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Proving Rho Meson is a Dynamical Gauge Boson of Hidden Local Symmetry

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Abstract: The rho meson has long been successfully identified with a dynamical gauge boson of Hidden Local Symmetry (HLS) $H_{\rm local}$ in the non-linear sigma model G/H gauge equivalent to the model having the symmetry $G_{\rm global} \times H_{\rm local}$, with $G = [SU(2)_L \times SU(2)_R] \simeq O(4)$, $H = SU(2)_V \simeq O(3)$. However, under a hitherto unproven assumption that its kinetic term is dynamically generated, together with an ad hoc choice of the auxiliary field parameter "$a = 2$", we prove this assumption, thereby solving the long-standing mystery. The rho meson kinetic term is generated simply by the large N limit of the Grassmannian model $G/H = O(N)/[O(N-3) \times O(3)]$ gauge equivalent to $O(N)_{\rm global} \times [O(N-3) \times O(3)]_{\rm local}$, extrapolated to $N = 4$, $O(4)_{\rm global} \times O(3)_{\rm local}$, with all the phenomenologically successful "$a = 2$ results", i.e., ρ-universality, KSRF relation, and the Vector Meson Dominance, realized *independently of the parameter "a"*. This in turn establishes validity of the large N dynamics at the quantitative level directly by the experiments. The relevant cutoff reads $\Lambda \simeq 4\pi F_\pi$ for $N = 4$, which is regarded as a matching scale of the HLS as a "magnetic dual" to QCD. Skyrmion is stabilized by such a dynamically generated rho meson without recourse to the underlying QCD, a further signal of the duality. The unbroken phase with a massless rho meson may be realized as a novel chiral-restored hadronic phase in the hot/dense QCD.

Keywords: hidden local symmetry; rho meson; large N limit; dynamical generation; grassmannian manifold; kawarabayashi–suzuki–ryazuddin–fayyazuddin relation; vector meson dominance; rho meson universality; skyrmion; second order phase transition

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1. Introduction

Since its proposal [1,2] (for reviews see [3–5]), identifying the rho meson as a dynamical gauge boson of Hidden Local Symmetry (HLS) $H_{\rm local}$ has been widely accepted in the model as having the symmetry $G_{\rm global} \times H_{\rm local}$, with $G = [SU(2)_L \times SU(2)_R] \simeq O(4)$ and $H = SU(2)_V \simeq O(3)$, where its Lagrangian consists of two independent invariants $\mathcal{L}_{\rm HLS} = \mathcal{L}_A + a\mathcal{L}_V$, with a arbitrary parameter. This gauge is equivalent to the non-linear sigma model, $\mathcal{L}_{\rm CCWZ}$, *à* la Callan–Coleman–Wess–Zumino (CCWZ) [6,7] based on the manifold G/H. In the absence of the kinetic term of the HLS gauge boson, it is merely an auxiliary field such that $\mathcal{L}_V = 0$, and $\mathcal{L}_{\rm HLS} = \mathcal{L}_A = \mathcal{L}_{\rm CCWZ}$ after gauge fixing.

Once we *assume*, however, that its kinetic term, $\mathcal{L}_{\rm kinetic}$, is generated at the quantum level by the dynamics of the non-linear sigma model itself, thereby put by hand to the Lagrangian, $\mathcal{L}_{\rm HLS} \Rightarrow \mathcal{L}_A + a\mathcal{L}_V + \mathcal{L}_{\rm kinetic}$, novel physics come out [1–5]: All the successful phenomenological results, such as the universality of the rho meson coupling (ρ universality), the Kawarabayashi–Suzuki–Ryazuddin–Fayyazuddin (KSRF) relation, and the Vector Meson Dominance (VMD), are derived for a *particular parameter choice* $a = 2$ in the resultant Lagrangian (*at tree level*), in such a way that $a\mathcal{L}_V$ becomes the HLS gauge-invariant mass terms of the ρ meson, which contains ρ mass, ρ couplings, additional π self-couplings, etc. For application of the HLS to the nuclear physics, see [8].

For all the phenomenological success of the HLS model of the rho meson, however, the basic assumption of the dynamical origin of the kinetic term and the particular parameter choice $a = 2$ has never been proved within the dynamics of the HLS model itself.

Alternatively, we may simply assume that the kinetic term is already generated by the underlying theory, QCD in the case at hand, regarding the HLS model as a low-energy effective theory, including loop corrections in the sense of the derivative expansion [4]. In this case, however, the parameter a (the renormalized one) is a completely free parameter to be adjusted to $a = 2$ by hand within the HLS model framework. It can only be determined with additional information, by "Wilsonian matching" with the QCD parameters at UV scale Λ as the input, $a(\Lambda^2) \sim 1$, which then predicts $a(\mu^2 = M_\pi^2 = 0) = 2$ as the infrared value (massless π on-shell) through the (one-loop) renormalization group for $F_\pi^2(\mu^2 = 0)$ due to the π loop alone in the HLS model itself (see [4] and references cited therein).

In this paper, we resolve this long-standing mystery as simply a *consequence of the non-perturbative dynamics of the large N limit of the non-linear sigma model* based on the Grassmannian manifold $G/H = O(N)/[O(N-p) \times O(p)$, with $p = 3 =$ fixed, which is reduced to the relevant case $G/H = O(4)/O(3) \simeq SU(2)_L \times SU(2)_R/SU(2)_V$ for the extrapolation $N \to 4$, with the rho meson being the dynamical gauge boson of $O(3)_{\rm local} \simeq [SU(2)_V]_{\rm local}$. Preliminary results of this paper were given as the supplementary ones in [9], which is mainly addressed to a subject on a possible dynamical gauge boson of HLS within the Higgs sector of the Standard Model, different from the present one, that of the QCD, but the details of the relevant calculations in the present paper may be found in [9].

It is in fact well known that the HLS gauge bosons in many non-linear sigma models, such as the CP^{N-1} model with $G/H = U(N)/[U(N-1) \times U(1)] \simeq SU(N)/[SU(N-1) \times U(1)]$ gauge equivalent to the model $SU(N)_{\rm global} \times U(1)_{\rm local}$, do acquire a kinetic term for the $U(1)_{\rm local}$ gauge boson at the quantum level in the large N limit [3,4,10–18]. It was further shown (in a context irrelevant to rho meson physics, however) that the HLS gauge bosons $O(p)_{\rm local}$ and $U(p)_{\rm local}$ in the Grassmannian models $G/H = O(N)/[O(N-p) \times O(p)$ [19] and $G/H = U(N)/[U(N-p) \times U(p)]$, respectively [19,20], are dynamically generated in the large N limit. However, it was shown only in a specific parameterization "covariant derivative" type which is just *a particular* $a = 2$ *choice* from the onset (see [9] and Equation (10)).

Here, we show *not only the generation of the* $O(p)_{\rm local}$ *gauge boson* ρ_μ *but also that all the successful "*$a = 2$ *results" are direct consequences of the pure dynamics at the quantum level of the large N limit* for the *arbitrary value* of a, thereby resolving the long-standing mystery of the rho meson simply on the firm dynamical base. This in turn provides yet another experimental verification of the large N reliability, this time even quantitatively, not just qualitatively. (It is known that the large N results remain qualitatively true even for the smallest value $N = 2$ in the CP^{N-1} model, as checked by the equivalent $O(3)$ model exactly solvable in two dimensions [14]).

2. Grassmaniann N Extension

Let us define the generic HLS base [2,3] of an $N \times N$ real matrix field $\xi(x) = \xi(\check{\rho}^{(p)}) \cdot \xi(\check{\rho}^{(N-p)}) \cdot \xi(\pi)$, which transforms under $G_{\rm global} \times H_{\rm local} = O(N)_{\rm global} \times [O(N-p) \times O(p)]_{\rm local}$ as $\xi(x) \to h(x) \cdot \xi(x) \cdot g^{-1}$ with $h(x) \in [O(N-p) \times O(p)]_{\rm local}$, $g \in O(N)_{\rm global}$, where $\xi(\pi) = e^{i\pi_a(x)X_a/f_\pi}$ is the CCWZ base for $G/H = O(N)/[O(N-p) \times O(p)]$, with f_π being the (*bare/tree-level*) decay constant of the NG boson π, while $\xi(\check{\rho}^{(p)}) = e^{i\check{\rho}^{(p)}(x)/f_\rho^{(p)}}$ and $\xi(\check{\rho}^{(N-p)}) = e^{i\check{\rho}^{(N-p)}(x)/f_\rho^{(N-p)}}$, with $\check{\rho}^{(p)} = \check{\rho}_a^{(p)} S_a^{(p)}$ and $\check{\rho}^{(N-p)} = \check{\rho}_a^{(N-p)} S_a^{(N-p)}$ being the would-be NG bosons to be absorbed into the HLS gauge bosons $\rho_\mu^{(p)}$ and $\rho_\mu^{(N-p)}$ of $O(p)_{\rm local}$ and $O(N-p)_{\rm local}$, with the (*bare/tree-leve*) decay constant $f_\rho^{(p)}$ and $f_\rho^{(N-p)}$, respectively:

$$\xi(x) = \xi(\pi) \quad \left(\text{unitary gauge, } \check{\rho}^{(N-p)}(x) = \check{\rho}^{(p)}(x) = 0\right). \tag{1}$$

Here, the generators read $X_a \in \mathcal{G} - \mathcal{H}$, $S_a^{(p)} \in \mathcal{H}^{(p)} = \mathcal{O}(p)$, $S_a^{(N-p)} \in \mathcal{H}^{N-p} = \mathcal{O}(N-p)$, with $\text{tr}(T_a T_b) = 2\delta_{ab}$, $\text{tr}(S_a X_b) = 0$, $T_a = \{S_a, X_a\} = -T_a^t$.

