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Non-Classical Symmetry Analysis of a Class of Nonlinear Lattice Equations

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Abstract: In this paper, a non-classical symmetry method for obtaining the symmetries of differential–difference equations is proposed. The non-classical symmetry method introduces an additional constraint known as the invariant surface condition, which is applied after the infinitesimal transformation. By solving the governing equations that satisfy this condition, we can obtain the corresponding reduced equation. This allows us to determine the non-classical symmetry of the differential–difference equation. This method avoids the complicated calculation involved in extending the infinitesimal generator and allows for a wider range of symmetry forms. As a result, it enables the derivation of a greater number of differential–difference equations. In this paper, two kinds of (2+1)-dimensional Toda-like lattice equations are taken as examples, and their corresponding symmetric and reduced equations are obtained using the non-classical symmetry method.

Keywords: non-classical symmetry; Lattice equation; Lie symmetry; differential–difference equation

1. Introduction

A lattice system is a mathematical representation of a network or lattice of particles. It can be used to describe various phenomena, such as the vibrations of atoms and molecules in a matrix or the oscillations of particles connected by a spring [1]. Nonlinear lattice dynamics has a wide range of applications in physics, biology, statistical physics, acoustics, optics, condensed matter physics, and other fields [2,3]. The mathematical model of a lattice system often takes the form of nonlinear differential–difference equations [4].

The Lie symmetry method is not only an effective approach for obtaining exact solutions of partial differential equations but also for differential–difference equations [5,6]. The Lie symmetry analysis method has also been proposed for finding similarity reduction and exact solutions of nonlinear evolution equations [7]. In 1993, Levi [8] proposed the classical Lie symmetry method for differential–difference equations. The Toda equation is a classical model of differential–difference equation [9–12], whose symmetries have been studied [13] and the differential–difference Lie symmetry method was applied to solve a class of Toda-like lattice equations [14]. In 2022, a survey of the connection between orthogonal polynomials, Toda lattices and related lattices, and Painlevé equations (discrete and continuous) was given [15].

In 1969, Bluman and Cole [16] generalized the classical symmetry method of Lie to a non-classical method. They proposed the non-classical symmetry method for the first time and obtained a new exact solution of the one-dimensional heat conduction equation using this new method. For a partial differential equation, we know that the given system of partial differential equations is invariant under symmetry in the usual symmetry method. In the non-classical method [17,18], not only is the given system invariant, but the invariant surface conditions and their differential consequences are also invariant under the corresponding symmetry. Moreover, the over-determined system of partial differential equations is nonlinear in the non-classical method. This implies that by using non-classical

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symmetry analysis, one could obtain new symmetries that go beyond those obtained by using the classical symmetry method. Bluman and Tian [19] utilized the non-classical symmetry method to solve the nonlinear Kompaneets equation. Xin et al. [20] obtained nonlocal symmetries and solutions for the (2+1) dimension integrable Burgers equation. ARRIGO et al. [21] discussed nonclassical symmetry solutions and the methods between Bluman–Cole and Clarkson–Kruskal. Subhankar and Raja Sekhar [22] studied nonclassical symmetry analysis and conservation laws of a one-dimensional macroscopic production model and the evolution of nonlinear waves.

We consider the invariance of a given differential equation

$$\Delta = \Delta(x, t, u, u^{(1)}, \cdots, u^{(n)}) = 0, \tag{1}$$

together with its invariant surface condition

$$\Delta_0 = \xi(x, t, u)u_x + \eta(x, t, u)u_t - \phi(x, t, u) = 0, \tag{2}$$

where $u = u(x, t)$. Equation (2) describes the solution surface, which remains invariant under a one parameter group of transformations with the infinitesimal generator

$$\Gamma = \sum_{i=1}^{p} \xi_i(x, u)\frac{\partial}{\partial x_i} + \phi(x, u)\frac{\partial}{\partial u}. \tag{3}$$

$\Gamma^{(m)}$ is the m-order prolongation of Γ

$$\Gamma^{(m)} = \Gamma + \sum_{i=1}^{p}\sum_{M} \phi^{M}_{[x_i]}(x, u^{(m)})\frac{\partial}{\partial u_{x_i}}, \tag{4}$$

where

$$\phi^{M}_{[x_i]}(x, u) = D_M(\phi_{[x_i]} - \sum_{i=1}^{p} \xi_i u_{x_i}) + \sum_{i=1}^{p} \xi_i u_{x_i, M}, \tag{5}$$

with $M = (m_1, \cdots, m_i)$, $1 \le i \le p$.

It is assumed that Equation (1) is invariant under the action of infinitesimal generators (3) and invariant surface conditions (2), if and only if the system of governing equations is satisfied

$$\Gamma^{(n)}\Delta|_{\{\Delta=0, \Delta_0=0\}} = 0, \tag{6}$$

In this way, the non-classical symmetry of Equation (1) can be obtained.

