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A Note on Incomplete Fibonacci–Lucas Relations

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Abstract: We define the incomplete generalized bivariate Fibonacci p-polynomials and the incomplete generalized bivariate Lucas p-polynomials. We study their recursive relations and derive an interesting relationship through their generating functions. Subsequently, we prove an incomplete version of the well-known Fibonacci–Lucas relation and make some extensions to the relation involving incomplete generalized bivariate Fibonacci and Lucas p-polynomials. An argument about going from the regular to the incomplete Fibonacci–Lucas relation is discussed. We provide a relation involving the incomplete Leonardo and the incomplete Lucas–Leonardo p-numbers as an illustration.

Keywords: Fibonacci–Lucas relation; bivariate Fibonacci p-polynomials; incomplete generalized bivariate Fibonacci p-polynomials

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1. Introduction

In [1], Filipponi investigated and obtained many properties of the incomplete Fibonacci numbers and the incomplete Lucas numbers. For a real number x, $\lfloor x \rfloor$ denotes the least integer greater than or equal to x. For any positive integer n, the incomplete Fibonacci numbers $F_n(s)$ are defined as

$$F_n(s) = \sum_{j=0}^{s} \binom{n-1-j}{j}, \tag{1}$$

where s is an integer with $0 \le s \le \lfloor \frac{n-1}{2} \rfloor$. Similarly, the incomplete Lucas numbers $L_n(s)$ are defined by

$$L_n(s) = \sum_{j=0}^{s} \frac{n}{n-j} \binom{n-j}{j}, \tag{2}$$

where $0 \le s \le \lfloor \frac{n}{2} \rfloor$. Note that $F_n(\lfloor \frac{n-1}{2} \rfloor)$ is equal to the original Fibonacci number F_n, and $L_n(\lfloor \frac{n}{2} \rfloor) = L_n$ is the Lucas number, and this is a part of the reason that the name "incomplete" is used. Additionally, for our convenience, let $L_0(0) = 2$. Some special cases of (1) and (2) are

$$F_n(0) = L_n(0) = 1, \text{for all } n \ge 1,$$

and

$$F_n(1) = n-1, \text{for all } n \ge 3;\ L_n(1) = n+1, \text{for all } n \ge 2.$$

Sury [2] proved the well-known Fibonacci–Lucas relation:

$$2^{n+1}F_{n+1} = 2^0L_0 + 2^1L_1 + \cdots + 2^nL_n = \sum_{i=0}^{n} 2^iL_i \quad (n \ge 0). \tag{3}$$

Chung [3] proved a more general relation for the sequence of W-polynomials and w-polynomials. Chung, Yao, and Zhou [4] extended Sury's formula (3) in both a regular and an alternating form to Fibonacci k-step and Lucas k-step polynomials.

Equivalently, we rewrite (3) as

$$2^{n+1}F_{n+1}(\lfloor \frac{n}{2} \rfloor) = \sum_{i=0}^{n} 2^i L_i(\lfloor \frac{i}{2} \rfloor),$$

since $F_n(\lfloor \frac{n-1}{2} \rfloor) = F_n$ and $L_n(\lfloor \frac{n}{2} \rfloor) = L_n$. Now we can extend the Fibonacci–Lucas relation to the incomplete version:

$$2^{n+1}F_{n+1}(s) = 2^{2s+1}F_{2s+1}(s) + \sum_{i=2s+1}^{n} 2^i L_i(s), \tag{4}$$

for all positive integer n and $0 \leq s \leq \lfloor \frac{n}{2} \rfloor$. To see (4), it is suffice to show that, for all s with $0 \leq s \leq \lfloor \frac{n}{2} \rfloor$,

$$2^{n+1}F_{n+1}(s) = \sum_{i=0}^{2s} 2^i L_i(\lfloor \frac{i}{2} \rfloor) + \sum_{i=2s+1}^{n} 2^i L_i(s). \tag{5}$$

By the well-known Fibonacci–Lucas relation (3), we have

$$\sum_{i=0}^{2s} 2^i L_i(\lfloor \frac{i}{2} \rfloor) = 2^{2s+1}F_{2s+1}(s).$$

Identity (5) can be proved by induction on n, or it can be proved directly as follows. See also (3.13) in [1].

Lemma 1. *For any positive integer n, we have*

$$L_n(s) + F_n(s) = 2F_{n+1}(s),$$

where $0 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$.

Proof. By the definitions of (1) and (2), for s with $0 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$, we have

$$L_n(s) + F_n(s) = \sum_{j=0}^{s} \left[\binom{n-1-j}{j} + \frac{n}{n-j}\binom{n-j}{j} \right] = 2\sum_{j=0}^{s} \binom{n-j}{j} = 2F_{n+1}(s).$$

□

Now, from the previous lemma, we know that the right-hand side of (5) is equal to

$$2^{2s+1}F_{2s+1}(s) + \sum_{i=2s+1}^{n} 2^i(2F_{i+1}(s) - F_i(s)),$$

where $0 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$. Note that $F_{2s+1}(s) = F_{2s+1}$ and $L_{2s}(s) = L_{2s}$. Hence, we have

$$\begin{aligned} &2^{2s+1}F_{2s+1}(s) + \sum_{i=2s+1}^{n} 2^{i+1}F_{i+1}(s) - \sum_{i=2s+1}^{n} 2^i F_i(s) \\ &= 2^{2s+1}F_{2s+1} + 2^{n+1}F_{n+1}(s) - 2^{2s+1}F_{2s+1} \\ &= 2^{n+1}F_{n+1}(s). \end{aligned}$$

If n is even and $s = \lfloor \frac{n}{2} \rfloor$, (5) holds obviously. Thus (5) or the incomplete Fibonacci–Lucas relation (4) follows.