To study the large N limit in the Grassmannian models, including the CP^{N-1} model, it is customary to parameterize the HLS base as ($p \times N$ degrees of freedom of $\phi_{i\beta}$ consist of $p \times (N-p)$ of π, $p \times (p-1)/2$ of $\check{\rho}$, and $p \times (p+1)/2$ of the constraints):

$$\begin{aligned}
\xi(x)_{\alpha\beta} &= \frac{G}{N}\begin{pmatrix} \phi_{i,\beta}(x) \\ \Phi_{k,\beta}(x) \end{pmatrix}, \alpha=(i,k),\ \beta=(j,l),\ i,j=1,\cdots,p\,;k,l=p+1,\cdots N\,, \quad (2)\\
\xi^t\cdot\xi &= \frac{G}{N}\left(\phi^t\phi+\Phi^t\Phi\right)=\mathbb{1}\,,\quad \frac{G}{N}\equiv\frac{1}{f_\pi{}^2}\,,\\
\xi\cdot\xi^t &= \frac{G}{N}\begin{pmatrix} \phi\phi^t & \phi\Phi^t \\ \Phi\phi^t & \Phi\Phi^t \end{pmatrix}=\begin{pmatrix} \mathbb{1}_{p\times p} & 0 \\ 0 & \mathbb{1}_{(N-p)\times(N-p)} \end{pmatrix}=\mathbb{1}, \quad (3)
\end{aligned}$$

where $G\equiv N/f_\pi{}^2$ is the (bare) coupling constant to be fixed in the large N limit (s.t. $f_\pi{}^2=\mathcal{O}(N)$).

The covariantized Maurer–Cartan one-form reads:

$$\hat{\alpha}_\mu \equiv \frac{1}{i}D_\mu\xi\cdot\xi^t=\frac{G}{iN}\begin{pmatrix} \partial_\mu\phi-i\rho_\mu^{(p)}\phi \\ \partial_\mu\Phi-i\rho_\mu^{(N-p)}\Phi \end{pmatrix}\cdot\left(\phi^t\ \Phi^t\right)=\hat{\alpha}_{\mu,\perp}+\hat{\alpha}_{\mu,||}, \quad (4)$$

where $\hat{\alpha}_{\mu,\perp}\equiv\frac{1}{2}\mathrm{tr}\left(\hat{\alpha}_\mu X^a\right)X^a$, $\hat{\alpha}_{\mu,||}\equiv\frac{1}{2}\mathrm{tr}\left(\hat{\alpha}_\mu S^a\right)S^a$ are

$$\begin{aligned}
\hat{\alpha}_{\mu,\perp} &= \alpha_{\mu,\perp}=\begin{pmatrix} 0 & \frac{G}{iN}\partial_\mu\phi\cdot\Phi^t \\ \frac{G}{iN}\partial_\mu\Phi\cdot\phi^t & 0 \end{pmatrix},\\
\hat{\alpha}_{\mu,||} &= \begin{pmatrix} \frac{G}{iN}\partial_\mu\phi\cdot\phi^t-\rho_\mu^{(p)} & 0 \\ 0 & \frac{G}{iN}\partial_\mu\Phi\cdot\Phi^t-\rho_\mu^{(N-p)} \end{pmatrix},
\end{aligned}$$

all transforming homogeneously as

$$\left(\hat{\alpha}_{\mu,\perp},\hat{\alpha}_{\mu,||}\right) \rightarrow h(x)\cdot\left(\hat{\alpha}_{\mu,\perp},\hat{\alpha}_{\mu,||}\right)\cdot h^{-1}(x)\,, \quad (5)$$

with $h(x)\in\mathcal{H}$ for $H=[O(N-p)\times O(p)]_{\mathrm{local}}$.

Thus, the HLS Lagrangian consists of three independent invariants at the lowest derivative [9]:

$$\mathcal{L}^{(N,p)} = \mathcal{L}_A+a^{(p)}\mathcal{L}_V^{(p)}+a^{(N-p)}\mathcal{L}_V^{(N-p)}\,, \quad (6)$$

where

$$\begin{aligned}
\mathcal{L}_A &= \frac{f_\pi{}^2}{4}\mathrm{tr}\left(\hat{\alpha}_{\mu,\perp}^2\right)=-\frac{G}{2N}\mathrm{tr}\left(\phi^t\partial_\mu\phi\cdot\Phi^t\partial^\mu\Phi\right)\\
&= \frac{1}{2}\mathrm{tr}\left(\partial_\mu\phi\partial^\mu\phi^t+\frac{G}{N}\left(\phi\partial_\mu\phi^t\right)^2\right)\\
(\text{unitary gauge}) &\longrightarrow \mathcal{L}_{\mathrm{CCWZ}}=\frac{f_\pi{}^2}{4}\mathrm{tr}\left(\alpha_{\mu,\perp}^2(\pi)\right)^2=\frac{1}{2}\left(\partial_\mu\pi_a\right)^2+\cdots\,, \quad (7)
\end{aligned}$$

with (unitary-gauge) $\alpha_{\mu,\perp}\to\alpha_{\mu,\perp}(\pi)=\partial_\mu\xi(\pi)\cdot\xi^t(\pi)=\partial_\mu\xi(\pi)\cdot\xi^\dagger(\pi)$, and

$$\begin{aligned}
a^{(p)}\mathcal{L}_V^{(p)} &= \frac{a^{(p)}f_\pi{}^2}{4}\mathrm{tr}\left(\left[\hat{\alpha}_{\mu,||}^{(p)}\right]^2\right)=\frac{1}{2}\mathrm{tr}\left[\frac{a^{(p)}}{2}\cdot\frac{N}{G}\left(\rho_\mu^{(p)}-i\frac{G}{N}\phi\partial_\mu\phi^t\right)^2\right]\\
&= \frac{(f_\rho^{(p)})^2}{4}\mathrm{tr}\left[\rho_\mu^{(p)}-\frac{\partial_\mu\check{\rho}^{(p)}}{f_\rho^{(p)}}-\frac{i\left[\partial_\mu\pi,\pi\right]}{2f_\pi{}^2}+\cdots\right]^2, \quad (8)
\end{aligned}$$

where we should impose a bare/tree relation between the two decay constants,

$$(f_\rho^{(p)})^2 = a^{(p)} f_\pi{}^2 = a^{(p)} \frac{N}{G}, \tag{9}$$

to normalize the kinetic term of the would-be NG boson $\check{\rho}^{(p)}$ to the canonical form, and similarly for $a^{(N-p)} \mathcal{L}_V^{(N-p)}$. In the unitary gauge, $\check{\rho}^{(p)} = 0$, Equation (8) reads the mass term of $\rho_\mu^{(p)}$ as usual in the HLS formalism, and so does $a^{(N-p)} \mathcal{L}_V^{(N-p)}$, the mass term of $\rho_\mu^{(N-p)}$.