The question that naturally arises is whether the non-classical method can be extended to discuss discrete equations. To the best of our knowledge, this question has not been reported so far. Based upon the discrete Lie symmetry method [23], we extend the infinitesimal generator and its prolongation to the discrete performance to analyze the non-classical symmetries of differential–difference equations in this Letter.

In Section 2, the non-classical symmetry method for differential–difference equations is introduced. In Sections 3 and 4, the non-classical symmetry method of differential–difference equations is applied to study two types of (2+1)-dimensional Toda-like lattice equations. The determinant equations of the equation are derived, and the corresponding reduced equations are obtained by solving the determinant equations. Conclusions are finally presented in Section 5.

2. Non-Classical Symmetry of Differential–Difference Equations

In this paper, we will introduce the non-classical symmetric method using an example of a differential–difference equation of order m that contains both continuous variables $x = (x_1, x_2, \cdots, x_m)$ and a discrete variable n. The equation is formulated as follows:

$$\Delta = \Delta_n^{(m)} \equiv \Delta(x, n, u(n+k)|_{k=-a}^{b}, u_{x_i}(n+k)|_{k=-a_i}^{b_i}, u_{x_i x_j}(n+k)|_{k=-a_{ij}}^{b_{ij}}) = 0, \tag{7}$$

where all $a, b, a_i, b_i, a_{ij}, b_{ij}$ are finite non-negative integers, $m = \mathrm{Max}\{b, b_i, b_{ij}\}$. The infinitesimal generators acting on Equation (7) are as follows:

$$\Gamma = \sum_{i=1}^{p} \xi_i(x, u(n)) \frac{\partial}{\partial x_i} + \phi(x, u(n), n) \frac{\partial}{\partial u(n)}. \tag{8}$$

$\Gamma^{(m)}$ is the m-order prolongation of Γ, which is formed as

$$\Gamma^{(m)} = \Gamma + \sum_{i=1}^{p} \sum_{M} \phi^{M}_{[x_i]}(x, u^m(n), n) \frac{\partial}{\partial u_{x_i}(n)}, \tag{9}$$

where

$$\phi^{M}_{[x_i]}(x, u(n), n) = D_M(\phi_{[x_i]}(n) - \sum_{i=1}^{p} \xi_i u_{x_i}(n)) + \sum_{i=1}^{p} \xi_i u_{x_i, M}(n), \tag{10}$$

with $M = (m_1, \cdots, m_i)$, $1 \leq i \leq p$.

To better understand the non-classical symmetry of differential–difference equations, consider a differential–difference equation involving $x_1 = x$ and $x_2 = t$ as an example.

$$\Delta \equiv \Delta(x, t, u(n), u(n+1), \cdots, u_x(n), u_x(n+1), \cdots, u_t(n), u_t(n+1), \cdots, u_{xt}(n), \cdots) = 0, \tag{11}$$

Then the infinitesimal transformation acting on this equation is

$$\begin{aligned} x^* &= x + \xi(x, t, u(n))\varepsilon + O(\varepsilon^2), \\ t^* &= t + \eta(x, t, u(n))\varepsilon + O(\varepsilon^2), \\ u^* &= u + \phi(x, t, u(n))\varepsilon + O(\varepsilon^2). \end{aligned} \tag{12}$$

The infinitesimal transformation is derived from the following equation

$$u(x + \varepsilon\xi, t + \varepsilon\eta) = u(x, t, u(n)) + \phi(x, t, u(n))\varepsilon + O(\varepsilon^2). \tag{13}$$

The Equation (13) is about taking the derivative ε and expanding it; thus, it can be written as follows

$$\Delta_0 = \xi(x, t, u(n))u_x(n) + \eta(x, t, u(n))u_t(n) - \phi(x, t, u(n), n) = 0. \tag{14}$$

Equation (14) is called the invariant surface condition of differential–difference equation.

According to the classical Lie symmetry method, the differential–difference Equation (11) is assumed to be invariant under the infinitesimal transformations Δ_0. The non-classical symmetry method adds invariant surface conditions on this basis, which is also invariant under infinitesimal transformations; that is, the governing equations are satisfied for Δ and Δ_0 at the same time:

$$\Gamma^{(2)} \Delta|_{\{\Delta=0, \Delta_0=0\}} = 0, \tag{15}$$

where the infinitesimal generator is

$$\Gamma = \xi(x, t, u(n)) \frac{\partial}{\partial x} + \eta(x, t, u(n)) \frac{\partial}{\partial t} + \phi(x, t, u(n), n) \frac{\partial}{\partial u(n)}. \tag{16}$$