In 2012, Tasci, Cetin Firengiz, and Tuglu [5] investigated the incomplete bivariate Fibonacci and Lucas p-polynomials:

$$F_{n;p}^{(s)}(x,y) = \sum_{j=0}^{s} \binom{n-jp-1}{j} x^{n-j(p+1)-1} y^{j},$$

where $0 \le s \le \lfloor \frac{n-1}{p+1} \rfloor$, and

$$L_{n;p}^{(s)}(x,y) = \sum_{j=0}^{s} \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^{j},$$

where $0 \le s \le \lfloor \frac{n}{p+1} \rfloor$. When $s = \lfloor \frac{n-1}{p+1} \rfloor$, $F_{n;p}^{(s)}(x,y)$ reduces to the bivariate Fibonacci p-polynomials $F_{n;p}(x,y)$. That is,

$$F_{n;p}(x,y) = \sum_{j=0}^{\lfloor \frac{n-1}{p+1} \rfloor} \binom{n-jp-1}{j} x^{n-j(p+1)-1} y^{j}.$$

In [5], some basic properties and generating functions of the incomplete bivariate Fibonacci and Lucas p-polynomials are given. In this note, we define the incomplete generalized bivariate Fibonacci and Lucas p-polynomials as below. For any two integers $p \ge 1, n \ge 0$ and any two polynomials $h(x,y), \ell(x,y)$ with real coefficients, define

$$F_{n;p}^{(s)}(h(x,y), \ell(x,y)) = \sum_{j=0}^{s} \binom{n-jp-1}{j} h^{n-j(p+1)-1}(x,y) \ell^{j}(x,y),$$

where $0 \le s \le \lfloor \frac{n-1}{p+1} \rfloor$, and

$$L_{n;p}^{(s)}(h(x,y), \ell(x,y)) = \sum_{j=0}^{s} \frac{n}{n-jp} \binom{n-jp}{j} h^{n-j(p+1)}(x,y) \ell^{j}(x,y),$$

where $0 \le s \le \lfloor \frac{n}{p+1} \rfloor$. Given $s \ge 0$ and $p \ge 1$, we let $F_{n;p}^{(s)}(h(x,y), \ell(x,y)) = 0$ if $n < s(p+1)+1$ and $L_{n;p}^{(s)}(h(x,y), \ell(x,y)) = 0$ if $n < s(p+1)$. From now on, we write $F_{n;p}^{(s)}(h,\ell)$ for $F_{n;p}^{(s)}(h(x,y), \ell(x,y))$ and similar to $L_{n;p}^{(s)}(h,\ell)$ if there is no misunderstanding.

We are at the stage of stating the following main theorem.

Theorem 1. *For any integers $p \ge 1$ and $n \ge 0$, any real nonzero number r, and any polynomials $h(x,y), \ell(x,y) \in \mathbb{R}[x,y]$, we have a relation involving the incomplete generalized bivariate Fibonacci and Lucas p-polynomials,*

$$\begin{aligned} r^{n+1} p h(x,y) F_{n+1;p}^{(s)}(h,\ell) = {} & r^{s(p+1)+1} p h(x,y) F_{s(p+1)+1;p}^{(s)}(h,\ell) + \sum_{i=s(p+1)+1}^{n} r^{i} L_{i;p}^{(s)}(h,\ell) \\ & + (rph(x,y) - p - 1) \sum_{i=s(p+1)+1}^{n} r^{i} F_{i+1;p}^{(s)}(h,\ell), \end{aligned}$$

where s is any integer with $0 \le s \le \lfloor \frac{n}{p+1} \rfloor$.

We replace r in Theorem 1 with $-1/r$ (since $r \neq 0$) to obtain an alternating relation involving the incomplete generalized bivariate Fibonacci and Lucas p-polynomials:

$$
\begin{aligned}
(-1)^n h(x,y) F^{(s)}_{n+1;p}(h,\ell) = (-1)^{s(p+1)} r^{n-s(p+1)} h(x,y) F^{(s)}_{s(p+1)+1;p}(h,\ell) \\
+ \sum_{i=s(p+1)+1}^{n} (-1)^i r^{n-i} \Big[L^{(s)}_{i+1;p}(h,\ell) + r h(x,y) F^{(s)}_{i;p}(h,\ell) \\
- (p+1)\ell(x,y) F^{(s-1)}_{i-p+1;p}(h,\ell) \Big].
\end{aligned}
$$

In the case of $h(x,y) = x$ and $\ell(x,y) = y$ in Theorem 1, we have a relation and an alternating relation involving the incomplete bivariate Fibonacci and Lucas p-polynomials, respectively.