Here, we note that in contrast to the $O(p)_{\text{local}}$ gauge boson, the kinetic term for the $O(N-p)_{\text{local}}$ gauge boson, carrying index running $1, \cdots, N-p$, thus subject to all the planar diagram contributions in the large N limit, is *not* dynamically generated and ($O(N-p)_{\text{local}}$ does not exist for $N = 4, p = 3$ anyway), and stays as an auxiliary field (i.e., $\mathcal{L}_V^{(N-p)} = 0$) as was the case in the previous calculations for CP^{N-1} and the Grassmannian models. The $SU(N-1)_{\text{local}}$ gauge boson in the CP^{N-1} model with $G/H = SU(N)/[SU(N-1) \times U(1)]$, which carries the index running through $1, \cdots, N-1$, is not dynamically generated in the large N limit, in contrast to the $U(1)_{\text{local}}$ part [3,4,10–18]. The same is true for $G/H = O(N)/[O(N-p) \times O(p)]$ and $G/H = U(N)/[U(N-p) \times U(p)]$, with $O(N-p)_{\text{local}}$ and $U(N-p)_{\text{local}}$, respectively [19,20]. Similarly, a popular N extension $G/H = O(N)/O(N-1)$ gauge equivalent to the model $O(N)_{\text{global}} \times O(N-1)_{\text{local}}$ has no dynamical gauge boson for $O(N-1)_{\text{local}}$ and is irrelevant to the rho meson.

Then, without loss of generality, the starting Lagrangian Equations (6)–(8), is simplified as $a^{(p)} \mathcal{L}_V^{(p)} \equiv a\mathcal{L}_V$, $a^{(N-p)} \mathcal{L}_V^{(N-p)} = 0$, $\rho_\mu \equiv \rho_\mu^{(p)}$, $\check{\rho} = \check{\rho}^{(p)}$, $f_\rho^2 \equiv (f_\rho^{(p)})^2 = a f_\pi^2$, etc.:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_A + a\mathcal{L}_V &= \frac{1}{2}\text{tr}\left[\left(\partial_\mu \phi \partial^\mu \phi^t\right) + \frac{1}{2} \cdot \frac{aN}{G} \rho_\mu^2 - ia\rho^\mu \phi \partial_\mu \phi^t\right)\right] \\ &+ \frac{1}{2}\text{tr}\left[\left(1 - \frac{a}{2}\right) \frac{G}{N} (\phi \partial_\mu \phi^t)^2 - \eta\left(\phi\phi^t - \frac{N}{G}\mathbb{1}\right)\right], \end{aligned} \tag{10}$$

where tr and $\mathbb{1}$ should read $\text{tr}_{p\times p}$ and $\mathbb{1}_{p\times p}$, respectively, and $(\rho_\mu)_{ij} = \rho_\mu^a (S^a)_{ij}$ with $\text{tr}(S^a S^b) = 2\delta^{ab}$, and the $p \times p$ matrix Lagrange multiplier $\eta_{i,j}(x)$ is used for the constraint Equation (3) as in the standard large N arguments of CP^{N-1} [3,4,10–18] and other Grassmannian models [19,20]. (In the broken phase this is simply equivalent to the constraint Equation (3), while in the unbroken phase the multiplier is only a correct description. See later discussions.)

For $N = 4, p = 3$ Equation (10) with $O(4)_{\text{global}} \times O(3)_{\text{local}}$ is identical to the standard HLS Lagrangian [1–5] for the rho meson with $[SU(2)_L \times SU(2)_R]_{\text{global}} \times [SU(2)_V]_{\text{local}}$. It is now clear [9] that Equation (10) coincides with that of the conventional "covariant derivative type" Lagrangian [19] for a particular choice $a = 2$, with $\phi\phi^t = (N/G)\mathbb{1}$ (see Equation (3)).

From Equation (10), the effective potential in the large N limit for $\langle \boldsymbol{\phi}_{i,\beta}(x) \rangle = \sqrt{N} v(\delta_{i,j}, 0)$ (we took $v \neq 0$ real, i.e., the unitary gauge $\check{\rho}(x) = 0$), and $\langle \eta_{i,j}(x) \rangle = \eta\, \delta_{i,j}$, takes the form (in D dimensions):

$$\frac{V_{\text{eff}}(v, \eta)}{Np/2} = \eta\left(v^2 - \frac{1}{G}\right) + \int \frac{d^D k}{i(2\pi)^D} \ln\left(k^2 - \eta\right), \tag{11}$$

where the (a-dependent) 1-PI contributions are sub-leading in the large N limit (this observation is due to H. Ohki) and, therefore, the result is *independent of the parameter a*, in precisely the same form as that of the conventional "covariant derivative" parameterization of CP^{N-1} and the Grassmannian models corresponding to $a = 2$ [3,4,10–20], and hence yields the same gap equation:

$$\begin{aligned}\frac{1}{Np}\frac{\partial V_{\rm eff}}{\partial v} &= 2\eta v = 0\,, \\ \frac{1}{Np}\frac{\partial V_{\rm eff}}{\partial \eta} &= v^2 - \frac{1}{G} + \frac{1}{G_{\rm crit}} - v_\eta^2 = 0\,, \end{aligned} \tag{12}$$

with (for cutoff Λ)

$$\begin{aligned}\frac{1}{G_{\rm crit}} &\equiv \int \frac{d^Dk}{i(2\pi)^D}\frac{1}{-k^2} = \frac{1}{\left(\frac{D}{2}-1\right)\Gamma(\frac{D}{2})}\frac{\Lambda^{D-2}}{(4\pi)^{\frac{D}{2}}}, \\ v_\eta^2 &\equiv \int \frac{d^Dk}{i(2\pi)^D}\left(\frac{1}{-k^2} - \frac{1}{\eta - k^2}\right) = \frac{\Gamma(2-\frac{D}{2})}{\frac{D}{2}-1}\cdot\frac{\eta^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}}.\end{aligned}$$

The cutoff Λ can be removed for $2 \le D < 4$ (the theory is renormalizable), introducing the renormalized coupling at renormalization point μ as $1/G^{(R)}(\mu) \equiv 1/G - \int \frac{d^Dk}{i(2\pi)^D}\frac{1}{\mu^2-k^2} \equiv \mu^{D-2}/g^{(R)}(\mu)$, $1/G^{(R)}_{\rm crit} \equiv \int \frac{d^Dk}{i(2\pi)^D}\left(\frac{1}{-k^2} - \frac{1}{\mu^2-k^2}\right) = \frac{\Gamma(2-D/2)}{(D/2-1)}\cdot\frac{\mu^{D-2}}{(4\pi)^{D/2}} \equiv \mu^{D-2}/g^{(R)}_{\rm crit}$, s.t., $1/G - 1/G_{\rm crit} = \mu^{D-2}\left(1/g^{(R)}(\mu) - 1/g^{(R)}_{\rm crit}\right)$ in the gap equation. The renormalized coupling $g^{(R)}(\mu)$ has an ultraviolet fixed point at $g^{(R)}_{\rm crit}$, $0 \le g^{(R)}_{\rm crit} = (4\pi)^{D/2}(D/2-1)/\Gamma(2-D/2) < \infty\,(2 \le D < 4)$, with the beta function $\beta(g^{(R)}(\mu)) = \mu\partial g^{(R)}(\mu)/\partial\mu = -(D-2)g^{(R)}(\mu)[g^{(R)}(\mu) - g^{(R)}_{\rm crit}]/g^{(R)}_{\rm crit}$. While for $D = 4$ the theory is not renormalizable, $1/g^{(R)}_{\rm crit} \sim \Gamma(2-D/2)/(4\pi)^2\Big|_{D\to 4} \sim \ln(\Lambda^2/\mu^2)/(4\pi)^2$, with the remaining log divergence identified in the cutoff notation.

The gap equation implies as usual the *second-order phase transition* between two phases, the weak coupling phase with the symmetry spontaneously broken which is the same as the classical level, and the strong coupling phase with that spontaneously unbroken which is a new phase at quantum level:

$$\text{(i)}\quad G < G_{\rm cr} : v \neq 0\,,\ \eta = 0\,(\text{broken phase})$$
$$v^2 = \frac{1}{G} - \frac{1}{G_{\rm crit}} > 0\,, \tag{13}$$
$$\text{(ii)}\quad G > G_{\rm crit} : v = 0\,,\ \eta \neq 0\,(\text{unbroken phase})$$
$$v_\eta^2 = \frac{1}{G_{\rm crit}} - \frac{1}{G} > 0\,, \tag{14}$$

with the phase transition point $v = \eta = 0$. We may define a full decay constant F_π at quantum level in the large N limit:

$$\begin{aligned} F_\pi^2 \equiv Nv^2 = N\left(\frac{1}{G} - \frac{1}{G_{\rm crit}}\right) &= {f_\pi}^2 - \frac{N}{\left(\frac{D}{2}-1\right)\Gamma(\frac{D}{2})}\frac{\Lambda^{D-2}}{(4\pi)^{\frac{D}{2}}} \\ &\longrightarrow {f_\pi}^2 - N\frac{\Lambda^2}{(4\pi)^2}\ (D\to 4)\,. \end{aligned} \tag{15}$$

Thisindicates that approaching from the broken phase to the critical point, $F_\pi^2 \to 0$, is due to the power divergence of $1/G_{\rm crit}$ (quadratic divergence for $D = 4$), similarly to the "Wilsonian matching" of the HLS model with the underlying QCD at the UV scale Λ [4].

