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The Relationship between the Box Dimension of Continuous Functions and Their (k, s)-Riemann–Liouville Fractional Integral

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Abstract: This article is a study on the (k, s)-Riemann–Liouville fractional integral, a generalization of the Riemann–Liouville fractional integral. Firstly, we introduce several properties of the extended integral of continuous functions. Furthermore, we make the estimation of the Box dimension of the graph of continuous functions after the extended integral. It is shown that the upper Box dimension of the (k, s)-Riemann–Liouville fractional integral for any continuous functions is no more than the upper Box dimension of the functions on the unit interval $I = [0, 1]$, which indicates that the upper Box dimension of the integrand $f(x)$ will not be increased by the σ-order (k, s)-Riemann–Liouville fractional integral ${}^{s}_{k}D^{-\sigma} f(x)$ where $\sigma > 0$ on I. Additionally, we prove that the fractal dimension of ${}^{s}_{k}D^{-\sigma} f(x)$ of one-dimensional continuous functions $f(x)$ is still one.

Keywords: fractal dimension; continuous functions; (k, s)-Riemann–Liouville fractional integral

1. Introduction

Fractional calculus, including fractional differentiation and integration, is generally recognized. After centuries of development, it has been discovered that fractional calculus can solve some non-classical problems in scientific theory and engineering applications. Moreover, it has broad significance in inequalities of mathematics [1,2], nuclear and particle physics [3] and elsewhere on account of the valuable results found when fractional calculus is applied to several practical problems. Additionally, using modern functional analysis techniques, Alsaedi [4] developed the existence theory of fractional differential equations and proved that a second-order ordinary differential equation with non-local fractional integrals has symmetric solutions. In [5], the paper extended fractional calculus for hypergeometric functions with high symmetry. For more information on applications in mathematics theory, readers are encouraged to see Refs. [6–8].

In the field of fractal geometry, scholars agree with the fact that the roughness of the graph of fractal functions will change with the end of integration or differentiation of the functions. This change can be measured by the fractal dimension of the graph of fractal functions, which will decrease after fractional integration and increase after fractional differentiation. On the basis of this widely recognized fact, research on the relationship of the fractal dimension between arbitrary fractal functions and their fractional calculus, whether from its own theoretical explorations or applications in other disciplines, has attracted more and more attention from relevant researchers. The connection between the Box dimension of linear fractal interpolation functions and the fractional order has been investigated in [9]. For Besicovitch functions, research on this corresponding relationship has been discussed in Refs. [10–13]. Additionally, Liang [14] proved that the parallel relationship between a self-affine fractal function and its fractional calculus is linear. Besides these functions with specific expressions, it is worth mentioning that the previous article [15] has already explored such relationship for general continuous functions and it puts forward that the fractal dimension has the same order of variation as fractional calculus. After a

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certain amount of research on special functions, a summary conjecture of the expression was proposed by Liang [16] as follows:

Conjecture 1. *Suppose that $f(x)$ is a continuous function defined on the unit interval $I = [0, 1]$. Let*

$$\mathbb{G}(f, I) = \{(x, f(x)) \ : \ x \in I\}$$

denote to be the graph of $f(x)$ on I. Assume that $D^{-\sigma}f(x)$ and $D^{\sigma}f(x)$ are σ-order fractional integration and differentiation of $f(x)$, respectively. Then, the following assertions hold:

1. *If $\overline{\dim}_B\mathbb{G}(f, I) = \alpha \in (1, 2)$, then*

$$\overline{\dim}_B\mathbb{G}(D^{-\sigma}f, I) \leq \overline{\dim}_B\mathbb{G}(f, I) - \sigma,\ \alpha - \sigma \geq 1, \tag{1}$$

$$\overline{\dim}_B\mathbb{G}(D^{\sigma}f, I) \leq \overline{\dim}_B\mathbb{G}(f, I) + \sigma,\ \alpha + \sigma \geq 1. \tag{2}$$

2. *If the Box dimension of $\mathbb{G}(f, I)$ exists and equals $\alpha \in (1, 2)$, then*

$$\dim_B \mathbb{G}(D^{-\sigma}f, I) \leq \dim_B \mathbb{G}(f, I) - \sigma,\ \alpha - \sigma \geq 1, \tag{3}$$

$$\dim_B \mathbb{G}(D^{\sigma}f, I) \leq \dim_B \mathbb{G}(f, I) + \sigma,\ \alpha + \sigma \geq 1. \tag{4}$$

In the past 20 years, research on the fractal dimension of fractional calculus has mostly concentrated on the special functions, such as Weierstrass functions and the Hölder functions. We refer the readers to Refs. [17–20] for more details. More recent work about the fractal dimensions of the graph of continuous functions can be found in [21–25].

The motivation for this paper is multifaceted. Surfaces of fractal functions often exhibit extremely strong irregularity, which can be used to fit biological electrical signal records, flow curves in network analysis, and even natural river trends. In mathematics, some sets and functions can form fractal images through fractal calculus. This indicates a closed relationship between fractal calculus and the fractal dimension theory. Additionally, Conjecture 1 shows that fractional calculus and integer calculus exhibit the same order of variation in their effects on continuous functions. In order not to be limited to an integral form in Conjecture 1, promoting various types of fractional integrals is a novel direction and necessary measure, which acts as the second motivation of our article based on a extend integral herein. It is well known that the Riemann–Liouville fractional integral [26] and the Hadamard fractional integral [27] have been widely used in the field of the fractal dimension theory for many years. In early research, Katugampola [28] generalized the two widely used integrals into a new fractional integral form, which is applied to the Lebesgue measurable space as a generalized fractional integration operator. Furthermore, Sarikaya [29] established a new fractional integral that extends all the fractional integrals mentioned above, which is named the (k, s)-Riemann–Liouville fractional integral (for short, (k, s)-RLFI) and defined as follows:

Definition 1 ((k, s)-RLFI)**.** *The (k,s)-Riemann–Liouville fractional integral of a continuous function $f(x)$ of order $\sigma > 0$ is given by*

$${}_k^sD^{-\sigma}f(x) = \frac{(s+1)^{1-\frac{\sigma}{k}}}{k\Gamma_k(\sigma)} \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \tag{5}$$

where $k > 0$, $s \in \mathbb{R} \setminus \{-1\}$ and $x \in I$.