For the convenience of later calculation, the first order and second order continuation are given respectively

$$\begin{aligned} \Gamma^{(1)} &= \Gamma + \sum_{i=n-L}^{n+M} \phi_{[x]}(x, t, u(i), i) \frac{\partial}{\partial u_x(i)} + \sum_{i=n-L}^{n+M} \phi_{[t]}(x, t, u(i), i) \frac{\partial}{\partial u_t(i)}, \\ \Gamma^{(2)} &= \Gamma^{(1)} + \sum_{i=n-L}^{n+M} \phi_{[xx]}(x, t, u(i), i) \frac{\partial}{\partial u_{xx}(i)} + \sum_{i=n-L}^{n+M} \phi_{[xt](i)}(x, t, u(i), i) \frac{\partial}{\partial u_{xt}(i)} \\ &\quad + \sum_{i=n-L}^{n+M} \phi_{[tt]}(x, t, u(i), i) \frac{\partial}{\partial u_{tt}(i)}, \end{aligned} \tag{17}$$

where $n-L \leq i \leq n+M$, L, M are non-negative integers, for short $\xi(x,t,u(n)) = \xi$, $\eta(x,t,u(n)) = \eta$, $\phi(x,t,u(i),i) = \phi(i)$; hence,

$$\phi_{[x]}(i) = D_x(\phi(i) - \xi u_x(i) - \eta u_t(i)) + \xi u_{xx}(i) + \eta u_{xt}(i), \tag{18}$$

$$\phi_{[t]}(i) = D_t(\phi(i) - \xi u_x(i) - \eta u_t(i)) + \xi u_{xt}(i) + \eta u_{tt}(i), \tag{19}$$

$$\phi_{[xx]}(i) = D_{xx}(\phi(i) - \xi u_x(i) - \eta u_t(i)) + \xi u_{xxx}(i) + \eta u_{xxt}(i), \tag{20}$$

$$\phi_{[xt]}(i) = D_{xt}(\phi(i) - \xi u_x(i) - \eta u_t(i)) + \xi u_{xxt}(i) + \eta u_{ttx}(i), \tag{21}$$

$$\phi_{[tt]}(i) = D_{tt}(\phi(i) - \xi u_x(i) - \eta u_t(i)) + \xi u_{xtt}(i) + \eta u_{ttt}(i). \tag{22}$$

We substituted (18)–(22) into the governing Equation (17) by solving the determining Equation (15); that is, the coefficient equations of the derivative of u to x, t, the expression about ξ, η, ϕ can be obtained and its corresponding characteristic equation can also be obtained under the integral of the invariant surface conditions

$$\frac{\mathrm{d}x}{\xi(x,t,u(n))} = \frac{\mathrm{d}t}{\eta(x,t,u(n))} = \frac{\mathrm{d}u_n}{\phi(x,t,u(n),n)}. \tag{23}$$

By solving this characteristic equation, the invariants of Equation (1) can be found. In this article, we only need to consider two cases under the invariant surface condition (8): $\eta \equiv 1$ and $\eta \equiv 0, \xi \equiv 1$.

The invariant surface condition Δ_0 introduced a new correlation equation for the derivative of Equation (11). We substitute the invariant surface condition Δ_0 into the governing system (15), replacing the related terms and further reducing the governing system (15) to obtain the determining system of sum ξ, η and ϕ. The system of determining equations obtained by the non-classical symmetric method is usually nonlinear, while solving the system of determining equations gives a reduced equation with the same solution as the control equation. Finally, the group invariant solutions of Equation (11) can be obtained by solving the reduced equations. This is the non-classical symmetric approach to differential–difference equations. In simple terms, the non-classical symmetry method seeks the invariance of the original equation under the invariant surface condition.

Summarizing the above, two theorems are obtained.

Theorem 1. *Suppose $\sigma(U)$ is a symmetry of the following differential–difference equations,*

$$\Delta(n,t,\delta\partial U) = 0, \tag{24}$$

then U is the group invariant solution of the equation corresponding to the invariant group of $\sigma(U)$ if and only if

$$\begin{cases} \Delta(n,t,\delta\partial U) = 0, \\ \sigma(U) = 0. \end{cases} \tag{25}$$

where δ is a difference operator and ∂ is a differetial operator.

Theorem 2. *Suppose (24) is a system of differential–difference equations defined on $X \times t \times U$ with a maximum rank. G is the group of local transformations acting on $X \times t \times U$, and if any of the generators of G are true when*

$$\Gamma^{(n)}(\Delta)|_{\{\Delta=0,\Delta_0=0\}} = 0, \tag{26}$$

then G is the invariant group of this system of equations, where $n \in X$.

Theorem 2 can be called the Lie criterion for non-classical Lie symmetry.

3. Non-Classical Symmetry of (2+1)-Dimensional Toda Lattice Equations

The (2+1)-dimensional Toda equation is a completely integrable differential–difference equation with a Lax pair, an infinitely multiple conservation law, a Backlund transform, a soliton solution, and the general characteristics of integrable models. The study of the Toda equation is of great significance in the study of fully integrable nonlinear systems. The (2+1)-dimensional Toda lattice Equation [8] has the following form

$$\Delta = u_{xt}(n) - e^{u(n-1)-u(n)} + e^{u(n)-u(n+1)}, \tag{27}$$

where $u(n)$ is the function of x, t.