3. Dynamical Generation of Rho Meson

Now the (amputated) two-point function of ρ_μ in the large N limit takes the form:

$$\Gamma^{(\rho)}_{\mu\nu}(q) = \left(\frac{a}{2}\right)\left(\frac{N}{G}\right) g_{\mu\nu} + \left(\frac{a}{2}\right)^2 B_{\mu\lambda}(q)\cdot C^{\lambda}_{\nu}(q),$$
$$C_{\mu\nu}(q) = g_{\mu\nu} + \left(\frac{a}{2}-1\right)\frac{G}{N} B_{\mu\lambda}(q)\cdot C^{\lambda}_{\nu}(q)\,, \tag{16}$$

where the four-ϕ vertex $\left(\frac{a}{2}-1\right)\frac{G}{N}$ in our Lagrangian Equation (10) (second line) gives rise to an infinite sum of the bubble graph contribution $B_{\mu\nu}(q)$;

$$\begin{aligned}\frac{1}{N} B_{\mu\nu}(q) &= \frac{1}{2}\int \frac{dk^D}{i(2\pi)^D}\frac{(2k+q)_\mu(2k+q)_\nu}{(k^2-\eta)((k+q)^2-\eta)} \\ &= q^2 f(q^2,\eta)\cdot\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \left(v^2 - \frac{1}{G}\right)\cdot g_{\mu\nu}\,,\end{aligned} \tag{17}$$

with

$$f(q^2,\eta) \equiv -\frac{\Gamma(2-\frac{D}{2})}{2(4\pi)^{\frac{D}{2}}\Gamma(2)}\int_0^1 dx \frac{(1-2x)^2}{[x(1-x)q^2+\eta]^{2-\frac{D}{2}}}\,,$$

which reads for $D \to 4$ ($\epsilon \equiv 2 - D/2 \to 0$ and $1/A^\epsilon \simeq 1 - \epsilon \ln A$):

$$\begin{aligned} f(q^2,0) &= -\frac{1}{2}\cdot\frac{1}{3(4\pi)^2}\cdot\left[\ln\left(\frac{\Lambda^2}{q^2}\right)+\frac{8}{3}\right], \\ f(0,\eta) &= -\frac{1}{2}\cdot\frac{1}{3(4\pi)^2}\cdot\left[\ln\left(\frac{\Lambda^2}{\eta}\right)\right],\end{aligned} \tag{18}$$

where we have used the gap equation Equation (12) and identified $\Gamma(\epsilon) \simeq 1/\epsilon \to \ln\Lambda^2$. The finite part common to both phases is included in the definition of the cutoff Λ, while the part $+8/3$ is an extra one in the broken phase $\eta \equiv 0$, similarly to that in the CP^{N-1} [18].

3.1. $a = 2$ Case

Note that for $a = 2$, we have $C_{\mu\nu} = g_{\mu\nu}$ in Equation (16), which yields $\Gamma^{(\rho)}_{\mu\nu}(q)$:

$$\frac{\Gamma^{(\rho)}_{\mu\nu}(q)}{N} = \left(\frac{1}{G}\right) g_{\mu\nu} + \frac{B_{\mu\lambda}(q)}{N}\cdot g^{\lambda}_{\nu}(q) = \left(q^2 f(q^2,\eta)+v^2\right)\cdot\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) + v^2\cdot\frac{q_\mu q_\nu}{q^2}\,, \tag{19}$$

the well-known form of one-loop dominance in the large N limit in the conventional "covariant derivative" parameterization for CP^{N-1} model and other Grassmannian models [3,4,10–20].

For the broken phase $v \neq 0$, $\eta = 0$, this is readily inverted to yield the ρ_μ propagator for $a = 2$: $\langle\rho_\mu\rho_\nu\rangle(q) \equiv \langle\rho^{ij}_\mu\rho^{ji}_\nu\rangle(q) = 2\langle\rho^a_\mu\rho^a_\nu\rangle(q)$:

$$\begin{aligned}\langle\rho_\mu\rho_\nu\rangle(q) &= -\Gamma^{(\rho)}_{\mu\nu}(q)^{-1} = \frac{1}{N}\frac{-f^{-1}(q^2,0)}{q^2+f^{-1}(q^2,0)v^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) - \frac{1}{N}\frac{1}{v^2}\frac{q_\mu q_\nu}{q^2} \\ &= \frac{1}{N}\frac{-f^{-1}(q^2,0)}{q^2+f^{-1}(q^2,0)v^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{-f^{-1}(q^2,0)v^2}\right) = 2\Delta_{\mu\nu}(q), \\ \Delta_{\mu\nu}(q) &\equiv \frac{g^2_{\rm HLS}(q^2)}{q^2-M^2_\rho(q^2)}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{M^2_\rho(q^2)}\right), \\ & M^2_\rho(q^2) \equiv -f^{-1}(q^2,0)v^2 = g^2_{\rm HLS}(q^2)\cdot 2F^2_\pi, \\ & g^{-2}_{\rm HLS}(q^2) \equiv -2Nf(q^2,0) = \frac{N}{3(4\pi)^2}\left[\ln\frac{\Lambda^2}{q^2}+\frac{8}{3}\right],\end{aligned} \tag{20}$$

which is the form of the unitary gauge (we took the unitary gauge $\hat{\rho}(x) = 0$ with $v \neq 0 =$ real), with the physical pole position and the on-shell HLS coupling given as $q^2 = M_\rho^2 = -f^{-1}(M_\rho^2, 0)v^2 = g_{\rm HLS}^2(M_\rho^2) \cdot 2F_\pi^2$ and $g_{\rm HLS}^2 \equiv g_{\rm HLS}^2(M_\rho^2)$, respectively. The relation implies the rho meson mass is generated by the Higgs mechanism

$$M_\rho^2 = g_{\rm HLS}^2 \cdot F_\rho^2 \,, \quad F_\rho^2 = 2 \cdot F_\pi^2 \,, \tag{21}$$

where F_ρ is the decay constant of the would-be NG boson $\hat{\rho}$ (absorbed into the rho meson in the unitary gauge) at the quantum level, which is to be compared with the tree-level relation Equation (9), with $a = a^{(p)} = 2$. Equation (20) suggests even more general results than the on-shell relation Equation (21):

$$M_\rho^2(q^2) = g_{\rm HLS}^2(q^2) \cdot F_\rho^2 \,, \quad F_\rho^2 = 2 \cdot F_\pi^2 \,. \tag{22}$$

We shall come back to this point later when discussing VMD.

The q^2 dependence of $M_\rho^2(q^2)$ and $g_{\rm HLS}^2(q^2)$ may be regarded as the running mass and the (*asymptotically non-free/infrared free*) running coupling. The resultant rho meson mass relation $M_\rho^2 = -f^{-1}(M_\rho^2, 0)v^2 = g_{\rm HLS}^2(M_\rho^2) \cdot 2F_\pi^2$ is independent of N and can be extrapolated into $N \to 4$ with $p = 3$ for the actual rho meson.

We thus *establish the dynamic generation of the rho meson as the HLS gauge boson for* $a = 2$ [9] in exactly the same way as in the CP^{N-1} model and other Grassmannian models.

In the unbroken phase, $v = 0, \eta \neq 0$, on the other hand, $\Gamma_{\mu\nu}^{(\rho)}(q)$ in Equation (19) is transverse, implying the HLS is an *unbroken gauge symmetry*. Although not invertible as it stands, it is of course inverted by fixing the gauge as usual, to get the *massless* propagator $\langle \rho_\mu \rho_\nu \rangle(q) = g_{\rm HLS}^2(q^2, \eta) \cdot \frac{g_{\mu\nu}}{q^2} +$ gauge term, with $g_{\rm HLS}^{-2}(q^2, \eta) \equiv -2Nf(q^2, \eta) \simeq -2Nf(0, \eta) \equiv g_{\rm HLS}^{-2}(\eta)$ which is analytic at $q^2 = 0$. η-dependence may be regarded as the running of the coupling, asymptotically non-free/infrared free, $g_{\rm HLS}^2(\eta) \to 0\,(\eta \to 0)$, the same as that in broken phase, see Equation (18). *Without gauge symmetry* ($a = 0$), $\langle \alpha_{\mu,||}\, \alpha_{\nu,||} \rangle(q)$ is *ill defined in the unbroken phase* $v = 0$, where the factor $g_{\mu\lambda} + \frac{G}{N} B_{\mu\lambda}$ is pure transverse and *not invertible*, in accordance with the Weinberg–Witten theorem [21] in the absence of massless spin $J \geq 1$ particles in the positive definite Hilbert space (no gauge symmetry). The situation is also the same as the CP^{N-1} and the Grassmannian models. Note also that the *massless rho meson is stable*, since it does not decay into the pions which are no longer the NG bosons and have non-zero mass degenerate with $\check{\rho}$ (no longer the would-be NG boson) and other degrees of freedom of $\phi_{i,\beta}$ (corresponding to the six constraints in the broken phase, in addition to the 3 π's and 3 $\check{\rho}$'s for $N = 4, p = 3$), $M_\pi^2 = M_{\check{\rho}}^2 = \cdots = \eta \neq 0$. Note that the phase transition is of the second order with $v = \eta = 0$ and all the spectra are decoupled (free) massless particles: $M_\rho^2 = M_{\check{\rho}}^2 = M_\pi^2 = 0$ at the phase transition point $G = G_{\rm crit}$ (conformal).