The extended integral can be traced back to the discussion on the extension of the classical Gamma function named the k-Gamma function in [30], which is defined as

$$\Gamma_k(x) = \int_0^\infty \tau^{x-1} e^{\frac{-\tau^k}{k}} d\tau \tag{6}$$

where $x > 0$ and $k > 0$. Up to now, (k,s)-RLFI has demonstrated wide applicability and flexibility in various fields. A weighted version of the (k,s)-Riemann–Liouville fractional operator has been given in [31]. The most remarkable feature of the paper is the result of applications on fractional kinetic equations. Based on (k,s)-RLFI, Tomar [32] put forward some new definitions in the field of probability theory, like (k,s)-Riemann–Liouville fractional variance and expectation functions; meanwhile, some generalized integral inequalities are presented and applied. However, there is not much research on the fractal dimension theory of (k,s)-RLFI. Recently, Priya and Uthayakumar [33] observed that the Hausdorff dimension and the Box dimension of the graph of a continuous function under (k,s)-RLFI are both one. Moreover, the linear relationship between the fractal dimension of (k,s)-RLFI of the Weierstrass functions and the fractional order has been discussed in [34]. In this work, we will provide a discussion about the fractal dimension of the graph of (k,s)-RLFI of general continuous functions.

The operation of the present paper is organized followed: Section 1 mainly covers the development process of (k,s)-RLFI, a new type of fractional integral and the main work in the relevant literature. Then, in Section 2, we begin with some limitations and assumptions and recall some fundamental definitions. After that, we investigated some analysis properties about (k,s)-RLFI in Section 3. On the basis of these properties and Section 2, we demonstrate the main results of this article in Section 4. Furthermore, a concrete example is provided in Section 5 to illustrate our main results. Finally, we summarize the conclusion we draw in our article in Section 6.

2. Preliminaries

This section puts forward some definitions of the fractal dimension. Significantly, a key lemma has been proved for the main results of this paper. In order to simplify the proof of this paper, we begin with some limitations and assumptions, as follows:

(1) Any functions mentioned in this article are continuous, and we denote all of them as $C(I)$ on I;
(2) For any function $f(x) \in C(I)$, it is reasonable to assume $f(x) \geq 0$ according to Proposition 1;
(3) For convenience, all C mentioned in this article are constants, which can represent different positive values without causing objection;
(4) If $f(x)$ is continuous or bounded on I, there exists a positive constant number Q such that $|f(x)| \leq Q$;
(5) For any $\delta > 0$, assume that I is divided into $m = [\delta^{-1}]$ sub-intervals with equal width δ, i.e, $m = \inf\{M \in \mathbb{N} : M \geq \delta^{-1}\}$;
(6) Set $\Delta_p = [p\delta, (p+1)\delta],\ p = 0, 1, 2, \ldots, m-1$. Sometimes, write

$$\int_{\Delta_p} f(x)dx = \int_{p\delta}^{(p+1)\delta} f(x)dx; \tag{7}$$

(7) For any continuous function $f(x)$ and a closed interval $[x_1, x_2]$, we write $R_{f,[x_1,x_2]}$ for the maximum range of $f(x)$ over the interval as

$$\mathcal{R}_{f,[x_1,x_2]} = \sup_{x_1 \leq x < y \leq x_2} |f(x) - f(y)|; \tag{8}$$

Now, we recall two widely used definitions of the fractal dimension, which are defined as follows:

Definition 2 ([35]). *Suppose that E is a non-empty subset of $\mathbb{R}^2$ and let $N_\delta(E)$ be the smallest number of sets of diameter at most δ which can cover E. Then, the lower and upper Box dimensions of E are defined as*

$$\underline{\dim}_B E = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \tag{9}$$

and

$$\overline{\dim}_B E = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}. \tag{10}$$

If these two equations are equal, we refer to the common value as the Box dimension of E

$$\dim_B E = \lim_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}. \tag{11}$$

Definition 3 ([35]). *Suppose that E is any subset of $\mathbb{R}^2$ and h is a non-negative real number. For any $\delta > 0$, the h-dimensional Hausdorff measure of E is defined as*

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(E) \tag{12}$$

where

$$\mathcal{H}^h_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^h : \{U_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } E \right\}.$$

Remark 1. *The diameter of E is the greatest distance apart of any pair of points in E, i.e.,*

$$|E| = \sup_{x,y \in E} \|x - y\|;$$

Remark 2. *Let $\{U_i\}$ be a countable collection of sets of diameter at most δ that cover E for each i, that is, $\{U_i\}$ is a δ-cover of E.*

Definition 4 ([35]). *Let $A \subset \mathbb{R}^2$ and $h \geq 0$. The Hausdorff dimension of E is defined as*

$$\dim_H(E) = \inf\left\{h : \mathcal{H}^h(E) = 0\right\} = \sup\left\{h : \mathcal{H}^h(E) = \infty\right\}. \tag{13}$$

To ensure the completeness of the process of the following proofs in this paper, we provide two basic theorems as follows:

Theorem 1 (The Mean Value Theorem). *If a function $f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b), there is at least one point ϵ ($a \leq \epsilon \leq b$) so that the following equation holds:*

$$f(a) - f(b) = f'(\epsilon)(b - a) \tag{14}$$

Theorem 2 (The Fubini's Theorem). *If a binary function $f(x,y)$ is integrable on a rectangular region $R_1 \times R_2$, then the following equation holds for all x or y:*

$$\int_{R_1} \left(\int_{R_2} f(x,y)dy \right) dx = \int_{R_2} \left(\int_{R_1} f(x,y)dx \right) dy. \tag{15}$$

3. Analysis Properties of (k,s)-RLFI

This section shows several analysis properties of continuous functions under (k,s)-RLFI, such as boundedness and continuity.

