_{i \in I}$.
Then, we see that $Int(\chi^{\xi}) = \left\{\left\langle \breve{\aleph}, \sup\left\{\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}, \in \beth\left\{\nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\} \right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$ and hence $(Int(\chi^{\xi}))^c = \left\{\left\langle \breve{\aleph}, \in \beth\left\{\nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}, \sup\left\{\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\} \right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$. Since
$\chi^{\zeta^c} = \left\{\left\langle \breve{\aleph}, \nu^{\eta}{}_{\chi^{\zeta}}(\breve{\aleph}), \mu^{\eta}{}_{\chi^{\zeta}}(\breve{\aleph}) \right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$ and $\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}) \leq \mu^{\eta}{}_{\chi^{\zeta}}(\breve{\aleph}), \nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}) \geq \nu^{\eta}{}_{\chi^{\zeta}}(\breve{\aleph})$ for

each $i \in I$, we obtain that
$\left\{\chi^{\zeta}{}_i = \left\{\left\langle \breve{\aleph}, \mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph}), \nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\rangle : \breve{\aleph} \in \overline{\Im}\right\}\right\}_{i \in I}$ is the family of FFSs containing χ^{ζ^c}, i.e., $Cl\left(\chi^{\zeta^c}\right) = \left\{\left\langle \breve{\aleph}, \in \sqsupset\left\{\nu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}, \sup\left\{\mu^{\eta}{}_{\chi^{\zeta}{}_i}(\breve{\aleph})\right\}\right\rangle : \breve{\aleph} \in \overline{\Im}\right\}$. Therefore, $Cl\left(\chi^{\zeta^c}\right) = (Int(\chi^{\zeta}))^c$ immediately.

(b) This is analogous to **(a)**. □

Definition 11. *A Fermatean fuzzy number (FFN) or Fermatean fuzzy point (FF point)* $\aleph = (\mu^{\eta}, \nu^{\eta})$ *is said to be contained in FFS* $\mathcal{K}$

$$\mathcal{K} = \{\langle \rho, \mu^{\eta}{}_{\mathcal{K}}(\rho), \nu^{\eta}{}_{\mathcal{K}}(\rho)\rangle : \rho \in \overline{\Im}\}$$

written as $\aleph \in \mathcal{K}$, *if* $\mu^{\eta} \le \mu^{\eta}{}_{\mathcal{K}}$ *and if* $\nu^{\eta} \ge \nu^{\eta}{}_{\mathcal{K}}, \forall \rho \in \overline{\Im}$.

Definition 12. *An FFN* $\aleph = (\mu^{\eta}, \nu^{\eta})$ *contained in an FFS* $\mathcal{K}$ *is said to be an FF interior point if there exists FFOS* $\mathcal{U}$ *such that,* $\aleph \in \mathcal{U} \subseteq \mathcal{K}$. *Then,* $\mathcal{K}$ *is called an FF neighbourhood of FFN* $\aleph$. *Note that* $\aleph$ *in the FF interior point of FFS* $\mathcal{K}$ *if and only if* $\mathcal{K}$ *is an FF neighbourhood of FFN* $\aleph$.

Theorem 5. *Let* $(\overline{\Im}, \hbar^{\daleth})$ *be an FFTS.*
(i) If ϕ *and* φ *are the neighbourhoods of FFN* $\aleph$, *then* $\phi \bigcap \varphi$ *and* $\phi \bigcup \varphi$ *are also neighbourhoods of* $\aleph$.
(ii) If ψ *is a neighbourhood of FFN* $\aleph$, *then each FF superset* δ *of* ψ *is also a neighbourhood of* $\aleph$.

Proposition 4. *Let* $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ *and* $\left(\overline{\Im}, \hbar^{\daleth}{}_2\right)$ *be two FFTSs and* $\sqsupset : \overline{\Im} \to \breve{\amalg}$ *be an FF mapping. Then, the following are equivalent to each other:*

a $\sqsupset$ *is FF continuous mapping;*
b $\sqsupset[Cl(\chi^{\zeta})] \subseteq Cl(\sqsupset[\chi^{\zeta}])$ *for each FFS*χ^{ζ} *in* $\overline{\Im}$;
c $Cl\left(\sqsupset^{-1}[K]\right) \subseteq \sqsupset^{-1}[Cl(K)]$ *for each FFSK in* $\breve{\amalg}$;
d $\sqsupset^{-1}[Int(K)] \subseteq Int\left(\sqsupset^{-1}[K]\right)$ *for each FFSK in* $\breve{\amalg}$.

Proof. (a) ⇒ (b) Let $\sqsupset : \overline{\Im} \to \breve{\amalg}$ be FF continuous mapping and χ^{ζ} be an FFS over $\overline{\Im}$. Then, $\sqsupset[\chi^{\zeta}] \subseteq Cl(\sqsupset[\chi^{\zeta}])$ and $\chi^{\zeta} \subseteq \sqsupset^{-1}[Cl(\sqsupset[\chi^{\zeta}])]$. Since $Cl(\sqsupset[\chi^{\zeta}])$ is an FFCS in $\breve{\amalg}$ and $\sqsupset$ is FF continuous mapping, $\sqsupset^{-1}[Cl(\sqsupset[\chi^{\zeta}])]$ is an FFCS in $\overline{\Im}$. On the other hand, if $Cl(\chi^{\zeta})$ is the smallest FFCS containing χ^{ζ}, then $Cl(\chi^{\zeta}) \subseteq \sqsupset^{-1}[Cl(\sqsupset[\chi^{\zeta}])]$ and so, $\sqsupset[Cl(\chi^{\zeta})] \subseteq Cl(\sqsupset[\chi^{\zeta}])$.

(b) ⇒ (c) Suppose that $\chi^{\zeta} = \sqsupset^{-1}[K]$. From (b), $\sqsupset[Cl(\chi^{\zeta})] = \sqsupset\left[Cl\left(\sqsupset^{-1}[K]\right)\right] \subseteq Cl(\sqsupset[\chi^{\zeta}]) = Cl\left(\sqsupset\left[\sqsupset^{-1}[K]\right]\right) \subseteq Cl(K)$. Then, $Cl\left(\sqsupset^{-1}[K]\right) = Cl(\chi^{\zeta}) \subseteq \sqsupset^{-1}[\sqsupset[Cl(\chi^{\zeta})]] \subseteq \sqsupset^{-1}[Cl(K)]$.

(c) ⇒ (d) Since Int $(K) = (Cl(K^c))^c$, then $Cl\left(\sqsupset^{-1}[K]\right) = Cl(\chi^{\zeta}) \subseteq \sqsupset^{-1}[\sqsupset[Cl(\chi^{\zeta})]] \subseteq \sqsupset^{-1}[Cl(K)]$.

Assume that G is an FFOS in $\breve{\amalg}$. Then, Int $(G) = G$. From (d), $\sqsupset^{-1}[G] = \sqsupset^{-1}[Int(G)] \subseteq Int\left(\sqsupset^{-1}[G]\right) \subseteq \sqsupset^{-1}[G]$. Therefore, $\sqsupset$ is an FF continuous mapping. □

Definition 13. *Let* $(\overline{\Im}, \hbar^{\daleth})$ *be an FFTS.*
(i) A subfamily Γ *of* $\hbar^{\daleth}$ *is called an FF basis (FFB) for* $\hbar^{\daleth}$ *if for each* $\chi^{\zeta} \in \hbar^{\daleth}$, *there exists* $\Gamma' \subseteq \Gamma$ *such that* $\chi^{\zeta} = \cup\Gamma'$.
(ii) A collection Φ *of some FFSs on* $\overline{\Im}$ *is called an FF subbase (FFSB) for some FFT* $\hbar^{\daleth}$ *if the finite intersections of members of* Φ *form an FF basis for* $\hbar^{\daleth}$.

Theorem 6. $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ *and* $\left(\overline{\Im}, \hbar^{\daleth}{}_2\right)$ *are two FFTSs and* $\sqsupset : \overline{\Im} \to \breve{\amalg}$ *is an FF mapping. Then,*

1. $\sqsupset$ *is an FF continuous mapping iff for each* $\tilde{F} \in \Gamma$, *we have* $\sqsupset^{-1}[\tilde{F}]$ *as an FF open subset of* $\overline{\Im}$ *such that* Γ *is an FF basis for* $\hbar^{\daleth}{}_2$.

2. *$\sqsupset$ is an FF continuous mapping iff for each $K \in \chi^{\zeta}$, we have $\sqsupset^{-1}[K]$ as an FF open subset of $\overline{\Im}$ such that Φ is an FF subbase for $\hbar^{\daleth}{}_2$.*

Proof. (i) Let $\sqsupset$ be an FF continuous mapping. Since each $\tilde{F} \in \Gamma \subseteq \hbar^{\daleth}{}_2$ and $\sqsupset$ is an FF continuous mapping, then $\sqsupset^{-1}[\tilde{F}] \in \hbar^{\daleth}{}_1$.
Conversely, suppose that Γ is an FF basis for $\hbar^{\daleth}{}_2$ and $\sqsupset^{-1}[\tilde{F}] \in \hbar^{\daleth}{}_1$ for each $\tilde{F} \in \Gamma$. Then, for an arbitrary FFOS $\chi^{\zeta} \in \hbar^{\daleth}{}_2$,

$$\sqsupset^{-1}[\chi^{\zeta}] = \sqsupset^{-1}\left[\cup_{\tilde{F}\in\Gamma}\tilde{F}\right] = \bigcup_{\tilde{F}\in\Gamma}\sqsupset^{-1}[\tilde{F}] \in \hbar^{\daleth}{}_1.$$

That is, $\sqsupset$ is an FF continuous mapping.
(ii) Let $\sqsupset$ be an FF continuous mapping. Since each $K \in \Phi \subseteq \hbar^{\daleth}{}_2$ and $\sqsupset$ is an FF continuous mapping, then $\sqsupset^{-1}[K] \in \hbar^{\daleth}{}_1$.