The invariant surface condition is

$$\Delta_0 = \xi(x,t,u(n))u_x(n) + \eta(x,t,u(n))u_t(n) - \phi(x,t,u(n),n). \tag{28}$$

As a brief note $\xi(x,t,u(n)) = \xi$, $\eta(x,t,u(n)) = \eta$, $\phi(x,t,u(n),n) = \phi(n)$, the system of equations determining the non-classical symmetry of Equation (27) is obtained from the following governing Equation (15).

The infinitesimal generator Γ is

$$\Gamma = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial t} + \phi(n)\frac{\partial}{\partial u(n)}, \tag{29}$$

Its second-order prolongation $\Gamma^{(2)}$ is

$$\begin{aligned}
\Gamma^{(2)} \quad &= \phi(n-1)\frac{\partial}{\partial u(n-1)} + \phi(n)\frac{\partial}{\partial u(n)} + \phi(n+1)\frac{\partial}{\partial u(n+1)} \\
&+\phi_x(n-1)\frac{\partial}{\partial u_x(n-1)} + \phi_x(n)\frac{\partial}{\partial u_x(n)} + \phi_x(n+1)\frac{\partial}{\partial u_x(n+1)} \\
&+\phi_t(n-1)\frac{\partial}{\partial u_t(n-1)} + \phi_t(n)\frac{\partial}{\partial u_t(n)} + \phi_t(n+1)\frac{\partial}{\partial u_t(n+1)} \\
&+\phi_{xx}(n-1)\frac{\partial}{\partial u_{xx}(n-1)} + \phi_{xx}(n)\frac{\partial}{\partial u_{xx}(n)} + \phi_{xx}(n+1)\frac{\partial}{\partial u_{xx}(n+1)} \\
&+\phi_{xt}(n-1)\frac{\partial}{\partial u_{xt}(n-1)} + \phi_{xt}(n)\frac{\partial}{\partial u_{xt}(n)} + \phi_{xt}(n+1)\frac{\partial}{\partial u_{xt}(n+1)} \\
&+\phi_{tt}(n-1)\frac{\partial}{\partial u_{tt}(n-1)} + \phi_{tt}(n)\frac{\partial}{\partial u_{tt}(n)} + \phi_{xt}(n+1)\frac{\partial}{\partial u_{tt}(n+1)},
\end{aligned} \tag{30}$$

where

$$\phi_x(n-1)\frac{\partial}{\partial u_x(n-1)} = \phi_x(n)\frac{\partial}{\partial u_x(n)} = \phi_x(n+1)\frac{\partial}{\partial u_x(n+1)} = 0, \tag{31}$$

$$\phi_t(n-1)\frac{\partial}{\partial u_t(n-1)} = \phi_t(n)\frac{\partial}{\partial u_t(n)} = \phi_t(n+1)\frac{\partial}{\partial u_t(n+1)} = 0, \tag{32}$$

$$\phi_{xx}(n-1)\frac{\partial}{\partial u_{xx}(n-1)} = \phi_{xx}(n)\frac{\partial}{\partial u_{xx}(n)} = \phi_{xx}(n+1)\frac{\partial}{\partial u_{xx}(n+1)} = 0, \tag{33}$$

$$\phi_{tt}(n-1)\frac{\partial}{\partial u_{tt}(n-1)} = \phi_{tt}(n)\frac{\partial}{\partial u_{tt}(n)} = \phi_{tt}(n+1)\frac{\partial}{\partial u_{tt}(n+1)} = 0, \tag{34}$$

$$\phi_{xt}(n-1)\frac{\partial}{\partial u_{xt}(n-1)} = \phi_{xt}(n+1)\frac{\partial}{\partial u_{xt}(n+1)} = 0, \tag{35}$$

$$\begin{aligned}\phi_{[xt]}(n) &= \phi_{xt}(n)\frac{\partial}{\partial u_{xt}(n)}\\ &= D_{xt}(\phi(n)-\xi u_x(n)-\eta u_t(n))+\xi u_{xxt}(n)+\eta u_{xtt}(n)\\ &= \phi_{xt}+\phi_{ux}u_t+(\phi_{uu}u_t)u_x+\phi_u u_{xt}-(\xi_{xt}+u_t\xi_{xu})u_t-\xi_x u_{tt}\\ &-\xi_u u_{xt}u_t-\xi_u u_x u_{tt}-(\xi_{ut}+\xi_{uu}u_t)u_x u_t-(\eta_{xt}+\eta_{xu}u_t)u_x\\ &-2\eta_u u_x u_{xt}-(\eta_{ut}+\eta_{uu}u_t)u_x^2-(\xi_t+\xi_u u_t)u_{xt}\\ &-(\eta_t+\eta_u u_t)u_{xx}-\eta_x u_{xt}.\end{aligned} \tag{36}$$