3.2. Case for Arbitrary Value of a

Since we have established the dynamical generation of the rho meson for $a = 2$, the next question is wether the conclusion is dependent on the specific value of $a = 2$. Here, we show that the result is independent of a.

For the generic case for arbitrary a, the large N dominant diagrams are not just the one-loop but do include *an infinite sum of the bubble diagrams coming from the extra four-vertex* $\left(\frac{a}{2} - 1\right)\frac{G}{N}$ as in Equation (16). $C_{\mu\nu}$ in Equation (16) is solved straightforwardly though tediously (see [9] for details): From Equations (16) and (17) we have

$$
\begin{aligned}
\frac{a}{2}C_{\mu\nu}(q) &= \frac{a}{2}\left[g_{\mu\nu} + \left(1-\frac{a}{2}\right)\frac{G}{N}B_{\mu\nu}(q)\right]^{-1} \\
&= \left[1-\left(1-\frac{2}{a}\right)G\left(v^2+q^2 f\right)\right]^{-1}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) \\
&\quad +\left[1-\left(1-\frac{2}{a}\right)Gv^2\right]^{-1}\frac{q_\mu q_\nu}{q^2}, \\
\frac{\Gamma^{(\rho)}_{\mu\nu}(q)}{N} &= \frac{2}{G}\left(1-\frac{2}{a}\right)^{-1}\left[\frac{a}{2}C_{\mu\nu}-g_{\mu\nu}\right], \\
&= \left[\frac{f^{-1}}{q^2+v^2 f^{-1}}-\left(1-\frac{2}{a}\right)G\right]^{-1}\cdot\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) \\
&\quad +\left[\frac{1}{v^2}-\left(1-\frac{2}{a}\right)G\right]^{-1}\cdot\frac{q_\mu q_\nu}{q^2}, \quad f\equiv f(q^2,\eta).
\end{aligned} \tag{23}
$$

This of course is reduced to Equation (19) for $a=2$.

We finally arrive at the *dynamically generated propagating HLS gauge boson for any a*, whose propagator in the broken phase and takes the same form of the unitary gauge as that for $a=2$ except for the contact term (to be discussed later): [9]

$$
\begin{aligned}
\langle\rho_\mu\rho_\nu\rangle(q) = -\Gamma^{(\rho)}_{\mu\nu}(q)^{-1} &= \left[\frac{-f^{-1}}{q^2+v^2 f^{-1}}+\left(1-\frac{2}{a}\right)G\right]\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) \\
&\quad -\left[\frac{1}{v^2}-\left(1-\frac{2}{a}\right)G\right]\frac{q_\mu q_\nu}{q^2} \\
&= \left(1-\frac{2}{a}\right)\frac{G}{N}g_{\mu\nu}+2\Delta_{\mu\nu}(q).
\end{aligned} \tag{24}
$$

which is reduced to Equation (20) for $a=2$. Again, the mass relation from $\Delta_{\mu\nu}$ in the last line of Equation (20) is independent of N and thus safely extrapolated to the realistic rho meson, $N\to 4$ with $p=3(=\text{fixed})$.

Here, the physical pole position $q^2 = M_\rho^2 = g^2_{\text{HLS}}(M_\rho^2)\cdot 2F_\pi^2$ and the on-shell HLS coupling $g^2_{\text{HLS}} \equiv g^2_{\text{HLS}}(M_\rho^2)$ are both *independent of a*;

$$
M_\rho^2 = g^2_{\text{HLS}}\cdot 2F_\pi^2, \quad F_\rho^2 = 2\cdot F_\pi^2, \quad (a-\text{independent}), \tag{25}
$$

which is the same as Equation (21) but now it is an *a-independent* result, in contrast to that of the bare quantities at tree level Equation (9): $f_\rho^2 = a{f_\pi}^2$. This relation is a reminiscence of the KSRF II relation, $M_\rho^2 = 2g^2_{\rho\pi\pi}F_\pi^2$, where $g_{\rho\pi\pi}$ is the $\rho\pi\pi$ coupling. In fact, in the next section we will show $g_{\rho\pi\pi} = g_{\text{HLS}}$ ("rho-universality") independently of a, and thus derive the KSRF II relation (as well as KSRF I) independently of a.

Note that the *a-dependence is exactly cancelled in the physical part* $\Delta_{\mu\nu}(q)$ as it should be, since a is actually a redundant parameter for the auxiliary field ρ_μ. While the $a-$dependence remains in the *unphysical contact term* $-\frac{2G}{aN}g_{\mu\nu}$ which corresponds to the tree ρ_μ "propagator" with tree mass $\frac{aN}{2G}$, it is an artifact in using the auxiliary field ρ_μ for the composite field $\alpha_{\mu,||} = i\frac{G}{N}\phi\partial_\mu\phi^t$ whose two-point function is independent of a and exists even for $a=0$ (without HLS!) *in the broken phase*. They satisfy an exact relation via Ward–Takahashi identity [9] :

$$
\langle\rho_\mu\rho_\nu\rangle(q) = \langle\alpha_{\mu,||}\,\alpha_{\nu,||}\rangle(q) - \frac{2G}{aN}g_{\mu\nu}\,. \tag{26}
$$

This implies that both are identical at a limit $a\to\infty$, which is relevant to the physics of the off-shell rho meson, like the skyrmion physics.

Moreover, the whole contact term is cancelled in the $\pi\pi$ scattering (VMD). The $\pi\pi$ scattering amplitude $T_{\mu\nu}(q)$ is given as

$$2T_{\mu\nu}(q) = -\frac{G}{N}g_{\mu\nu} + \langle\alpha_{\mu,||}\,\alpha_{\nu,||}\rangle(q) = 2\Delta_{\mu\nu}(q)\,, \quad \text{(VMD)}. \tag{27}$$

where the $g_{\mu\nu}$ term is from the tree vertex, while $\langle\alpha_{\mu,||}\,\alpha_{\nu,||}\rangle(q)$ is only from the loop contributions (bubble sum) dominant in the large N limit, giving the result consistent with the Ward–Takahashi identity, Equation (26), with Equation (24):

$$\begin{aligned}
\langle\alpha_{\mu,||}\,\alpha_{\nu,||}\rangle(q) &= \left(i\frac{G}{N}\right)^2\langle\phi\partial_\mu\phi^t\;\phi\partial_\nu\phi^t\rangle(q) \\
&= \left(i\frac{G}{N}\right)^2\left[B_{\mu\nu}(q) + B_{\mu\lambda}(q)\cdot\left(-\frac{G}{N}\right)B^\lambda_\nu(q) + \cdots\right] \\
&= \left(i\frac{G}{N}\right)^2 B^\lambda_\mu(q)\left[g_{\lambda\nu} + \frac{N}{G}\langle\alpha_{\mu,||}\,\alpha_{\nu,||}\rangle(q)\right] = \left(g_{\mu\lambda} + \frac{G}{N}B_{\mu\lambda}\right)^{-1}\left(i\frac{G}{N}\right)^2 B^\lambda_\nu(q) \\
&= \frac{G}{N}g_{\mu\nu} + 2\Delta_{\mu\nu}(q),
\end{aligned} \tag{28}$$

with $B_{\mu\nu}(q)$ given in Equation (17). The four-ϕ vertex (both for tree and loop) here is different from Equation (16) for the ρ_μ case: $(\frac{a}{2}-1)\frac{G}{N} + (-\frac{a}{2})\frac{G}{N} = -\frac{G}{N}$ (the additional second term is from the tree rho contribution $(-\frac{ia}{2})(\frac{a}{2}\frac{N}{G})^{-1}(-\frac{ia}{2})$), the same as that for $a = 0$ (original non-linear sigma model without HLS) as it should be independent of the auxiliary field. Then, the contact term $\frac{G}{N}g_{\mu\nu}$ is precisely canceled in $T_{\mu\nu}$, namely the *VMD for arbitrary value of a*, now with the generalized relation Equation (22) not just the on-shell one Equation (21). This is compared with the conventional HLS approach where the VMD for $\pi\pi$ scattering is realized only for $a = 4/3$ (not $a = 2$!!) [4].