Theorem 3. *Let $f(x)$ be bounded on I; then, the (k,s)-Riemann–Liouville fractional integral of $f(x)$ shows the boundedness wherein $\sigma > 0$, $k > 0$ and $s > -1$.*

Proof. Since $|f(x)| \le Q$, then we have

$$\begin{aligned}
\left|{}_k^s D^{-\sigma} f(x)\right| &= \left|\frac{(s+1)^{1-\frac{\sigma}{k}}}{k\Gamma_k(\sigma)} \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau\right| \\
&\le \frac{(s+1)^{1-\frac{\sigma}{k}}}{k\Gamma_k(\sigma)} \int_0^x \left|\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s\right| |f(\tau)| d\tau \\
&\le \frac{Q(s+1)^{1-\frac{\sigma}{k}}}{k\Gamma_k(\sigma)} \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s d\tau \\
&\le \frac{Q(s+1)^{1-\frac{\sigma}{k}}}{k\Gamma_k(\sigma)} \frac{k}{\sigma(s+1)} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}} \Big|_x^0 \\
&= \frac{Q(s+1)^{-\frac{\sigma}{k}}}{\sigma\Gamma_k(\sigma)} x^{(s+1)\frac{\sigma}{k}} \quad (x \in I) \\
&\le C.
\end{aligned}$$

Therefore, (k,s)-RLFI of $f(x)$ is bounded. □

Theorem 4. *Suppose that $k > 0$, $s > 0$ and $\sigma > 0$. For any functions $f(x) \in C(I)$, its (k,s)-Riemann–Liouville fractional integral ${}_k^s D^{-\sigma} f(x)$ is continuous on I where $0 < \frac{\sigma}{k} < 1$.*

Proof. Let $0 \le x < x + \varepsilon \le 1$ where ε is a positive number that tends to 0. Then,

$$\begin{aligned}
&\frac{k\Gamma_k(\sigma)}{(s+1)^{1-\frac{\sigma}{k}}} \left|{}_k^s D^{-\sigma} f(x+\varepsilon) - {}_k^s D^{-\sigma} f(x)\right| \\
&= \left| \int_0^{x+\varepsilon} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau - \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right| \\
&= \left| \int_0^{\varepsilon} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau + \int_{\varepsilon}^{x+\varepsilon} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right. \\
&\quad \left. - \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right| \\
&\le \int_0^{\varepsilon} \left| \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s \right| |f(\tau)| d\tau + \left| \int_0^x \left[(x+\varepsilon)^{s+1} - (\tau+\varepsilon)^{s+1}\right]^{\frac{\sigma}{k}-1} (\tau+\varepsilon)^s \right. \\
&\quad \left. \times f(\tau+\varepsilon) d\tau - \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right| \\
&\le Q \int_0^{\varepsilon} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s d\tau + \int_0^x \left| \left[(x+\varepsilon)^{s+1} - (\tau+\varepsilon)^{s+1}\right]^{\frac{\sigma}{k}-1} (\tau+\varepsilon)^s \right. \\
&\quad \left. \times f(\tau+\varepsilon) d\tau - \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) \right| d\tau \\
&=: I_1 + I_2.
\end{aligned}$$

Since $f(x)$ is continuous and $\varepsilon \to 0$, then $x + \varepsilon \to x$, $\tau + \varepsilon \to \tau$ and $f(x+\varepsilon) \to f(x)$.

$$\begin{aligned}
I_1 &= Q \int_0^{\varepsilon} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s d\tau \\
&= -\frac{kQ}{\sigma(s+1)} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}} \Big|_0^{\varepsilon} \\
&= \frac{kQ}{\sigma(s+1)} \left\{ (x+\varepsilon)^{(s+1)\frac{\sigma}{k}} - \left[(x+\varepsilon)^{s+1} - \varepsilon^{s+1}\right]^{\frac{\sigma}{k}} \right\} \\
&\to 0. \quad (\varepsilon \to 0)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
I_2 &= \int_0^x \left| \left[(x+\varepsilon)^{s+1} - (\tau+\varepsilon)^{s+1} \right]^{\frac{\sigma}{k}-1} (\tau+\varepsilon)^s f(\tau+\varepsilon) d\tau - \left(x^{s+1} - \tau^{s+1} \right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) \right| d\tau \\
&\to \int_0^x \left| \left(x^{s+1} - \tau^{s+1} \right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau - \left(x^{s+1} - \tau^{s+1} \right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) \right| d\tau \quad (\varepsilon \to 0) \\
&= 0.
\end{aligned}
$$

Therefore, we obtain that ${}_k^sD^{-\sigma} f(x+\varepsilon) \to {}_k^sD^{-\sigma} f(x)$ when $\varepsilon \to 0$, which means that ${}_k^sD^{-\sigma} f(x)$ is continuous. □

Theorem 5. *Suppose k, s, σ_1 and σ_2 are any real numbers. For any $f(x) \in C(I)$, it holds that*

$$
({}_k^sD^{-\sigma_1} {}_k^sD^{-\sigma_2}) f(x) = {}_k^sD^{-(\sigma_1+\sigma_2)} f(x). \tag{16}
$$

Proof. From (15) of Theorem 2, we have

$$
\begin{aligned}
({}_k^sD^{-\sigma_1} {}_k^sD^{-\sigma_2}) f(x) &= \frac{(s+1)^{1-\frac{\sigma_1}{k}}}{k\Gamma_k(\sigma_1)} \int_0^x \left(x^{s+1} - \tau_2^{s+1} \right)^{\frac{\sigma_1}{k}-1} \tau_2^s \\
&\quad \times \left(\frac{(s+1)^{1-\frac{\sigma_2}{k}}}{k\Gamma_k(\sigma_2)} \int_0^x \left(\tau_2^{s+1} - \tau_1^{s+1} \right)^{\frac{\sigma_2}{k}-1} \tau_1^s f(\tau_1) d\tau_1 \right) d\tau_2 \\
&= \frac{(s+1)^{2-\frac{\sigma_1}{k}-\frac{\sigma_2}{k}}}{k^2\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)} \int_0^x \tau_1^s f(\tau_1) \times \\
&\quad \left[\int_0^x \left(x^{s+1} - \tau_2^{s+1} \right)^{\frac{\sigma_1}{k}-1} \tau_2^s \left(\tau_2^{s+1} - \tau_1^{s+1} \right)^{\frac{\sigma_2}{k}-1} d\tau_2 \right] d\tau_1.
\end{aligned}
$$

