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Existence and Uniqueness of Solutions of Hammerstein-Type Functional Integral Equations

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Abstract: The authors deal with nonlinear and general Hammerstein-type functional integral equations (HTFIEs). The first objective of this work is to apply and extend Burton's method to general and nonlinear HTFIEs in a Banach space via the Chebyshev norm and complete metric. The second objective of the paper is to extend and improve some earlier results to nonlinear HTFIEs. The authors prove two new theorems with regard to the existence and uniqueness of solutions (EUSs) of HTFIEs via a technique called progressive contractions, which belongs to T. A. Burton, and the Chebyshev norm and complete metric.

Keywords: existence and uniqueness of solutions; nonlinear HTFIE; fixed point; progressive; contraction; Chebyshev norm; complete metric

MSC: 45D05; 45G10; 45J05; 45H10; 47H10

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1. Introduction

Integral equations (IEs) have wide applications in numerous scientific areas. In fact, IEs occur widely in diverse areas of applied mathematics and physics. They offer a powerful technique for solving a variety of real-world problems; see the books of Burton [1], Corduneanu [2], Rahman [3], and Wazwaz [4]. Next, one obvious reason for using IEs rather than differential equations is also that all of the conditions specifying the initial value problems or boundary value problems for a differential equation can often be condensed into a single IE. This is an important advantage during research. Hence, the EUSs of HTFIEs deserve to be investigated. For the sake of the brevity, we would not like to outline more information.

The aim of this work is to obtain some new results with regard to EUSs for general and nonlinear HTFIEs. The approach of this work is based on verifying the existence of a unique solution of HTFIEs on an interval with a short length. To reach our aim, we convert the considered HTFIEs to a new starting time such that the solution on another small length interval is fitted onto the first solution and so forth.

We will now outline some earlier works with regard to EUSs and some other qualitative properties of solutions of IEs and integro-differential equations (IDEs). The information to be given may be useful for readers and researchers working on the related topics.

In 2016, Burton [5] benefited from the method of "direct fixed point mappings" by considering progressive contractions. Hence, the authors [5] obtained sufficient conditions under which the following scalar IDE has a unique solution on $[0, E]$, $E > 0$, $E \in \mathbb{R}$:

$$\frac{dx}{dt} = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds, \tag{1}$$

where $f,\ g \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$, and $A \in C((0,\infty), \mathbb{R})$.

In [5], it is also asserted that the vector case of IDE (1) can be considered in an analogous way.

In 2017, Burton [6] took into consideration a scalar fractional differential equation of the Riemann–Liouville type. Burton [6] discussed the existence of a global solution to that equation via a method called progressive contractions.

In 2017, Burton and Purnaras [7] considered the following IE incorporating a variable time delay:

$$x(t) = L(t) + g(t, x(t)) + \int_0^t A(t-s)[f(s, x(s)) + f(s, x(s - r(s)))]ds, \tag{2}$$

where $L \in C([0,\infty), \mathbb{R}), r \in C([0,\infty), (0,\infty)), f,\ g \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$ and $A \in C((0,\infty), \mathbb{R})$. Using the method of progressive contractions, Burton and Purnaras [7] discussed the EUSs of IE (2) with a variable time delay.

In 2018, Burton and Purnaras [8] considered an IE as follows:

$$x(t) = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds, \tag{3}$$

where $g \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$, $f \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$, and $A \in C((0,\infty), \mathbb{R})$, and the kernel A is also locally integrable. In Burton and Purnaras [8], it is assumed that the function g is a contraction, the function f satisfies the Lipschitz condition in x, the function A is continuous, and $\int_0^t |A(u)|du \to 0$ as $t \to \infty$. Burton and Purnaras [8] proved that IE (3) has a unique solution in a closed interval case according to the fixed point method.

In 2019, Burton [9] presented a simple method called a progressive contraction to get a unique solution of an IE, which is given by

$$x(t) = g(t, x(t)) + \int_0^t A(t-s) f(t, s, x(s)) ds. \tag{4}$$

The constructed progressive contraction, which belongs to Burton [9], also allows the existence of solutions of IE (4) on $\mathbb{R}_+$, $\mathbb{R}_+ = [0,\infty)$.

In 2019, Burton and Purnaras [8] studied IEs which have the form of

$$x(t) = g(t, x(t)) + \int_0^t A(t,s) v(t, s, x(s)) ds. \tag{5}$$

Burton and Purnaras [10] offered an elementary alternative to measures of non-compactness and Darbo's theorem by using progressive contractions. This method yields a unique fixed point, unlike Darbo's theorem. The authors also offered a technique for finding the mapping set with regard to IE (5).

We should mention that IE (4) has a convolution-type kernel $A(t-s)$. However, IE (5) has a nonconvolution-type kernel $A(t,s)$. Hence, IE (4) and IE (5) are different.

In 2019, using methods of fixed point theory, Burton and Purnaras [11] studied an IDE as follows:

$$\frac{dx}{dt} = g(t, x(t)) + f(t, x(t)) \int_0^t A(t,s) v(t, s, x(s)) ds. \tag{6}$$

Burton and Purnaras [11] then used a technique called "progressive contractions" to prove that the product term on the right-hand side of the IDE (6) is again a contraction over sufficiently small intervals. In this way, Burton and Purnaras [11] obtained a unique global solution using the solutions over successive intervals.

In 2021, Filip and Rus [12] constructed conditions such that they allow the IE

$$x(t) = g(t) + \left[\int_a^c K(t, s, x(s)) + \int_a^t H(t, s, x(s)) \right] ds \tag{7}$$

to admit a unique solution in the space $C([0, b], B)$, where B is a Banach space. Filip and Rus [12] also gave an iterative algorithm for IE (7).