Corollary 1. *For any integers $p \geq 1$ and $n \geq 0$ and any real nonzero number r, we have*

$$
\begin{aligned}
r^{n+1} p x F^{(s)}_{n+1;p}(x,y) = r^{s(p+1)+1} p x F^{(s)}_{s(p+1)+1;p}(x,y) + \sum_{i=s(p+1)+1}^{n} r^i L^{(s)}_{i;p}(x,y) \\
+ (rpx - p - 1) \sum_{i=s(p+1)+1}^{n} r^i F^{(s)}_{i+1;p}(x,y),
\end{aligned}
$$

and

$$
\begin{aligned}
(-1)^n x F^{(s)}_{n+1;p}(x,y) = (-1)^{s(p+1)} r^{n-s(p+1)} x F^{(s)}_{s(p+1)+1;p}(x,y) \\
+ \sum_{i=s(p+1)+1}^{n} (-1)^i r^{n-i} \Big[L^{(s)}_{i+1;p}(x,y) + r x F^{(s)}_{i;p}(x,y) - (p+1) y F^{(s-1)}_{i-p+1;p}(x,y) \Big],
\end{aligned}
$$

where s is any integer with $0 \leq s \leq \lfloor \frac{n}{p+1} \rfloor$.

In the case of $h(x,y)$ being just a polynomial of x, say $h(x)$, and $\ell(x,y) = 1 = p$ in Theorem 1, we obtain a relation involving the incomplete $h(x)$-Fibonacci $F^{(s)}_n(h(x))$ and the incomplete $h(x)$-Lucas polynomials $L^{(s)}_n(h(x))$ [6].

Corollary 2. *For any integer $n \geq 0$, any real nonzero number r, and a polynomial $h(x)$ with a real coefficient, we have*

$$
\begin{aligned}
r^{n+1} h(x) F^{(s)}_{n+1}(h(x)) = r^{2s+1} h(x) F^{(s)}_{2s+1}(h(x)) + \sum_{i=2s+1}^{n} r^i L^{(s)}_i(h(x)) \\
+ (rh(x) - 2) \sum_{i=2s+1}^{n} r^i F^{(s)}_{i+1}(h(x)),
\end{aligned}
$$

where s is any integer with $0 \leq s \leq \lfloor \frac{n}{2} \rfloor$.

In the case of $h(x,y) = \ell(x,y) = 1$ and $p = 1$ in Theorem 1, we have the following generalized incomplete Fibonacci–Lucas relation.

Corollary 3. *For any integer $n \geq 0$ and any real nonzero number r, we have*

$$
r^{n+1} F_{n+1}(s) = r^{2s+1} F_{2s+1} + \sum_{i=2s+1}^{n} r^i L_i(s) + (r-2) \sum_{i=2s+1}^{n} r^i F_{i+1}(s),
$$

where s is any integer with $0 \leq s \leq \lfloor \frac{n}{2} \rfloor$.

Of course, from Corollary 3, we recover the incomplete Fibonacci–Lucas relation (4) when $r = 2$.

This note is organized as follows. In Section 2, we establish an inter-relationship between the incomplete generalized bivariate Fibonacci p-polynomials and the incomplete generalized bivariate Lucas p-polynomials and investigate some properties of these polynomials. Afterwards, we derive both of the two generating functions, and from these, we can obtain an interesting relationship between the two generating functions (Proposition 6). We then give proof of our main theorem (Theorem 1). In Section 3, we discuss the regular generalized bivariate Fibonacci and Lucas p-polynomials and obtain a potential connection between the regular (complete) and incomplete Fibonacci–Lucas relation. We also discuss, as an example, a relation involving the Leonardo p-numbers and the Lucas–Leonardo p-numbers. We show a procedure for how to obtain such a relation in an incomplete version from a regular (complete) form. A summary and conclusion will be given in Section 4.

2. Some Properties and Proofs

In this section, let p be a positive integer and $n \geq 0$ be an integer. We note that, from the definitions of the incomplete generalized bivariate Fibonacci and Lucas p-polynomials,

$$F^{(0)}_{n;p}(h,\ell) = h^{n-1}(x,y), F^{(1)}_{n;p}(h,\ell) = h^{n-1}(x,y) + (n-p-1)h^{n-p-2}(x,y)\ell(x,y),$$

and

$$L^{(0)}_{n;p}(h,\ell) = h^{n}(x,y), L^{(1)}_{n;p}(h,\ell) = h^{n}(x,y) + nh^{n-p-1}(x,y)\ell(x,y).$$

Proposition 1. *The incomplete generalized bivariate Fibonacci p-polynomials satisfy a nonhomogeneous recurrence relation:*

$$\begin{aligned} F^{(s)}_{n;p}(h,\ell) &= h(x,y)F^{(s)}_{n-1;p}(h,\ell) + \ell(x,y)F^{(s)}_{n-p-1;p}(h,\ell) \\ &\quad - \binom{n-(s+1)p-2}{s} h^{n-(s+1)(p+1)-1}(x,y)\ell^{s+1}(x,y), \end{aligned} \tag{6}$$

for all $n \geq p+2$ and $0 \leq s \leq \lfloor \frac{n-p-2}{p+1} \rfloor$.