Let $y = \frac{\tau_2^{s+1} - \tau_1^{s+1}}{x^{s+1} - \tau_1^{s+1}}$; then, $(x^{s+1} - \tau_1^{s+1}) dy = (s+1) \tau_2^s d\tau_2$. Therefore, we obtain the following changes in the internal integral of the above equation:

$$
\begin{aligned}
&\int_0^x \left(x^{s+1} - \tau_2^{s+1} \right)^{\frac{\sigma_1}{k}-1} \tau_2^s \left(\tau_2^{s+1} - \tau_1^{s+1} \right)^{\frac{\sigma_2}{k}-1} d\tau_2 \\
=& \frac{x^{s+1} - \tau_1^{s+1}}{s+1} \int_0^1 \left(x^{s+1} - \tau_2^{s+1} \right)^{\frac{\sigma_1}{k}-1} \left(\tau_2^{s+1} - \tau_1^{s+1} \right)^{\frac{\sigma_2}{k}-1} dy \\
=& \frac{(x^{s+1} - \tau_1^{s+1})^{\frac{\sigma_1+\sigma_2}{k}-1}}{s+1} \int_0^1 \left(\frac{\tau_2^{s+1} - \tau_1^{s+1}}{x^{s+1} - \tau_1^{s+1}} \right)^{\frac{\sigma_2}{k}-1} \left(\frac{x^{s+1} - \tau_1^{s+1}}{x^{s+1} - \tau_1^{s+1}} \right)^{\frac{\sigma_1}{k}-1} dy \\
=& \frac{(x^{s+1} - \tau_1^{s+1})^{\frac{\sigma_1+\sigma_2}{k}-1}}{s+1} \int_0^1 y^{\frac{\sigma_2}{k}-1} (1-y)^{\frac{\sigma_1}{k}-1} dy \\
=& \frac{(x^{s+1} - \tau_1^{s+1})^{\frac{\sigma_1+\sigma_2}{k}-1}}{s+1} \frac{\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)}{\Gamma_k(\sigma_1+\sigma_2)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
({}_k^sD^{-\sigma_1} {}_k^sD^{-\sigma_2}) f(x) &= \frac{(s+1)^{2-\frac{\sigma_1}{k}-\frac{\sigma_2}{k}}}{k^2\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)} \int_0^x \tau_1^s f(\tau_1) \left[\frac{(x^{s+1} - \tau_1^{s+1})^{\frac{\sigma_1+\sigma_2}{k}-1}}{s+1} \frac{\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)}{\Gamma_k(\sigma_1+\sigma_2)} \right] d\tau_1 \\
&= \frac{(s+1)^{1-\frac{\sigma_1}{k}-\frac{\sigma_2}{k}}}{k^2\Gamma_k(\sigma_1+\sigma_2)} \int_0^x (x^{s+1} - \tau_1^{s+1})^{\frac{\sigma_1+\sigma_2}{k}-1} \tau_1^s f(\tau_1) d\tau_1 \\
&= {}_k^sD^{-(\sigma_1+\sigma_2)} f(x).
\end{aligned}
$$

The proof of Theorem 5 is complete. □

Remark 3. *Assume that k, s, σ_1 and σ_2 are real numbers. For $f(x) \in C(I)$, it holds that*

$$({}_k^sD^{-\sigma_1}{}_k^sD^{-\sigma_2})f(x) = ({}_k^sD^{-\sigma_2}{}_k^sD^{-\sigma_1})f(x).$$

4. Main Results

In this section, we obtain the main results of our article. To prove the main result, we first provide a key lemma. Then, the relationship between the Box dimension of $\mathbb{G}(f, I)$ and $\mathbb{G}({}_k^sD^{-\sigma}f, I)$ can be discussed based on Definition 2, Lemma 1 and Lemma 2. Moreover, we calculate that the fractal dimension of one-dimensional continuous functions under (k,s)-RLFI in this section.

Lemma 1 ([35])**.** *Let $f(x) \in C(I)$. If $N_{f,\delta}$ is the smallest number of squares of the δ-mesh that intersect $\mathbb{G}(f, I)$; then,*

$$\delta^{-1}\sum_{p=0}^{m-1}\mathcal{R}_{f,\Delta_p} \leq N_{f,\delta} \leq 2m + \delta^{-1}\sum_{p=0}^{m-1}\mathcal{R}_{f,\Delta_p}. \tag{17}$$

Lemma 2. *Let $0 < \delta < 1$, and m is the least integer no less than δ^{-1}. Assume $x,\ y \in \Delta_p = [p\delta, (p+1)\delta]$ $(p = 0, 1 \cdots, m-1)$ and $\tau \in \Delta_q = [q\delta, (q+1)\delta]$ $(q = 0, 1 \cdots, p-1)$, then*

$$\int_{q\delta}^{(q+1)\delta}\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s d\tau \leq C\delta^{\frac{\sigma}{k}(s+1)}(p-q)^{\frac{\sigma}{k}(s+1)-1} \tag{18}$$

when σ, k, s are positive numbers and $\frac{\sigma}{k} \in (0,1)$.