Substituting (31)–(36) into $\Gamma^{(2)}$ (30), the control system (15) will take the following form:

$$\begin{aligned}\Gamma^{(2)}\Delta|_{\{\Delta=0,\Delta_0=0\}} &= \phi(n-1)e^{u(n-1)-u(n)}+\phi(n)(e^{u(n-1)-u(n)}-e^{u(n)-u(n+1)})\\ &+\phi(n+1)e^{u(n)-u(n+1)}+\phi_{xt}+\phi_{xu}(\phi(n)-\xi u_x)+\phi_{ut}u_x\\ &-\xi_{xu}(\phi(n)-\xi u_x)^2-\xi_{ut}u_x(\phi(n)-\xi u_x)-\xi_{uu}u_x(\phi(n)-\xi u_x)^2\\ &-\xi_t(e^{u(n-1)-u(n)}-e^{u(n)-u(n+1)})+\phi_{uu}u_x(\phi(n)-\xi u_x)\\ &-\xi_u u_x[\phi_t-\xi_t u_x-\xi(e^{u(n-1)-u(n)}-e^{u(n)-u(n+1)})]\\ &+\phi_u(e^{u(n-1)-u(n)}-e^{u(n)-u(n+1)})-\xi_{xt}(\phi(n)-\xi u_x)\\ &-\xi_x(\phi_t-\xi_t u_x-\xi)(e^{u(n-1)-u(n)}-e^{u(n)-u(n+1)})\\ &-2\xi_u(e^{u(n-1)-u(n)}-e^{u(n)-u(n+1)})(\phi(n)-\xi u_x)\\ &=0,\end{aligned} \tag{37}$$

According to the non-classical symmetry method, the non-classical symmetries of the (2+1) dimensional Toda-like lattice Equation (27) only need to be discussed in two cases; one is $\eta \equiv 1$ and the other case is $\eta \equiv 0,\ \xi \equiv 1$:

Case 1. When $\eta \equiv 1$, the system of equations of ξ and ϕ is determined by

$$\begin{cases}\phi_{ut}+\phi\phi_{uu}-\phi\xi_{xt}-\phi\xi_{ux}-\xi_x\phi_t-\phi\xi_{uu}+\phi\xi_x\xi_{uu}=0,\\ \xi_t\xi_u+\xi\xi_{ut}-\xi\xi_{xu}-\xi\phi_{uu}=0,\\ \phi_u+\xi\xi_x-\phi\xi_u-\xi_t=0,\\ \phi(n+1)-\phi=0,\\ \phi-\phi(n-1)=0,\\ \xi\xi_{uu}=0,\\ \xi\xi_u=0.\end{cases} \tag{38}$$

Using the symbolic computing software Maple to solve Equation (38), the solutions are

$$\begin{cases}\xi=A(x),\\ \phi=B(t),\end{cases} \tag{39}$$

where $A(x)$ is an arbitrary function about x, $B(t)$ is an arbitrary function of t.

Accordingly, the non-classical symmetry of Equation (27) is given by

$$Y_1=A(x)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+B(t)\frac{\partial}{\partial u(n)}. \tag{40}$$

Choose $A(x)=ax^2+bx+c,\ B(t)=kt+d$, where $k,\ a,\ b,\ c$ and d are arbitrary constants as an example. Substituting the non-classical symmetry Y_1 to the invariant surface condition (28), the following system of equations, also called symmetry reduction, is obtained:

$$\begin{cases}u_t(n)=kt+d-(ax^2+bx+c)u_x(n),\\ u_{xt}(n)=e^{u(n-1)-u(n)}-e^{u(n)-u(n+1)}.\end{cases} \tag{41}$$

By simplifying Equation (41), a reduced equation of the same dimension as Equation (27) can be obtained

$$(2ax+b)u_x(n)+(ax^2+bx+c)u_{xx}(n)+e^{u(n-1)-u(n)}-e^{u(n)-u(n+1)}=0. \tag{42}$$

If we choose $a = 1,\ b = 0$ and $c = 0$, Equation (42) will be simplified to

$$2xu_x(n) + x^2u_{xx}(n) + e^{u(n-1)-u(n)} - e^{u(n)-u(n+1)} = 0. \tag{43}$$

Case 2. When $\eta \equiv 0$ *and* $\xi \equiv 1$, the determining system of equations of ϕ is

$$\begin{cases} \phi_u + \phi(n-1) + \phi = 0, \\ \phi(n+1) - \phi - \phi_u = 0, \\ \phi_{ux} + \phi\phi_{uu} = 0, \\ \phi_{xt} + \phi\phi_{ux} = 0, \end{cases} \tag{44}$$

Using the symbolic computing software Maple to solve the Equation (44), we can obtain

$$\phi = C(t), \tag{45}$$

where $C(t)$ is an arbitrary function of t.