Proof. For $n \geq p+2$ and $0 \leq s \leq \lfloor \frac{n-p-2}{p+1} \rfloor$, we have

$$\begin{aligned} & F^{(s)}_{n;p}(h,\ell) - h(x,y)F^{(s)}_{n-1;p}(h,\ell) \\ &= \sum_{j=0}^{s} \left[\binom{n-jp-1}{j} - \binom{n-jp-2}{j} \right] h^{n-j(p+1)-1}(x,y)\ell^{j}(x,y) \\ &= \sum_{j=1}^{s} \binom{n-jp-2}{j-1} h^{n-j(p+1)-1}(x,y)\ell^{j}(x,y) \\ &= \sum_{j=0}^{s-1} \binom{n-(j+1)p-2}{j} h^{n-(j+1)(p+1)-1}(x,y)\ell^{j+1}(x,y) \\ &= \ell(x,y)F^{(s)}_{n-p-1;p}(h,\ell) - \binom{n-(s+1)p-2}{s} h^{n-(s+1)(p+1)-1}(x,y)\ell^{s+1}(x,y). \end{aligned}$$

□

It is easy to see that the recurrence relation (6) can be written in a homogeneous form:

$$F^{(s+1)}_{n;p}(h,\ell) = h(x,y)F^{(s+1)}_{n-1;p}(h,\ell) + \ell(x,y)F^{(s)}_{n-p-1;p}(h,\ell), \quad \text{for } 0 \leq s \leq \lfloor \frac{n-p-2}{p+1} \rfloor.$$

Proposition 2. *For all integer $t \geq 0$, and $0 \leq s \leq \lfloor \frac{n-t-p-1}{p+1} \rfloor$, we have*

$$\sum_{j=0}^{t} \binom{t}{j} h^j(x,y)\ell^{t-j}(x,y)F^{(s+j)}_{n+p(j-1);p}(h,\ell) = F^{(s+t)}_{n+(p+1)t-p;p}(h,\ell).$$

Proof. For the case $t = 0$, the identity holds trivially. Assume that the desired identity holds for some $t > 0$. Now, for $0 \leq s \leq \lfloor \frac{n-t-p-1}{p+1} \rfloor$,

$$\begin{aligned}
&\sum_{j=0}^{t+1} \binom{t+1}{j} h^j(x,y)\ell^{t+1-j}(x,y)F^{(s+j)}_{n+p(j-1);p}(h,\ell) \\
&= \sum_{j=0}^{t+1} \left[\binom{t}{j} + \binom{t}{j-1}\right] h^j(x,y)\ell^{t+1-j}(x,y)F^{(s+j)}_{n+p(j-1);p}(h,\ell) \\
&= \sum_{j=0}^{t} \binom{t}{j} h^j(x,y)\ell^{t+1-j}(x,y)F^{(s+j)}_{n+p(j-1);p}(h,\ell) \\
&\qquad + \sum_{j=1}^{t+1} \binom{t}{j-1} h^j(x,y)\ell^{t+1-j}(x,y)F^{(s+j)}_{n+p(j-1);p}(h,\ell) \\
&= \ell(x,y) \sum_{j=0}^{t} \binom{t}{j} h^j(x,y)\ell^{t-j}(x,y)F^{(s+j)}_{n+p(j-1);p}(h,\ell) \\
&\qquad + h(x,y) \sum_{j=0}^{t} \binom{t}{j} h^j(x,y)\ell^{t-j}(x,y)F^{(s+1+j)}_{n+pj;p}(h,\ell) \\
&= \ell(x,y)F^{(s+t)}_{n+(p+1)t-p;p}(h,\ell) + h(x,y)F^{(s+t+1)}_{n+(p+1)t;p}(h,\ell) \\
&= F^{(s+t+1)}_{n+(p+1)(t+1)-p;p}(h,\ell).
\end{aligned}$$

Thus, by induction on t, the desired identity follows for all $t \geq 0$. □

Similarly, we obtain the recurrence relation for the incomplete generalized bivariate Lucas p-polynomials,

$$L^{(s+1)}_{n;p}(h,\ell) = h(x,y)L^{(s+1)}_{n-1;p}(h,\ell) + \ell(x,y)L^{(s)}_{n-p-1;p}(h,\ell),$$

where $n \geq p+1$ and $0 \leq s \leq \lfloor \frac{n-p-1}{p+1} \rfloor$. Equivalently, a nonhomogeneous recursion is given by

$$\begin{aligned}
L^{(s)}_{n;p}(h,\ell) &= h(x,y)L^{(s)}_{n-1;p}(h,\ell) + \ell(x,y)L^{(s)}_{n-p-1;p}(h,\ell) \\
&\quad - \frac{n-p-1}{n-(s+1)p-1}\binom{n-(s+1)p-1}{s} h^{n-(s+1)(p+1)}(x,y)\ell^{s+1}(x,y).
\end{aligned}$$

By a similar argument to the proof of Proposition 2, we obtain the following result.