Proof. Applying Theorem 1,

$$\begin{aligned}
&\int_{q\delta}^{(q+1)\delta}\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s d\tau \\
&= \frac{k}{\sigma(s+1)}\left\{\left[x^{s+1} - (q\delta)^{s+1}\right]^{\frac{\sigma}{k}} - \left[x^{s+1} - ((q+1)\delta)^{s+1}\right]^{\frac{\sigma}{k}}\right\} \\
&= \frac{k}{\sigma(s+1)}\left[((q+1)\delta)^{s+1} - (q\delta)^{s+1}\right]\xi^{\frac{\sigma}{k}-1} \\
&\leq C\delta^{s+1}(q+1)^s\left[x^{s+1} - ((q+1)\delta)^{s+1}\right]^{\frac{\sigma}{k}-1} \\
&\leq C\delta^{s+1}(q+1)^s[x - (q+1)\delta]^{\frac{\sigma}{k}-1}\beta^{s(\frac{\sigma}{k}-1)} \\
&\leq C\delta^{\frac{\sigma}{k}+s}(q+1)^s(p-q-1)^{\frac{\sigma}{k}-1}[(q+1)\delta]^{s(\frac{\sigma}{k}-1)} \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)}(q+1)^{s\frac{\sigma}{k}}(p-q-1)^{\frac{\sigma}{k}-1} \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)}(p-q)^{\frac{\sigma}{k}(s+1)-1}
\end{aligned}$$

where

$$\begin{cases}
\xi \in \left[x^{s+1} - ((q+1)\delta)^{s+1}, x^{s+1} - (q\delta)^{s+1}\right], \\
\beta \in [(q+1)\delta, x], \\
\frac{\sigma}{k} \in (0,1), \\
s > 0.
\end{cases}$$

Therefore, Lemma 2 holds. □

Theorem 6. *Let $f(x) \in C(I)$. For the (k,s)-Riemann–Liouville fractional integral of $f(x)$, it holds that*

$$\overline{\dim}_B\mathbb{G}({}_k^sD^{-\sigma}f, I) \leq \overline{\dim}_B\mathbb{G}(f, I) \tag{19}$$

where $k > 0$, $s > 0$ *and* $\sigma > 0$ *such that* $\frac{\sigma}{k} \in (0,1)$.

Proof. Firstly, we estimate the oscillation of ${}^{s}_{k}D^{-\sigma}f(x)$ on Δ_p, indicated by $\mathcal{R}_{{}^{s}_{k}D^{-\sigma}f,\Delta_p}$. At this step, there are three parts that we need to consider separately. Since $f(x) \in C(I)$, we choose to represent the maximum and minimum values of $f(x)$ on Δ_p as $M_{p,\delta}$ and $m_{p,\delta}$, respectively. Then, from (8) and $f(x) \geq 0$, we observe that

$$\mathcal{R}_{f,\Delta_p} = M_{p,\delta} - m_{p,\delta}.$$

Let $p\delta \leq x < y \leq (p+1)\delta$,

$$\begin{aligned}
&\frac{k\Gamma_k(\sigma)}{(s+1)^{1-\frac{\sigma}{k}}}\left|{}^{s}_{k}D^{-\sigma}f(x) - {}^{s}_{k}D^{-\sigma}f(y)\right| \\
&= \left|\int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau - \int_0^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau\right| \\
&= \left|\int_0^{p\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau - \int_0^{p\delta} \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau\right. \\
&\quad \left. + \int_{p\delta}^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau - \int_{p\delta}^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau\right| \\
&\leq \int_0^{p\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s M_{q,\delta}d\tau - \int_0^{p\delta} \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s m_{q,\delta}d\tau \\
&\quad + \left|\int_{p\delta}^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau - \int_{p\delta}^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau\right| \\
&= \sum_{q=0}^{p-1}\int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s M_{q,\delta}d\tau - \sum_{q=0}^{p-1}\int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s m_{q,\delta}d\tau \\
&\quad + \sum_{q=0}^{p-1}\int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s m_{q,\delta}d\tau - \sum_{q=0}^{p-1}\int_{q\delta}^{(q+1)\delta} \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s m_{q,\delta}d\tau \\
&\quad + \left|\int_{p\delta}^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau - \int_{p\delta}^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau\right| \\
&\leq \sum_{q=0}^{p-1}\int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s \left(M_{q,\delta} - m_{q,\delta}\right)d\tau \\
&\quad + \sum_{q=0}^{p-1}\int_{q\delta}^{(q+1)\delta} \left[\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s - \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s\right] m_{q,\delta}d\tau \\
&\quad + \max\left\{\int_{p\delta}^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau,\ \int_{p\delta}^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1}\tau^s f(\tau)d\tau\right\} \\
&=: H_1 + H_2 + H_3.
\end{aligned}$$

From Lemma 2, we can obtain that

$$\begin{aligned}
H_1 &= \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s \left(M_{q,\delta} - m_{q,\delta}\right) d\tau \\
&= \left(M_{q,\delta} - m_{q,\delta}\right) \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s d\tau \\
&\leq \sum_{q=0}^{p-1} C\delta^{\frac{\sigma}{k}(s+1)} (p-q)^{\frac{\sigma}{k}(s+1)-1} \left(M_{q,\delta} - m_{q,\delta}\right) \\
&= \sum_{q=0}^{p-1} C\delta^{\frac{\sigma}{k}(s+1)} (p-q)^{\frac{\sigma}{k}(s+1)-1} \left(M_{q,\delta} - m_{q,\delta}\right) \\
&= C\delta^{\frac{\sigma}{k}(s+1)} \sum_{q=0}^{p} (p-q+1)^{\frac{\sigma}{k}(s+1)-1} \mathcal{R}_{f,\Delta_q}.
\end{aligned}$$

Since $|f(x)| \leq Q$, then $m_{q,\delta},\ M_{p,\delta} \leq Q$. For H_2, it follows from (18) that