In 2022, Burton and Purnaras [13] studied quadratic IEs, which have the form

$$x(t) = g(t, x(t)) + f(t, x(t)) \int_0^t A(t - s)v(t, s, x(s))ds. \tag{8}$$

Burton and Purnaras [13] suggested a new direction in fixed point theory such that in classical fixed point theorems, one has a mapping P of a closed convex set in a Banach space into itself. Instead of this case, in Burton and Purnaras [13], for the map P defined by the right-hand side of IE (8), a closed bounded convex nonempty set P is constructed such that $P : G \to G^0$, where G^0 is the interior of P.

In 2022, Ilea et al. [14] studied the functional IE of the form:

$$x(t) = g(t) + \left[\int_a^t K(t, s, B(x(s))) + \int_a^c H(t, s, A(x(s))) \right] ds. \tag{9}$$

Ilea et al. [14] constructed new results with regard to the EUSs and convergence of the relative consecutive approximations for IE (9) via the improved fibre contraction principle of Petruşel et al. [15]. Ilea et al. [14] also obtained Gronwall and comparison-type results in the Banach space.

In 2021, Gubran et al. [16] introduced a new class of contractions called a mixed type of weak and F-contractions.

Recently, Ansari et al. [17] derived some fixed point results depending upon certain contractive mappings. Ansari et al. [17] also established the EUSs of certain nonlinear IEs.

Furthermore, several attractive outcomes concerning applications of fixed point methods, various fixed point theorems, and EUSs, several other qualitative properties with regard to certain IEs and IDEs can be found in the books of Abbas et al. [18] and Burton [19], and the papers of Chauhan et al. [20], Deep et al. [21], Graef et al. [22], Lungu and Rus [23], Khan et al. [24,25], Tunç and Tunç [26,27], Tunç et al. [27,28], and their references.

Next, some other interesting results with regard to fixed point theorems in fuzzy metric spaces, elementary fixed point theory for mappings, etc., can be found in the papers of Imdad et al. [29], Sessa and Akkouchi [30], Sessa et al. [31], Jafarzadeh et al. [32], and Nazi et al. [33].

Indeed, the main motivation for this work comes from the scientific sources and works summarized above, and in particular, it comes from the results of Ilea and Otrocol [34,35]. As for the essential key source for motivation of this paper, in 2020, Ilea and Otrocol [34] extended the technique of Burton [9]. Namely, Ilea and Otrocol [34] considered the following IEs in a Banach space:

$$x(t) = \int_0^t K(t, s, x(s))ds \tag{10}$$

and

$$x(t) = g(t, x(t)) + \int_0^t f(t, s, x(s))ds. \tag{11}$$

Ilea and Otrocol [34] proved that both of these IEs (10) and (11) have a unique solution in the spaces $C(\mathbb{R}^+, B)$, $C([0,b], B)$ and $C([0,b) \times B, B)$, respectively, where $b \in \mathbb{R}$, $b > 0$, and B is a Banach space. The main results of Ilea and Otrocol [34], with regard to the uniqueness solutions of these equations, have been obtained in the spaces $C([0,\infty), B)$, $C([0,b], B)$, and $C([0,b) \times B, B)$, respectively, using the Chebyshev norm.

In this paper, firstly, instead of IEs of Ilea and Otrocol [34], we will consider the following nonlinear and general HTFIE in Banach space:

$$x(t) = F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t [K(t, s, x(s)) + Q(t, s, q(x(s)))] ds, \qquad (12)$$

where $x \in C([0,b], B)$, $t, s \in [0,b]$, $s \le t$, $F, q \in C(B, B)$, $G, H \in C([0,b] \times B, B)$ and $K, Q \in C([0,b] \times [0,b] \times B, B)$.

To the best knowledge of the authors of this paper, according to the above literature information and relative international databases, the EUSs of HTFIE (12) in a Banach space via the technique called Burton's progressive contractions, the Chebyshev norm, and complete metric have not been discussed up to now. However, Tunç and Tunç [26] and Castro and Simões [36] obtained some new results with regard to the Ulam stabilities of certain-delay Hammerstein-type integral equations in the bounded and infinite interval cases. The considered delay of Hammerstein-type integral equations and the topic of Tunç and Tunç [26] and Castro and Simões [36] are not related to those of this paper. Indeed, HTFIE (12) is a new mathematical model and it can be seen that HTFIE (1) is different from all the above mathematical models as IEs and IDEs and those can be found in the international data base. Next, from the above information, it can also be seen that direct fixed-point mappings, classical fixed-point theorems via suitable mappings of a closed convex set in a Banach space, iterative algorithms, fibre contraction principle, etc., have been utilized to provide the existence and uniqueness of solutions of the considered IEs and IDEs. In this paper, as in Ilea and Otrocol [34], we used Burton's progressive contractions in Banach spaces, the Chebyshev norm, and complete metric to discuss the EUSs of the HTFIEs taken for study. The technique used in the proofs of this paper is different from those are available in the earlier literature, and it is also a very effective technique and approach to achieve the aim of this paper. In fact, the outcomes of this study advance and improve that of Burton [9], Ilea and Otrocol [34], and others that can be found in the reference part of this paper.

In this paper, we will utilize the Bielecki norm $\| .\|_\tau$, which is given by

$$\|x\|_\tau = \max|x(t)|\exp(-\tau t), \ \tau > 0,$$

and the Chebyshev norm, which is given by

$$\|x\|_\infty = \sup_{t \in [0,b]} |x(t)|.$$

Throughout the rest of the paper, let $C([.], B)$ denote $C([0,b], B)$. Next, when it is needed for the sake of the brevity, we also denote $x(t)$ and $y(t)$ by x and y, respectively.

The problem here is that we will obtain the main and application results of this work using the Chebyshev norm.

