

Article

On an Umbral Point of View of the Gaussian and Gaussian-like Functions

Giuseppe Dattoli ¹, Emanuele Di Palma ¹  and Silvia Licciardi ^{2,*} 

¹ ENEA—Frascati Research Center, Via Enrico Fermi 45, 00044 Rome, Italy; pinodattoli@libero.it (G.D.); emanuele.dipalma@enea.it (E.D.P.)

² Department of Engineering, University of Palermo, Viale delle Scienze, 90128 Palermo, Italy

* Correspondence: silviakant@gmail.com or silvia.licciardi@unipa.it

Abstract: The theory of Gaussian functions is reformulated using an umbral point of view. The symbolic method we adopt here allows an interpretation of the Gaussian in terms of a Lorentzian image function. The formalism also suggests the introduction of a new point of view of trigonometry, opening a new interpretation of the associated special functions. The $\text{Erfi}(x)$, is, for example, interpreted as the “sine” of the Gaussian trigonometry. The possibilities offered by the Umbral restyling proposed here are noticeable and offered by the formalism itself. We mention the link between higher-order Gaussian trigonometric functions, Hermite polynomials, and the possibility of introducing new forms of distributions with longer tails than the ordinary Gaussians. The possibility of framing the theoretical content of the present article within a redefinition of the hypergeometric function is eventually discussed.

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1. Introduction

Umbral methods (*UM*) [1,2] realize a common environment in which the properties of different functions can be studied using a specific image realization [3,4]. The Bessel functions have indeed been studied using the Gaussian as the associated image function [2]. The trigonometric functions [2,5,6], too, have been framed within a context having the Gaussian as a pivot reference function. Within this context, a natural transition between circular and Bessel function has been obtained, and the spherical Bessel functions have been shown to be the common thread between the two families of functions.

The usefulness of the umbral methods, as intended in this article, has been further corroborated by recent articles on a symbolic treatment of Hermite polynomials [7], where the authors have proven interesting properties of three-variable Hermite polynomials. The main break-through contained in [7] can be summarized by saying that this family of polynomials can be viewed as a suitable form of the Newton binomial. Moreover, a further application of the method discussed in [8] has opened new possibilities for the study of non-standard Hermite polynomial-generating functions [9]. The interest for the methods we have outlined is growing even more due to its relevant association with the monomiality principle [10–12], which opens new and fascinating speculations on the possibility of obtaining a cross-bridge with old symbolic methods, important precursors of the modern umbral and algebraic theories [2,13].

The paradigm we have followed is that of providing a kind of “downgrade” of the level of the complexity of the function itself by reducing a higher-order transcendental function to a lower order. The simplification occurs through the introduction of a set of

operators with specific algebraic properties, whose role has been clarified and made more rigorous within the well-established properties of the Borel transform [14,15].

The distinctive feature of the umbral theory used in this article and illustrated in [2] is the use of what has been defined as the image function. This namely involves the introduction of a function of elementary nature, allowing the “downgrade” of a higher transcendental to an elementary transcendental, which simplifies the study of the properties of more complicated forms. The adoption of this strategy has been applied to Bessel functions, whose image is a Gaussian. This point of view has opened significant opportunities for simplification in the study of the relevant properties.

The procedure we have just summarized has revealed interesting aspects of special functions, unveiling elements of superposition hardly achievable by other means. In this paper, we use the same point of view by exploring the consequences of the “downgrading” of the Gaussian function to the status of a rational function.

The starting point of our study is the reduction of the Gaussian to a Lorentz function, and we will see how this opens a view on other families of functions, leading to generalized forms of Gaussian trigonometric-like functions. The paper is organized as outlined below.

In Section 2, we clarify how a Lorentzian can be chosen as the relevant umbral image and explore further associated umbral forms, which naturally leads to the sine and cosine Gaussian functions.

In Section 3, we extend the formalism to the study of higher-order trigonometric-like functions and introduce further generalizations.

Finally, Section 4 contains comments, including applications and possible links of the previous conclusions with Lèvy distributions.

2. Gaussian Functions, Lorentzian Functions, and Associated Trigonometric Functions

We make use of umbral operators to construct images of special and ordinary functions. The underlying formalism revealed quite powerful-to-deal-with computational details (difficult to accomplish with ordinary means) and disclosed intimate relationships between different forms of special functions.

In this section, we show the consequences deriving from the umbral restyling of the Gaussian function in terms of the Lorentzian function.

Proposition 1. *We write the Gaussian function as*

$$e^{-x^2} = \frac{1}{1 + \hat{c} x^2} \varphi_0, \quad (1)$$

where the umbral operator \hat{c} is such that

$$\hat{c}^\alpha \varphi_0 = \frac{1}{\Gamma(\alpha + 1)}, \quad (2)$$

with $\Gamma(\cdot)$ being the Euler gamma function and α representing any real or complex number.

Proof. According to the umbral point of view (see [2] for a rigorous treatment of the umbral methods),

$$e^{-x^2} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r!} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{\Gamma(r+1)} = \sum_{r=0}^{\infty} (-\hat{c})^r x^{2r} \varphi_0 = \frac{1}{1 + \hat{c} x^2} \varphi_0. \quad (3)$$

□

Corollary 1. A consequence of Equation (1) is that the primitive of the Gaussian function, too, can be formally expressed in terms of an elementary function, namely (we use the notation $\tan^{-1}(x)$ to indicate $\arctan(x)$)

$$\int e^{-x^2} dx = \hat{c}^{-\frac{1}{2}} t g^{-1}(\hat{c}^{\frac{1}{2}} x) \varphi_0. \tag{4}$$

Proposition 2. It is possible to cast Equation (1) in the form

$$e^{-x^2} := C_g(x) = \frac{1}{2} \left(\frac{1}{1 - i \hat{c}^{\frac{1}{2}} x} + \frac{1}{1 + i \hat{c}^{\frac{1}{2}} x} \right) \varphi_0 \tag{5}$$

and define the associated function as

$$S_g(x) := \frac{1}{2i} \left[\frac{1}{1 - i \hat{c}^{\frac{1}{2}} x} - \frac{1}{1 + i \hat{c}^{\frac{1}{2}} x} \right] \varphi_0 = \frac{2}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^r (r+1)! (2x)^{2r+1}}{[2(r+1)]!}. \tag{6}$$

The above functions (5) and (6) will be referred as the cosine and sine Gaussian functions, respectively.