$$\begin{aligned}
H_2 &= \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left[\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s - \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s \right] m_{q,\delta} d\tau \\
&= \sum_{q=0}^{p-1} \frac{k}{\sigma(s+1)} \left\{ \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}} \Big|_{(q+1)\delta}^{q\delta} - \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}} \Big|_{(q+1)\delta}^{q\delta} \right\} m_{q,\delta} \\
&= \frac{k}{\sigma(s+1)} \sum_{q=0}^{p-1} \left\{ \left[\left[x^{s+1} - (q\delta)^{s+1}\right]^{\frac{\sigma}{k}} - \left[x^{s+1} - ((q+1)\delta)^{s+1}\right]^{\frac{\sigma}{k}} \right] \right. \\
&\quad \left. - \left[\left[y^{s+1} - (q\delta)^{s+1}\right]^{\frac{\sigma}{k}} - \left[y^{s+1} - ((q+1)\delta)^{s+1}\right]^{\frac{\sigma}{k}} \right] \right\} m_{q,\delta} \\
&\leq C \left\{ \left| (x^{s+1})^{\frac{\sigma}{k}} - (y^{s+1})^{\frac{\sigma}{k}} \right| + \left| \left[x^{s+1} - (p\delta)^{s+1}\right]^{\frac{\sigma}{k}} - \left[y^{s+1} - (p\delta)^{s+1}\right]^{\frac{\sigma}{k}} \right| \right\} \max_{0 \leq q < p} m_{q,\delta} \\
&\leq C \left[2 \left| (x^{s+1})^{\frac{\sigma}{k}} - (y^{s+1})^{\frac{\sigma}{k}} \right| \right] Q \\
&\leq C \left[((p+1)\delta)^{\frac{\sigma}{k}(s+1)} - (p\delta)^{\frac{\sigma}{k}(s+1)} \right] Q \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)} p^{\frac{\sigma}{k}(s+1)-1} Q.
\end{aligned}$$

For H_3,

$$\begin{aligned}
H_3 &= \max\left\{ \int_{p\delta}^{x} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau,\ \int_{p\delta}^{y} \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right\} \\
&\leq \int_{p\delta}^{(p+1)\delta} \left[((p+1)\delta)^{s+1} - \tau^{s+1} \right]^{\frac{\sigma}{k}-1} \tau^s M_{p,\delta} d\tau \\
&= \frac{k}{\sigma(s+1)} \left[((p+1)\delta)^{s+1} - (p\delta)^{s+1} \right]^{\frac{\sigma}{k}} M_{p,\delta} \\
&\leq C (p^s \delta^{s+1})^{\frac{\sigma}{k}} M_{p,\delta} \\
&\leq C p^{s\frac{\sigma}{k}} \delta^{\frac{\sigma}{k}(s+1)} Q.
\end{aligned}$$

Hence, combining H_1, H_2 and H_3,

$$\begin{aligned}
\mathcal{R}_{{}_k^sD^{-\sigma}f,\Delta_p} &= \sup_{x,y\in\Delta_p} \left| {}_k^sD^{-\sigma}f(x) - {}_k^sD^{-\sigma}f(y) \right| \\
&\leq H_1 + H_2 + H_3 \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)} \left[\sum_{q=0}^{p} (p-q+1)^{\frac{\sigma}{k}(s+1)-1} \mathcal{R}_{f,\Delta_q} + p^{\frac{\sigma}{k}(s+1)-1}Q + p^{s\frac{\sigma}{k}}Q \right] \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)} \sum_{q=0}^{p} (p-q+1)^{\frac{\sigma}{k}(s+1)-1} \mathcal{R}_{f,\Delta_q}.
\end{aligned}$$

Next, we calculate $N_{{}_k^sD^{-\sigma}f,\delta}$, the size of δ-mesh squares intersecting $\mathbb{G}({}_k^sD^{-\sigma}f, I)$. From (17) of Lemma 1, it follows that

$$\begin{aligned}
N_{{}_k^sD^{-\sigma}f,\delta} &\leq \sum_{p=0}^{m-1} \left\{ 2 + \delta^{-1}\mathcal{R}_{{}_k^sD^{-v}f,\Delta_p} \right\} \\
&\leq \sum_{p=0}^{m-1} \left\{ 2 + \delta^{-1}C\delta^{\frac{\sigma}{k}(s+1)} \left[\sum_{q=0}^{p} (p-q+1)^{\frac{\sigma}{k}(s+1)-1} \mathcal{R}_{f,\Delta_q} \right] \right\} \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)-1} \sum_{q=0}^{m-1} q^{\frac{\sigma}{k}(s+1)-1} \sum_{p=0}^{m-1} \mathcal{R}_{f,\Delta_p} \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)} m^{\frac{\sigma}{k}(s+1)} \delta^{-1} \sum_{p=0}^{m-1} \mathcal{R}_{f,\Delta_p} \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)} \delta^{-\frac{\sigma}{k}(s+1)} \delta^{-1} \sum_{p=0}^{m-1} \mathcal{R}_{f,\Delta_p} \\
&\leq CN_{f,\delta}.
\end{aligned}$$

Ultimately, by Definition 2,

$$\begin{aligned}
\overline{\dim}_B \mathbb{G}({}_k^sD^{-\sigma}f, I) &= \varlimsup_{\delta\to 0} \frac{\log N_{{}_k^sD^{-\sigma}f,\delta}}{-\log\delta} \leq \varlimsup_{\delta\to 0} \frac{\log CN_{f,\delta}}{-\log\delta} \\
&= \varlimsup_{\delta\to 0} \frac{\log N_{f,\delta}}{-\log\delta} = \overline{\dim}_B \mathbb{G}(f, I).
\end{aligned}$$

We complete the proof of this theorem. □

Analogous results can be found in Refs. [36,37]. They have shown that the Box dimensions of the graphs of continuous functions on I do not increase after the Riemann–Liouville fractional integral and the Hadamard fractional integral, which means that the result of this article is applicable to the other types of fractional operators.

Next, we will provide a discussion of one-dimensional continuous functions in the following theorems. For the purpose of proving the theorems, we obtain the basic following lemma and propositions for continuous functions:

Lemma 3 ([35])**.** *For any* $f(x) \in C[0,1]$*, we have*

$$1 \leq \dim_H \mathbb{G}(f, I) \leq \underline{\dim}_B \mathbb{G}(f, I) \leq \overline{\dim}_B \mathbb{G}(f, I).$$

Proposition 1 ([35])**.** *Suppose that* $f(x) \in C(I)$ *and* $n \in \mathbb{R} \setminus \{0\}$.