The remaining sections of this work include the following contents: The first main outcome of this study, i.e., Theorem 1, is put forward for HTFIE (12) in Section 2. Section 3 includes an application result of the Burton method, i.e., Theorem 2, to the case of functional HTFIEs. As for the other two sections, Sections 4 and 5 include the discussion and conclusion of this work, respectively.

2. Burton Method in the Case of HTFIEs

We will now present the first main outcome of this study with regard to the uniqueness of solutions of nonlinear and general HTFIE (1) in a Banach space. The approach of proof is based on the progressive contraction, the Chebyshev norm, and complete metric (see Burton [9], Ilea and Otrocol [34]). The first outcome of this study is arranged hereinafter in Theorem 1.

We now consider HTFIE (12) with the following conditions:

$$
\begin{aligned}
&(As1)\ F \in C(B,B), G,\ H \in C([0,b] \times B, B),\\
&K,\ Q \in C([0,b] \times [0,b] \times B, B);\\
&(As2)\ |F(\omega) - F(\rho)| \leq F_b|\omega - \rho|, \forall \omega,\ \rho \in B,\\
&|G(t,\omega) - G(t,\rho)| \leq G_b|\omega - \rho|, \forall t \in [0,b], \omega,\ \rho \in B,\\
&|H(t,\omega)| \leq H_b, \forall t \in [0,b], \omega \in B,\\
&|K(t,s,\omega) - K(t,s,\rho)| \leq K_b|\omega - \rho|, \forall t,\ s \in [0,b], \omega,\ \rho \in B,\\
&|Q(t,s,q(\omega)) - Q(t,s,q(\rho))| \leq Q_b|q(\omega) - q(\rho)|, \forall t,\ s \in [0,b],\\
&\omega,\ \rho \in B,\\
&|q(\omega) - q(\rho)| \leq q_b|\omega - \rho|, \forall \omega,\ \rho \in B,
\end{aligned}
$$

where F_b, G_b, H_b, K_b, Q_b and q_b are positive constants.

Theorem 1. *If the conditions $(As1)$ and $(As2)$ are satisfied, then HTFIE (12) has a unique solution in the space $C([.], B)$ provided that $F_b + G_b + H_b(K_b + Q_b q_b)b < 1$.*

Proof. For the proof, it is sufficient to prove the EUSs of HTFIE (12) in $C([.], B)$, where $b \in \mathbb{R}, b > 0$. □

We describe a mapping T from $C([.], B)$ to $C([.], B)$. The mapping T is defined as

$$T(x)(t) = F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t [K(t,s,x(s)) + Q(t,s,q(x(s)))]ds.$$

To prove this theorem, we follow the method of Burton [5]. Hence, letting

$$[0,b] = \bigcup_{k=0}^{m-1} \left[\frac{kb}{m}, \frac{(k+1)b}{m}\right], m \in \mathbb{N}^*,$$

we will separate $[0,b]$ into m equal splits to represent the terminal marks by

$$0, \frac{b}{m}, \frac{2b}{m}, \frac{3b}{m}, \ldots, b.$$

We will actualize the proof step by step as in the following lines, respectively.

Step (1a). Presume that $(M_1,\ \|.\|_1)$ is a complete metric space (CM space), which consists of the functions $x \in C\left(\left[0, \frac{b}{m}\right], \mathbb{R}\right)$. Here, $\|.\|_1$ denotes the Chebyshev metric such that

$$\|x(t)\|_i = \max_{t \in [0, \frac{ib}{m}]} |x(t)|,$$

where $i = 1, 2, \ldots, m-1$.

We describe the transformation

$$T_1 : M_1 \to M_1 \text{ together with } x \in M_1$$

such that

$$(T_1x)(t) = F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t [K(t, s, x(s)) + Q(t, s, q(x(s)))]ds,$$

$$t \in \left[0,\ \frac{b}{m}\right].$$

Then, letting $x,\ y \in M_1$, $t \in \left[0,\ \frac{b}{m}\right]$ and using $(As1)$, $(As2)$, we find

$$\begin{aligned}
&|(T_1x)(t) - (T_1y)(t)| \\
&\quad = \left| F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t [K(t, s, x(s)) + Q(t, s, q(x(s)))]ds \right. \\
&\quad \left. - F(y(t)) - G(t, y(t)) - H(t, y(t)) \int_0^t [K(t, s, y(s)) + Q(t, s, q(y(s)))]ds \right| \\
&\quad \leq |F(x) - F(y)| + |G(t, x) - G(t, y)| \\
&\quad + H_b \int_0^t |K(t, s, x(s)) - K(t, s, y(s))| ds \\
&\quad + H_b \int_0^t |Q(t, s, q(x(s))) - Q(t, s, q(y(s)))| ds \\
&\quad \leq (F_b + G_b)|x - y| + H_b K_{b,1} \int_0^t |x(s) - y(s)| ds \\
&\quad + H_b Q_{b,1} \int_0^t |q(x(s)) - q(y(s))| ds \\
&\quad \leq (F_b + G_b)|x - y| + H_b K_{b,1} \tfrac{b}{m} \max_{t \in [0, \frac{b}{m}]} |x(t) - y(t)| \\
&\quad + H_b Q_{b,1} q_{b,1} \tfrac{b}{m} \max_{t \in [0, \frac{b}{m}]} |x(t) - y(t)| \\
&\quad \leq (F_b + G_b)|x - y| \qquad + H_b (K_{b,1} + Q_{b,1} q_{b,1}) \tfrac{b}{m} \|x - y\|_1 \\
&\qquad\qquad \leq \left[F_b + G_b + H_b (K_{b,1} + Q_{b,1} q_{b,1}) \tfrac{b}{m} \right] \|x - y\|_1.
\end{aligned}$$