Proposition 3. *For all integer $t \geq 0$, and $0 \leq s \leq \lfloor \frac{n-t-p}{p+1} \rfloor$, we have*

$$\sum_{j=0}^{t} \binom{t}{j} h^j(x,y)\ell^{t-j}(x,y)L^{(s+j)}_{n+p(j-1);p}(h,\ell) = L^{(s+t)}_{n+(p+1)t-p;p}(h,\ell).$$

There is an identity between the incomplete generalized bivariate Fibonacci p-polynomials and the incomplete generalized bivariate Lucas p-polynomials.

Lemma 2. *For any integer s with* $0 \le s \le \lfloor \frac{n-p-1}{p+1} \rfloor$*, we have*

$$L_{n;p}^{(s+1)}(h,\ell) = F_{n+1;p}^{(s+1)}(h,\ell) + p\ell(x,y)F_{n-p;p}^{(s)}(h,\ell). \tag{7}$$

Proof. It can be deduced directly from the definition.

$$\begin{aligned} L_{n;p}^{(s+1)}(h,\ell) - F_{n+1;p}^{(s+1)}(h,\ell) &= \sum_{j=0}^{s+1}\left[\frac{n}{n-jp}\binom{n-jp}{j} - \binom{n-jp}{j}\right]h^{n-j(p+1)}(x,y)\ell^j(x,y) \\ &= \sum_{j=1}^{s+1}\frac{jp}{n-jp}\binom{n-jp}{j}h^{n-j(p+1)}(x,y)\ell^j(x,y) \\ &= \sum_{j=1}^{s+1}p\binom{n-jp-1}{j-1}h^{n-j(p+1)}(x,y)\ell^j(x,y) \\ &= p\ell(x,y)F_{n-p;p}^{(s)}(h,\ell). \end{aligned}$$

□

We define the generating function of the incomplete generalized bivariate Fibonacci p-polynomial $F_{n;p}^{(s)}(h,\ell)$ by

$$R_p^{(s)}(h,\ell;z) = \sum_{n=0}^{\infty} F_{n;p}^{(s)}(h,\ell)z^n.$$

Since $F_{n;p}^{(s)}(h,\ell) = 0$ for $n < s(p+1)+1$, we see that

$$R_p^{(s)}(h,\ell;z) = z^{s(p+1)+1}\sum_{j=0}^{\infty} F_{s(p+1)+1+j;p}^{(s)}(h,\ell)z^j.$$

Let $G_p^{(s)}(h,\ell;z) = \sum_{j=0}^{\infty} F_{s(p+1)+1+j;p}^{(s)}(h,\ell)z^j$ and then $R_p^{(s)}(h,\ell;z) = z^{s(p+1)+1}G_p^{(s)}$ $(h,\ell;z)$.

Proposition 4. *The generating function* $R_p^{(s)}(h,\ell;z)$ *of the incomplete generalized bivariate Fibonacci p-polynomials is given by*

$$\begin{aligned} R_p^{(s)}(h,\ell;z) =& z^{s(p+1)+1}\Bigg[F_{(s+1)(p+1);p}^{(s)}(h,\ell)z^p + \sum_{j=0}^{p-1}(1-h(x,y)z)F_{s(p+1)+1+j;p}^{(s)}(h,\ell)z^j \\ &- \frac{z^{p+1}\ell^{s+1}(x,y)}{(1-h(x,y)z)^{s+1}}\Bigg](1-h(x,y)z-\ell(x,y)z^{p+1})^{-1}. \end{aligned}$$

Proof. We write

$$\begin{aligned} &G_p^{(s)}(h,\ell;z) - F_{s(p+1)+1;p}^{(s)}(h,\ell) - F_{s(p+1)+2;p}^{(s)}(h,\ell)z - \cdots - F_{s(p+1)+p+1;p}^{(s)}(h,\ell)z^p \\ &= \sum_{j=p+1}^{\infty} F_{s(p+1)+1+j;p}^{(s)}(h,\ell)z^j. \end{aligned}$$

In light of (6), the above right-hand side is equal to

$$\begin{aligned} \sum_{j=p+1}^{\infty}\Bigg[&h(x,y)F_{s(p+1)+j;p}^{(s)}(h,\ell) + \ell(x,y)F_{s(p+1)+j-p;p}^{(s)}(h,\ell) \\ &-\binom{s+j-p-1}{s}h^{j-p-1}(x,y)\ell^{s+1}(x,y)\Bigg]z^j, \end{aligned}$$

or

$$h(x,y)z\left[G_p^{(s)}(h,\ell;z)-\sum_{j=0}^{p-1}F_{s(p+1)+1+j;p}^{(s)}(h,\ell)z^j\right]+\ell(x,y)z^{p+1}G_p^{(s)}(h,\ell;z)$$
$$-\frac{z^{p+1}\ell^{s+1}(x,y)}{(1-h(x,y)z)^{s+1}}.$$