Conversely, assume that Φ is an FF subbase for $\hbar^{\daleth}{}_2$ and $\sqsupset^{-1}[K] \in \hbar^{\daleth}{}_1$ for each $K \in \Phi$. Then, for an arbitrary FFOS $\chi^{\zeta} \in \hbar^{\daleth}{}_2$,

$$\begin{aligned}\sqsupset^{-1}[\chi^{\zeta}] &= \sqsupset^{-1}\left[\cup_{i_j\in I}\left(K_{i_1} \cap K_{i_2} \cap \ldots \cap K_{i_n}\right)\right] \\ &= \bigcup_{i_j\in I}\left(\sqsupset^{-1}[K_{i_1}] \cap \sqsupset^{-1}[K_{i_2}] \cap \ldots \cap \sqsupset^{-1}[K_{i_n}]\right) \in \hbar^{\daleth}{}_1\end{aligned}$$

That is, $\sqsupset$ is an FF continuous mapping. □

Definition 14. *Let $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ and $\left(\overline{\Im}, \hbar^{\daleth}{}_2\right)$ be two FFTSs and $\sqsupset : \overline{\Im} \to \breve{\amalg}$ be an FF mapping. Then:*
(i) $\sqsupset$ is called an FF open function if $\sqsupset[\chi^{\zeta}]$ is an FFOS over $\breve{\amalg}$ for every FFOS χ^{ζ} over $\overline{\Im}$. (ii) $\sqsupset$ is called an FF closed function if $\sqsupset[K]$ is an FFCS over $\breve{\amalg}$ for every FFCS K over $\overline{\Im}$.

Example 3. *Let $\overline{\Im} = \{\breve{\alpha}^{\gamma}_1, \breve{\alpha}^{\gamma}_2, \breve{\alpha}^{\gamma}_3\}$ and $\breve{\amalg} = \{y_1, y_2, y_3\}$. Consider the following families of FF sets $\hbar^{\daleth}{}_1 = \{0_{\overline{\Im}}, 1_{\overline{\Im}}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_2, \chi^{\zeta}{}_3, \chi^{\zeta}{}_4\}$ and $\hbar^{\daleth}{}_2 = \{0_{\breve{\amalg}}, 1_{\breve{\amalg}}, S_1, S_2, S_3, S_4\}$ where*

$$\begin{aligned}\chi^{\zeta}{}_1 &= \{\langle\breve{\alpha}^{\gamma}_1, 0.27, 0.43\rangle, \langle\breve{\alpha}^{\gamma}_2, 0.57, 0.17\rangle, \langle\breve{\alpha}^{\gamma}_3, 0.57, 0.47\rangle\} \\ \chi^{\zeta}{}_2 &= \{\langle\breve{\alpha}^{\gamma}_1, 0.57, 0.47\rangle, \langle\breve{\alpha}^{\gamma}_2, 0.77, 0.27\rangle, \langle\breve{\alpha}^{\gamma}_3, 0.67, 0.57\rangle\} \\ \chi^{\zeta}{}_3 &= \{\langle\breve{\alpha}^{\gamma}_1, 0.57, 0.38\rangle, \langle\breve{\alpha}^{\gamma}_2, 0.77, 0.17\rangle, \langle\breve{\alpha}^{\gamma}_3, 0.67, 0.47\rangle\} \\ \chi^{\zeta}{}_4 &= \{\langle\breve{\alpha}^{\gamma}_1, 0.27, 0.40\rangle, \langle\breve{\alpha}^{\gamma}_2, 0.57, 0.27\rangle, \langle\breve{\alpha}^{\gamma}_3, 0.57, 0.57\rangle\}\end{aligned}$$

$$\begin{aligned}S_1 &= \{\langle y_1, 0.57, 0.17\rangle, \langle y_2, 0.27, 0.43\rangle, \langle y_3, 0.57, 0.47\rangle\}, \\ S_2 &= \{\langle y_1, 0.77, 0.27\rangle, \langle y_2, 0.57, 0.47\rangle, \langle y_3, 0.67, 0.57\rangle\}, \\ S_3 &= \{\langle y_1, 0.77, 0.17\rangle, \langle y_2, 0.57, 0.38\rangle, \langle y_3, 0.67, 0.47\rangle\}, \\ S_4 &= \{\langle y_1, 0.57, 0.27\rangle, \langle y_2, 0.27, 0.40\rangle, \langle y_3, 0.57, 0.57\rangle\},\end{aligned}$$

It is clear that $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ and $\left(\overline{\Im}, \hbar^{\daleth}{}_2\right)$ are FFTSs. If FF mapping $\sqsupset : \overline{\Im} \to \breve{\amalg}$ is defined as

$$\begin{aligned}\sqsupset(\breve{\alpha}^{\gamma}_1) &= y_2 \\ \sqsupset(\breve{\alpha}^{\gamma}_2) &= y_1 \\ \sqsupset(\breve{\alpha}^{\gamma}_3) &= y_3\end{aligned}$$

Then $\sqsupset$ is an FF open function. However, $\sqsupset$ is not an FF closed function on FFTSs $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$.

Theorem 7. *Let $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ and $\left(\overline{\Im}, \hbar^{\daleth}{}_2\right)$ be two FFTSs and $\sqsupset : \overline{\Im} \to \breve{\amalg}$ be an FF mapping. Then:*

1. *$\sqsupset$ is an FF open function if $\sqsupset[Int(\chi^{\zeta})] \subseteq Int(\sqsupset[\chi^{\zeta}])$ for each FF set χ^{ζ} over $\overline{\Im}$.*

2. $\beth$ *is an FF closed function if* $Cl(\beth[\chi^\zeta]) \subseteq \beth[Cl(\chi^\zeta)]$ *for each FF set* χ^ζ *over* $\overline{\Im}$.

Proof. (1) Let $\beth$ be an FF open function and χ^ζ be an FFS over $\overline{\Im}$. Then, $Int(\chi^\zeta)$ is an FFOS and $Int(\chi^\zeta) \subseteq \chi^\zeta$. Since $\beth$ is an FF open function, $\beth[Int(\chi^\zeta)]$ is an FFOS over $\breve{\Pi}$ and $\beth[Int(\chi^\zeta)] \subseteq \beth[\chi^\zeta]$. Thus, $\beth[Int(\chi^\zeta)] \subseteq Int(\beth[\chi^\zeta])$ is obtained.
Conversely, suppose that χ^ζ is any FFOS over $\overline{\Im}$. Then, $\chi^\zeta = Int(\chi^\zeta)$. From the condition of theorem, we have $\beth[Int(\chi^\zeta)] \subseteq Int(\beth[\chi^\zeta])$. Then, $\beth[\chi^\zeta] = \beth[Int(\chi^\zeta)] \subseteq Int(\beth[\chi^\zeta]) \subseteq \beth[\chi^\zeta]$. This implies that $\beth[\chi^\zeta] = Int(\beth[\chi^\zeta])$. That is, $\beth$ is an FF open function.
(2) Let $\beth$ be an FF closed function and χ^ζ be a FFS over $\overline{\Im}$. Since $\beth$ is an FF closed function, then $\beth[Cl(\chi^\zeta)]$ is an FFCS over $\breve{\Pi}$ and $\beth[\chi^\zeta] \subseteq \beth[Cl(\chi^\zeta)]$. Thus, $Cl(\beth[\chi^\zeta]) \subseteq \beth[Cl(\chi^\zeta)]$ is obtained.

Conversely, assume that χ^ζ is any FFCS over $\overline{\Im}$. Then, $\chi^\zeta = Cl(\chi^\zeta)$. From the condition of theorem, we have $Cl(\beth[\chi^\zeta]) \subseteq \beth[Cl(\chi^\zeta)] = \beth[\chi^\zeta] \subseteq Cl(\beth[\chi^\zeta])$. This means that, $Cl(\beth[\chi^\zeta]) = \beth[\chi^\zeta]$. That is, $\beth$ is an FF closed function. □

Definition 15. *Let* $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ *and* $\left(\overline{\Im}, \hbar^{\daleth}{}_2\right)$ *be two FFTSs and* $\beth : \overline{\Im} \to \breve{\Pi}$ *be an FF mapping. Then,* $\beth$ *is a called an FF homeomorphism if:*
(i) $\beth$ *is a bijective mapping;*
(ii) $\beth$ *is an FF continuous mapping;*
(iii) $\beth^{-1}$ *is an FF continuous mapping.*

Theorem 8. *Let* $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ *and* $\left(\overline{\Im}, \hbar^{\daleth}{}_2\right)$ *be two FFTSs and* $\beth : \overline{\Im} \to \breve{\Pi}$ *be an FF mapping. Then, the following conditions are equivalent:*
(a) $\beth$ *is an FF homeomorphism;*
(b) $\beth$ *is an FF continuous mapping and FF open function; (c)* $\beth$ *is an FF continuous mapping and FF closed function.*

Proof. The proof can be easily obtained by using the previous theorems on continuity, openness and closedness are omitted. □

4. Ff Connectedness

In this section, we define the generalised concept of IF-connected TS and provide the related results with illustrations.

Definition 16. *Let A be an FF subset in* $(\overline{\Im}, \hbar^{\daleth}{}_{\overline{\Im}})$.
(a) If there exist FFOSs $\mathcal{U}^\varsigma$ *and* $\mathcal{V}^\tau$ *in* $\overline{\Im}$ *satisfying the following properties, then* A^ζ *is called FF* $\mathfrak{C}_i$*-disconnected* $(i = 1, 2, 3, 4)$ *:*

C_1 $A^\zeta \subseteq \mathcal{U}^\varsigma \cup \mathcal{V}^\tau, \mathcal{U}^\varsigma \cap \mathcal{V}^\tau \subseteq A^{\zeta^c}, A^\zeta \cap \mathcal{U}^\varsigma \neq 0_x, A^\zeta \cap \mathcal{V}^\tau \neq 0_x;$

C_2 $A^\zeta \subseteq \mathcal{U}^\varsigma \cup \mathcal{V}^\tau, A^\zeta \cap \mathcal{U}^\varsigma \cap \mathcal{V}^\tau \neq 0_x, A^\zeta \cap \mathcal{U}^\varsigma \neq 0_x, A^\zeta \cap \mathcal{V}^\tau \neq 0_x;$

C_3 $A^\zeta \subseteq \mathcal{U}^\varsigma \cup \mathcal{V}^\tau, \mathcal{U}^\varsigma \cap \mathcal{V}^\tau \subseteq A^{\zeta^c}, \mathcal{U}^\varsigma \nsubseteq A^{\zeta^c}, \mathcal{V}^\tau \nsubseteq A^{\zeta^c};$

C_4 $A^\zeta \subseteq \mathcal{U}^\varsigma \cup \mathcal{V}^\tau, A^\zeta \cap \mathcal{U}^\varsigma \cap \mathcal{V}^\tau \neq 0_x, \mathcal{U}^\varsigma \nsubseteq A^{\zeta^c}, \mathcal{V}^\tau \nsubseteq A^{\zeta^c}.$

(b) A^ζ *is said to be FF* $\mathfrak{C}_i$*-connected* $(i = 1, 2, 3, 4)$ *if* A^ζ *is not FF* $\mathfrak{C}_i$*-disconnected* $(i = 1, 2, 3, 4)$.