We then have the effective action with kinetic term of rho meson ρ_μ, which becomes that of the composite field $\alpha_{\mu,||}$ at $a \to \infty$ as implied in Equation (26):

$$\mathcal{L}^{\text{eff}}_{\text{kinetic}} = -\frac{1}{4g^2_{\text{HLS}}}\cdot\frac{1}{2}\text{tr}\rho^2_{\mu\nu}, \tag{29}$$

$$\longrightarrow -\frac{1}{4g^2_{\text{HLS}}}\cdot\frac{1}{2}\text{tr}\alpha^2_{\mu\nu,||} \quad (a\to\infty), \tag{30}$$

where $\alpha_{\mu\nu,||} \equiv \partial_\mu\alpha_{\nu,||} - \partial_\nu\alpha_{\mu,||} - i\left[\alpha_{\mu,||},\alpha_{\nu,||}\right]$, with $f_\pi^2 = N/G \Rightarrow F_\pi^2 = Nv^2$. For $N = 4$, $p = 3$. Equation (30) is precisely the Skyrme term, $\frac{1}{32e^2_{\text{Skyrme}}}\text{tr}_{SU(2)}[L_\mu, L_\nu]^2$, with $e^2_{\text{Skyrme}} = g^2_{\text{HLS}}$, in the $SU(2)$ basis, where we have $\alpha_{\mu\nu,||} = i[\alpha_{\mu,\perp},\alpha_{\nu,\perp}]$ and $L_\mu \equiv \partial_\mu U\cdot U^\dagger$, $U = \xi^2(\pi) = e^{i\pi^a\tau^a/F_\pi}$ [22].

In the conventional HLS arguments with $a = 2$ and assumed rho meson kinetic term, the kinetic term can still stabilize the skyrmion but only for the low-energy region $q^2 \ll M_\rho^2 = \mathcal{O}(a)$, where the rho meson mass term effectively becomes the constraint $\rho_\mu = \alpha_{\nu,||}$ [22]. In the large N limit here, on the other hand, with the induced rho meson kinetic term we can freely take the value a at any q^2 region including $q^2 > M_N^2$ to describe the nucleon with mass m_N identified with the skyrmion, *without affecting all the $a-$independent successful relations of the rho meson* to be shown in the next section. Thus, the system (π,ρ,N) can be well described by the non-linear sigma model at large N without ad hoc rho meson kinetic term, with the free parameter choice of a just for the detailed description of the nucleon.

As seen from Equation (20), the HLS coupling depends on the cutoff Λ as it should, since the non-linear sigma model is a non-renormalizable model for $D = 4$ (see [18] for other formulation). From Equation (20), with $N = 4$ and $q^2 = M_\rho^2 \simeq (770\,\text{MeV})^2$,

$F_\pi \simeq 92\,\mathrm{MeV}$, we have $\Lambda = e^{-4/3} \cdot M_\rho \cdot e^{12\pi^2 F_\pi^2/M_\rho^2} \simeq 1.1$ GeV $\simeq 4\pi F_\pi$, roughly the validity scale of the chiral perturbation theory. As an asymptotically non-free theory the kinetic term vanishes $1/g^2_{\rm HLS}(q^2 = \mu^2) = -2Nf(\mu^2, 0) \to 0$ $(\mu^2 \to \tilde{\Lambda}^2)$ at the Landau pole $\mu = \tilde{\Lambda} = e^{4/3}\Lambda \simeq 4.2$ GeV $\gg \Lambda \gg M_\rho$, where the ρ_μ returns to an auxiliary field as a static composite of π, the situation sometimes referred to as "compositeness condition" [23] advocated in a reformulation of the top quark condensate model [24]. In this viewpoint, the HLS gauge bosons as bound states of π's develop the kinetic term as we integrate the higher frequency modes in the large N limit from Λ^2 down to the scale μ^2 in the sense of the Wilsonian renormalization group [4].

This also implies $g^2_{\rm HLS}(q^2 = \mu^2) \to 0$ $(\mu^2/\tilde{\Lambda}^2 \to 0)$ at approaching the phase transition point $F_\pi^2 = Nv^2 \to 0$ $(G \to G_{\rm cr}-)$. Thus, the rho meson in the broken phase, with M_ρ close enough to the phase transition point, $M_\rho/\tilde{\Lambda}, M_\rho/\Lambda \to 0$, is to be identified with a gauge boson. Since the phase transition is second order, the *HLS as a gauge symmetry is crucial not only in the unbroken phase but also in the broken phase near the phase transition point*. (Just on the phase transition point all the spectra become massless and free, i.e., trivially scale-symmetric.) The result $g^2_{\rm HLS} \to 0$ and $M_\rho^2 \to M_\pi^2 (\equiv 0)$ near the phase transition point in the broken phase is similar to the Vector Manifestation (see [4] and references cited therein), *both not precisely on the phase transition point* where ρ and π are just decoupled massless free particles $M_\rho^2 = M_\pi^2 = 0$. The latter is based on the one-loop "Wilsonian Matching" with QCD at Λ where the kinetic term is given with the parameter $a = a(\Lambda^2) \simeq 1$, which then runs down as $a(\mu^2) = F_\rho^2(\mu^2)/F_\pi^2(\mu^2) \sim 1$ $(M_\rho^2 < \mu^2 < \Lambda^2)$ (with ρ loop) and further down to π on shell $a(0) = F_\rho^2(M_\rho^2)/F_\pi^2(M_\pi^2 = 0) = 2$ (with the ρ loop decoupled for F_π^2 in $\mu^2 < M_\rho^2$), in contrast to the present case which is for any a at all orders in the large N limit without ρ loop at all.

4. Successful "*a* = 2" Relations Realized for Any *a*

Now we derive all the phenomenologically successful relations for the rho meson independently of a.

The large N Green function for $\rho\pi\pi$ is given as a bubble sum, which takes the *$a-$independent form of VMD*, as shown in Appendix D of Ref. [9]:

$$\begin{aligned}
\langle \rho_\mu(q)\phi(k)\phi(k+q)\rangle &= \langle \rho_\mu\rho_\nu\rangle(q) \cdot \Gamma^{\rho\pi\pi,\nu}(q,k,q+k)\Big|^{k^2=(k+q)^2=0}_{\phi-\text{amputated}} \\
&= \left[\langle \rho_\mu\rho_\nu\rangle(q) - g_{\mu\nu}\cdot\left(1-\frac{2}{a}\right)\frac{G}{N}\right]\cdot(q+2k)^\nu \\
&= 2\Delta_{\mu\nu}(q)\cdot(q+2k)^\nu,
\end{aligned} \tag{31}$$

where the $\rho\pi\pi$ vertex is given as

$$\begin{aligned}
\Gamma^{\rho\pi\pi,\nu}(q,k,q+k)\Big|^{k^2=(k+q)^2=0}_{\phi-\text{amputated}} &= \frac{a}{2}\left[g_{\mu\nu} + B_{\mu\lambda}(q)C^\lambda_\nu(q)\cdot\left(\frac{a}{2}-1\right)\frac{G}{N}\right]\cdot(q+2k)^\nu \\
&= \left[g_{\mu\nu} + \Gamma^{(\rho)}_{\mu\nu}(q)\left(1-\frac{2}{a}\right)\frac{G}{N}\right]\cdot(q+2k)^\nu,
\end{aligned} \tag{32}$$

with $\Gamma^{(\rho)}_{\mu\nu}(q) = -\langle\rho_\mu\rho_\nu\rangle(q)^{-1}$ given in Equation (16). Then the a-dependence and the contact term are all cancelled out. It is also shown by the Ward–Takahashi identity $0 = \int \mathcal{D}\phi \frac{\delta}{\delta\rho_\mu(x)}\left(\phi(y)\phi(z)\cdot e^{iS[\phi,\rho_\mu]}\right) = \int\mathcal{D}\phi\left(\frac{aN}{2G}\right)\left(\rho_\mu(x) - \alpha_{\mu,||}(x)\right)\cdot\phi(y)\phi(z)\cdot e^{iS[\phi,\rho_\mu]}$, such that

$$\langle \rho_\mu(q)\phi(k)\phi(k+q)\rangle = \langle \alpha_{\mu,||}(q)\phi(k)\phi(q+k)\rangle \tag{33}$$

for $\mathcal{L}$ given in Equation (10) even without explicit calculations, since $\langle \alpha_{\mu,||}\phi\phi\rangle$ is obviously independent of a.