This implies that

$$\left(1-h(x,y)z-\ell(x,y)z^{p+1}\right)G_p^{(s)}(h,\ell;z)=F_{(s+1)(p+1);p}^{(s)}(h,\ell)z^p$$
$$+\sum_{j=0}^{p-1}(1-h(x,y)z)F_{s(p+1)+1+j;p}^{(s)}(h,\ell)z^j-\frac{z^{p+1}\ell^{s+1}(x,y)}{(1-h(x,y)z)^{s+1}}.$$

Since $R_p^{(s)}(h,\ell;z)=z^{s(p+1)+1}G_p^{(s)}(h,\ell;z)$, the proof is done. □

Let the generating function of the incomplete Fibonacci numbers $F_n(s)$ be

$$R_s(z)=\sum_{n=0}^{\infty}F_n(s)z^n.$$

According to Proposition 4, the special case $p=1$ and $h(x,y)=\ell(x,y)=1$ gives

$$R_s(z)=\frac{[F_{2s+1}(s)+(F_{2s+2}(s)-F_{2s+1}(s))z]z^{2s+1}}{1-z-z^2}-\frac{z^{2s+3}}{(1-z-z^2)(1-z)^{s+1}}.$$

Because $F_{2s+1}(s)$ is the Fibonacci number F_{2s+1} and also $F_{2s+2}(s)=F_{2s+2}$, we obtain the following corollary.

Corollary 4. *Let $R_s(z)$ be the generating function of the incomplete Fibonacci numbers $F_n(s)$. We have*

$$R_s(z)=\frac{(F_{2s+1}+F_{2s}z)z^{2s+1}}{1-z-z^2}-\frac{z^{2s+3}}{(1-z-z^2)(1-z)^{s+1}}.$$

We now define the generating function of the incomplete generalized bivariate Lucas p-polynomials $L_{n;p}^{(s)}(h,\ell)$ by $T_p^{(s)}(h,\ell;z)=\sum_{n=0}^{\infty}L_{n;p}^{(s)}(h,\ell)z^n$. By (7), we have

$$\begin{aligned}T_p^{(s)}(h,\ell;z)&=\sum_{n=0}^{\infty}\left[F_{n+1;p}^{(s)}(h,\ell)+p\ell(x,y)F_{n-p;p}^{(s-1)}(h,\ell)\right]z^n\\&=\frac{1}{z}R_p^{(s)}(h,\ell;z)+p\ell(x,y)z^pR_p^{(s-1)}(h,\ell;z).\end{aligned}$$

From this, we further have the following result.

Proposition 5. *The generating function $T_p^{(s)}(h,\ell;z)$ of the incomplete generalized bivariate Lucas p-polynomials is given by*

$$T_p^{(s)}(h,\ell;z)=z^{s(p+1)}\left[L_{s(p+1)+p;p}^{(s)}(h,\ell)z^p+\sum_{j=0}^{p-1}(1-h(x,y)z)L_{s(p+1)+j;p}^{(s)}(h,\ell)z^j\right.$$
$$\left.-\frac{(1+p-ph(x,y)z)z^{p+1}\ell^{s+1}(x,y)}{(1-h(x,y)z)^{s+1}}\right](1-h(x,y)z-\ell(x,y)z^{p+1})^{-1}.$$

Corollary 5. *Let $T_s(z)$ be the generating function of the incomplete Lucas numbers $L_n(s)$. We have*

$$T_s(z) = \sum_{n=0}^{\infty} L_n(s)z^n = \frac{(L_{2s} + L_{2s-1}z)z^{2s}}{1 - z - z^2} - \frac{(2 - z)z^{2s+2}}{(1 - z - z^2)(1 - z)^{s+1}}.$$

From the previous two propositions, we cancel all inhomogeneous terms of the two representations of the generating function and consider

$$zT_p^{(s)}(h, \ell; z) - (1 + p - ph(x, y)z)R_p^{(s)}(h, \ell; z).$$

After careful calculation, we obtain the following result.

Proposition 6. *Notations as above, for all $s \geq 1$, we have*

$$zT_p^{(s)}(h, \ell; z) = (1 + p - ph(x, y)z)R_p^{(s)}(h, \ell; z) - ph(x, y)F_{s(p+1);p}^{(s-1)}(h, \ell)z^{s(p+1)+1}. \quad (8)$$

We remark here. We use only relation (6) (see Proposition 1) when proving Proposition 6, and do not use relation (7) in Lemma 2. To see (7) for another proof, we compare the coefficient z^{n+1} on both sides of Equation (8) to obtain

$$L_{n;p}^{(s)}(h, \ell) = (1 + p)F_{n+1;p}^{(s)}(h, \ell) - ph(x, y)F_{n;p}^{(s)}(h, \ell), \quad (9)$$

where $0 \leq s \leq \lfloor \frac{n-1}{p+1} \rfloor$. Replacing s in (9) with $s + 1$ and using the recurrence relation of $F_{n;p}^{(s+1)}(h, \ell)$, we obtain relation (7):

$$L_{n;p}^{(s+1)}(h, \ell) = F_{n+1;p}^{(s+1)}(h, \ell) + p\ell(x, y)F_{n-p;p}^{(s)}(h, \ell).$$

In the very special case $p = 1$ and $h(x, y) = \ell(x, y) = 1$ of (8), we obtain

$$zT_s(z) = (2 - z)R_s(z) - z^{2s+1}F_{2s}.$$

Comparing the coefficients of z^{n+1} on both sides of the above equation, we obtain

$$L_n(s) = 2F_{n+1}(s) - F_n(s),$$

which is indeed Lemma 1.