It is clear that, in FFTSs, we have the following implications:

$$\begin{array}{ccc} \mathfrak{C}_1\text{-connectedness} & \to & \mathfrak{C}_2\text{-connectedness} \\ \downarrow & & \downarrow \\ \mathfrak{C}_3\text{-connectedness} & \to & \mathfrak{C}_4\text{-connectedness.} \end{array}$$

Example 4. *Let* $\overline{\Im} = \{\breve{\alpha}_1^{\gamma}, \breve{\alpha}_2^{\gamma}, \breve{\alpha}_3^{\gamma}\}$*. Consider the following family of FF sets*

$$\chi^{\zeta}{}_1 = \{\langle \breve{\alpha}_1^{\gamma}, 0.50, 0.20\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.50, 0.40\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.40, 0.40\rangle\},$$
$$\chi^{\zeta}{}_2 = \{\langle \breve{\alpha}_1^{\gamma}, 0.40, 0.50\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.60, 0.30\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.20, 0.30\rangle\},$$
$$\chi^{\zeta}{}_3 = \{\langle \breve{\alpha}_1^{\gamma}, 0.50, 0.20\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.60, 0.30\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.40, 0.30\rangle\},$$
$$\chi^{\zeta}{}_4 = \{\langle \breve{\alpha}_1^{\gamma}, 0.40, 0.50\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.50, 0.40\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.20, 0.40\rangle\}.$$

Then, $\hbar^{\daleth} = \{1_{\overline{\Im}}, 0_{\overline{\Im}}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_2, \chi^{\zeta}{}_3, \chi^{\zeta}{}_4\}$ *is an FFTS on* $\overline{\Im}$*, and consider the FFSE given below*

$$E = \{\langle \breve{\alpha}_1^{\gamma}, 0.60, 0.20\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.50, 0.20\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.40, 0.30\rangle\},$$

in $\overline{\Im}$*. Then, E is FF* $\mathfrak{C}_1$*-connected, and E is also FF*$\mathfrak{C}_2$*-connected, FF*$\mathfrak{C}_3$*-connected, and FF*$\mathfrak{C}_4$*-connected.*

Example 5. *Consider the FFTS* $(\overline{\Im}, \hbar^{\daleth}{}_{\overline{\Im}})$ *given in Example 4 and consider the FFS F given below*

$$F = \{\langle \breve{\alpha}_1^{\gamma}, 0.20, 0.40\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.30, 0.60\rangle, \langle \breve{\alpha}_3^{\gamma}, 0.20, 0.40\rangle\},$$

One can verify whether F is FF$\mathfrak{C}_1$*-disconnected and hence not FF*$\mathfrak{C}_1$*-connected.*

Definition 17. *Let* $(\overline{\Im}, \hbar^{\daleth}{}_{\overline{\Im}})$ *be an FFTS:*

(i) $\overline{\Im}$ *is said to be FFC*$_5$*-disconnected if there exists an FFOS and FFCS G such that* $G \neq 1_x$ *and* $G \neq 0_x$*.*

(ii) $\overline{\Im}$ *is said to be FF*$\mathfrak{C}_5$*-connected if it is not FF*$\mathfrak{C}_5$*-disconnected.*

Example 6. *Let* $\overline{\Im} = \{1, 2\}$ *and define the FF subsets* $A^{\zeta}, S^{\varsigma}, C, andD$ *as follows ;*
Let $\overline{\Im} = \{\breve{\alpha}_1^{\gamma}, \breve{\alpha}_2^{\gamma}, \breve{\alpha}_3^{\gamma}\}$*. Consider the following family of FF sets:*

$$\chi^{\zeta}{}_1 = \{\langle \breve{\alpha}_1^{\gamma}, 0.40, 0.30\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.20, 0.70\rangle\},$$
$$\chi^{\zeta}{}_2 = \{\langle \breve{\alpha}_1^{\gamma}, 0.30, 0.40\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.70, 0.20\rangle\},$$
$$\chi^{\zeta}{}_3 = \{\langle \breve{\alpha}_1^{\gamma}, 0.30, 0.40\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.20, 0.70\rangle\},$$
$$\chi^{\zeta}{}_4 = \{\langle \breve{\alpha}_1^{\gamma}, 0.40, 0.30\rangle, \langle \breve{\alpha}_2^{\gamma}, 0.70, 0.20\rangle\},$$

Then, the family $\hbar^{\daleth} = \{1_{\overline{\Im}}, 0_{\overline{\Im}}, \chi^{\zeta}{}_1, \chi^{\zeta}{}_2, \chi^{\zeta}{}_3, \chi^{\zeta}{}_4\}$ *is an FFTS on* $\overline{\Im}$ *and* $(\overline{\Im}, \hbar^{\daleth})$ *is an FF*$\mathfrak{C}_5$*-disconnected, since* A^{ζ} *is a nonzero FFOS and FFCS in* $\overline{\Im}$*.*

Definition 18. *Let* $(\overline{\Im}, \hbar^{\daleth}{}_{\overline{\Im}})$ *be an FFTS:*
(i) $\overline{\Im}$ *is called FF disconnected if there exist FFOSs* $A^{\zeta} \neq 0_x$ *and* $S^{\varsigma} \neq 0_x$ *such that* $A^{\zeta}\mathcal{U}^{\varsigma}S^{\varsigma} = 1_x$ *and* $A^{\zeta} \cap S^{\varsigma} = 0_x$*.*
(ii) $\overline{\Im}$ *is called FF connected if* $\overline{\Im}$ *is not FF disconnected.*

Proposition 5. *FF* $\mathfrak{C}_5$*-connectedness implies FF connectedness.*

Proposition 6. *Let* $\left(\overline{\Im} and \hbar^{\daleth}{}_1\right)$, $(\breve{\amalg}, \hbar^{\daleth}{}_2)$ *be two FFTSs and let* $f : \overline{\Im} \to \breve{\amalg}$ *be an FF continuous surjection. If* $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ *is FF connected, then so is* $(\breve{\amalg}, \hbar^{\daleth}{}_2)$*.*

Proof. On the contrary, suppose that $(\breve{\amalg}, \hbar^{\daleth}{}_2)$ is FF disconnected. Then, there exist FFOSs $A^{\zeta} \neq 0_{\breve{\amalg}}, S^{\varsigma} \neq 0_{\breve{\amalg}}$ in $\breve{\amalg}$ such that $A^{\zeta}\mathcal{U}^{\varsigma}S^{\varsigma} = 1_y, A^{\zeta} \cap S^{\varsigma} = 0_{\breve{\amalg}}$. Now, we see that $\mathcal{U}^{\varsigma} = f^1(A^{\zeta}), \mathcal{V}^{\tau} = f^1(S^{\varsigma})$ are FFOSs in $\overline{\Im}$ since f is FF continuous. From $A^{\zeta} \neq 0_{\breve{\amalg}}$, we obtain $\mathcal{U}^{\varsigma} = f^1(A^{\zeta}) \neq 0_x$. Similarly, $\mathcal{V}^{\tau} \neq 0_x$. Hence, $A^{\zeta}\mathcal{U}^{\varsigma}S^{\varsigma} = 1_y => f^1(A^{\zeta})f^1(S^{\varsigma}) = f^1(1_y) = 1_x => \mathcal{U}_u^{\varsigma}\mathcal{V}^{\tau} = 1_{\overline{\Im}}; A^{\zeta} \cap S^{\varsigma} = 0_{\breve{\amalg}} \Rightarrow f^1(A^{\zeta}) \cap f^1(S^{\varsigma}) = f^1(0_{\breve{\amalg}}) = 0_{\overline{\Im}} \Rightarrow \mathcal{U}^{\varsigma} \cap \cap S^{\varsigma} = 0_{\breve{\amalg}} \Rightarrow f^1(A^{\zeta}) \cap f^1(S^{\varsigma}) = f^1(0_{\breve{\amalg}}) = 0_{\overline{\Im}} \Rightarrow \mathcal{U}^{\varsigma} \cap$ □

Corollary 1. *Let* $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right), \left(\check{\Pi}, \hbar^{\daleth}{}_2\right)$ *be two FFTSs and let* $f : \overline{\Im} \to \check{\Pi}$ *be an FF continuous surjection. If* $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ *is FF* $\mathfrak{C}_5$*-connected, then so is* $\left(\check{\Pi}, \hbar^{\daleth}{}_2\right)$.

Definition 19. *An FFTS* $(\overline{\Im}, \hbar^{\daleth})$ *is said to be FF strongly connected, if there exists nonzero FFCSs* A^{ζ} *and* S^{ς} *such that* $\mu^{\eta}{}_{A^{\zeta}} + \mu^{\eta}{}_{S^{\varsigma}} \leq 1$ *and* $\vartheta_{A^{\zeta}} + \vartheta_{S^{\varsigma}} \geq 1$.

Proposition 7. *Let* $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right), \left(\check{\Pi}, \hbar^{\daleth}{}_2\right)$ *be two FFTSs and let* $f : \overline{\Im} \to \check{\Pi}$ *be an FF continuous surjection. If* $\left(\overline{\Im}, \hbar^{\daleth}{}_1\right)$ *is FF strongly connected, then so is* $\left(\check{\Pi}, \hbar^{\daleth}{}_2\right)$.