Hence, we have

$$\max_{t \in [0, \frac{b}{m}]} |(T_1x)(t) - (T_1y)(t)| \leq \left[F_b + G_b + H_b (K_{b,1} + Q_{b,1} q_{b,1}) \frac{b}{m} \right] \|x - y\|_1.$$

Then, it follows that

$$\|T_1(x) - T_1(y)\|_1 \leq \left[F_b + G_b + H_b (K_{b,1} + Q_{b,1} q_{b,1}) \frac{b}{m} \right] \|x - y\|_1.$$

Then, the mapping T_1 is a contraction. Hence, it follows that this contraction has a unique fixed point x_1^* on the interval $\left[0, \frac{b}{m}\right]$ such that

$$\begin{aligned}
&(T_1x_1^*)(t) = x_1^*(t) = F(x_1^*(t)) + G(t, x_1^*(t)) \\
&+ H(t, x_1^*(t)) \int_0^t \left[K(t, s, x_1^*(s))) + Q(t, s, q(x_1^*(s))) \right] ds, t \in \left[0, \tfrac{b}{m}\right].
\end{aligned} \tag{13}$$

Step (2a). Presume that $(M_2, \|.\|_2)$ is a CM space, which consists of the functions $x \in C\left(\left[0, \frac{2b}{m}\right], \mathbb{R}\right)$ together with the same metric as that in Step (1a) and

$$x(t) = x_1^*(t) \text{ on } \left[0, \frac{b}{m}\right].$$

We describe a transformation

$$T_2 : M_2 \to M_2 \text{ together with } x \in M_2$$

such that

$$(T_2x)(t) = F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t [K(t, s, x(s)) + Q(t, s, q(x(s)))]ds.$$

We should note that for $t \in \left[0, \frac{b}{m}\right]$ and $x \in M_2$, it follows that $x = x_1^*$ is a fixed point. Then, according to (13), we can derive that

$$(T_2x)(t) = \begin{cases} x_1^*(t), \ t \in \left[0, \frac{b}{m}\right] \\ F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t [K(t, s, x(s)) + Q(t, s, q(x(s)))]ds, \\ \ t \in \left[\frac{b}{m}, \frac{2b}{m}\right] \end{cases}$$

$$= \begin{cases} x_1^*(t), \ t \in \left[0, \frac{b}{m}\right] \\ F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^{\frac{b}{m}} [K(t, s, x_1^*(s)) + Q(t, s, q(x_1^*(s)))]ds \\ +H(t, x(t)) \int_{\frac{b}{m}}^t [K(t, s, x(s)) + Q(t, s, q(x_1(s)))]ds, \ t \in \left[\frac{b}{m}, \frac{2b}{m}\right]. \end{cases}$$

Then, letting $x, \ y \in M_2$, we arrive at

$$\begin{aligned} |T_2(x)(t) - T_2(y)(t)| &\leq (F_b + G_b)|x - y| + H_b \int_{\frac{b}{m}}^t |K(t, s, x(s)) - K(t, s, y(s))|ds \\ &+ H_b \int_{\frac{b}{m}}^t |Q(t, s, q(x(s))) - Q(t, s, q(y(s)))|ds \\ &\leq (F_b + G_b)|x - y| + H_b K_{b,2} \int_{\frac{b}{m}}^t |x(s) - y(s)|ds \\ &+ H_b Q_{b,2} \int_{\frac{b}{m}}^t |q(x(s)) - q(y(s))|ds \\ &\leq (F_b + G_b)|x - y| + H_b K_{b,2} \frac{b}{m} \max_{t \in [\frac{b}{m}, \frac{2b}{m}]} |x(t) - y(t)| \\ &+ H_b Q_{b,2} q_{b,2} \frac{b}{m} \max_{t \in [\frac{b}{m}, \frac{2b}{m}]} |x(t) - y(t)| \\ &\leq (F_b + G_b)|x - y| + H_b K_{b,2} \frac{b}{m} \max_{t \in [0, \frac{2b}{m}]} |x(t) - y(t)| \\ &+ H_b Q_{b,2} q_{b,2} \frac{b}{m} \max_{t \in [0, \frac{2b}{m}]} |x(t) - y(t)|. \end{aligned}$$

Hence, we have

$$\max_{t\in[0,\frac{2b}{m}]}|(T_2x)(t)-(T_2y)(t)|\leq(F_b+G_b)|x-y|+H_bK_{b,2}\tfrac{b}{m}\max_{t\in[0,\frac{2b}{m}]}|x(t)-y(t)|$$
$$+H_bQ_{b,2}q_{b,2}\tfrac{b}{m}\max_{t\in[0,\frac{2b}{m}]}|x(t)-y(t)|.$$

By virtue of the above inequality, we conclude that

$$\|T_2(x)-T_2(y)\|_2\leq\left[F_b+G_b+H_b(K_{b,2}+Q_{b,2}q_{b,2})\frac{b}{m}\right]\|x-y\|_2$$

such that T_2 is a contraction on the interval $\left[0,\frac{2b}{m}\right]$ with a unique fixed point x_2^* on the same entire interval, i.e., $\left[0,\frac{2b}{m}\right]$. Hence, clearly, x_2^* is a unique and continuous solution of (13) such that $x_2^*(t)=x_1^*(t)$ on the interval $\left[0,\frac{b}{m}\right]$.