Accordingly, the non-classical symmetry of Equation (27) is given by

$$Y_2 = \frac{\partial}{\partial x} + C(t)\frac{\partial}{\partial u(n)}. \tag{46}$$

If we choose $C(t) = pt + q$, with arbitrary constants p and q as an example, and substitute the non-classical symmetry Y_2 to the invariant surface condition (28), we obtain the following system of equations:

$$\begin{cases} u_x(n) = pt + q, \\ u_{xt}(n) = e^{u(n-1)-u(n)} - e^{u(n)-u(n+1)}, \end{cases} \tag{47}$$

By simplifying the Equation (47), a reduced equation of the same dimension as Equation (17) can be obtained

$$p - e^{u(n-1)-u(n)} + e^{u(n)-u(n+1)} = 0. \tag{48}$$

With the help of Maple, the reduced Equation (48) could be solved under the initial condition $u(0) = 0, u(1) = 1$ and $p = 0$; that is,

$$u(n) = n. \tag{49}$$

By using the non-classical symmetry method, the two symmetries Y_1 and Y_2 obtained are about the change of the functions $A(x)$, $B(t)$ and $C(t)$, respectively. According to the different functions selected, the corresponding reduced equations of different forms can be obtained.

The above solving process can be summarized as the following theorem.

Theorem 3. *(2+1)-dimensional Toda Equation (27) has two non-classical symmetries, as follows*

$$Y_1 = A(x)\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + B(t)\frac{\partial}{\partial u(n)}, \quad \eta \equiv 1,$$

$$Y_2 = \frac{\partial}{\partial x} + C(t)\frac{\partial}{\partial u(n)}, \quad \eta \equiv 0,\ \xi \equiv 1.$$

The two symmetries Y_1 and Y_2 are changed according to $A(x)$, $B(t)$ and $C(t)$ individually. Depending on which is selected between $A(x)$, $B(t)$ and $C(t)$, different reduced equations can be obtained.

In [23], four Lie point symmetries were given

$$X_1 = \partial_t,\ X_2 = t\partial_{u(n)},\ X_3 = \partial_{u(n)},\ X_4 = t\partial_t + 2n\partial_{u(n)}.$$

It could be easily seen that $X_i(i=1,2,4)$ was included by Y_1.

4. Non-Classical Symmetry of Another Type of (2+1)-Dimensional Toda Lattice Equations

The other (2+1)-dimensional Toda lattice equation has the following form:

$$\Delta = u_{xt}(n) - [u_t(n)+1][u(n-1) - 2u(n) + u(n+1)] = 0. \tag{50}$$

The invariant surface condition is

$$\Delta_0 = \xi(x,t,u(n))u_x(n) + \eta(x,t,u(n))u_t(n) - \phi(x,t,u(n),n) = 0. \tag{51}$$

The system of equations determining the non-classical symmetry of Equation (50) is obtained from the following governing Equation (15). Where the infinitesimal generator Γ and its second-order prolongation are

$$\Gamma = \xi(x,t,u(n))\frac{\partial}{\partial x} + \eta(x,t,u(n))\frac{\partial}{\partial t} + \phi(x,t,u(n),n)\frac{\partial}{\partial u(n)},$$

$$\begin{aligned}\Gamma^{(2)} = {}& \phi(n-1)\frac{\partial}{\partial u(n-1)} + \phi(n)\frac{\partial}{\partial u(n)} + \phi(n+1)\frac{\partial}{\partial u(n+1)} \\ &+\phi_{[t]}(n)\frac{\partial}{\partial u_t(n)} + \phi_{[xt]}(n)\frac{\partial}{\partial u_{xt}(n)},\end{aligned} \tag{52}$$

where $\xi(x,t,u(n))$=ξ, $\eta(x,t,u(n))$=η, $\phi(x,t,u(n),n)$=$\phi(n)$ for short. Hence,

$$\phi_{[t]}(n) = D_t(\phi(n) - \xi u_x(n) - \eta u_t(n)) + \xi u_{xt}(n) + \eta u_{tt}(n), \tag{53}$$

$$\phi_{[xt]}(n) = D_{xt}(\phi(n) - \xi u_x(n) - \eta u_t(n)) + \xi u_{xxt}(n) + \eta u_{ttx}(n). \tag{54}$$

The non-classical symmetries of the (2+1) dimensional Toda-like lattice Equation (50) in two cases, $\eta \equiv 1$ and $\eta \equiv 0, \xi \equiv 1$, are discussed as follows:

Case 3. When $\eta \equiv 1$, the system of determining equations about ξ and $\phi(n)$, is

$$\left\{\begin{array}{l} \phi_{xt}(n) + \phi(n)\phi_{ux}(n) - \phi(n-1) + 2\phi(n) \\ -\phi(n+1) - \phi(n)\phi(n-1) - \phi(n)\phi(n+1) + 2\phi^2(n) = 0, \\ \phi_{ut}(n) - \xi_{xt}(n) - \phi(n)\xi_{ux} + \phi(n)\phi_{uu}(n) - \xi\phi_{ux}(n) + \xi\phi(n-1) \\ -2\xi\phi(n) + \xi\phi(n+1) = 0, \\ \phi_t(n) + \xi_x + \phi(n)\xi_x - \phi_u(n) - \phi(n)\phi_u(n) = 0, \\ \xi\xi_{ux} - \xi\phi_{uu}(n) - \xi_{ut} - \phi(n)\xi_{uu} = 0, \\ \xi_t - \phi(n)\xi_u - 2\xi_u + \xi\xi_x = 0, \\ \xi_t + \phi(n)\xi_u = 0, \\ \xi\xi_{uu} = 0, \\ \xi\xi_u = 0. \end{array}\right. \tag{55}$$

Using the symbolic computing software Maple to solve the Equation (55), we will obtain

$$\left\{\begin{array}{l} \xi(x,t,u(n)) = 0, \\ \phi(x,t,u(n)) = a, \end{array}\right. \tag{56}$$

$$\left\{\begin{array}{l} \xi(x,t,u(n)) = 0, \\ \phi(x,t,u(n)) = D(x), \end{array}\right. \tag{57}$$

$$\left\{\begin{array}{l} \xi(x,t,u(n)) = b, \\ \phi(x,t,u(n)) = \dfrac{E(x)+t+u(n)}{F(x)-t}, \end{array}\right. \tag{58}$$

where a, b are arbitrary constants, and $D(x)$, $E(x)$ and $F(x)$ are arbitrary functions of x.

Accordingly, the non-classical symmetry of Equation (50) are represented as

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t} + a\frac{\partial}{\partial u(n)}, \\ Y_2 &= \frac{\partial}{\partial t} + D(x)\frac{\partial}{\partial u(n)}, \\ Y_3 &= b\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{E(x)+t+u(n)}{F(x)-t}\frac{\partial}{\partial u(n)}. \end{aligned} \tag{59}$$

Substituting $a = -\frac{1}{2}$ into the non-classical symmetry Y_1 and applying it to the invariant surface condition (51) for example, we obtain the following system of equations

$$\begin{cases} u_t(n) = -\frac{1}{2}, \\ u_{xt}(n) = [u_t(n)+1][u(n-1) - 2u(n) + u(n+1)]. \end{cases} \tag{60}$$

By simplifying the Equation (60), a reduced equation of the same dimension as Equation (50) can be obtained

$$u(n-1) - 2u(n) + u(n+1) = 0. \tag{61}$$

With the help of Maple, the reduced Equation (61) could be solved under the initial condition $u(0) = 1, u(1) = 2$, that is

$$u(n) = n+1. \tag{62}$$

If we choose to substitute $D(x) = x$ into the non-classical symmetry Y_2 to the invariant surface condition (51) as an example, the following system of equations are obtained:

$$\begin{cases} u_t(n) = x, \\ u_{xt}(n) = [u_t(n)+1][u(n-1) - 2u(n) + u(n+1)]. \end{cases} \tag{63}$$

By simplifying the Equation (63), a reduced equation of the same dimension as Equation (50) can be obtained:

$$(x+1)[u(n-1) - 2u(n) + u(n+1)] - 1 = 0. \tag{64}$$

Substituting $b = -1$ and $E(x) = 2x, F(x) = x$ into the non-classical symmetry Y_2 and applying it to the invariant surface condition (51), the following system of equations is as follows:

$$\begin{cases} u_t(n) = u_x(n) + \dfrac{2x+t+u(n)}{x-t}, \\ u_{xt}(n) = [u_t(n)+1][u(n-1) - 2u(n) + u(n+1)]. \end{cases} \tag{65}$$

By simplifying the Equation (65), a reduced equation of the same dimension as Equation (50) can be obtained

$$\begin{aligned} &(x-t)^2 u_{xx}(n) + 3t + u(n) \\ &-[(x-t)^2 u_x(n) + 3x^2 - 3xt + (x-t)u(n)][u(n-1) - 2u(n) + u(n+1)] = 0. \end{aligned} \tag{66}$$

Case 4. When $\eta \equiv 0, \xi \equiv 1$, the system of determining equations about $\phi(n)$ is

$$\begin{cases} \phi_{ux}(n) + \phi(n)\phi_{uu}(n) - \phi(n-1) + 2\phi(n) - \phi(n+1) = 0, \\ \phi_{xt}(n) + \phi(n)\phi_{ut}(n) - \phi(n-1) + 2\phi(n) - \phi(n+1) = 0, \\ \phi_u(n) - \phi_t(n) = 0. \end{cases} \tag{67}$$

Using the symbolic computing software Maple to solve the Equation (67), then

$$\phi = G(x) + H(t+u(n)), \tag{68}$$

where $G(x)$ is an arbitrary function of x, and $H(t+u(n))$ is an arbitrary function of $(t+u(n))$.