Proof. This is analogous to the proof of Proposition 6. It is clear that, in FFTSs, strong FF connectedness does not imply FF $\mathfrak{C}_5$-connectedness, and the same is true for its converse. □

5. Fermatean Fuzzy α-Continuity

Definition 20. *An FFS* $\chi^{\zeta} =< x, \mu^{\eta}{}_{\chi^{\zeta}}, \gamma_{\chi^{\zeta}} >$ *of an FFTS* $(\overline{\Im}, \hbar^{\daleth})$ *is called an FF* α *open set if* $\chi^{\zeta} \subseteq Int(Cl(Int(\chi^{\zeta})))$. *An FFS whose complement is an FF* α *open set (FFαOS) is called an FF* α *closed set (FFαCS).*

Proposition 8. *Let* $(\overline{\Im}, \hbar^{\daleth})$ *be an FFTS. Then, the arbitrary union of FFαOS is an FFαOS and an arbitrary intersection of FFαCSs is FFαCS.*

Proof. Let $\{\chi^{\zeta}{}_i = \langle x, \mu^{\eta}{}_{\chi^{\zeta}}, \gamma_{\chi^{\zeta}} > | i \in I\}$ be a family of FFαOSs. Then, for each $i \in I, \chi^{\zeta}{}_i \subseteq Int(Cl(\text{ Int }(\chi^{\zeta}{}_i)))$. Thus, $\cup\chi^{\zeta}{}_i \subseteq \cup Int((Cl(\text{ Int }(\chi^{\zeta}{}_i)))) \subseteq Int(\cup Cl(\text{ Int }(\chi^{\zeta}{}_i)))) \subseteq Int(Cl(\cup Int(\chi^{\zeta}{}_i)))) \subseteq Int(Cl(Int(\cup\chi^{\zeta}{}_i))))$
Hence, $\cup\chi^{\zeta}{}_i$ is an FFαOS set. If we take the complement of this part, the following will be proven (i.e., the arbitrary intersection of FFαOS is also an FFαOS).

Every FFOS is an FFαOS and every FFCS is an FFαCS but the converse is not true. □

Definition 21. *The FF* α *closure of an FFS* χ^{ζ} *in an FFTS* $(\overline{\Im}; \hbar^{\daleth})$ *represented as* $Cl_{\alpha}(\chi^{\zeta})$ *and defined by* $Cl_{\alpha}(\chi^{\zeta}) = \bigcap\{C_i | C_i \text{is an FF}\alpha C \text{ set and } \chi^{\zeta} \subseteq C_i\}$

Proposition 9. *In an FFTS* $(\overline{\Im}, \hbar^{\daleth})$, *an FFS* χ^{ζ} *is FFαC if and only if* $\chi^{\zeta} = Cl_{\alpha}(\chi^{\zeta})$.

Proof. Assume that χ^{ζ} is an $FF\alpha C$ set. Then,
$\chi^{\zeta} \in \{C_i | C_i$ is a $FF\alpha C$ set and $\chi^{\zeta} \subseteq C_i\}$, so $\chi^{\zeta} = \cap\{C_i | C_i$ is a $FF\alpha C$ and $\chi^{\zeta} \subseteq C_i\}$
$= Cl_{\alpha}(\chi^{\zeta})$.
Conversely, consider $\chi^{\zeta} = Cl_{\alpha}(\chi^{\zeta})$,
$\chi^{\zeta} \in \{C_i | C_i$ is a $FF\alpha C$ set and $\chi^{\zeta} \subseteq C_i\}$
Thus, χ^{ζ} is an FFα-closed set. □

Proposition 10. *In an FFTS* $(\overline{\Im}, \hbar^{\daleth})$, *the following hold for q-ROα-closure:*

(1) $Cl_{\alpha}(\underline{0}) = \underline{0}$;
(2) $Cl_{\alpha}(\chi^{\zeta})$ *is a q-ROαC in* $(\overline{\Im}, \hbar^{\daleth})$ *for every FFS* χ^{ζ} *in* $\overline{\Im}$;
(3) $Cl_{\alpha}(\chi^{\zeta}) \subseteq Cl_{\alpha}(R)$ *whenever* $\chi^{\zeta} \subseteq R$ *for every* χ^{ζ} *and* R *in* $\overline{\Im}$;
(4) $Cl_{\alpha}(Cl_{\alpha}(\chi^{\zeta})) = Cl_{\alpha}(\chi^{\zeta})$ *for every FFS* χ^{ζ} *in* $\overline{\Im}$.

Proof. (1) The proof is obvious;

(2) By preposition, χ^{ζ} is FFαC iff the $\chi^{\zeta} = Cl_{\alpha}(\chi^{\zeta})$ we obtain $Cl_{\alpha}(\chi^{\zeta})$ is an FFαC for every χ^{ζ} in $\overline{\Im}$.

(3) By the same preposition, we obtain $\chi^{\zeta} = Cl_{\alpha}(\chi^{\zeta})$ and $R = Cl_{\alpha}(R)$. whenever $\chi^{\zeta} \subseteq R$, we have $Cl_{\alpha}(\chi^{\zeta}) \subseteq Cl_{\alpha}(R)$.

(4) Let χ^{ζ} be an FFFS in $\overline{\Im}$. We know that $\chi^{\zeta} = Cl_{\alpha}(\chi^{\zeta})$, $Cl_{\alpha}(\chi^{\zeta}) = Cl_{\alpha}\left(Cl_{\alpha}(\chi^{\zeta})\right)$. Thus, $Cl_{\alpha}\left(Cl_{\alpha}(\chi^{\zeta})\right) = Cl_{\alpha}(\chi^{\zeta})$ for every χ^{ζ} in $\overline{\Im}$. $\square$

Definition 22. *Let* $\left(\overline{\Im}, \hbar^{\daleth}{}_{\overline{\Im}}\right)$ *and* $(\breve{\Pi}, \hbar^{\daleth}{}_{\breve{\Pi}})$ *be FFTSs. A mapping* $\sqsupset : \overline{\Im} \to \breve{\Pi}$ *is named FFα-continuous* $(FF\alpha CN)$ *if the inverse image of each FFOS of* $\breve{\Pi}$ *is an FFαO set in* $\overline{\Im}$.

Theorem 9. *Let* $\sqsupset : \left(\overline{\Im}, \hbar^{\daleth}{}_{\overline{\Im}}\right) \to (\breve{\Pi}, \hbar^{\daleth}{}_{\breve{\Pi}})$ *be a mapping from a* $FFTS\left(\overline{\Im}, \hbar^{\daleth}{}_{\overline{\Im}}\right)$ *to an* $FFTS(\breve{\Pi}, \hbar^{\daleth}{}_{\breve{\Pi}})$. *If* $\sqsupset$ *is FFα-continues, then:*
(1) $\sqsupset(Cl(Int(Cl(\chi^{\zeta})))) \subseteq Cl(\sqsupset(\chi^{\zeta}))$ *for all FFS* χ^{ζ} *in* $\overline{\Im}$.
(2) $Cl\left(Int\left(\sqsupset^{-1}(\tilde{F})\right)\right) \subseteq \sqsupset^{-1}(Cl(\tilde{F}))$ *for all* $\tilde{F}$ *in* $\breve{\Pi}$.

6. Fermatean Fuzzy Codas Approach

Combinative distance-based assessment (CODAS) is a method for evaluating the similarity between two sets of data. This method is particularly useful in applications where the datasets are complex and may not be easily compared using traditional methods such as Euclidean distance. One of the main advantages of CODAS is its ability to handle data with missing or incomplete values. This is important because, in many real-world applications, data are often missing or incomplete due to various reasons such as errors in data collection or missing information. CODAS can handle such data using a combination of different distance measures which can be customised based on the needs of the application. CODAS is also highly flexible and can be applied to a wide range of data types, including numerical, categorical, and ordinal data. This makes it an ideal method for applications where datasets may contain a mix of different data types.

The CODAS method is a useful tool for evaluating the similarity between two sets of data, particularly when the data are complex or contain missing or incomplete values. It is flexible and can be applied to a wide range of data types, making it a valuable tool for many different applications.

The fundamental task in general MCDM problems is that of selecting one or even more options from a set of available alternatives based on numerous criteria. The CODAS method is a comparatively recent MCDM method introduced by Ghorabaee et al. [46] in 2016. Ghorabaee et al. [47] also extended the CODAS approach to the fuzzy set. We extended the CODAS approach to FFSs with an application to supplier selection. To begin, in contrast to the vast majority of existing group decision-making techniques, which assume either a known reputation vector or equal expert weights, the experts' reputation is determined by their qualifications and experience. Second, the Fermatean fuzzy direct rating approach is used to establish the relative relevance of criteria based on the expert group's evaluation preferences. Thirdly, the Fermatean fuzzy CODAS approach is used to construct alternative orderings based on their assessment scores. Assume that there are n alternatives given as $\aleph^{\daleth} = \{\aleph_1^{\daleth}, \ldots, \aleph_i^{\daleth}, \ldots, \aleph_n^{\daleth}\}(n \geq 2)$ and $\aleph^{\urcorner} = \left\{\aleph_1^{\urcorner}, \ldots, \aleph_j^{\urcorner}, \ldots, \aleph_m^{\urcorner}\right\}(m \geq 2)$ is the finite set of m criteria. Suppose that $\wp^{\mho} = \{\wp_1^{\mho}, \ldots, \wp_e^{\mho}, \ldots, \wp_z^{\mho}\}(z \geq 2)$ constitute the assemblage of invited DMs. The FF-CODAS approach consists of the following steps.

Step 1: Determine the reputation of the experts:

$$\widehat{\Delta}_e = \operatorname{avg}\left(\Theta_e^{(1)}, \Theta_e^{(2)}\right) = \left(\frac{\mu^{\eta}{}_{\Theta_e^{(1)}} + \mu^{\eta}{}_{\Theta_e^{(2)}}}{2}, \frac{\nu^{\eta}{}_{\Theta_e^{(1)}} + \nu^{\eta}{}_{\Theta_e^{(2)}}}{2}\right), \quad e = 1, 2, \ldots, z \tag{1}$$

here, $\widehat{\Delta}_e$ indicates the FF average reputation of the invited DM $\wp_e^{\mho}$. $\Theta_e^{(1)}$ and $\Theta_e^{(2)}$ are FFNs that express the education and expertise of the invited DM $\wp_e^{\mho}$, respectively. Table 2 shows an FF linguistic scale that can be used to distinguish specialists based on their credentials and expertise.