Step (3a). Presume that $(M_3,\ \|.\|_3)$ is a CM space, which consists of the functions $x\in C\left(\left[0,\frac{3b}{m}\right],\mathbb{R}\right)$ together with the same metric as that in Step (1a) and

$$x(t)=x_2^*(t)\text{ on }\left[0,\ \frac{2b}{m}\right].$$

We designate a mapping

$$T_3:M_3\to M_3\ \text{ together with } x\in M_3$$

such that

$$(T_3x)(t)=F(x(t))+G(t,x(t))+H(t,x(t))\int_0^t[K(t,s,x(s))+Q(t,s,q(x(s)))]ds.$$

Similar to the former steps, it can be easily shown that the mapping T_3 is a contraction. This contraction has a unique fixed point x_3^* on the interval $\left[0,\frac{3b}{m}\right]$. Indeed, x_3^* is a unique continuous solution of (13) on the interval $\left[0,\frac{3b}{m}\right]$ with $x_3^*(t)=x_2^*(t)$ on the interval $\left[0,\frac{2b}{m}\right]$. Hence the operator T_3 is well defined and a contraction on the interval $\left[0,\frac{3b}{m}\right]$.

Next, benefiting from the method of mathematical induction, we obtain that T_m is a contraction and hence we get a unique and continuous solution on the interval $[0,b]$. For the case $2<i<m-1$, let x_{i-1}^* be the unique solution of (13) on the interval $\left[0,\ \frac{(i-1)b}{m}\right]$. Next, presume that $(M_i,\ \|.\|_i)$ is a CM space, which consists of the functions $x\in C\left(\left[0,\frac{ib}{m}\right],\mathbb{R}\right)$ together with the supremum metric, such that

$$x(t)=x_{i-1}^*(t)\text{ on the interval }\left[0,\ \frac{(i-1)b}{m}\right].$$

We now describe a mapping

$$T_i:M_i\to M_i\ \text{ together with } x\in M_i$$

such that

$$(T_i x)(t) = F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t [K(t, s, x(s)) + Q(t, s, q(x(s)))] ds$$

$$= \begin{cases} x_{i-1}^*(t), \ t \in \left[0, \frac{(i-1)b}{m}\right] \\ F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t K(t, s, x(s)) ds \\ +H(t, x(t)) \int_0^t Q(t, s, q(x(s))) ds, \ t \in \left[\frac{(i-1)b}{m}, \frac{ib}{m}\right] \end{cases}$$

$$= \begin{cases} x_{i-1}^*(t), \ t \in \left[0, \frac{(i-1)b}{m}\right] \\ F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^{\frac{(i-1)b}{m}} \left[K(t, s, x_{i-1}^*(s)) + Q(t, s, q(x_{i-1}^*(s)))\right] ds \\ +H(t, x(t)) \int_{\frac{(i-1)b}{m}}^t [K(t, s, x(s)) + Q(t, s, q(x(s)))] ds, \ t \in \left[\frac{(i-1)b}{m}, \frac{ib}{m}\right]. \end{cases}$$

We will now prove that T_i is a contraction. Hence, let $x, \ y \in M_i$ and $t \in \left[0, \frac{ib}{m}\right]$. Next, since $x(t) = y(t) = x_{i-1}^*(t)$ on the interval $\left[0, \frac{(i-1)b}{m}\right]$, then

$$\begin{aligned} |T_i(x)(t) - T_i(y)(t)| &\le (F_b + G_b)|x - y| + H_b \int_{\frac{(i-1)b}{m}}^t |K(t, s, x(s)) - K(t, s, y(s))| ds \\ &+ H_b \int_{\frac{(i-1)b}{m}}^t |Q(t, s, q(x(s))) - Q(t, s, q(y(s)))| ds \\ &\le (F_b + G_b)|x - y| + H_b K_{b,i} \int_{\frac{(i-1)b}{m}}^t |x(s) - y(s)| ds \\ &+ H_b Q_{b,i} \int_{\frac{(i-1)b}{m}}^t |q(x(s)) - q(y(s))| ds \\ &\le (F_b + G_b)|x - y| + H_b K_{b,i} \frac{b}{m} \max_{t \in [\frac{(i-1)b}{m}, \frac{ib}{m}]} |x(t) - y(t)| \\ &+ H_b Q_{b,i} q_{b,i} \frac{b}{m} \max_{t \in [\frac{(i-1)b}{m}, \frac{ib}{m}]} |x(t) - y(t)| \\ &\le (F_b + G_b)|x - y| + H_b K_{b,i} \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)| \\ &+ H_b Q_{b,i} q_{b,i} \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)|. \end{aligned}$$

Hence, we have

$$\begin{aligned} \max_{t \in [0, \frac{ib}{m}]} |(T_i x)(t) - (T_i y)(t)| \le (F_b + G_b)|x - y| + H_b K_{b,i} \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)| \\ + H_b Q_{b,i} q_{b,i} \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)|. \end{aligned}$$

Thus, it follows that

$$\|T_i(x) - T_i(y)\|_i \le \left[F_b + G_b + H_b(K_{b,i} + Q_{b,i} q_{b,i}) \frac{b}{m}\right] \|x - y\|_i.$$

According to the above outcomes, we conclude that T_i is a contraction of the interval $\left[0, \frac{ib}{m}\right]$ including a unique fixed point x_i^*, and these outcomes implement the proof of Theorem 1.

3. Applications of Burton Method to the Case of HTFIEs

We now take into account the following HTFIE:

$$x(t) = g(t, x(t)) + h(t, x(t)) \int_0^t [p(t, s, x(s)) + r(t, s, q(x(s)))]ds, \tag{14}$$

where $t \in [0, b)$, $x \in C([0, b), \mathbb{R})$, $0 \leq s \leq t < b$, $q \in C(\mathbb{R}, \mathbb{R})$, $g,\ h \in C([0, b) \times \mathbb{R}, \mathbb{R})$ and $p,\ r \in C([0, b) \times [0, b) \times \mathbb{R}, \mathbb{R})$ are given functions. We discuss the solution in the set $C([0, b], \mathbb{R})$ and consider the Chebyshev norm. Next, as before, letting

$$[0, b] = \bigcup_{k=0}^{m-1} \left[\frac{kb}{m}, \frac{(k+1)b}{m}\right], m \in \mathbb{N}^*,$$