Proof. By the use of standard algebraic manipulations, Equation (1) becomes

$$e^{-x^2} = \frac{1}{1 + \hat{c} x^2} \varphi_0 = \frac{1}{2} \left(\frac{1}{1 - i \hat{c}^{\frac{1}{2}} x} + \frac{1}{1 + i \hat{c}^{\frac{1}{2}} x} \right) \varphi_0 := C_g(x),$$

and the associated function $S_g(x)$ is

$$\begin{aligned} S_g(x) &:= \frac{1}{2i} \left[\frac{1}{1 - i \hat{c}^{\frac{1}{2}} x} - \frac{1}{1 + i \hat{c}^{\frac{1}{2}} x} \right] \varphi_0 = \sum_{r=0}^{\infty} (-1)^r \hat{c}^{r+\frac{1}{2}} x^{2r+1} \varphi_0 \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{\Gamma(r + \frac{3}{2})} = \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^r r! (2x)^{2r+1}}{(2r+1)!} = e^{-x^2} \operatorname{Erfi}(x). \end{aligned} \tag{7}$$

It is worth noting that the Gaussian sine does not represent something new in the scenario of special functions and that it can be indeed interpreted in terms of the Dawson integral $F(x)$, namely [16]

$$\begin{aligned} S_g(x) &= \frac{2}{\sqrt{\pi}} F(x), \\ F(x) &= e^{-x^2} \int_0^x e^{y^2} dy = \frac{\sqrt{\pi}}{2} e^{-x^2} \operatorname{Erfi}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{2n+1}}{(2n+1)!!}. \end{aligned} \tag{8}$$

In passing, we noted that the ‘‘Gaussian trigonometric identity’’ is

$$S_g(x)^2 + C_g(x)^2 = e^{-2x^2} (1 + \operatorname{Erfi}(x)^2). \tag{9}$$

□

The geometrical nature of trigonometric functions is expressed through their representative circles, given e.g., by the plot of cosine vs. sine functions. The same holds for the Gaussian trigonometry we have outlined. In Figure 1, we show $C_g(x)$ vs. $S_g(x)$ resulting in an egg-like form, characterizing the ‘‘circle’’ of the Gaussian trigonometry, and in Figure 2, we also report the behavior of $C_g(x)$ and $S_g(x)$ vs. x . The plotted functions are odd and even functions, but they are not periodic since they vanish at $x \rightarrow \pm\infty$.

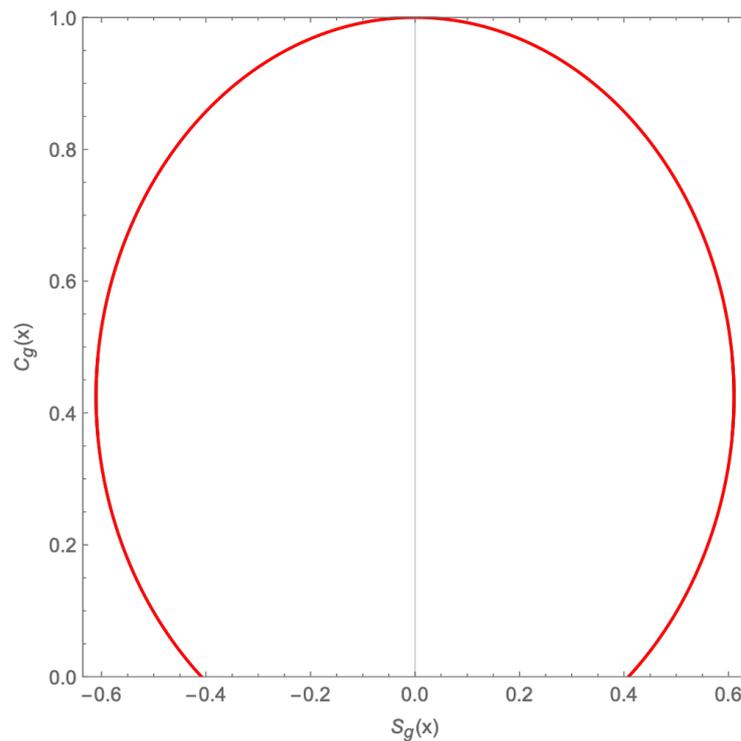


Figure 1. Gaussian trigonometric “circumference”: egg-shaped curve. $C_g(x)$ vs. $S_g(x)$.

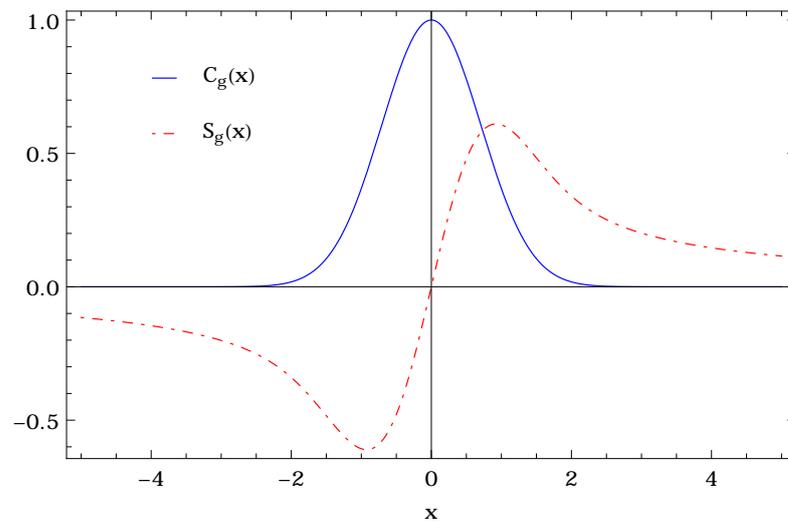


Figure 2. $C_g(x)$ and $S_g(x)$ vs. x .