Table 2. FF linguistic scale to distinct DMs

Qualifications	Experience (Years)	FFNs
F.Sc. (Higher Secondary)	$[6.5, 10)$	$(0.150, 0.900)$
B.Sc. (Graduate)	$[10, 15.5)$	$(0.250, 0.700)$
M.Sc. (Master's)	$[15.5, 20)$	$(0.500, 0.500)$
M.S./M.Phil. (Postgraduate)	$[20, 35)$	$(0.700, 0.250)$
Ph.D. (Doctorate)	≥ 35	$(0.900, 0.150)$

Step 2: Normalise the importance of the DMs:

$$\gimel_e = \frac{\text{score}^P\left(\widehat{\Delta}_e\right)}{\sum_{t=1}^{z} \text{score}^P\left(\widehat{\Delta}_t\right)} = \frac{1 + \mu\eta^3_{\widehat{\Delta}_e} - \nu\eta^3_{\widehat{\Delta}_e}}{\sum_{t=1}^{z}\left(1 + \mu\eta^3_{\widehat{\Delta}_t} - \nu\eta^3_{\widehat{\Delta}_t}\right)}, \quad e = 1, \ldots, z \tag{2}$$

here, $\gimel = (\gimel_1, \ldots, \gimel_e, \ldots, \gimel_z)^T$ is the importance vector of the DMs, with $\gimel_e \in [0, 1]$ and $\sum_{e=1}^{z} \gimel_e = 1$.

Step 3: Evaluate the criteria importance matrices $V^e = \left[V_j^e\right]_{m \times 1}$:

$$\aleph_1^{\urcorner}(\mu\eta^e_{V_1}, \nu\eta^e_{V_1})[t]\aleph_2^{\urcorner}(\mu\eta^e_{V_2}, \nu\eta^e_{V_2})\vdots\vdots\vdots\vdots\aleph_m^{\urcorner}(\mu\eta^e_{V_m}, \nu\eta^e_{V_m})[b]$$

where $V_j^e = \left(\mu\eta^e_{V_j}, \nu\eta^e_{V_j}\right)(j = 1, \ldots, m; e = 1, \ldots, z)$ is an FFN representing the important assessment of the criterion $\aleph_j^{\urcorner}$ provided by the DM $\wp_e^{\mho}$. It is defined by utilising an FF linguistic importance scale, which is shown in Table 3 and can be used to offer expert criteria importance preferences.

Table 3. FF linguistic scale to evaluate criteria importance.

Linguistic Term	FFN
Extremely unimportant (EU)	$(0.100, 0.975)$
Not important (NI)	$(0.200, 0.850)$
Slightly important (SI)	$(0.350, 0.700)$
Moderately important (MI)	$(0.550, 0.500)$
Important (I)	$(0.700, 0.350)$
Very important (VI)	$(0.850, 0.200)$
Extremely important (EI)	$(0.975, 0.100)$

Step 4: Compute the consolidated criterion significance matrix:

$$\begin{aligned}
\widehat{W} &= \left[\widehat{W}_j\right]_{m \times 1} : \\
\widehat{W}_j &= \left(\mu\eta_{\widehat{W}_j}, \nu\eta_{\widehat{W}_j}\right) \\
&= FFWG_{\gimel}\left(V_j^1, \ldots, V_j^e, \ldots, V_j^z\right) = \bigotimes_{e=1}^{z}\left(V_j^e\right)^{\gimel_e}
\end{aligned}$$

$$= \left(\prod_{e=1}^{z} \mu\eta^{\gimel_e}_{V_j^e}, \sqrt[3]{1 - \prod_{e=1}^{z}\left(1 - \nu\eta^3_{V_j^e}\right)^{\gimel_e}}\right), \quad j = 1, \ldots, m \tag{3}$$

where $\widehat{W}_j = \left(\mu\eta_{\widehat{W}_j}, \nu\eta_{\widehat{W}_j}\right)$ is the FF aggregated importance evaluation of the criterion $\aleph_j^{\urcorner}$ given by the DMs.

Step 5: Normalise the aggregated criteria importance:

$$\beth_j = \frac{\text{score}^P\left(\widehat{W}_j\right)}{\sum_{l=1}^{m} \text{score}^P\left(\widehat{W}_l\right)} = \frac{1 + \mu\eta^3_{\widehat{W}_j} - \nu\eta^3_{\widehat{W}_j}}{\sum_{l=1}^{m}\left(1 + \mu\eta^3_{\widehat{W}_l} - \nu\eta^3_{\widehat{W}_l}\right)}$$

where $\beth = (\beth_1, \ldots, \beth_j, \ldots, \beth_m)^T$ is the importance vector of the criteria, with $\beth_j \in [0,1] (j = 1, \ldots, n)$ and $\sum_{j=1}^{n} \beth_j = 1$.

Step 6: Obtain the decision matrices $\Gamma^e = \left[\Gamma^e_{ij}\right]_{n \times m}$:

where $\Gamma^e_{ij} = \left(\mu\eta_{\Gamma^e_{ij}}, \nu\eta_{\Gamma^e_{ij}}\right)(i = 1, \ldots, n; j = 1, \ldots, m; e = 1, \ldots, z\ )$ is an FFN that represents the assessment of the alternative $\aleph^{\daleth}_i$ with respect to the criterion $\aleph^{\urcorner}_j$ given by the invited expert $\wp^{\mho}_e$. It is defined using a Fermatean fuzzy linguistic assessment scale. The nine-point Fermatean fuzzy linguistic scale presented in Table 4 can be used to present the alternative assessment preferences of experts.

Table 4. FF linguistic scale to evaluate alternatives.

Linguistic Term	FFNs
Inadequate (I)	$(0.100, 0.975)$
Very poor (VP)	$(0.200, 0.900)$
Poor (P)	$(0.300, 0.800)$
Medium poor (MP)	$(0.400, 0.650)$
Medium (M)	$(0.550, 0.500)$
Medium good (MG)	$(0.650, 0.400)$
Good (G)	$(0.800, 0.300)$
Very good (VG)	$(0.900, 0.200)$
Exceptional (E)	$(0.975, 0.100)$

Step 7: Determine the aggregated decision matrix $G = \left[G_{ij}\right]_{n \times m}$:

$$\begin{aligned} G_{ij} &= \text{FFWG}_{\beth}\left(\Gamma^1_{ij}, \ldots, \widehat{\Gamma}^e_{ij}, \ldots, \Gamma^k_{ij}\right) \\ &= \left(\prod_{e=1}^{z} \mu\eta^{\beth_e}_{\Gamma^e_{ij}}, \sqrt[3]{1 - \prod_{e=1}^{z}\left(1 - \nu\eta_{\Gamma^e_{ij}}\right)^{\beth_e}}\right) \end{aligned} \tag{4}$$

where the aggregation is determined by applying the "Fermatean fuzzy weighted geometric (FFWG) operator" [48] and $G_{ij} = \left(\mu\eta_{G_{ij}}, \nu\eta_{G_{ij}}\right)$ is the Fermatean fuzzy aggregated assessment of the alternative $\aleph^{\daleth}_i$ with respect to the criterion $\aleph^{\urcorner}_j$ given by the experts.

Step 8: Determine the normalised decision matrix $\widehat{R} = \left[\widehat{R}_{ij}\right]_{n \times m}$:

$$\widehat{R}_{ij} = \begin{cases} (G_{ij})^c; & | \aleph^{\urcorner}_j \in C^- \\ G_{ij}; & | \aleph^{\urcorner}_j \in C^+. \end{cases} \tag{5}$$

where $\widehat{R}_{ij} = \left(\mu\eta_{\widehat{R}_{ij}}, \nu\eta_{\widehat{R}_{ij}}\right)$ denotes the FF normalised assessment of the alternative $\aleph^{\daleth}_i$ with respect to the criterion $\aleph^{\urcorner}_j$ given by the experts, $C^+ \subseteq C$ is the set of benefit criteria, $C^- \subseteq C$ is the set of cost criteria, and $C^+ \cup C^- = C$. Only alternative assessments with respect to cost criteria are transformed by utilising the complement operation.

Step 9: Determine the Fermatean fuzzy negative-ideal solution (FFNIS).

$$\begin{aligned}\widehat{S}_j^- &= \left(\mu^{\eta}{}_{\widehat{S}_j^-}, \nu^{\eta}{}_{\widehat{S}_j^-}\right) = \widehat{R_j^-} \mid \text{score}\left(\widehat{R_j^-}\right) \\ &= \min_{1 \le i \le n}\left[\text{score}\left(R_{ij}\right)\right], j = 1, \ldots, m\end{aligned}$$

$\widehat{S^-} = \left\{\widehat{S_1^-}, \ldots, \widehat{S_j^-}, \ldots, \widehat{S_m^-}\right\}$ is a collection of FFNs that represent the FFNIS and $\widehat{R}_j^-$ is an FFN with the lowest score function value of alternatives with respect to the criterion $\aleph_j^{\daleth}$.