We may introduce "renormalized" field $\rho_\mu^{(R)} \equiv g_{\rm HLS}^{-1}(q^2)\cdot\rho_\mu$ by rescaling the "kinetic term" to the canonical one, i.e., $\Delta_{\mu\nu}^{(R)}(q) \equiv g_{\rm HLS}^{-2}(q^2)\cdot\Delta_{\mu\nu}(q)$ and $\langle\rho_\mu^{(R)}\phi\phi\rangle = \langle\alpha_{\mu,||}^{(R)}\phi\phi\rangle = 2g_{\rm HLS}(q^2)\cdot\Delta_{\mu\nu}^{(R)}(q)\cdot(q+2k)^\nu$, which is compared with the definition of $g_{\rho\pi\pi}(q^2)$, $\langle\rho_\mu^{(R)}\phi\phi\rangle \equiv 2g_{\rho\pi\pi}(q^2)\cdot\Delta_{\mu\nu}^{(R)}(q)\cdot(q+2k)^\nu$, resulting in the ρ universality *independently of a*:

$$g_{\rho\pi\pi}(q^2) = g_{\rm HLS}(q^2) \quad (\rho \text{ universality}). \tag{34}$$

It then leads to the KSRF relations (generalized for $\forall q^2$ as in Equation (22)) *independently of a*:

$$\begin{aligned} g_\rho(q^2) &= M_\rho(q^2)F_\rho = 2g_{\rho\pi\pi}(q^2)F_\pi^2 \ (\text{KSRF I}), &(35)\\ M_\rho^2(q^2) &= 2g_{\rho\pi\pi}^2(q^2)F_\pi^2 \ (\text{KSRF II}), &(36)\end{aligned}$$

with $\langle 0|J_\mu^{\rm em}|\rho^{(R)}(q^2)\rangle \equiv g_\rho(q^2)\epsilon_\mu(q) = M_\rho(q^2)F_\rho\epsilon_\mu(q)$, where $M_\rho^2(q^2)$ and $g_{\rm HLS}(q^2)$ in $\Delta_{\mu\nu}(q)$ are given in Equation (20). This is consistent with the fact that the KSRF I relation is a low-energy theorem of the HLS valid for any a [3], which is proved to all order of loop expansion [25].

The VMD for the electromagnetic form factor $F_{\mathcal{B}\pi\pi}(q^2)$ also follows $a-$independently, similarly to the VMD in the $\pi\pi$ scattering shown in Equation (27). Here, the photon field $\mathcal{B}_\mu$ is introduced by gauging $H_{\rm global}$, $D_\mu\phi \Rightarrow \partial_\mu\phi - i\rho_\mu\phi + i\phi\mathcal{B}_\mu$ in Equation (10) with $\alpha_{\mu,||} = i\frac{G}{N}\phi\partial_\mu\phi^t$. It has contributions from the $\mathcal{B}_\mu - \rho_\mu$ mixing and from the "direct coupling" to $\alpha_{\mu,||}$ (with the tree contact term canceled by the bubble sum as in the $\pi\pi$ scattering), both coupled to the identical VMD Green functions $\langle\rho_\mu^{(R)}\phi\phi\rangle = \langle\alpha_{\mu,||}^{(R)}\phi\phi\rangle$, Equation (33), in a linear combination to cancel the a dependence:

$$\begin{aligned} 2F_{\mathcal{B}\pi\pi}(q^2)(q+2k)_\mu &= \langle J_\mu^{\rm em}(q)\,\phi(k)\,\phi(k+q)\rangle\Big|_{\phi-\text{amputated}}^{k^2=(k+q)^2=0}\\ &= -g_\rho(q^2)\left[\frac{a}{2}\langle\rho_\mu^{(R)}\,\phi\,\phi\rangle + \left(1-\frac{a}{2}\right)\langle\alpha_{\mu,||}^{(R)}\,\phi\,\phi\rangle\right](q+2k)^\nu\\ &= 2\cdot\left(-M_\rho^2(q^2)\cdot\Delta_{\mu\nu}^{(R)}(q)\right)\cdot(q+2k)^\nu. \end{aligned} \tag{37}$$

Namely,

$$F_{\mathcal{B}\pi\pi}(q^2) = \frac{M_\rho^2(q^2)}{M_\rho^2(q^2)-q^2}, \quad F_{\mathcal{B}\pi\pi}(0) = 1. \tag{38}$$

Thus the *VMD is realized independently of a*.

Although it takes the same form as the naive VMD, $F_{\mathcal{B}\pi\pi}(q^2) \approx \frac{M_\rho^2}{M_\rho^2-q^2}$ near the on-shell, $q^2 \approx M_\rho^2$, $M_\rho(q^2)$ here has log q^2 dependence as in Equation (20). Actually, such a q^2 dependence is necessary for the modern version of the VMD in both the space-like and the time-like momentum regions, see, e.g., [26–28].

We now have the effective action at arbitrary value of a for the rho meson after rescaling the ρ_μ field $\rho_\mu \to g_{\rm HLS}\rho_\mu$ with $g_{\rm HLS} = g_{\rho\pi\pi}$ and $\mathcal{B}_\mu \to e\mathcal{B}_\mu$:

$$\mathcal{L}_\rho^{\rm eff} = \frac{1}{2}\cdot 2F_\pi^2\left[g_{\rm HLS}^2\left((\rho_\mu^+)^2+(\rho_\mu^-)^2\right) + (\rho_\mu^0,\mathcal{B}_\mu)\begin{pmatrix} g_{\rm HLS}^2 & e\,g_{\rm HLS}\\ e\,g_{\rm HLS} & e^2\end{pmatrix}\begin{pmatrix}\rho^{0\mu}\\ \mathcal{B}^\mu\end{pmatrix}\right] - \frac{1}{2}{\rm tr}\rho_{\mu\nu}^2. \tag{39}$$

After diagonalization this gives for arbitrary a:

$$M_{\rho^0}^2 = \left(g_{\rm HLS}^2 + e^2\right)\cdot 2F_\pi^2 = M_{\rho^\pm}^2 + 2e^2F_\pi^2\,, \tag{40}$$

in perfect agreement with the experiment, the same result as that of the conventional HLS prediction with assumed kinetic term of rho meson for $a = 2$ [4].

However, in contrast to the above a-independent results of the rho meson, mostly on-shell and some off-shell results, the quantities related to the off-shell rho meson in principle do depend on a as the rho meson propagator in Equation (24) has the a-dependent contact term. A typicality of such is the skyrmion physics to be stabilized by the Skyrme term as we mentioned in the last section.

5. Conclusions and Discussions

To conclude, we have proved that the rho meson is a dynamical gauge boson of the HLS $O(3)_{\rm local} \simeq [SU(2)_V]_{\rm local}$ by the large N dynamics of the model $G/H = O(N)/[O(N-3) \times O(3)]$, with all the successful "$a = 2$ results" being realized purely dynamically *independently of* N for *any value of* a, thus safely extrapolated to $N = 4$, $O(4)/O(3) \simeq O(4)_{\rm global} \times O(3)_{\rm local} \simeq [SU(2)_L \times SU(2)_R]_{\rm global} \times [SU(2)_V]_{\rm local}$.

The "$a = 2$ results" originally obtained for particular choice of $a = 2$ [1–4] are now clear to be artifacts of the combined use of the a-dependent *tree-level* rho meson mass term and the *ad hoc added kinetic term* which was *assumed* to be generated at the quantum level *without affecting the pole structure* of the dynamically generated propagator. Actually, as we demonstrated, the tree-level parameter is no longer the true one of the pole at quantum level when the kinetic term is generated, namely the pole position (and residue as well) of the full propagator is shifted from the tree level one in such a way that the a-dependence is totally canceled out. Actually, the parameter *a is a redundant parameter for the auxiliary field* ρ_μ and is irrelevant to the physical results at quantum level as it should be for the correct calculations. The results of the present paper revealed that it is indeed the case in the large N limit.

Further implications of the results are as follows: Once the rho kinetic term is generated, Equation (29), it stabilizes the Skyrmion without ad hoc Skyrme term, with a free parameter a, and hence the non-linear sigma model in the large N limit perfectly describes via HLS the low energy QCD for π, ρ, N at the scale $\lesssim \Lambda \simeq 4\pi f_\pi$ without explicit recourse to the QCD.