we will separate $[0, b]$ into m equal splits such that the terminal marks are represented as

$$0, \frac{b}{m}, \frac{2b}{m}, \frac{3b}{m}, \ldots, b.$$

Theorem 2. *We assume there exist positive constants g_b, h_b, $L_{p,k}$, $L_{r,k}$, q_b such that the below conditions $(C1)$, $(C2)$ hold and $g_b + h_b\Big(L_{p,k}(b) + L_{r,k}(b)q_b(b)\Big)b < 1$:*

$$\begin{aligned}
&(C1) g,\ h \in C([0, b) \times \mathbb{R}, \mathbb{R}), p,\ r \in C([0, b) \times [0, b) \times \mathbb{R}, \mathbb{R}),\\
&(C2) |g(t, \omega) - g(t, \rho)| \leq g_b|\omega - \rho|, g_b \in (0, 1), \forall t \in [0, b), \omega,\ \rho \in \mathbb{R},\\
&|h(t, \omega)| \leq h_b, \forall t \in [0, b), \omega \in \mathbb{R},\\
&|p(t, s, \omega) - p(t, s, \rho)| \leq L_{p,k}(b)|\omega - \rho|, \forall t,\ s \in [0, b), \omega,\ \rho \in \mathbb{R},\\
&|r(t, s, q(\omega)) - r(t, s, q(\rho))| \leq L_{r,k}(b)|q(\omega) - q(\rho)|, \forall t,\ s \in [0, b),\\
&\omega,\ \rho \in \mathbb{R},\\
&|q(\omega) - q(\rho)| \leq q_b|\omega - \rho|, \forall \omega,\ \rho \in \mathbb{R}\ .
\end{aligned}$$

Then, HTFIE (14) admits a unique solution in $C([0, b) \times B, B)$.

Proof. Letting

$$[0, b] = \bigcup_{k=0}^{m-1} \left[\frac{kb}{m}, \frac{(k+1)b}{m}\right], m \in \mathbb{N}^*,$$

we will separate $[0, b]$ into m equal splits such that the terminal marks are represented as

$$0, \frac{b}{m}, \frac{2b}{m}, \frac{3b}{m}, \ldots, b.$$

We will complete this proof step by step as in the following lines, respectively. □

Step (1b). Presume that $(M_1, \|.\|_1)$ is a CM space including the functions $x \in C\left(\left[0, \frac{b}{m}\right], \mathbb{R}\right)$ together with the Chebyshev metric $\|.\|_1$. This metric is described by

$$\|x(t)\|_i = \max_{t \in [0, \frac{ib}{m}]} |x(t)|,$$

where $i = 1, 2, \ldots, m-1$.

We describe a transformation

$$T_1 : M_1 \to M_1 \text{ together with } x \in M_1$$

such that

$$(T_1x)(t) = g(t, x(t)) + h(t, x(t))\int_0^t [p(t, s, x(s)) + r(t, s, q(x(s)))]ds,\\ t \in \left[0,\ \tfrac{b}{m}\right].$$

Then, letting $x,\ y \in M_1$, $t \in \left[0,\ \frac{b}{m}\right]$ and using $(C1)$, $(C2)$, we get

$$\begin{aligned}
&|(T_1x)(t) - (T_1y)(t)| \\
&= \Bigg|g(t, x(t)) + h(t, x(t))\int_0^t [p(t, s, x(s)) + r(t, s, q(x(s)))]ds \\
&- g(t, y(t)) - h(t, y(t))\int_0^t [p(t, s, y(s)) + r(t, s, q(y(s)))]ds\Bigg| \\
&\le |g(t, x) - g(t, y)| \\
&+ h_b\int_0^t |p(t, s, x(s)) - p(t, s, y(s))|ds \\
&+ h_b\int_0^t |r(t, s, q(x(s))) - r(t, s, q(y(s)))|ds \\
&\le g_b|x - y| + h_b L_{p,1}(b)\int_0^t |x(s) - y(s)|ds \\
&+ h_b L_{r,1}(b) q_{b,1}(b)\int_0^t |x(s) - y(s)|ds \\
&\le g_b|x - y| + h_b L_{p,1}(b)\tfrac{b}{m}\max_{t\in[0,\frac{b}{m}]} |x(t) - y(t)| \\
&+ h_b L_{r,1}(b) q_{b,1}(b)\tfrac{b}{m}\max_{t\in[0,\frac{b}{m}]} |x(t) - y(t)| \\
&\le \left[g_b + h_b\big(L_{p,1}(b) + L_{r,1}(b)q_{b,1}(b)\big)\tfrac{b}{m}\right]\|x - y\|_1.
\end{aligned}$$

Thus, it follows that

$$\max_{t\in[0,\frac{b}{m}]} |(T_1x)(t) - (T_1y)(t)| \le \left[g_b + h_b\big(L_{p,1}(b) + L_{r,1}(b)q_{b,1}(b)\big)\frac{b}{m}\right]\|x - y\|_1.$$

Hence, we have

$$\|T_1(x) - T_1(y)\|_1 \le \left[g_b + h_b\big(L_{p,1}(b) + L_{r,1}(b)q_{b,1}(b)\big)\frac{b}{m}\right]\|x - y\|_1.$$

Then, the mapping T_1 is a contraction. This contraction has a unique fixed point x_1^* on the interval $\left[0, \frac{b}{m}\right]$ such that

$$\begin{aligned}
&(T_1x_1^*)(t) = x_1^*(t) = g(t, x_1^*(t)) \\
&+ h(t, x_1^*(t))\int_0^t \left[p(t, s, x_1^*(s))) + r(t, s, q(x_1^*(s)))\right]ds, t \in \left[0, \tfrac{b}{m}\right].
\end{aligned} \tag{15}$$