Step 10: Calculate the weighted Euclidean distance (H_i) and weighted Hamming distance (E_i) of the alternatives from the FFNIS given in Equations (6) and (7), respectively.

$$H_i\left(\aleph_i^{\gimel}, \widehat{S}^-\right) = \frac{1}{2}\sum_{j=1}^{n}\omega_j\left(\left|\mu^{\eta}{}^3_{\widehat{R}_{ij}} - \mu^{\eta}{}^3_{\widehat{R}_j^-}\right| + \left|\nu^{\eta}{}^3_{\widehat{R}_{ij}} - \nu^{\eta}{}^3_{\widehat{S}_j^-}\right| + \left|\pi^3_{\widehat{R}_{ij}} - \pi^3_{\widehat{S}_j^-}\right|\right) \tag{6}$$

$$E_i\left(\aleph_i^{\gimel}, \widehat{S}^-\right) = \sqrt{\frac{1}{2}\sum_{j=1}^{n}\omega_j\left[\left(\mu^{\eta}{}^3_{\widehat{R}_{ij}} - \mu^{\eta}{}^3_{\widehat{R}_j^-}\right)^2 + \left(\nu^{\eta}{}^3_{\widehat{R}_{ij}} - \nu^{\eta}{}^3_{\widehat{S}_j^-}\right)^2 + \left(\pi^3_{\widehat{R}_{ij}} - \pi^3_{\widehat{S}_j^-}\right)^2\right]} \tag{7}$$

Step 11: Construct the relative assessment matrix $\text{P} = [\text{P}_{it}]_{n \times n}$:

$$\text{P}_{it} = E_i\left(\aleph_i^{\gimel}, \widehat{S}^-\right) - E_t\left(\aleph_t^{\gimel}, \widehat{S}^-\right) + \Phi_{it}\left(E_i\left(\aleph_i^{\gimel}, \widehat{S}^-\right) - E_t\left(\aleph_t^{\gimel}, \widehat{S}^-\right)\right) \cdot \left[H_i\left(\aleph_i^{\gimel}, \widehat{S}^-\right) - H_t\left(\aleph_t^{\gimel}, \widehat{S}^-\right)\right]$$

$i, t = 1, \ldots, n$, where Φ is a threshold function to recognise the equality of Euclidean distance measures of two alternatives. It is defined as follows

$$\Phi_{it}\left(E_i\left(\aleph_i^{\gimel}, \widehat{S}^-\right) - E_t\left(\aleph_t^{\gimel}, \widehat{S}^-\right)\right) = \begin{cases} 1 & \phi \le \left|E_i\left(\aleph_i^{\gimel}, \widehat{S}^-\right) - E_t\left(\aleph_t^{\gimel}, \widehat{S}^-\right)\right| \\ \\ 0 & \phi > \left|E_i\left(\aleph_i^{\gimel}, \widehat{S}^-\right) - E_t\left(\aleph_t^{\gimel}, \widehat{S}^-\right)\right| \end{cases} \tag{8}$$

where ϕ is the threshold parameter.

Step 12: Calculate the assessment scores and rank the alternatives:

$$R_i = \sum_{t=1}^{n}\text{P}_{it}, \quad i = 1, 2, \ldots, n \tag{9}$$

where R_i represents the assessment scores of the alternative $\aleph_i^{\gimel}$. The alternatives are ranked according to the decreasing values of their assessment score. The highest score is the most desirable alternative.

7. Case Study

Sustainable supplier selection (SSS) is the process of evaluating and choosing suppliers based on their ability to meet the needs of an organisation while also considering the environmental and social impact of their practices. This approach to sourcing is becoming increasingly important as organisations recognise the need to minimise their environmental footprint and promote ethical business practices.

There are several reasons for which SSS is important. Firstly, it helps organisations meet their sustainability goals and reduce their environmental impact. By choosing suppliers that are committed to sustainability, organisations can minimise the environmental impact of their supply chain and reduce their greenhouse gas emissions. This can help organisations meet their sustainability targets and reduce their carbon footprint. Secondly, SSS can help organisations reduce their risk. By choosing suppliers that are committed to sustainability, organisations can reduce the risk of supply chain disruptions caused by environmental disasters or social unrest. This is particularly important for organisations that rely on global supply chains, as these may be vulnerable to risks such as natural disasters or political instability. Thirdly, SSS can help organisations build and maintain a positive reputation. Consumers and other stakeholders are increasingly concerned about the environmental and social impact of the products and services they consume, and are more likely to choose companies that are transparent about their supply chain practices and that have a strong commitment to sustainability. By choosing sustainable suppliers, organisations can demonstrate their commitment to sustainability and build trust among their stakeholders.

There are several approaches that organisations can take to implementing SSS. One approach is to incorporate sustainability criteria into the supplier selection process. This can involve evaluating suppliers based on their environmental performance, labour practices, and social impact. Organisations can use tools such as sustainability assessment frameworks or rating systems to evaluate suppliers based on these criteria.

Another approach is to work with suppliers to improve their sustainability practices. This can involve setting sustainability targets for suppliers and providing them with support to meet these targets. Organisations can also work with suppliers to implement sustainability initiatives, such as reducing waste or increasing the use of renewable energy. SSS is often approached as an MCDM problem, as it involves the evaluation of multiple criteria such as cost, quality, delivery performance, and sustainability. MCDM methods are used to weigh these criteria and determine the most suitable supplier based on the organisation's specific needs and priorities.

SSS is an important aspect of corporate responsibility and can be approached as a MCDM problem. By choosing suppliers that are committed to sustainability and incorporating sustainability criteria into the supplier selection process, organisations can minimise their environmental impact, reduce risk, and build a positive reputation. Using MCDM methods, organisations can evaluate multiple criteria and choose suppliers that best meet their needs while also considering the environmental and social impacts of their practices.

Supplier selection is a critical process for firms to conduct in order to maintain a competitive edge and achieve their supply chain objectives. According to industry statistics, manufacturers spend up to 70% of their total product costs on products and services, while high-technology businesses spend up to 80% of their total product costs on goods and services. To properly manage this strategically critical purchasing function, it is crucial to choose the most appropriate strategy and parameters for the situation. In today's dynamic business environment, all the aspects of delivering goods must be considered, including reliability, versatility, and fast response, through the successful structure and implementation of the distribution chain. Vendor assessment is a critical part of the supply network, as it has an impact on the organisation's long objectives and productivity. Manufacturers have a variety of qualities and shortcomings that must be carefully evaluated by purchasers before they are ranked according to certain criteria. As a result, each choice must be integrated by weighing the performance of various suppliers at each supply chain level.

The problem becomes much more acute in manufacturing plants because considerable amounts of effort and money are spent on acquiring. Reliable vendors assist organisations in achieving the highest levels of their manufacturing strategy while also supplying practitioners with the greatest number of benefits. Supplier selection is regarded as a challenging task due to the high number of variables and the interactions among them. In general, the supply chain supplier selection problem is a group decision-making problem with a large number of criteria that must be met. Because group decision making involves human judgement, precise facts are insufficient to convey these judgments, which are based on human preferences. The more pragmatic approach is to make judgments based on language rather than numerical qualities. As a result, linguistic variables are utilised to assess the grades and weights given to the problem's criteria.

Supplier selection is an MCDM dilemma that is influenced by a number of competing considerations, including cost, reliability, and execution. Dickson conducted a study questionnaire-based distributed to 273 purchasing professionals, which resulted in the identification of 23 different regularly utilised criteria for the supplier selection problem. Dickson came to the conclusion that quality, delivery, and performance history are the most essential criteria out of the 23 elements considered [49]. Numerous strategies have been developed over the years to efficiently handle the challenge. In the literature, methods such as "analytic hierarchy process" (AHP), "analytic network process" (ANP), "linear programming" (LP), "mathematical programming", "multi-objective programming", "data envelopment analysis" (DEA), "neural networks" (NN), "case-based reasoning" (CBR), and "fuzzy set theory" (FST) have been used [50]. Additionally, the integration of many approaches has been developed by academics, and the integration capitalises on the strengths of each method while compensating for its flaws.

7.1. Criterion for SSS

There are several criteria that organisations can use to evaluate the sustainability of their suppliers. These criteria can be grouped into three main categories: environmental, social, and economic.

7.1.1. Environmental Criteria

- Carbon emissions: Organisations can evaluate suppliers based on their carbon emissions and the steps they are taking to reduce them. This can include evaluating the energy efficiency of their facilities, their use of renewable energy sources, and their transportation practices.
- Resource use: Organisations can evaluate suppliers based on their use of natural resources such as water and raw materials and their efforts to conserve these resources.
- Waste reduction: Organisations can evaluate suppliers based on their waste reduction efforts, including the recycling of materials and the implementation of zero waste initiatives.
- Environmental compliance: Organisations can evaluate suppliers based on their compliance with environmental regulations and their efforts to minimise the environmental impact of their operations.

7.1.2. Social Criteria

- Labour practices: Organisations can evaluate suppliers based on their treatment of employees, including their working conditions, wages, and benefits. This can also include evaluating the suppliers' policies on issues such as diversity, equity, and inclusion.
- Community involvement: Organisations can evaluate suppliers based on their involvement in and impact on the local community, including their charitable activities and efforts to address community needs.
- Human rights: Organisations can evaluate suppliers based on their respect for human rights and their efforts to prevent human rights abuses in their operations.

7.1.3. Economic Criteria

- Cost: Organisations can evaluate suppliers based on the cost of their products or services and the value they provide.
- Quality: Organisations can evaluate suppliers based on the quality of their products or services and their ability to meet the needs of the organisation.
- Delivery performance: Organisations can evaluate suppliers based on their ability to deliver products or services on time and in the required quantities.
- Innovation: Organisations can evaluate suppliers based on their ability to bring innovative products or services into the market and their willingness to collaborate on new product development.

There are several criteria that organisations can use to evaluate the sustainability of their suppliers. These criteria can be grouped into environmental, social, and economic categories and can be customised based on the specific needs and priorities of the organisation.

7.2. Decision-Making Application

A high-tech industrial business wishes to find an appropriate material supplier from whom to procure critical components for future products. Following preliminary screening, five candidates ($\aleph_1^{\daleth}$, $\aleph_2^{\daleth}$, $\aleph_3^{\daleth}$, $\aleph_4^{\daleth}$, and $\aleph_5^{\daleth}$) will be evaluated further. To pick the best acceptable supplier, a committee of four DMs, $\wp_1^{\mho}$, $\wp_2^{\mho}$, $\wp_3^{\mho}$, and $\wp_4^{\mho}$, has been constituted. Seven criteria are taken into account, as given in Table 5.

Table 5. Criterion for the assessment.

	Criteria
$\aleph_1^{\daleth}$	Quality
$\aleph_2^{\daleth}$	Technical support
$\aleph_3^{\daleth}$	Performance history
$\aleph_4^{\daleth}$	Cost
$\aleph_5^{\daleth}$	Reputation in industry
$\aleph_6^{\daleth}$	Risk factor
$\aleph_7^{\daleth}$	Professionalism

Step 1: Four DMs participated in the provided case study. The five-point FF linguistic scale was applied to various DMs given in Table 6. The table contains FFNs that denote the experts' credentials and expertise. Then, utilising Equation (1) and the related FFNs, a Fermatean fuzzy average distinction of an expert is calculated, as given in Table 7.

Table 6. Information about the DMs.

DMs	Qualifications	Experience (Years)	Gender
$\wp_1^{\mho}$	Ph.D.	15	Male
$\wp_2^{\mho}$	M.S.	19	Female
$\wp_3^{\mho}$	Ph.D.	23	Male
$\wp_4^{\mho}$	Ph.D.	35	Male

Table 7. Information about the DMs in terms of FFNs.