The dynamically generated kinetic term, with the induced gauge coupling $g^2_{\rm HLS}(q^2)$ being asymptotically non-free/infrared free in both broken and unbroken phases, has a cutoff $\Lambda \simeq 4\pi f_\pi \gg M_\rho$ (and Landau pole $\tilde{\Lambda}$), so that the rho meson is sitting *near the second order phase transition point* as a composite HLS *gauge boson* to be matched with the underlying QCD. This implies [9] that the large N dynamics reveals the HLS as a "magnetic gauge theory" (infrared free in both phases) dual to the underlying QCD as the "electric gauge theory" [4,29–31], similarly to the Seiberg duality in the SUSY QCD [32].

If the HLS as the unbroken magnetic gauge theory is realized, say in hot/dense QCD, we would have a new possibility for the chiral symmetry restored hadronic phase having a massless rho meson and massive $\pi, \check{\rho}$ [9], which is contrasted with $M_\rho^2 \to M_\pi^2 (\equiv 0)$ near the phase transition point in the broken phase (not precisely on the phase transition point) similarly to the "Vector Manifestation" as described in the text.

It was frequently emphasized that the large N results are valid even for the small N at least qualitatively [14]. The result of the present paper is a yet another proof of this statement, and even more, quantitatively not just qualitatively, in perfect agreement with the experimental facts of the rho meson.

This further implies the dynamical HLS bosons in other system described by the large N Grassmannian models. A notable case of such is the Standard Model (SM) Higgs Lagrangian, re-parameterized [5] as a scale-invariant version of the model $G/H = O(4)/(3) \simeq O(4)_{\rm global} \times O(3)_{\rm local}$, is precisely the same as the rho meson case, except for an extra mode, pseudo-dilaton (SM Higgs boson) to make the model (approximately) scale-invariant (Having no indices running through N, it is irrelevant to the SM rho physics in the large N limit) [9]. This justifies the basic assumption [33] that there exists a rho meson-like vector boson within the SM ("SM rho") which stabilizes a skyrmion ("SM skyrmion") as a candidate for the dark matter existing even within the SM.

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References

1. Bando, M.; Kugo, T.; Uehara, S.; Yamawaki, K.; Yanagida, T. Is rho meson a dynamical gauge boson of hidden local symmetry? *Phys. Rev. Lett.* **1985**, *54*, 1215. [CrossRef] [PubMed]
2. Bando, M.; Kugo, T.; Yamawaki, K. On the vector mesons as dynamical gauge bosons of hidden local symmetries. *Nucl. Phys.* **1985**, *259*, 493. [CrossRef]
3. Bando, M.; Kugo, T.; Yamawaki, K. Nonlinear Realization and Hidden Local Symmetries. *Phys. Rept.* **1988**, *164*, 217. [CrossRef]
4. Harada, M.; Yamawaki, K. Hidden local symmetry at loop: A New perspective of composite gauge boson and chiral phase transition. *Phys. Rept.* **2003**, *381*, 1. [CrossRef]
5. Yamawaki, K. Hidden Local Symmetry and Beyond. *Int. J. Mod. Phys. E* **2017**, *26*, 1740032. [CrossRef]
6. Coleman, S.R.; Wess, J.; Zumino, B. Structure of phenomenological Lagrangians. I. *Phys. Rev.* **1969**, *177*, 2239. [CrossRef]
7. Callan, C.G., Jr.; Coleman, S.R.; Wess, J.; Zumino, B. Structure of phenomenological Lagrangians. II. *Phys. Rev.* **1969**, *177*, 2247. [CrossRef]
8. Holt, J.W.; Rho, M.; Weise, W. Chiral symmetry and effective field theories for hadronic, nuclear and stellar matter. *Phys. Rep.* **2016**, *621*, 2–75. [CrossRef]
9. Yamawaki, K. Dynamical Gauge Boson of Hidden Local Symmetry within the Standard Model. *arXiv* **2018**, arXiv:1803.07271.
10. Eichenherr, H. SU(N) Invariant Nonlinear Sigma Models. *Nucl. Phys. B* **1978**, *146*, 215; Erratum in *Nucl. Phys. B* **1979**, *155*, 544. [CrossRef]
11. Golo, V.L.; Perelomov, A.M. Solution of the Duality Equations for the Two-Dimensional SU(N) Invariant Chiral Model. *Phys. Lett.* **1978**, *79B*, 112. [CrossRef]
12. D'Adda, A.; Luscher, M.; Vecchia, P.D. A 1/n Expandable Series of Nonlinear Sigma Models with Instantons. *Nucl. Phys. B* **1978**, *146*, 63. [CrossRef]
13. D'Adda, A.; Vecchia, P.D.; Luscher, M. Confinement and Chiral Symmetry Breaking in CP**n-1 Models with Quarks. *Nucl. Phys. B* **1979**, *152*, 125. [CrossRef]
14. Witten, E. Instantons, the Quark Model, and the 1/n Expansion. *Nucl. Phys. B* **1979**, *149*, 285. [CrossRef]
15. Arefeva, I.Y.; Azakov, S.I. Renormalization and Phase Transition in the Quantum Cp**(n-1) Model (d = 2, 3). *Nucl. Phys. B* **1980**, *162*, 298. [CrossRef]
16. Haber, H.E.; Hinchliffe, I.; Rabinovici, E. The CP**(n-1) Model with Unconstrained Variables. *Nucl. Phys. B* **1980**, *172*, 458. [CrossRef]
17. Kugo, T.; Terao, H.; Uehara, S. Dynamical Gauge Bosons and Hidden Local Symmetries. *Prog. Theor. Phys. Suppl.* **1985**, *85*, 122. [CrossRef]
18. Weinberg, S. Effective field theories in the large N limit. *Phys. Rev. D* **1997**, *56*, 2303. [CrossRef]
19. Brezin, E.; Hikami, S.; Zinn-Justin, J. Generalized Nonlinear Σ Models with Gauge Invariance. *Nucl. Phys. B* **1980**, *165*, 528. [CrossRef]
20. Bando, M.; Taniguchi, Y.; Tanimura, S. Dynamical gauge boson and strong-weak reciprocity. *Prog. Theor. Phys.* **1997**, *97*, 665. [CrossRef]
21. Weinberg, S.; Witten, E. Limits on Massless Particles. *Phys. Lett.* **1980**, *96*, 59. [CrossRef]
22. Igarashi, Y.; Johmura, M.; Kobayashi, A.; Otsu, H.; Sato, T.; Sawada, S. Stabilization of Skyrmions via ρ Mesons. *Nucl. Phys. B* **1985**, *259*, 721. [CrossRef]
23. Bardeen, W.A.; Hill, C.T.; Lindner, M. Minimal Dynamical Symmetry Breaking of the Standard Model. *Phys. Rev. D* **1990**, *41*, 1647. [CrossRef] [PubMed]
24. Miransky, V.A.; Tanabashi, M.; Yamawaki, K. Dynamical Electroweak Symmetry Breaking with Large Anomalous Dimension and t Quark Condensate. *Phys. Lett. B* **1989**, *221*, 177. [CrossRef]
25. Harada, M.; Kugo, T.; Yamawaki, K. Proving the low-energy theorem of hidden local symmetry. *Phys. Rev. Lett.* **1993**, *71*, 1299. [CrossRef]
26. Gounaris, G.J.; Sakurai, J.J. Finite width corrections to the vector meson dominance prediction for rho —> e+ e-. *Phys. Rev. Lett.* **1968**, *21*, 244. [CrossRef]

27. Harada, M.; Schechter, J. Effects of symmetry breaking on the strong and electroweak interactions of the vector nonet. *Phys. Rev. D* **1996**, *54*, 3394. [CrossRef]
28. Benayoun, M.; David, P.; DelBuono, L.; Jegerlehner, F. Upgraded Breaking of the HLS Model: A Full Solution to the $\tau^- e^+ e^-$ and ϕ Decay Issues and Its Consequences on g-2 VMD Estimates. *Eur. Phys. J. C* **2012**, *72*, 1848. [CrossRef]
29. Harada, M.; Yamawaki, K. Conformal phase transition and fate of the hidden local symmetry in large N(f) QCD. *Phys. Rev. Lett.* **1999**, *83*, 3374. [CrossRef]
30. Komargodski, Z. Vector Mesons and an Interpretation of Seiberg Duality. *J. High Energy Phys.* **2011**, *1102*, 019. [CrossRef]
31. Kitano, R. Hidden local symmetry and color confinement. *J. High Energy Phys.* **2011**, *1111*, 124. [CrossRef]
32. Seiberg, N. Exact results on the space of vacua of four-dimensional SUSY gauge theories. *Phys. Rev. D* **1994**, *49*, 6857. [CrossRef] [PubMed]
33. Matsuzaki, S.; Ohki, H.; Yamawaki, K. Dark Side of the Standard Model: Dormant New Physics Awaken. *arXiv* **2016**, arXiv:1608.03691.