The existence of other differences between the circular functions and the possibility of introducing higher-order sine/cosine Gaussians, as well as the relevant link to the Hermite polynomials, will be discussed in the forthcoming section.

3. Higher-Order Gaussian Trigonometric Functions

After the introductory remarks of the previous section, we introduce higher-order forms of Gaussian trigonometric functions. To accomplish this task, we take advantage of well-known Laplace transform identities [4].

Proposition 3. *The use of the Laplace transform (LT) technique allows us to write*

$$C_g(x) = \int_0^\infty e^{-s} \cos(\sqrt{\hat{c}} x s) ds \varphi_0, \quad S_g(x) = \int_0^\infty e^{-s} \sin(\sqrt{\hat{c}} x s) ds \varphi_0. \quad (10)$$

Proof. The first equation simply follows by the Laplace transform $\frac{k}{k^2+a^2} = \int_0^\infty e^{-kt} \cos(at) dt$

$$C_g(x) = \frac{1}{1 + (\sqrt{\hat{c}} x)^2} \varphi_0 = \int_0^\infty e^{-s} \cos(\sqrt{\hat{c}} x s) ds \varphi_0.$$

The derivation of the second part of Equation (10) follows analogous lines. \square

Corollary 2. *The successive derivatives of the Gaussian trigonometric functions can be written in umbral form as*

$$\begin{aligned} C_g^{(m)}(x) &= \sqrt{\hat{c}^m} \int_0^\infty e^{-s} s^m \cos\left(\sqrt{\hat{c}} x s + m \frac{\pi}{2}\right) ds \varphi_0, \\ S_g^{(m)}(x) &= \sqrt{\hat{c}^m} \int_0^\infty e^{-s} s^m \sin\left(\sqrt{\hat{c}} x s + m \frac{\pi}{2}\right) ds \varphi_0. \end{aligned} \tag{11}$$

Corollary 3. *The Hermite polynomials are associated with the successive derivatives of the Gaussian $H_n(x)e^{-x^2} = (-1)^n \partial_x^n e^{-x^2}$. Therefore, we obtain the identity*

$$H_m(x) e^{-x^2} = (-1)^m \sqrt{\hat{c}^m} \int_0^\infty e^{-s} s^m \cos\left(\sqrt{\hat{c}} x s + m \frac{\pi}{2}\right) ds \varphi_0, \tag{12}$$

which can be exploited for an alternative definition of this family of polynomials.

Let us now return to the definition of the Gaussian sine, which, by using Equation (7), can also be specified as

$$S_g(x) = \frac{\hat{c}^{\frac{1}{2}} x}{1 + \hat{c} x^2} \varphi_0, \tag{13}$$

and provide the integral in the following example.

Example 1. $\forall x \in \mathbb{R}$

$$\int_0^x S_g(\xi) d\xi = -\frac{1}{4\sqrt{\pi}} \sum_{r=1}^\infty \frac{(-1)^r (2x)^{2r} (r-1)!}{r (2r-1)!}. \tag{14}$$

indeed

$$\begin{aligned} \int_0^x S_g(\xi) d\xi &= \frac{\hat{c}^{-\frac{1}{2}}}{2} \ln(1 + \hat{c} x^2) \varphi_0 = \frac{\hat{c}^{-\frac{1}{2}}}{2} (-1) \sum_{r=1}^\infty \frac{(-1)^r (\hat{c} x^2)^r}{r} \varphi_0 \\ &= -\frac{1}{2} \sum_{r=1}^\infty \frac{(-1)^r x^{2r}}{r \Gamma\left(r + \frac{1}{2}\right)} = -\frac{1}{4\sqrt{\pi}} \sum_{r=1}^\infty \frac{(-1)^r (2x)^{2r} (r-1)!}{r (2r-1)!}. \end{aligned} \tag{15}$$

Remark 1. *The Gaussian trigonometric functions can be accordingly derived from the real and imaginary parts of the function*

$$E_g(x) = \frac{1 + i \hat{c}^{\frac{1}{2}} x}{1 + \hat{c} x^2} \varphi_0. \tag{16}$$

Observation 1. *It is worth stressing that the two Gaussian trigonometric functions are linked by the Kramers–Kronig identity [17]*

$$S_g(x) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^\infty \frac{C_g(\xi)}{\xi - x} d\xi, \tag{17}$$

where \mathcal{P} is the Cauchy principal value (we recall that the Cauchy principal value is " $\mathcal{P} \int_{-\infty}^{\infty} \frac{C_g(\xi)}{\xi-x}$ $d\xi = \lim_{\varepsilon \rightarrow 0^+} \left(\int_{-\infty}^{x-\varepsilon} \frac{C_g(\xi)}{\xi-x} d\xi + \int_{x+\varepsilon}^{\infty} \frac{C_g(\xi)}{\xi-x} d\xi \right)$ ".).

Proposition 4. The successive derivative of the complex function $S_g(x)$ can be obtained from those of $E_g(x)$

$$S_g^{(m)}(x) = \frac{2^{m+1}}{\sqrt{\pi}} \sum_{r=\lceil \frac{m-1}{2} \rceil}^{\infty} \frac{(-1)^r (r+1)! (2r+1)! (2x)^{2r+1-m}}{[2(r+1)]! (2r+1-m)!}. \quad (18)$$

Proof. By exploiting Equation (16) and the identity based on the Laplace transform of the Lorentz function, we obtain

$$E_g^{(n)}(x) = \partial_x^n \int_0^{\infty} (1 + i\hat{c}^{\frac{1}{2}}x) e^{-s} e^{-s\hat{c}x^2} ds \varphi_0 = \operatorname{Re} [E_g^{(n)}(x)] + i \operatorname{Im} [E_g^{(n)}(x)]. \quad (19)$$

We study the individual parts using the two-variable Hermite polynomials [14,18]

$$H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2r} y^r}{(n-2r)! r!} \quad (20)$$

and their properties

$$\partial_x^n e^{ax^2} = H_n(2ax, a) e^{ax^2}, \quad (21)$$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x, y) = e^{xt+yt^2} \quad (22)$$