Step (2b). Presume that $(M_2, \|.\|_2)$ is a CM space, which consists of the functions $x \in C\left(\left[0, \frac{2b}{m}\right], \mathbb{R}\right)$ together with the same metric as that one in Step (1a) and

$$x(t) = x_1^*(t) \text{ on } \left[0, \frac{b}{m}\right].$$

We describe a map

$$T_2 : M_2 \to M_2, \text{ together } x \in M_2,$$

such that

$$(T_2x)(t) = g(t, x(t)) + h(t, x(t)) \int_0^t [p(t, s, x(s)) + r(t, s, q(x(s)))]ds. \tag{16}$$

We should note that for $t \in \left[0, \frac{b}{m}\right]$ and $x \in M_2$, it follows that $x = x_1^*$ is a fixed point. Then, according to (16) we can derive that

$$(T_2x)(t) = \begin{cases} x_1^*(t),\ t \in \left[0, \frac{b}{m}\right] \\ g(t, x(t)) + h(t, x(t)) \int_0^t [p(t, s, x(s)) + r(t, s, q(x(s)))]ds, \\ \ t \in \left[\frac{b}{m}, \frac{2b}{m}\right] \end{cases}$$

$$= \begin{cases} x_1^*(t),\ t \in \left[0, \frac{b}{m}\right] \\ g(t, x(t)) + h(t, x_1(t)) \int_0^{\frac{b}{m}} \left[p(t, s, x_1^*(s)) + r(t, s, q(x_1^*(s)))\right]ds \\ +h(t, x(t)) \int_{\frac{b}{m}}^t [p(t, s, x(s)) + r(t, s, q(x(s)))]ds,\ t \in \left[\frac{b}{m}, \frac{2b}{m}\right]. \end{cases}$$

Then, letting $x,\ y \in M_2$, we derive that

$$\begin{aligned} &|T_2(x)(t) - T_2(y)(t)| \le g_b|x - y| + h_b \int_{\frac{b}{m}}^t |p(t, s, x(s)) - p(t, s, y(s))|ds \\ &+h_b \int_{\frac{b}{m}}^t |r(t, s, q(x(s))) - r(t, s, q(y(s)))|ds \\ &\le g_b|x - y| + h_b L_{p,2}(b) \int_0^t |x(s) - y(s)|ds \\ &+h_b L_{r,2}(b) \int_0^t |q(x(s)) - q(y(s))|ds \\ &\le g_b|x - y| + h_b L_{p,2}(b) \frac{b}{m} \max_{t \in [\frac{b}{m}, \frac{2b}{m}]} |x(t) - y(t)| \\ &+h_b L_{r,2}(b) q_{b,2}(b) \frac{b}{m} \max_{t \in [\frac{b}{m}, \frac{2b}{m}]} |x(t) - y(t)| \\ &\le g_b|x - y| + h_b L_{p,2}(b) \frac{b}{m} \max_{t \in [0, \frac{2b}{m}]} |x(t) - y(t)| \\ &+h_b L_{r,2}(b) q_{b,2}(b) \frac{b}{m} \max_{t \in [0, \frac{2b}{m}]} |x(t) - y(t)|. \end{aligned}$$

Thus, we note that

$$|(T_2x)(t)-(T_2y)(t)|\leq g_b|x-y|+h_bL_{p,2}(b)\tfrac{b}{m}\max_{t\in[0,\frac{2b}{m}]}|x(t)-y(t)|$$
$$+h_bL_{r,2}(b)q_{b,2}(b)\tfrac{b}{m}\max_{t\in[0,\frac{2b}{m}]}|x(t)-y(t)|.$$

Hence, we have

$$\|T_2(x)-T_2(y)\|_2\leq\left[g_b+h_b\big(L_{p,2}(b)+L_{r,2}(b)q_{b,2}(b)\big)\frac{b}{m}\right]\|x-y\|_2.$$

As a consequence, we conclude that T_2 is a contraction with a unique fixed point x_2^* on the interval $\left[0,\frac{2b}{m}\right]$. Hence, clearly, x_2^* is a unique and continuous solution of (15) with $x_2^*(t)=x_1^*(t)$ on the interval $\left[0,\frac{b}{m}\right]$.

Step (3b). Presume that $(M_3,\ \|.\|_3)$ is a CM space, which includes the functions $x\in C\left(\left[0,\frac{3b}{m}\right],\mathbb{R}\right)$ together with the same metric as that in Step (1a) and

$$x(t)=x_2^*(t)\text{ on }\left[0,\ \frac{2b}{m}\right].$$

Similar to Step (2a), we can define a map T_3 and show that the map T_3 is a contraction with a unique fixed point x_3^* on the interval $\left[0,\frac{3b}{m}\right]$. Then, we can obtain a continuous and unique solution x_3^* on the interval $\left[0,\ \frac{b}{m}\right]$.

Finally, in what follows, benefiting from the method of mathematical induction, we define a mapping T_i and show that T_i is a contraction. Hence, we will obtain a unique and continuous solution on $[0,b]$.