We are now in a position to prove our main theorem.

Proof of Theorem 1. Our proof relies on Equation (9) and the similar argument for proving (4). Consider the telescoping sum

$$\sum_{i=m_1}^{m_2} \left[r^{i+1}F_{i+1;p}^{(s)}(h, \ell) - r^i F_{i;p}^{(s)}(h, \ell) \right] = r^{m_2+1}F_{m_2+1;p}^{(s)}(h, \ell) - r^{m_1}F_{m_1;p}^{(s)}(h, \ell),$$

where $0 \leq m_1 < m_2 \leq n$. Hence, for $0 \leq s \leq \lfloor \frac{n-1}{p+1} \rfloor$, we have

$$\begin{aligned} &r^{n+1}ph(x, y)F_{n+1;p}^{(s)}(h, \ell) - r^{s(p+1)+1}ph(x, y)F_{s(p+1)+1;p}^{(s)}(h, \ell) \\ &= \sum_{i=s(p+1)+1}^{n} r^{i+1}ph(x, y)F_{i+1;p}^{(s)}(h, \ell) - \sum_{i=s(p+1)+1}^{n} r^i ph(x, y)F_{i;p}^{(s)}(h, \ell). \end{aligned}$$

In light of (9), the right-hand side is equal to

$$\sum_{i=s(p+1)+1}^{n} r^{i+1} ph(x,y) F_{i+1;p}^{(s)}(h,\ell) - \sum_{i=s(p+1)+1}^{n} r^{i}\Big[(1+p)F_{i+1;p}^{(s)}(h,\ell) - L_{i;p}^{(s)}(h,\ell)\Big]$$
$$= \sum_{i=s(p+1)+1}^{n} r^{i} L_{i;p}^{(s)}(h,\ell) + (rph(x,y)-p-1)\sum_{i=s(p+1)+1}^{n} r^{i} F_{i+1;p}^{(s)}(h,\ell).$$

If n is a multiplier of $p+1$ and $s=\lfloor\frac{n}{p+1}\rfloor$, the desired relation holds obviously. This proves Theorem 1. □

3. From Complete to Incomplete

Actually, one may start with the regular Fibonacci–Lucas relation. Given two polynomials $h(x,y), \ell(x,y) \in \mathbb{R}[x,y]$, and an integer $p \geq 1$. Let the generalized bivariate Fibonacci and Lucas p-polynomials be defined by recursive relations:

$$F_{n;p}(h,\ell) = h(x,y)F_{n-1;p}(h,\ell) + \ell(x,y)F_{n-p-1;p}(h,\ell) \quad (n \geq p+1), \tag{10}$$

with initial conditions

$$F_{0;p}(h,\ell) = 0,\ F_{m;p}(h,\ell) = h^{m-1}(x,y), \ \text{ for } m = 1,2,\ldots,p,$$

and

$$L_{n;p}(h,\ell) = h(x,y)L_{n-1;p}(h,\ell) + \ell(x,y)L_{n-p-1;p}(h,\ell) \quad (n \geq p+1), \tag{11}$$

with initial conditions

$$L_{0;p}(h,\ell) = p+1,\ L_{m;p}(h,\ell) = h^{m}(x,y), \ \text{ for } m = 1,2,\ldots,p.$$

It is not difficult to derive the explicit formulas of these polynomials.

Proposition 7. *The explicit formula of generalized bivariate Fibonacci p-polynomials $F_{n;p}(h,\ell)$ is*

$$F_{n;p}(h,\ell) = \sum_{j=0}^{\lfloor\frac{n-1}{p+1}\rfloor} \binom{n-jp-1}{j} h^{n-j(p+1)-1}(x,y)\ell^{j}(x,y);$$

and the explicit formula of generalized bivariate Lucas p-polynomials $L_{n;p}(h,\ell)$ is given by

$$L_{n;p}(h,\ell) = \sum_{j=0}^{\lfloor\frac{n}{p+1}\rfloor} \frac{n}{n-jp}\binom{n-jp}{j} h^{n-j(p+1)}(x,y)\ell^{j}(x,y).$$

Notice that $F_{n;p}(h,\ell) = F_{n;p}^{(\lfloor\frac{n-1}{p+1}\rfloor)}(h,\ell)$ and $L_{n;p}(h,\ell) = L_{n;p}^{(\lfloor\frac{n}{p+1}\rfloor)}(h,\ell)$. One may obtain the relation between these polynomials easily.