DMs	Qualifications	Experience (Years)	Average	Positive Score
$\wp_1^{\mho}$	$(0.900, 0.150)$	$(0.250, 0.700)$	$(0.575, 0.425)$	1.113
$\wp_2^{\mho}$	$(0.700, 0.250)$	$(0.500, 0.500)$	$(0.600, 0.375)$	1.163
$\wp_3^{\mho}$	$(0.900, 0.150)$	$(0.700, 0.250)$	$(0.800, 0.200)$	1.504
$\wp_4^{\mho}$	$(0.900, 0.150)$	$(0.900, 0.150)$	$(0.900, 0.150)$	1.726

Step 2: The FF average reputations of DMs are normalised using Equation 2. Because a DM cannot have a negative reputation value, the positive score algorithm is employed to obtain a crisp average result. The obtained reputation vector of the DMs is $\gimel = (0.2021, 0.2112, 0.2732, 0.3135)$.

Step 3: The seven-point FF linguistic scale shown in Table 3 is used to determine the relative relevance of criteria. DMs examine predefined factors that influence the supplier evaluation process. Table 8 contains the ratings of the criteria's linguistic significance. Table 9 has seven matrices of criterion importance, one for each DM. These are generated using the matching FF linguistic importance scale and linguistic evaluations gathered from the field.

Table 8. DM evaluations of the criteria using linguistic terms.

Criterion	DMs			
	$\wp_1^{\mho}$	$\wp_2^{\mho}$	$\wp_3^{\mho}$	$\wp_4^{\mho}$
$\aleph_1^{\daleth}$	EI	VI	MI	I
$\aleph_2^{\daleth}$	VI	I	VI	MI
$\aleph_3^{\daleth}$	I	MI	VI	I
$\aleph_4^{\daleth}$	VI	VI	MI	VI
$\aleph_5^{\daleth}$	MI	I	VI	MI
$\aleph_6^{\daleth}$	I	MI	EI	VI
$\aleph_7^{\daleth}$	EI	I	VI	MI

Table 9. DM evaluations of the criteria in terms of corresponding FFNs.

Criterion	DMs			
	$\wp_1^{\mho}$	$\wp_2^{\mho}$	$\wp_3^{\mho}$	$\wp_4^{\mho}$
$\aleph_1^{\daleth}$	$(0.975, 0.100)$	$(0.850, 0.200)$	$(0.550, 0.500)$	$(0.700, 0.350)$
$\aleph_2^{\daleth}$	$(0.850, 0.200)$	$(0.700, 0.350)$	$(0.850, 0.200)$	$(0.550, 0.500)$
$\aleph_3^{\daleth}$	$(0.700, 0.350)$	$(0.550, 0.500)$	$(0.850, 0.200)$	$(0.700, 0.350)$
$\aleph_4^{\daleth}$	$(0.850, 0.200)$	$(0.850, 0.200)$	$(0.550, 0.500)$	$(0.850, 0.200)$
$\aleph_5^{\daleth}$	$(0.550, 0.500)$	$(0.700, 0.350)$	$(0.850, 0.200)$	$(0.550, 0.500)$
$\aleph_6^{\daleth}$	$(0.700, 0.350)$	$(0.550, 0.500)$	$(0.975, 0.100)$	$(0.850, 0.200)$
$\aleph_7^{\daleth}$	$(0.975, 0.100)$	$(0.700, 0.350)$	$(0.850, 0.200)$	$(0.550, 0.500)$

Step 4: Equation (3) aggregates the FF significance ratings of the parameters by taking into account the DMs' repute vector. Table 10 contains the calculated value.

Table 10. Aggregated FFNs for the criteria.

Criterion	Importance		
	Aggregated FFNs	**Positive Score**	**Normalised**
$\aleph_1^{\daleth}$	$(0.730090, 0.370315)$	1.33838	0.1440
$\aleph_2^{\daleth}$	$(0.711773, 0.376696)$	1.30715	0.1407
$\aleph_3^{\daleth}$	$(0.701479, 0.372318)$	1.29357	0.1392
$\aleph_4^{\daleth}$	$(0.754690, 0.346039)$	1.38840	0.1494
$\aleph_5^{\daleth}$	$(0.651828, 0.425725)$	1.19979	0.2150
$\aleph_6^{\daleth}$	$(0.773977, 0.339312)$	1.42458	0.1533
$\aleph_7^{\daleth}$	$(0.731785, 0.373507)$	1.33977	0.1442

Step 5: In this step, we normalise the FF aggregated importance evaluations of the criteria. Because criteria cannot have a negative significance, the positive score function is used to calculate the crisp aggregated values. Normalised values are given in Table 10.

Step 6: The alternatives are evaluated using the nine-point FF linguistic scale listed in Table 4. Table 11 contains the linguistic assessments of the options in relation to four DMs' decision criteria. Table 12 contains the initial decision matrices. These were designed using the FF linguistic assessment scale as a guide.

Table 11. DM evaluations for the alternatives in linguistic terms.

Experts	Alternatives	Criterion						
		$\aleph_1^{\urcorner}$	$\aleph_2^{\urcorner}$	$\aleph_3^{\urcorner}$	$\aleph_4^{\urcorner}$	$\aleph_5^{\urcorner}$	$\aleph_6^{\urcorner}$	$\aleph_7^{\urcorner}$
$\wp_1^{\mho}$	$\aleph_1^{\lrcorner}$	VG	G	MG	G	VG	VG	G
	$\aleph_2^{\lrcorner}$	MP	M	G	MG	VG	P	G
	$\aleph_3^{\lrcorner}$	MG	I	G	MP	M	VG	P
	$\aleph_4^{\lrcorner}$	G	I	MP	VG	M	G	MP
	$\aleph_5^{\lrcorner}$	P	MP	G	VG	M	VG	MP
$\wp_2^{\mho}$	$\aleph_1^{\lrcorner}$	VG	VG	G	MG	VG	G	MG
	$\aleph_2^{\lrcorner}$	M	G	MP	VG	M	P	G
	$\aleph_3^{\lrcorner}$	MP	I	VG	MP	M	P	VG
	$\aleph_4^{\lrcorner}$	M	MP	VP	G	M	VG	G
	$\aleph_5^{\lrcorner}$	G	M	MP	P	MP	P	VG
$\wp_3^{\mho}$	$\aleph_1^{\lrcorner}$	VG	G	VG	G	G	MG	G
	$\aleph_2^{\lrcorner}$	MP	M	VG	P	MP	MP	MG
	$\aleph_3^{\lrcorner}$	G	VG	MP	P	VG	VG	MG
	$\aleph_4^{\lrcorner}$	M	I	G	MP	VP	G	VG
	$\aleph_5^{\lrcorner}$	M	MP	P	VP	VG	I	G
$\wp_4^{\mho}$	$\aleph_1^{\lrcorner}$	VG	VG	G	MG	VG	G	MG
	$\aleph_2^{\lrcorner}$	M	G	MP	VG	M	P	G
	$\aleph_3^{\lrcorner}$	MP	I	VG	MP	M	P	VG
	$\aleph_4^{\lrcorner}$	M	MP	VP	G	M	VG	G
	$\aleph_5^{\lrcorner}$	G	M	MP	P	MP	P	VG

Table 12. DM evaluations for the alternative terms of the corresponding FFNs.

Experts	Alternatives	Criterion						
		$\aleph_1^{\urcorner}$	$\aleph_2^{\urcorner}$	$\aleph_3^{\urcorner}$	$\aleph_4^{\urcorner}$	$\aleph_5^{\urcorner}$	$\aleph_6^{\urcorner}$	$\aleph_7^{\urcorner}$
$\wp_1^{\mho}$	$\aleph_1^{\lrcorner}$	(0.900, 0.200)	(0.800, 0.300)	(0.650, 0.400)	(0.800, 0.300)	(0.900, 0.200)	(0.900, 0.200)	(0.800, 0.300)
	$\aleph_2^{\lrcorner}$	(0.400, 0.650)	(0.550, 0.500)	(0.800, 0.300)	(0.650, 0.400)	(0.900, 0.200)	(0.300, 0.800)	(0.800, 0.300)
	$\aleph_3^{\lrcorner}$	(0.650, 0.400)	(0.100, 0.975)	(0.800, 0.300)	(0.400, 0.650)	(0.550, 0.500)	(0.900, 0.200)	(0.300, 0.800)
	$\aleph_4^{\lrcorner}$	(0.800, 0.300)	(0.100, 0.975)	(0.400, 0.650)	(0.900, 0.200)	(0.550, 0.500)	(0.800, 0.300)	(0.400, 0.650)
	$\aleph_5^{\lrcorner}$	(0.300, 0.800)	(0.400, 0.650)	(0.800, 0.300)	(0.900, 0.200)	(0.550, 0.500)	(0.900, 0.200)	(0.400, 0.650)
$\wp_2^{\mho}$	$\aleph_1^{\lrcorner}$	(0.900, 0.200)	(0.900, 0.200)	(0.800, 0.300)	(0.650, 0.400)	(0.900, 0.200)	(0.800, 0.300)	(0.650, 0.400)
	$\aleph_2^{\lrcorner}$	(0.550, 0.500)	(0.800, 0.300)	(0.400, 0.650)	(0.900, 0.200)	(0.550, 0.500)	(0.300, 0.800)	(0.800, 0.300)
	$\aleph_3^{\lrcorner}$	(0.400, 0.650)	(0.100, 0.975)	(0.900, 0.200)	(0.400, 0.650)	(0.550, 0.500)	(0.300, 0.800)	(0.900, 0.200)
	$\aleph_4^{\lrcorner}$	(0.550, 0.500)	(0.400, 0.650)	(0.200, 0.900)	(0.800, 0.300)	(0.550, 0.500)	(0.900, 0.200)	(0.800, 0.300)
	$\aleph_5^{\lrcorner}$	(0.800, 0.300)	(0.550, 0.500)	(0.400, 0.650)	(0.300, 0.800)	(0.400, 0.650)	(0.300, 0.800)	(0.900, 0.200)
$\wp_3^{\mho}$	$\aleph_1^{\lrcorner}$	(0.900, 0.200)	(0.800, 0.300)	(0.900, 0.200)	(0.800, 0.300)	(0.800, 0.300)	(0.650, 0.400)	(0.800, 0.300)
	$\aleph_2^{\lrcorner}$	(0.400, 0.650)	(0.550, 0.500)	(0.900, 0.200)	(0.300, 0.800)	(0.400, 0.650)	(0.400, 0.650)	(0.650, 0.400)
	$\aleph_3^{\lrcorner}$	(0.800, 0.300)	(0.900, 0.200)	(0.400, 0.650)	(0.300, 0.800)	(0.900, 0.200)	(0.900, 0.200)	(0.650, 0.400)
	$\aleph_4^{\lrcorner}$	(0.550, 0.500)	(0.100, 0.975)	(0.800, 0.300)	(0.400, 0.650)	(0.200, 0.900)	(0.800, 0.300)	(0.900, 0.200)
	$\aleph_5^{\lrcorner}$	(0.550, 0.500)	(0.400, 0.650)	(0.300, 0.800)	(0.200, 0.900)	(0.900, 0.200)	(0.100, 0.975)	(0.800, 0.300)
$\wp_4^{\mho}$	$\aleph_1^{\lrcorner}$	(0.900, 0.200)	(0.900, 0.200)	(0.800, 0.300)	(0.650, 0.400)	(0.900, 0.200)	(0.800, 0.300)	(0.650, 0.400)
	$\aleph_2^{\lrcorner}$	(0.550, 0.500)	(0.800, 0.300)	(0.400, 0.650)	(0.900, 0.200)	(0.550, 0.500)	(0.300, 0.800)	(0.800, 0.300)
	$\aleph_3^{\lrcorner}$	(0.400, 0.650)	(0.100, 0.975)	(0.900, 0.200)	(0.400, 0.650)	(0.550, 0.500)	(0.300, 0.800)	(0.900, 0.200)
	$\aleph_4^{\lrcorner}$	(0.550, 0.500)	(0.400, 0.650)	(0.200, 0.900)	(0.800, 0.300)	(0.550, 0.500)	(0.900, 0.200)	(0.800, 0.300)
	$\aleph_5^{\lrcorner}$	(0.800, 0.300)	(0.550, 0.500)	(0.400, 0.650)	(0.300, 0.800)	(0.400, 0.650)	(0.300, 0.800)	(0.900, 0.200)