For the case $2<i<m-1$, let x_{i-1}^* be the unique solution of (4) on the interval $\left[0,\ \frac{(i-1)b}{m}\right]$. Next, presume that $(M_i,\ \|.\|_i)$ is a CM space, which consists of functions $x\in C\left(\left[0,\frac{ib}{m}\right],\mathbb{R}\right)$ together with the supremum metric such that

$$x(t)=x_{i-1}^*(t)\text{ on the interval }\left[0,\ \frac{(i-1)b}{m}\right].$$

We now describe a mapping

$$T_i:M_i\to M_i\ \text{ together }x\in M_i$$

such that

$$(T_ix)(t)=g(t,x(t))+h(t,x(t))\int_0^t[p(t,s,x(s))+r(t,s,q(x(s)))]ds$$
$$=\begin{cases}x_{i-1}^*(t),\ t\in\left[0,\frac{(i-1)b}{m}\right]\\ g(t,x(t))+h(t,x(t))\int_0^t p(t,s,x(s))ds\\ +h(t,x(t))\int_0^t r(t,s,q(x(s)))ds,\ t\in\left[\frac{(i-1)b}{m},\frac{ib}{m}\right]\end{cases}$$

$$=\begin{cases}x_{i-1}^*(t),\ t\in\left[0,\frac{(i-1)b}{m}\right]\\ g(t,x(t))+h(t,x(t))\int_0^{\frac{(i-1)b}{m}}\left[p(t,s,x_{i-1}^*(s))+r(t,s,q(x_{i-1}^*(s)))\right]ds\\ +h(t,x(t))\int_{\frac{(i-1)b}{m}}^t[p(t,s,x(s))+r(t,s,q(x(s)))]ds,\ t\in\left[\frac{(i-1)b}{m},\frac{ib}{m}\right].\end{cases}$$

We will now prove that T_i is a contraction. Let $x,\ y \in M_i$ and $t \in \left[0, \frac{ib}{m}\right]$. Since $x(t) = y(t) = x^*_{i-1}(t)$ on the interval $\left[0, \frac{(i-1)b}{m}\right]$, we obtain

$$
\begin{aligned}
&|T_i(x)(t) - T_i(y)(t)| \le g_b|x-y| \\
&+h_b \int_{\frac{(i-1)b}{m}}^{t} |p(t,s,x(s)) - p(t,s,y(s))|ds \\
&+h_b \int_{\frac{(i-1)b}{m}}^{t} |r(t,s,q(x(s))) - r(t,s,q(y(s)))|ds \\
&\le g_b|x-y| + h_b L_{p,i}(b) \int_{\frac{(i-1)b}{m}}^{t} |x(s) - y(s)|ds \\
&+h_b L_{r,i}(b) q_{b,i}(b) \int_{\frac{(i-1)b}{m}}^{t} |x(s) - y(s)|ds \\
&\le g_b|x-y| + h_b L_{p,i}(b) \frac{b}{m} \max_{t \in [\frac{(i-1)b}{m}, \frac{ib}{m}]} |x(t) - y(t)| \\
&+h_b L_{r,i}(b) q_{b,i}(b) \frac{b}{m} \max_{t \in [\frac{(i-1)b}{m}, \frac{ib}{m}]} |x(t) - y(t)| \\
&\le g_b|x-y| + h_b L_{p,i}(b) \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)| \\
&\qquad +h_b L_{r,i}(b) q_{b,i}(b) \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)|.
\end{aligned}
$$

Hence, we can conclude that

$$
\|T_i(x) - T_i(y)\|_i \le \left[g_b + h_b\left(L_{p,i}(b) + L_{r,i}(b) q_{b,i}(b)\right)\frac{b}{m}\right] \|x-y\|_i.
$$

According to the above result, we conclude that T_i is a contraction on the interval $\left[0, \frac{ib}{m}\right]$ including a unique fixed point x^*_i. This outcome is the end of the proof of Theorem 2.

4. Discussion

1. According to the literature review in the introduction section of this paper and that can be found in international data of relevant literature, the EUSs of HTFIEs (12) and (14) have not been discussed in Banach spaces by this time depending upon the technique called progressive contractions, which belongs to T. A. Burton, and the Chebyshev norm and complete metric. This is a new study for these qualitative concepts with regard to HTFIEs (12) and (14) in Banach spaces via progressive contractions, the Chebyshev norm, and complete metric.

2. When $F(x(t)) = 0$, $G(t,x(t)) = g(t,x(t))$, $H(t,x(t)) = 1$ and $Q(t,s,q(x(s))) = 0$, then HTFIE (12) is reduced to the first IE of Ilea and Otrocol [34], i.e.,

$$
x(t) = \int_0^t K(t,s,x(s))ds.
$$

When $h(t,x(t)) = 1$ $p(t,s,x(s)) = f(t,s,x(s))$ and $r(t,s,q(x(s))) = 0$, then HTFIE (14) is reduced to the second IE of Ilea and Otrocol [16], i.e.,

$$
x(t) = g(t,x(t)) + \int_0^t f(t,s,x(s))ds.
$$

For both of these cases, the conditions of Theorems 1 and 2 coincide with that of Theorems 2.1 and 3.1 of Ilea and Otrocol [34], respectively. Hence, our main results as Theorems 1 and 2 include and improve the main results of Ilea and Otrocol [34] (Theorems 2.1 and 3.1). Furthermore, we extended the Burton method [9] to a case where instead of scalar IEs, we considered two nonlinear and general HTFIEs in Banach spaces. This is a positive difference and new contribution with regard to the EUSs of nonlinear and general HTFIEs in Banach spaces.

3. The outcomes of this work leads new improvements and contributions from known IEs to nonlinear and general HTFIEs.

5. Conclusions

In this study, two nonlinear and general HTFIEs have been taken into consideration. The Burton method called progressive contractions has been applied to general and nonlinear HTFIEs in Banach spaces via the Chebyshev norm and compete metric to obtain the existence and uniqueness of solutions of these HTFIEs. In this study, two new theorems that have sufficient conditions have been proven with regard to the existence and uniqueness of solutions. Hence, we extended the Burton method, i.e., Burton's progressive contractions method, to general and nonlinear HTFIEs in Banach spaces. The outcomes of this paper are new and original, and they have new contributions to the existence and uniqueness theory of IEs and IDEs. As for future research directions, Burton's progressive contractions method can be extended to Caputo fractional order and Riemann–Liouville fractional order Hammerstein IEs, Hammerstein IDEs, etc., with and without delay. For the sake of brevity, we would not like to provide proper fractional differential equation models here.

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