Thus, we obtain

$$\begin{aligned} \operatorname{Re} [E_g^{(n)}(x)] &= C_g^{(n)}(x) = \int_0^{\infty} e^{-s} \partial_x^n e^{-s\hat{c}x^2} ds \varphi_0 \\ &= \int_0^{\infty} e^{-s} H_n(-2s\hat{c}x, -s\hat{c}) e^{-s\hat{c}x^2} ds \varphi_0 \end{aligned} \quad (23)$$

and

$$\begin{aligned} \operatorname{Im} [E_g^{(n)}(x)] &= S_g^{(n)}(x) = \hat{c}^{\frac{1}{2}} \int_0^{\infty} e^{-s} \partial_x^n (x e^{-s\hat{c}x^2}) ds \varphi_0 \\ &= \hat{c}^{\frac{1}{2}} \sum_{r=0}^n \binom{n}{r} (\partial_x^{n-r} x) \int_0^{\infty} e^{-s} H_r(-2s\hat{c}x, -s\hat{c}) e^{-s\hat{c}x^2} ds \varphi_0 \\ &= \frac{2^{n+1}}{\sqrt{\pi}} \sum_{r=\lceil \frac{n-1}{2} \rceil}^{\infty} \frac{(-1)^r (r+1)! (2r+1)! (2x)^{2r+1-n}}{[2(r+1)]! (2r+1-n)!}. \end{aligned} \quad (24)$$

□

Remark 2. This is an interesting result because $E_g(x)$, as commented on in the final section, is equivalent to the plasma dispersion function

The behavior of the successive derivatives of the Gaussian sine function is reported in Figure 3.

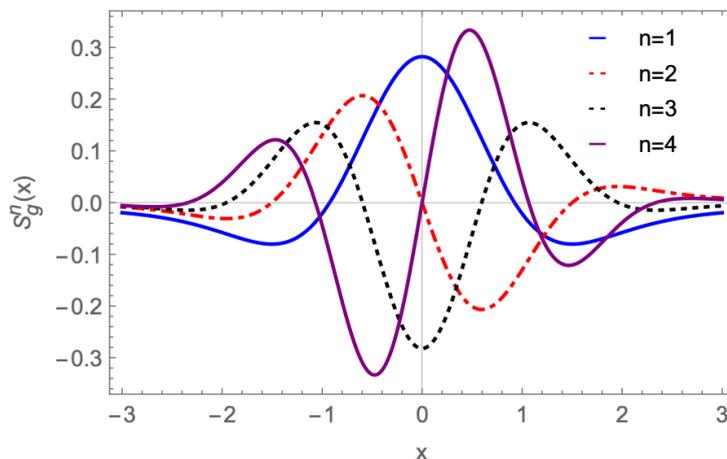


Figure 3. Normalized $S_g^{(n)}(x)$ for different values of n .

The successive derivatives of $S_g(x)$ are products of Gaussian and of Hermite polynomials. The relevant behavior is therefore not dissimilar from the Harmonic oscillator functions. For a comparison, see [19] for the relevant numerical and graphical outline.

4. New Forms of Gaussian-like Distributions

We extend here the methods we just outlined in the previous sections, introducing new Gaussian-like functions and studying their relevant properties. The use of the methods we have just touched upon displays a wide range of flexibility, as we are going to show in the examples below.

Example 2. We define the function by means of the series

$$e(-x^2|n) := \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(\frac{r}{n} + 1) x^{2r}}{r!}, \quad \forall x \in \mathbb{R}, \forall n \in \mathbb{N} : n > 1 \tag{25}$$

and ask whether its integral

$$I_e(n) := \int_{-\infty}^{\infty} e(-x^2|n) dx \tag{26}$$

can be calculated in analytical form. This can be checked straightforwardly by the use of the umbral technique discussed so far (conventional methods based on the Laplace transform technique can be used too but they are more involute). We first write Function (25) in a Gaussian-like form through the \hat{p} umbral operator and the following umbral image (it is easily proved through Equation (27) that $e^{-\hat{p}x^2} \gamma_0 = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r} \hat{p}^r}{r!} \gamma_0 = \text{sum}_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r!} \Gamma(\frac{r}{n} + 1) = e(-x^2|n)$.)

$$e(-x^2|n) = e^{-\hat{p}x^2} \gamma_0, \quad \hat{p}^\alpha \gamma_0 = \Gamma(\frac{\alpha}{n} + 1). \tag{27}$$

Then, we are allowed to evaluate the integral (26) as

$$I_e(n) = \int_{-\infty}^{\infty} e^{-\hat{p}x^2} dx \gamma_0 = \sqrt{\frac{\pi}{\hat{p}}} \gamma_0 = \Gamma\left(1 - \frac{1}{2n}\right) \sqrt{\pi}. \tag{28}$$

We note that by taking $n = 1$ in the previous equations, namely for $e(-x^2|1) = \frac{1}{1+x^2}$, Equation (28) yields, as it must be, $I_e(1) = \pi$.

The result reported in Equation (28) has been, as usual, checked numerically. The term-wise integration, which we have used as a benchmark, becomes difficult. The corresponding series converges very slowly for small values of n , and the integration becomes unstable. We have, therefore, tested Equation (28) by the use of the Ramanujan master theorem [20,21] and by other means discussed below.