(1) $\dim_B \mathbb{G}(nf, I) = \dim_B \mathbb{G}(f, I)$;
(2) $\dim_B \mathbb{G}(f+n, I) = \dim_B \mathbb{G}(f, I)$;
(3) *If* $f(x)$ *is a constant function, then* $\dim_B \mathbb{G}(f+n, I) = \dim_B \mathbb{G}(f, I) = 1$;

(4) $\quad 1 \leq \underline{\dim}_B \mathbb{G}(f,I) \leq \overline{\dim}_B \mathbb{G}(f,I) \leq 2$ *or* $1 \leq \dim_B \mathbb{G}(f,I) \leq 2$.

Theorem 7. *Suppose* $f(x) \in C(I)$ *and* $\overline{\dim}_B \mathbb{G}(f,I) = 1$; *then,*

$$\overline{\dim}_B \mathbb{G}({}_k^s D^{-\sigma} f, I) = 1 \tag{20}$$

where $\sigma > 0,\ s > 0, k > 0$ *and* $\frac{\sigma}{k} \in (0,1)$.

Proof. By Definition 2,

$$\overline{\dim}_B \mathbb{G}(f,I) = \overline{\lim_{\delta \to 0}} \frac{\log N_{f,\delta}}{-\log \delta} = 1.$$

Therefore,

$$\begin{aligned}\overline{\dim}_B \mathbb{G}({}_k^s D^{-\sigma} f, I) &= \overline{\lim_{\delta \to 0}} \frac{\log N_{{}_k^s D^{-\sigma} f,\delta}}{-\log \delta} \\ &\leq \overline{\lim_{\delta \to 0}} \frac{\log C N_{f,\delta}}{-\log \delta} \\ &= \overline{\lim_{\delta \to 0}} \frac{\log N_{f,\delta}}{-\log \delta} = 1.\end{aligned}$$

Meanwhile, from Proposition 1, we know the upper Box dimension of $\mathbb{G}({}_k^s D^{-\sigma} f, I)$ is no less than one as

$$\overline{\dim}_B \mathbb{G}({}_k^s D^{-\sigma} f, I) \geq 1,$$

which can lead to

$$\overline{\dim}_B \mathbb{G}({}_k^s D^{-\sigma} f, I) = 1.$$

The proof of Theorem 7 is complete. □

From Theorem 7, Lemma 3 and Proposition 1, we can derive the following conclusions.

Theorem 8. *Suppose that* $k > 0,\ s > 0$ *and* $\sigma > 0$. *Let* $f(x) \in C(I)$ *and* $\overline{\dim}_B \mathbb{G}(f,I) = 1$, *and we have*

$$\dim_H \mathbb{G}({}_k^s D^{-\sigma} f, I) = 1. \tag{21}$$

Corollary 1. *If* $f(x) \in C(I)$ *is a one-dimensional function and* $\frac{\sigma}{k} \in (0,1)$, *then*

$$\dim_B \mathbb{G}({}_k^s D^{-\sigma} f, I) = 1 \tag{22}$$

for any $\sigma > 0$, $s > 0$, *and* $k > 0$.

Among all the fractal functions, the most fundamental one may be a kind of continuous fractal function with the Box dimension one, which is also an important research object for the fractal functions. One-dimensional fractal functions often have the characteristics of infinite length and can be divided into bounded and unbounded variation functions. In fact, research on this type of continuous function has never been interrupted. The main reason is that continuous functions with unbounded variation exhibit the characteristics of fractal geometry, which are continuous but non-differentiable everywhere. In [38], Liang proved for the first time that the fractal dimension of continuous functions with bounded variation without specific expressions and their Riemann–Liouville fractional integral are both one. After that, the construction of continuous functions with unbounded variation and the combination of such functions with fractional calculus to study the fractal dimension become hot topics in the field of fractals. For example, a few one-dimensional functions with unbounded variation are constructed in Refs. [39,40]. The articles carefully researched the fractal dimension and its Riemann–Liouville fractional integral. Moreover, in [41], the authors conducted research on the Katugampola fractional calculus of one-dimensional continuous functions which are bounded variation. For this article, we further discuss the fractal dimensions of the one-dimensional continuous functions after the extended integral. On the basis of (k,s)-RLFI of one-dimensional continuous functions, researchers can continue to study continuous functions with the fractal dimension greater than one. Therefore, one-dimensional continuous functions that satisfy Theorem 7 and Corollary 1 are worth further exploration.

5. Example

In this section, we provide a concrete example to illustrate our results obtained in the past section.

Example 1. *Given* $0 < \alpha < 1$ *and* $\lambda \geq 2$*, the Weierstrass function is defined as*

$$W_{\alpha,\lambda}(x) = \sum_{j\geq 1} \lambda^{-\alpha j} \sin(\lambda^j x), \quad x \in I.$$

Then,

$$\dim_B \mathbb{G}(W_{\alpha,\lambda}, I) = 2 - \alpha.$$

Now, we present several graphs and numerical results for Example 1. Let $\alpha = 0.1$ and $\lambda = 2$. Figure 1 stands for the graph of $W_{0.1,2}(x)$ on I and Figure 2 represents the graph of ${}_k^s D^{-\sigma} W_{0.1,2}(x)$ on I when $k = s = 1$ and $\sigma = 0.5$. The figures intuitively indicate that the roughness of the Weierstrass function decreases after (k,s)-RLFI, followed by a decrease in the fractal dimension.

Take $k = s = 1$. Let σ be $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and 0.9, respectively. Table 1 shows the corresponding numerical results of the Box dimension of the graph of ${}_1^1 D^{-\sigma} W_{0.1,2}(x)$ on I by computing methods proposed in [42], which just corroborates our theoretical results obtained in Theorem 6.