Accordingly, the non-classical symmetry of Equation (50) is expressed as

$$Y_4 = \frac{\partial}{\partial x} + [G(x) + H(t+u)]\frac{\partial}{\partial u(n)}. \tag{69}$$

If we choose $G(x) = x$ and $H(t+u) = t+u(n)$ for Y_4 in the invariant surface condition (51), we obtain

$$\begin{cases} u_x(n) = x + t + u(n), \\ u_{xt}(n) = [u_t(n)+1][u(n-1) - 2u(n) + u(n+1)]. \end{cases} \tag{70}$$

By simplifying the Equation (70), a reduced equation of the same dimension as Equation (50) can be obtained

$$[u(n-1) - 2u(n) + u(n+1)] - 1 = 0. \tag{71}$$

With the help of Maple, the reduced Equation (71) could be solved under the initial condition $u(0) = 1$ and $u(1) = 1$; that is,

$$u(n) = \frac{1}{2}n^2 - \frac{n}{2} + 1 = \frac{1}{2}(n - \frac{1}{2})^2 + \frac{7}{8} \tag{72}$$

Figure 1 shows the image of $u(n)$ (72).

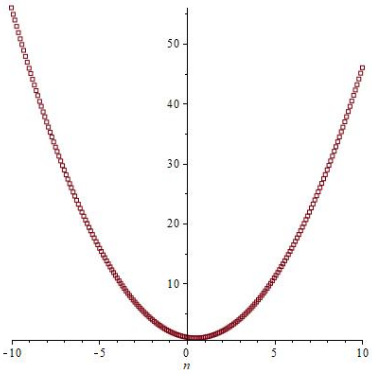

Figure 1. The plot of $u(n)$ (72).

The above solving process can be summarized as the following theorem:

Theorem 4. *The (2+1)-dimensional Toda equation (50) has four non-classical symmetries, as follows*

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t} + a\frac{\partial}{\partial u(n)}, \\ Y_2 &= \frac{\partial}{\partial t} + D(x)\frac{\partial}{\partial u(n)}, \\ Y_3 &= b\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{E(x)+t+u(n)}{F(x)-t}\frac{\partial}{\partial u(n)}, \\ Y_4 &= \frac{\partial}{\partial x} + [G(x) + H(t+u(n))]\frac{\partial}{\partial u(n)}. \end{aligned}$$

The four symmetries Y_1, Y_2, Y_3 and Y_4 are changed according to $a, b, D(x), E(x), F(x)$, $G(x)$ and $H(t+u(n))$ individually. Depending on which function is selected, different reduced equations can be obtained.

The main steps for solving lattice equations by using the differential–difference non-classical symmetry method are as follows:

Step 1: The non-classical Lie symmetry algorithm is used to calculate the Lie transform group, which can be described by its infinitesimal generator, which preserves the form of the equation on its extension space.

Step 2: Extend the Lie group and its infinitesimal generators to include the space of independent variables, function variables, and finite-order derivatives of function variables, in order to obtain the extended expressions.

Step 3: Assuming that the differential–difference equation is invariant under infinitesimal transformations, the invariant surface condition remains unchanged under infinitesimal transformations. As a result, the system of governing equations is obtained.

Step 4: Further reduce the system of governing equations to obtain a system of deterministic equations and a definite system of equations, with respect to infinitesimal generators.

Step 5: The non-classical symmetry of the differential–difference equation can be obtained by solving the system of equations using Maple 2020 software.

Step 6: Depending on the specific circumstances, various parameter equations can be chosen to derive low-dimensional reduction equations that yield the same solutions as the differential–difference equation.

5. Conclusions

In this paper, we combine the classical Lie-symmetry method of differential–difference equations proposed by Levi with the non-classical symmetry method of nonlinear partial differential equations proposed by Bluman. Our aim is to extend the non-classical symmetry method to differential–difference equations in order to get more symmetries of differential–difference equations. This method incorporates the invariant surface condition (2) into the classical Lie symmetry method in order to derive the governing equations of the studied differential–difference equation. By solving the governing equations under the $\eta \equiv 1$ or $\eta \equiv 0,\ \xi \equiv 1$, we can obtain its determination systems and then work out the non-classical symmetries of the equation and the corresponding symmetry reduction. The non-classical symmetries are only applicable to solutions that satisfy the invariant surface conditions. This implies that nonclassical symmetry only leaves a small sub-manifold of the solution manifold invariant. Finally, the effectiveness of this method is illustrated by considering examples of two types of (2+1)-dimensional Toda-like lattice equations. It can be seen that this method not only simplifies the calculation steps with the help of the invariant surface condition as an additional equation but also yields many new functional symmetries (40) and (46) of Toda-like lattice Equation (27), which cannot be obtained by the classical Lie symmetry method [23]. According to the selected arbitrary functions $A(x), B(t), C(t)$ of the infinitesimal generators, one can obtain the corresponding non-classical symmetries and the corresponding reduced equation of the (2+1)-dimensional Toda-like lattice equations.

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