Example 3. The same procedure yields the following further example:

$$\begin{aligned}
 I_{e^{a,b}}(n) &= \int_{-\infty}^{\infty} e(-ax^2 + bx | n) dx = \int_{-\infty}^{\infty} e^{-\hat{p}(ax^2 - bx)} dx \gamma_0 \\
 &= \sqrt{\frac{\pi}{a\hat{p}}} e^{\hat{p}\frac{b^2}{4a}} \gamma_0 = \sqrt{\frac{\pi}{a}} \sum_{r=0}^{\infty} \frac{\left(\frac{b^2}{4a}\right)^r}{r!} \hat{p}^{r-\frac{1}{2}} \gamma_0 = \sqrt{\frac{\pi}{a}} e^{-\frac{1}{2}} \left(\frac{b^2}{4a} | n\right), \tag{29} \\
 e_\nu(x | n) &= \sum_{r=0}^{\infty} \frac{x^r}{r!} \Gamma\left(\frac{r + \nu}{n} + 1\right).
 \end{aligned}$$

It is finally interesting to note that the use of the generating function

$$\begin{aligned}
 \sum_{s=0}^{\infty} \frac{t^s}{s!} H_s(x, y | n) &= e(xt + yt^2 | n), \\
 H_m(x, y | n) &= m! \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{x^{m-2r} y^r}{(m-2r)! r!} \Gamma\left(\frac{m-r}{n} + 1\right) \tag{30}
 \end{aligned}$$

allows the introduction of a further family of two-variable Hermite-like polynomials, whose properties will be explored elsewhere.

Example 4. It is evident that Function (25), which with increasing n becomes closer and closer to a Gaussian, can be exploited to model distributions with a tail longer than an ordinary Gaussian function. By setting

$$F(x; \sigma | n) := \frac{1}{\sqrt{2\pi} \Gamma\left(1 - \frac{1}{2n}\right) \sigma} e\left(-\frac{x^2}{2\sigma^2} | n\right), \tag{31}$$

we can evaluate the moments associated with its distribution using the generating function method (22). To this aim, we note that, $\forall d \in \mathbb{N}, \forall \sigma \in \mathbb{R} : \sigma \neq 0$,

$$\langle F(x; \sigma | n) \rangle_{(m, d)} := M_{(m, d)} = \int_{-\infty}^{\infty} (x + d)^m F(x, \sigma | n) dx, \quad \forall m \in \mathbb{Z}, \tag{32}$$

can be calculated by noting that

$$\begin{aligned}
 \sum_{m=0}^{\infty} \frac{t^m}{m!} M_{(m, d)} &= \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{t^m (x + d)^m}{m!} F(x, \sigma | n) dx \\
 &= \frac{e^{td}}{\sqrt{2\pi} \Gamma\left(1 - \frac{1}{2n}\right) \sigma} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{\hat{p}x^2}{2\sigma^2}} dx \gamma_0. \tag{33}
 \end{aligned}$$

The Gaussian integral on the r.h.s. of Equation (33) yields

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} M_{(m, d)} = \frac{e^{td} e^{\frac{t^2 \sigma^2}{2\hat{p}}}}{\Gamma\left(1 - \frac{1}{2n}\right)} \hat{p}^{-\frac{1}{2}} \gamma_0 \tag{34}$$

and the use of the Hermite-generating function (21) finally provides the result

$$\begin{aligned}
 M_{(m, d)} &= \frac{\hat{p}^{-\frac{1}{2}}}{\Gamma\left(1 - \frac{1}{2n}\right)} H_m\left(d, \frac{\sigma^2}{2} \hat{p}^{-1}\right) \gamma_0 \\
 &= \frac{m!}{\Gamma\left(1 - \frac{1}{2n}\right)} \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{d^{m-2r} \sigma^{2r}}{2^r r! (m-2r)!} \Gamma\left(1 - \frac{1}{2n} - \frac{r}{n}\right). \tag{35}
 \end{aligned}$$

The higher-order moments for $d = 0$ can accordingly be evaluated as

$$\langle F(x; \sigma | n) \rangle_m := M_m = \frac{1}{\Gamma\left(1 - \frac{1}{2n}\right)} \frac{m!}{\Gamma\left(\frac{m}{2} + 1\right)} \left(\frac{\sigma^2}{2}\right)^{\frac{m}{2}} \Gamma\left(1 - \frac{1}{2n} - \frac{m}{2n}\right). \tag{36}$$

The quasi-Gaussian distributions have only a finite number of non-diverging higher-order moments, compatible with the condition $\frac{m+1}{2n} < 1$. These distributions can be exploited to interpolate between Gaussian and Cauchy–Lorentz distributions.

Example 5. It is worth noting that the use of the integral representation of the Gamma function allows us to cast Equation (25) in the form

$$e\left(-x^2 | n\right) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r!} \int_0^{\infty} e^{-s} s^{\frac{r}{n}} ds = \int_0^{\infty} e^{-s} e^{-x^2 s^\alpha} ds, \quad \alpha = \frac{1}{n}. \tag{37}$$

By exploiting Property (21), the successive derivatives of $e\left(-x^2 | n\right)$ can then be written as

$$e^{(m)}\left(-x^2 | n\right) = \int_0^{\infty} e^{-s} H_m\left(-2 x s^\alpha, -s^\alpha\right) e^{-x^2 s^\alpha} ds, \quad \alpha = \frac{1}{n} \tag{38}$$

and can be exploited to establish families of functions providing a smooth transition, with increasing n , to the ordinary Hermite Gauss functions.

The integral transform (37) is an alternative to the umbral notation developed so far and shows noticeable features of interest, which will be touched upon below and will be discussed more carefully in a forthcoming investigation.

It is also worth noting that the use of the integral representation in Equation (37) yields almost straightforwardly the evaluation of its infinite integral, as shown below.

Example 6. We can accordingly write

$$I(\alpha) := \int_{-\infty}^{\infty} \int_0^{\infty} e^{-s-x^2 s^\alpha} ds dx. \tag{39}$$

Interchanging the integrals and using the ordinary Gaussian and the properties of the Gamma function [4], when $\text{Re}(\alpha) < 2$, we obtain

$$\begin{aligned} I(\alpha) &= \int_0^{\infty} e^{-s} \int_{-\infty}^{\infty} e^{-x^2 s^\alpha} dx ds = \int_0^{\infty} e^{-s} \sqrt{\frac{\pi}{s^\alpha}} ds \\ &= \sqrt{\pi} \int_0^{\infty} e^{-s} s^{-\frac{\alpha}{2}} ds = \sqrt{\pi} \Gamma\left(1 - \frac{\alpha}{2}\right) \end{aligned} \tag{40}$$

The same procedure can be exploited for the derivation of higher-order moments.