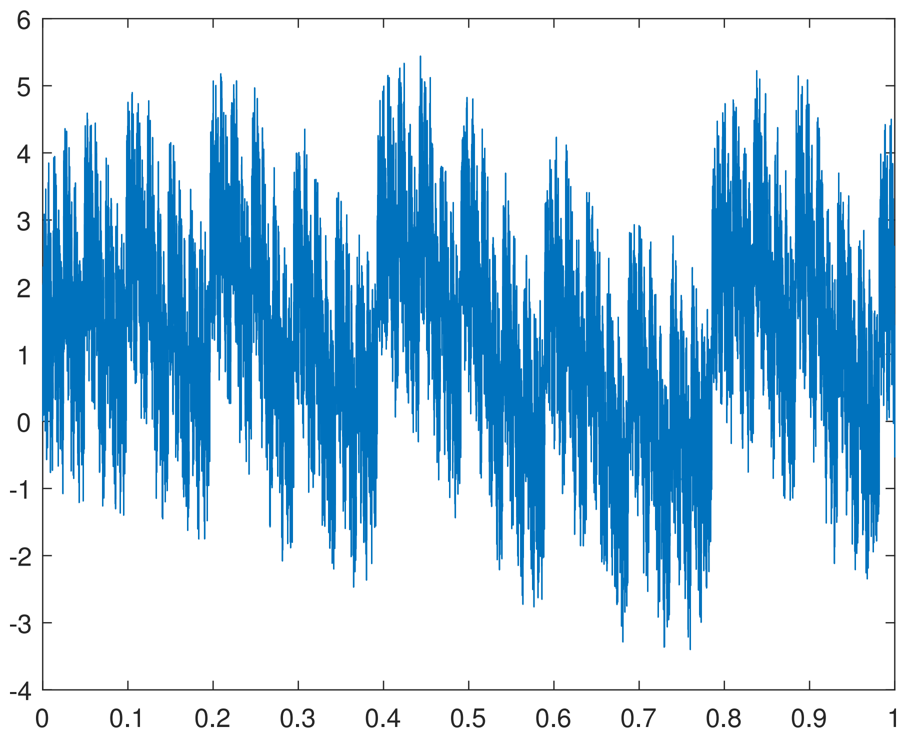

Figure 1. The graph of $W_{0.1,2}(x)$.

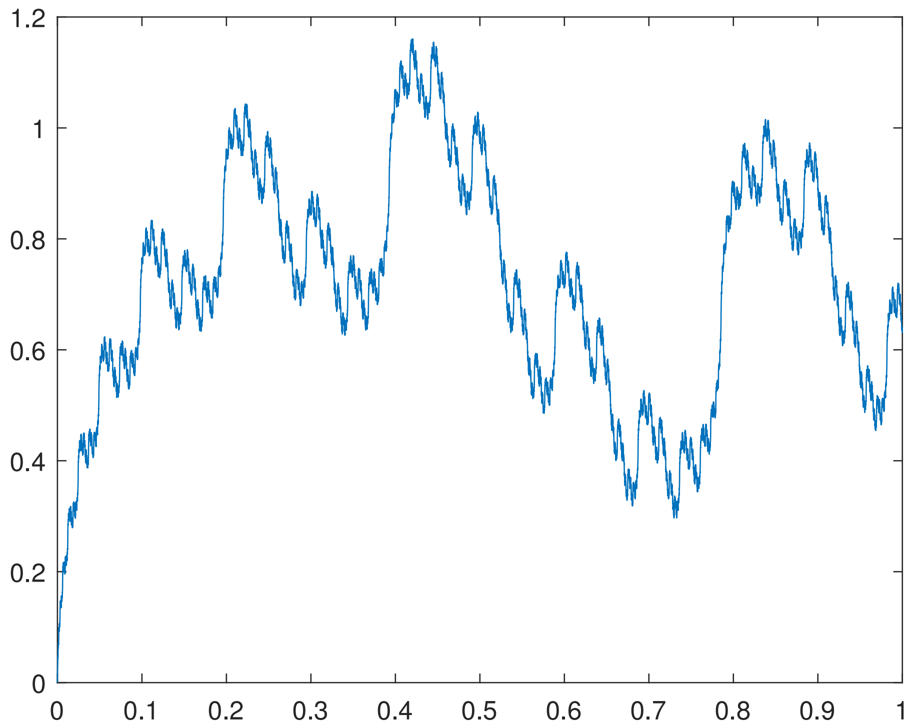

Figure 2. The graph of ${}_{k}^{s}D^{-\sigma}W_{0.1,2}(x)$.

Table 1. Numerical results of Example 1.

σ	$\dim_B \Gamma\left({}_{1}^{1}D^{-\sigma}W_{0.1,2}, I\right)$
0.1	1.7794
0.2	1.6772
0.3	1.5801
0.4	1.4825
0.5	1.3766
0.6	1.2703
0.7	1.1816
0.8	1.0792

6. Conclusions

The research on the fractal dimension theory of fractional calculus appeared in the 1990s. However, the subsequent research mostly focused on studying the fractal dimension of fractional integrals based on some special continuous functions, such as Weierstrass functions and Besicovitch functions. Until the emergence of Conjecture 1, the study of general continuous functions gradually attracted the attention of scholars. In the present article, we proved the upper Box dimension of the graph of continuous functions does not increase after (k,s)-RLFI. Simultaneously, we point out that the fractal dimension of one-dimensional continuous functions under (k,s)-RLFI remains unchanged. This result stems from the dimensionality reduction property of fractional integrals and the topological dimension of the continuous function itself being greater than one.

Moreover, there are still some points that need improvement in this article. The issue about the change in the fractal dimension of continuous functions after (k,s)-RLFI has not yet been solved completely. It is still important to find a way to demonstrate that the fractal dimension of continuous functions after (k,s)-RLFI can reach the upper bound of (1) or (3). In fact, we subjectively limit the range of parameters s, σ and k of (5) when estimating the Box dimension of $\mathbb{G}({}_{k}^{s}D^{-\sigma}f, I)$. Therefore, we naturally put forward a question, namely what is the relationship between the parameters of (5) and the change in the fractal dimension of $\mathbb{G}({}_{k}^{s}D^{-\sigma}f, I)$ if these parameters are not limited, and whether this relationship is linear like Conjecture 1? Furthermore, the main result has estimated the upper Box dimension of (k,s)-RLFI of a continuous function $f(x)$. Further exploration is needed for the study of the lower Box dimension of (k,s)-RLFI of $f(x)$. Moreover, we can continue to further consider the relationship between the Box dimension of $\mathbb{G}({}_k^sD^{-\sigma}f, I)$ and $\mathbb{G}(f, I)$ in the Hölder space in the future.

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