Step 7: Four decision matrices are aggregated using the FFWG operator specified in Equation (4), which take the DMs' reputational vector into consideration. Table 13 contains the derived FF aggregated assessments of the alternatives in relation to the criteria specified by four DMs.

Table 13. FF aggregated assessments of the alternatives.

Criterion	Alternatives				
	$\aleph_1^\daleth$	$\aleph_2^\daleth$	$\aleph_3^\daleth$	$\aleph_4^\daleth$	$\aleph_5^\daleth$
$\aleph_1^\urcorner$	(0.900, 0.200)	(0.473, 0.584)	(0.533, 0.557)	(0.593, 0.473)	(0.592, 0.562)
$\aleph_2^\urcorner$	(0.851, 0.258)	(0.669, 0.421)	(0.182, 0.948)	(0.207, 0.911)	(0.473, 0.584)
$\aleph_3^\urcorner$	(0.792, 0.309)	(0.574, 0.545)	(0.704, 0.459)	(0.336, 0.809)	(0.425, 0.676)
$\aleph_4^\urcorner$	(0.717, 0.360)	(0.624, 0.577)	(0.369, 0.693)	(0.677, 0.462)	(0.335, 0.804)
$\aleph_5^\urcorner$	(0.872, 0.236)	(0.557, 0.528)	(0.629, 0.455)	(0.417, 0.714)	(0.532, 0.563)
$\aleph_6^\urcorner$	(0.774, 0.322)	(0.325, 0.769)	(0.506, 0.681)	(0.851, 0.258)	(0.277, 0.872)
$\aleph_7^\urcorner$	(0.717, 0.359)	(0.755, 0.333)	(0.659, 0.536)	(0.718, 0.428)	(0.740, 0.419)

Step 8: Table 14 contains the normalised decision matrix. Equation (5) is used to determine it based on the aggregated decision matrix. The complement operation is used solely for the cost type attributes. Here, $\aleph_4^\urcorner$ and $\aleph_6^\urcorner$ are the cost-type attributes.

Table 14. Normalised assessment matrix.

Criterion	Alternatives				
	$\aleph_1^\daleth$	$\aleph_2^\daleth$	$\aleph_3^\daleth$	$\aleph_4^\daleth$	$\aleph_5^\daleth$
$\aleph_1^\urcorner$	(0.900, 0.200)	(0.473, 0.584)	(0.533, 0.557)	(0.593, 0.473)	(0.592, 0.562)
$\aleph_2^\urcorner$	(0.851, 0.258)	(0.669, 0.421)	(0.182, 0.948)	(0.207, 0.911)	(0.473, 0.584)
$\aleph_3^\urcorner$	(0.792, 0.309)	(0.574, 0.545)	(0.704, 0.459)	(0.336, 0.809)	(0.425, 0.676)
$\aleph_4^\urcorner$	(0.360, 0.717)	(0.577, 0.624)	(0.693, 0.369)	(0.462, 0.677)	(0.804, 0.335)
$\aleph_5^\urcorner$	(0.872, 0.236)	(0.557, 0.528)	(0.629, 0.455)	(0.417, 0.714)	(0.532, 0.563)
$\aleph_6^\urcorner$	(0.322, 0.774)	(0.769, 0.325)	(0.681, 0.506)	(0.258, 0.851)	(0.872, 0.277)
$\aleph_7^\urcorner$	(0.717, 0.359)	(0.755, 0.333)	(0.659, 0.536)	(0.718, 0.428)	(0.740, 0.419)

Step 9: To begin, the values of the FF normalised assessments' score functions are determined using the formulation of the FFNs' score function. Then, the FFNIS is calculated and provided as

$$\Big\{(0.473, 0.584), (0.182, 0.948), (0.336, 0.809), (0.462, 0.677), (0.417, 0.714), (0.258, 0.851), (0.659, 0.536)\Big\}.$$

Step 10: Evaluate the weighted Euclidean distances and weighted Hamming distances using Equations (6) and (7), given in Table 15.

Table 15. Weighted Euclidean distances and weighted Hamming distances.

Distance Measure	Alternatives				
	$\aleph_1^\daleth$	$\aleph_2^\daleth$	$\aleph_3^\daleth$	$\aleph_4^\daleth$	$\aleph_5^\daleth$
Weighted Euclidean	0.372646	0.266123	0.211559	0.0420982	0.278918
Weighted Hamming	0.599526	0.478071	0.364525	0.1012120	0.493522

Step 11: Construct the relative assessment matrix, which is given in Table 16. In the base case scenario, the threshold parameter $\phi > 0$ is set to 0.40.

Table 16. Relative assessment matrix.

Alternatives	$\aleph_1^{\daleth}$	$\aleph_2^{\daleth}$	$\aleph_3^{\daleth}$	$\aleph_4^{\daleth}$	$\aleph_5^{\daleth}$
$\aleph_1^{\daleth}$	0	0.106523	0.161087	0.330548	0.093728
$\aleph_2^{\daleth}$	0.106523	0	0.054564	0.224025	−0.012795
$\aleph_3^{\daleth}$	−0.161087	−0.054564	0	0.169461	−0.067359
$\aleph_4^{\daleth}$	−0.330548	−0.224025	−0.169461	0	−0.236820
$\aleph_5^{\daleth}$	−0.093728	0.012795	0.067359	-0.236820	0

Step 12: Calculate assessment scores and rank the alternatives using Equation (9) given in Table 17.

Table 17. Assessment scores and final ranking.

Alternatives	Assessment Score	Rank
$\aleph_1^{\daleth}$	0.691886	1
$\aleph_2^{\daleth}$	0.372317	2
$\aleph_3^{\daleth}$	−0.113549	4
$\aleph_4^{\daleth}$	−0.960854	5
$\aleph_5^{\daleth}$	0.250394	3

7.3. Comparison Analysis

We compare our findings to existing models to confirm their veracity and validity, as shown in Table 18.

Table 18. Comparison analysis.

Methods	Authors	Top Ranking
TOPSIS method	Aydemir and Gunduz [51]	$\aleph_1^{\daleth}$
VIKOR method	Gül [52]	$\aleph_1^{\daleth}$
CODAS method	Simic et. al. [53]	$\aleph_1^{\daleth}$

8. Conclusions

Numerous practitioners and academics have emphasised the benefits of supply chain management. Many organisations have learned that a well-managed supply chain system is a vital instrument for strengthening their competitive edge. Under these circumstances, establishing strong and long-term connections between customers and suppliers is important to the supply chain system's performance. As a result, the issue of supplier selection becomes the key concern when developing an effective supply chain system. We invented the Fermatean fuzzy "Combinative Distance-based Assessment" CODAS approach for the selection of optimum provider. To begin, in contrast to the overwhelming majority of existing group decision-making systems, which assume either a known reputation vector or equal expert weights, experts' reputation is established by their qualifications and experience. Second, the Fermatean fuzzy direct rating approach is utilised to establish the relative significance of criteria based on the expert group's evaluation preferences. Thirdly, the Fermatean fuzzy CODAS technique is utilised to generate alternative orderings based on their assessment scores. Topological data analysis (TDA) techniques are quickly gaining traction as methods for managing massive amounts of data. Moreover, numerous concepts relevant to FFTSs are explored in this paper. Examples are supplied for the FF interior, FF closure, and FF border of any FFS. The notions of FF base, FF subbase, FF continuous mapping, FF homeomorphism, FF open function, and FF closed function are also introduced, along with several crucial proofs. Furthermore, the novel idea of "Fermatean fuzzy α-continuous mapping" between FFTSs and "Fermatean fuzzy connectedness" is introduced and studied.

Author Contributions: M.B.: Methodology, Formal analysis, Writing—review & editing. H.M.A.F.: Methodology, Formal analysis, Writing—review & editing. M.R.: Investigation, Methodology, Supervision. N.J.: Methodology, Investigation. All authors made a significant scientific contribution to the research in the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data used to support the findings of the study are included within the article.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University, Abha, Saudi Arabia, for funding this work through a research group program under grant number R.G.P. 2/217/43.

Conflicts of Interest: The authors declare that they have no conflict of interest.

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