5. Applications and Final Comments

It is evident that the discussion of the properties of Gaussian trigonometric functions can be extended to the case of the distribution in Equation (25). We will not discuss these problems any more, leaving the subject for a forthcoming investigation. We will devote this concluding section to the relevance of the previous discussion to the link with known families of special functions.

We have already mentioned that the Gaussian sine is linked to the Dawson function through Equation (8), keeping the advantage from the relevant expression in terms of the confluent hypergeometric series ${}_1F_1(\cdot; \cdot; \cdot)$ [22]

$$S_g(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^x e^{y^2} dy = \frac{2}{\sqrt{\pi}} x e^{-x^2} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; x^2\right). \tag{41}$$

This identity offers a further direction along which we can extend the umbral point of view.

Example 7. Indeed, we can consider the following umbral interpretation of the confluent hypergeometric function ${}_1F_1(a; c; x)$ through the Pochhammer symbol $(y)_r = y(y + 1) \cdots (y + r - 1)$:

$$\begin{aligned}
 {}_1F_1(a; c; x) &= \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(c)_n n!} = \sum_{n=0}^{\infty} \hat{\kappa}^n \frac{x^n}{n!} \phi_0 = e^{\hat{\kappa}x} \phi_0, \\
 \hat{\kappa}^n \phi_0 &:= \hat{\kappa}_{a,c}^n \phi_0 = \frac{(a)_n}{(c)_n}.
 \end{aligned}
 \tag{42}$$

For umbral and Pochhammer properties, we note that

$$\hat{\kappa}^n \hat{\kappa}^m \phi_0 = \hat{\kappa}^{n+m} \phi_0 = \frac{(a)_{n+m}}{(c)_{n+m}} = \frac{(a)_n (a+n)_m}{(c)_n (c+n)_m}.
 \tag{43}$$

Using above relation, we can easily calculate higher orders of the derivatives of the hypergeometric function ${}_1F_1(a; c; x)$

$$\begin{aligned}
 \partial_x^s {}_1F_1(a; c; x) &= \hat{\kappa}^s e^{\hat{\kappa}x} \phi_0 = \hat{\kappa}^s \sum_{n=0}^{\infty} \hat{\kappa}^n \frac{x^n}{n!} \phi_0 = \sum_{n=0}^{\infty} \frac{(a)_s (a+s)_n x^n}{(c)_s (c+s)_n n!} \\
 &= \frac{(a)_s}{(c)_s} {}_1F_1(a+s; c+s; x).
 \end{aligned}
 \tag{44}$$

According to Equation (42), if we obtain $a = \frac{1}{2}$ and $c = \frac{3}{2}$, Equation (41) becomes

$$S_g(x) = \frac{2}{\sqrt{\pi}} x e^{-(1-\hat{\kappa})x^2} \phi_0.
 \tag{45}$$

This is a fairly straightforward form, which can be usefully applied to perform specific calculations involving this family of functions.

Regarding integrals involving the the Gaussian sine, we provide some examples, as shown below.

Example 8. From Equation (45) we obtain that, if $|\alpha| < 1$,

$$\begin{aligned}
 (1) \int_0^{\infty} S_g(x, \alpha) dx &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} x e^{-(1-\alpha\hat{\kappa})x^2} dx \phi_0 = \frac{1}{\sqrt{\pi}} \frac{1}{1-\alpha\hat{\kappa}} \phi_0 \\
 &= \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\infty} \alpha^r \hat{\kappa}^r \phi_0 = \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{\left(\frac{3}{2}\right)_r} \alpha^r = \frac{1}{\sqrt{\pi}} {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \alpha\right).
 \end{aligned}
 \tag{46}$$

$$\begin{aligned}
 (2) \int_{-\infty}^{\infty} \frac{S_g(x)}{x} dx &= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(1-\hat{\kappa})x^2} dx \phi_0 = \frac{2}{\sqrt{1-\hat{\kappa}}} \phi_0 \\
 &= 2 \sum_{r=0}^{\infty} \binom{r-\frac{1}{2}}{r} \hat{\kappa}^r \phi_0 = 2 \sum_{r=0}^{\infty} \binom{r-\frac{1}{2}}{r} \frac{\left(\frac{1}{2}\right)_r}{\left(\frac{3}{2}\right)_r} = 2 \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1\right) = \pi.
 \end{aligned}
 \tag{47}$$

Example 9. The Fresnel integrals can be written in terms of the hypergeometric functions ${}_1F_2(a; b, c; x)$ as [22]

$$\begin{aligned}
 C(x) &= \int_0^x \cos\left(\frac{\pi}{2}\eta^2\right) d\eta = x {}_1F_2\left(\frac{1}{4}; \frac{1}{2}, \frac{5}{4}; -\left(\frac{\pi}{4}x^2\right)^2\right), \\
 S(x) &= \int_0^x \sin\left(\frac{\pi}{2}\eta^2\right) d\eta = \frac{\pi}{2} \frac{x^3}{3} {}_1F_2\left(\frac{3}{4}; \frac{3}{2}, \frac{7}{4}; -\left(\frac{\pi}{4}x^2\right)^2\right), \\
 {}_1F_2(a; b, c; x) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n(c)_n} \frac{x^n}{n!}.
 \end{aligned}
 \tag{48}$$

According to the previous umbral discussion, the relevant images are Gaussian functions, and therefore, we find

$$\begin{aligned}
 C(x) &= x e^{-\left(\frac{\pi}{4}x^2\right)^2 \hat{\chi}_c} \phi_{0,c}, & \hat{\chi}_c^n &:= \frac{1}{4} \hat{\chi}_{\left(\frac{1}{2}, \frac{5}{4}\right)} \phi_{0,c} = \frac{\left(\frac{1}{4}\right)_n}{\left(\frac{1}{2}\right)_n \left(\frac{5}{4}\right)_n} \\
 S(x) &= \pi \frac{x^3}{3} e^{-\left(\frac{\pi}{4}x^2\right)^2 \hat{\chi}_s} \phi_{0,s}, & \hat{\chi}_s^n &:= \frac{3}{4} \hat{\chi}_{\left(\frac{3}{2}, \frac{7}{4}\right)} \phi_{0,s} = \frac{\left(\frac{3}{4}\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{7}{4}\right)_n}
 \end{aligned}
 \tag{49}$$

Example 10. It is evident that the previous results can be exploited to simplify computations involving this family of functions. For example, if we are interested in computing improper integrals involving the Fresnel functions, e.g.,

$$\int_0^{\infty} \frac{S(\xi)}{\xi^3} d\xi = \frac{\pi}{3} \int_0^{\infty} e^{-\left(\frac{\pi}{4}\xi^2\right)^2 \hat{\chi}_c} d\xi \phi_{0,s},
 \tag{50}$$

we can easily achieve our result by recalling that

$$\int_0^{\infty} e^{-ax^4} dx = \frac{1}{4} \Gamma\left(\frac{1}{4}\right) a^{-\frac{1}{4}}, \quad \text{Re}(a) > 0
 \tag{51}$$

and, by replacing the parameter "a" with $\left(\frac{\pi}{4}\xi^2\right)^2 \hat{\chi}_c$, we eventually obtain

$$\begin{aligned}
 \frac{\pi}{3} \int_0^{\infty} e^{-\left(\frac{\pi}{4}\xi^2\right)^2 \hat{\chi}_c} d\xi \phi_{0,c} &= \frac{\sqrt{\pi}}{6} \Gamma\left(\frac{1}{4}\right) \hat{\chi}_c^{-\frac{1}{4}} \phi_{0,s} \\
 &= \frac{\sqrt{\pi}}{6} \Gamma\left(\frac{1}{4}\right) \frac{\left(\frac{3}{4}\right)_{-\frac{1}{4}}}{\left(\frac{3}{2}\right)_{-\frac{1}{4}} \left(\frac{7}{4}\right)_{-\frac{1}{4}}}.
 \end{aligned}
 \tag{52}$$

The applicative framework of the results we have just obtained is indeed interesting. We would like to mention the Fried–Conte dispersion function $Z(x)$, often used in plasma physics [23], which, within the present context, is just written in terms of the complex function defined in Equation (16)

$$Z(x) = i\sqrt{\pi} E_g(x).
 \tag{53}$$

Before closing the paper, we would like to add a comment regarding the umbral definition of the so-called Lévy stable distributions [24–26] describing non-standard statistical effects in different phenomenological environments. Regarding the umbral form of this family of functions, we proceed as outlined below.

Example 11. The Lévy stable distribution $g_\alpha(x)$, in the present notation, can be defined as [2]

$$g_\alpha(x) = -\frac{1}{\pi x} e^{-\left(\frac{i}{x}\right)^\alpha} \varepsilon_0, \quad \forall \alpha \in \mathbb{R} : 0 < \alpha < 1, \quad \hat{f}^\beta \varepsilon_0 = \Gamma(\beta + 1) \sin(\pi \beta)
 \tag{54}$$

and its explicit expression in terms of infinite series can be written as

$$g_{\alpha}(x) = -\frac{1}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r x^{-\alpha r-1}}{r!} \Gamma(\alpha r + 1) \sin(\pi \alpha r). \quad (55)$$

Although it is of limited usefulness for accurate numerical computations, as underscored in [25], we have used this form because of its straightforward umbral version. By applying the same procedure leading to Equation (37), we find the following integral representation:

$$g_{\alpha}(x) = -\frac{1}{\pi} \text{Im} \left(\int_0^{\infty} e^{-sx} (-1)^{\alpha s^{\alpha}} ds \right). \quad (56)$$

These distributions have the remarkable property that all the moments $\langle x^{\mu} \rangle = \int_0^{\infty} x^{\mu} g_{\alpha}(x) dx$ are not defined for $\mu > \alpha$. By a direct application of our method, it is indeed easy to check that

$$\begin{aligned} \langle x^{\mu} \rangle &= -\frac{1}{\pi} \int_0^{\infty} x^{\mu-1} e^{-\left(\frac{x}{\alpha}\right)^{\alpha}} dx \varepsilon_0 = -\frac{1}{\pi \alpha} \Gamma\left(-\frac{\mu}{\alpha}\right) \hat{f}^{\mu} \varepsilon_0 \\ &= -\frac{1}{\pi \alpha} \Gamma\left(-\frac{\mu}{\alpha}\right) \Gamma(\mu + 1) \sin(\pi \mu) \\ &= \frac{\Gamma(\mu) \sin(\pi \mu)}{\sin\left(\pi \frac{\mu}{\alpha}\right) \Gamma\left(\frac{\mu}{\alpha}\right)}, \quad 0 < \mu < \alpha < 1. \end{aligned} \quad (57)$$

Example 12. Another important property [25],

$$\int_0^{\infty} e^{-px} g_{\alpha}(x) dx = e^{-p^{\alpha}}, \quad p > 0, \quad 0 < \alpha < 1, \quad (58)$$

namely, the fact that the Laplace transform of the Lévy stable distribution is the stretched exponential [27], is easily derived from Equation (56), as shown below.

$$\begin{aligned} \int_0^{\infty} e^{-px} g_{\alpha}(x) dx &= -\frac{\text{Im}}{\pi} \int_0^{\infty} e^{-px} \left[\int_0^{\infty} e^{-sx} (-1)^{\alpha s^{\alpha}} ds \right] dx \\ &= \frac{\text{Im}}{\pi} \int_0^{\infty} \frac{e^{-(-1)^{\alpha} s^{\alpha}}}{s + p} ds = e^{-p^{\alpha}}, \quad p > 0. \end{aligned} \quad (59)$$

Furthermore, the Laplace transform of the following modified form of $g_{\alpha}(x)$

$$g_{\alpha, \nu}(x) = -\frac{\text{Im}}{\pi} \int_0^{\infty} s^{\nu} e^{-sx} (-1)^{\alpha s^{\alpha}} ds \quad (60)$$

yields the Weibull distribution [28,29]

$$\int_0^{\infty} e^{-px} g_{\alpha, \alpha-1}(x) dx = p^{\alpha-1} e^{-p^{\alpha}}. \quad (61)$$

This article has outlined the study of the consequences that can be drawn from the umbral reformulation of the Gaussian functions in terms of an elementary rational function. The formalism has shown significant flexibilities, and we have been able to include in the present analysis different topics not of secondary interest. The introduction of the Gaussian “trigonometry” opens an interesting link with the theory of Harmonic oscillator functions and of the possibility of a generalization to multidimensional cases. Furthermore, the introduction of the complex function (16) offers interesting speculation on its relationship with the Kramers–Kronig relationship and its possible use in plasma physics [23] for the study of the plasma dispersion functions.

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References

1. Roman, S. *The Umbral Calculus*; Dover Publications: New York, NY, USA, 2005.
2. Licciardi, S.; Dattoli, G. *Umbral Calculus, a Different Mathematical Language*; World Scientific: Singapore, 2022.
3. Andrews, L.C. *Special Functions for Engineers and Applied Mathematicians*; Mc Millan: New York, NY, USA, 1985.
4. Abramovitz, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*; 9th Printing; Dover: New York, NY, USA, 1972.
5. Dattoli, G.; Di Palma, E.; Licciardi, S.; Sabia, E. From circular to Bessel functions: A transition through the umbral method. *Fractal Fract.* **2017**, *1*, 9. [[CrossRef](#)]
6. Dattoli, G.; Licciardi, S.; Pidotella, R.M. Theory of Generalized Trigonometric functions: From Laguerre to Airy forms. *J. Math. Anal. Applic.* **2018**, *468*, 103–115. [[CrossRef](#)]
7. Raza, N.; Zainab, U.; Araci, S.; Esi, A. Identities involving 3-variable Hermite polynomials arising from umbral method. *Adv. Differ. Equ.* **2020**, *640*, 1–16. [[CrossRef](#)]
8. Raza, N.; Zainab, U. Umbral treatment and lacunary generating function for Hermite polynomials. *Georgian Math. J.* **2023**. [[CrossRef](#)]
9. Gessel, I.M.; Jayawant, P. A Triple Lacunary Generating Function for Hermite Polynomials. *Electron. J. Comb.* **2005**, *12*, R00. [[CrossRef](#)] [[PubMed](#)]
10. Dattoli, G.; Licciardi, S. Monomiality and a New Family of Hermite Polynomials. *Symmetry* **2023**, *15*, 1254. [[CrossRef](#)]
11. Alyusof, R. Quasi-Monomiality Principle and Certain Properties of Degenerate Hybrid Special Polynomials. *Symmetry* **2023**, *15*, 407. [[CrossRef](#)]
12. Zainab, U.; Raza, N. Mittag-Leffler-Gould-Hopper polynomials: Symbolic Approach. *Authorea* **2023**. [[CrossRef](#)]
13. Dowker, J.S. Poweroids revisited—An old symbolic approach. *arXiv* **2013**, arXiv:1307.3150.
14. Babusci, D.; Dattoli, G.; Licciardi, S.; Sabia, E. *Mathematical Methods for Physics*; World Scientific: Singapore, 2019.
15. Dattoli, G.; Di Palma, E.; Sabia, E.; Licciardi, S. Products of Bessel Functions and Associated Polynomials. *Appl. Math. Comp.* **2015**, *266*, 507–514.
16. Nijimbere, V. Analytical and Asymptotic Evaluations of Dawson's Integral and Related Functions in Mathematical Physics. *J. Appl. Anal.* **2019**, *25*, 43–55. [[CrossRef](#)]
17. Jackson, J.D. *Classical Electrodynamics*, 3rd ed.; Wiley: New York, NY, USA, 1999; ISBN 0-471-30932-X.
18. Appell, P.; Kampé de Fériét, J. *Fonctions Hypergeometriques and Hyperspheriques. Polynomes d'Hermite*; Gauthiers-Villars: Paris, France, 1926.
19. Harmonic Oscillator (Notes on Quantum Mechanics). Last Updated Thursday. Available online: <http://quantum.bu.edu/notes/QuantumMechanics/HarmonicOscillator.pdf> (accessed on 30 November 2006).
20. Berndt, B. *Ramanujan's Notebooks; Part I*; Springer: New York, NY, USA, 1985.
21. Babusci, D.; Dattoli, G. On Ramanujan Master Theorem. *arXiv* **2011**, arXiv:1103.3947.
22. Nijimbere, V. Evaluation of the non-elementary integral $\int e^{\lambda x^a} dx$ and related integrals. *arXiv* **2017**, arXiv:1702.08438.
23. Fried, B.D.; Conte, S.D. *The Plasma Dispersion Function*; Academic Press: New York, NY, USA, 1961.
24. Kahane, P. *Lévy Flights and Related Topics in Physics*; Lecture Notes in Physics; Shlesinger, M.F., Zaslavsky, G.M., Frisch, U., Eds.; Springer: Berlin, Germany, 1995; Volume 45.
25. Penson, K.; Gorska, K. Exact and explicit probability densities for on-sided Levy stable distributions. *Phys. Rev. Lett.* **2010**, *105*, 210604. [[CrossRef](#)] [[PubMed](#)]
26. Anderssen, R.S.; Husain, S.A.; Loy, R.J. The Kohlrausch function: Properties and applications, *ANZIAM J.* **2004**, *45*, C800. [[CrossRef](#)]
27. Kohlrausch, R. Theorie des elektrischen Rückstandes in der Leidner Flasche. *Ann. Der Phys.* **1854**, *91*, 56–82, 179–213. [[CrossRef](#)]
28. Tsallis, C. Possible generalization of Boltzmann-Gibbs statistics. *J. Stat. Phys.* **1988**, *52*, 479–487. [[CrossRef](#)]
29. Tsallis, C. *Introduction to Non Extensive Statistical Mechanics: Approaching a Complex World*; Springer: New York, NY, USA, 2009.

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