Lemma 3. *Notations as above, we have*

$$L_{n;p}(h,\ell) = F_{n+1;p}(h,\ell) + p\ell(x,y)F_{n-p;p}(h,\ell). \tag{12}$$

Using relation (12), we can obtain a relation involving generalized bivariate Fibonacci p-polynomials $F_{n;p}(h,\ell)$ and generalized bivariate Lucas p-polynomials $L_{n;p}(h,\ell)$.

Theorem 2. *For any integer $n \geq 0$ and any real nonzero number r, we have*

$$r^{n+1}ph(x,y)F_{n+1;p}(h,\ell) = \sum_{i=0}^{n} r^i L_{i;p}(h,\ell) + (rph(x,y) - p - 1)\sum_{i=0}^{n} r^i F_{i+1;p}(h,\ell). \tag{13}$$

Proof. By (12) and the recurrence relation of generalized bivariate Fibonacci p-polynomials, the right-hand side of (13) is equal to

$$\begin{aligned}
&\sum_{i=0}^{n} r^i ph(x,y)\left[rF_{i+1;p}(h,\ell) - F_{i;p}(h,\ell)\right] \\
&= ph(x,y)\left[\sum_{i=0}^{n} r^{i+1}F_{i+1;p}(h,\ell) - \sum_{i=0}^{n} r^i F_{i;p}(h,\ell)\right] \\
&= r^{n+1}ph(x,y)F_{n+1;p}(h,\ell).
\end{aligned}$$

□

Indeed, Theorem 2 is a generalization of the well-known Fibonacci–Lucas relation (Sury's formula, see (3)). Theorem 2 also infers that two sums

$$\sum_{i=0}^{s(p+1)} r^i L_{i;p}^{(\lfloor \frac{i}{p+1} \rfloor)}(h,\ell) + (rph(x,y) - p - 1)\sum_{i=0}^{s(p+1)} r^i F_{i+1;p}^{(\lfloor \frac{i}{p+1} \rfloor)}(h,\ell)$$

must be equal to

$$ph(x,y)r^{s(p+1)+1}F^{(s)}_{s(p+1)+1;p}(h,\ell),$$

for s is any integer with $0 \leq s \leq \lfloor \frac{n}{p+1} \rfloor$. From this and using Equation (9) to make a telescopic sum, one can deduce the incomplete version of a generalized Fibonacci–Lucas relation, and obtain a relation involving the incomplete generalized bivariate Fibonacci and Lucas p-polynomial as Theorem 1.

Here is another example. For any integer $p \geq 1$, let the Leonardo p-numbers $\mathfrak{L}_{n;p}$ be defined by the following nonhomogeneous recurrence relation:

$$\mathfrak{L}_{n;p} = \mathfrak{L}_{n-1;p} + \mathfrak{L}_{n-p-1;p} + p, \tag{14}$$

for $n \geq p+1$ with initial conditions $\mathfrak{L}_{0;p} = \mathfrak{L}_{1;p} = \cdots = \mathfrak{L}_{p;p} = 1$. Tan and Leung [7] introduced the Leonardo p-sequence $\{\mathfrak{L}_{n;p}\}_{n\geq 0}$ as a generalization of classical Leonardo numbers. The Leonardo 1-numbers, simply denoted by $\mathfrak{L}_n$, are the classical Leonardo numbers that represent the number of vertices in the n-th Leonardo tree. That is, $\mathfrak{L}_n$ satisfies the relation:

$$\mathfrak{L}_n = \mathfrak{L}_{n-1} + \mathfrak{L}_{n-2} + 1,$$

for all integer $n \geq 2$ with initials $\mathfrak{L}_0 = \mathfrak{L}_1 = 1$. The first few terms of the classical Leonardo numbers are (OEIS:A001595, https://oeis.org/A001595 (accessed on 1 August 2023))

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, 753, 1219, 1973, 3193, 5167, 8361, 13, 529.$$

One can show easily that all classical Leonardo numbers are odd. Tan and Leung [7] investigated some basic properties of Leonardo p-numbers and derived some relations between the Leonardo p-numbers and the Fibonacci p-numbers (by letting $h(x,y) = \ell(x,y) = 1$ in (10)), such as

$$\mathfrak{L}_{n;p} = (p+1)F_{n+1;p} - p, \tag{15}$$

and in particular $\mathfrak{L}_n = 2F_{n+1} - 1$.

Now we define the companion of classical Leonardo numbers, $\mathfrak{K}_n$, the n-th Lucas–Leonardo number, which satisfies the relation

$$\mathfrak{K}_n = \mathfrak{K}_{n-1} + \mathfrak{K}_{n-2} + 1,$$

for all $n \geq 2$ with initial values $\mathfrak{K}_0 = 3, \mathfrak{K}_1 = 1$. Notice that $\mathfrak{K}_n + 1 = (\mathfrak{K}_{n-1} + 1) + (\mathfrak{K}_{n-2} + 1)$, which implies that $\mathfrak{K}_n = 2L_n - 1$. The sequence $\{\mathfrak{K}_n\}_{n\geq 0}$ begins with

$$3, 1, 5, 7, 13, 21, 35, 57, 93, 151, 245, 397, 643, 1041, 1685, \ldots$$

(OEIS:A022319, https://oeis.org/A022319 (accessed on 1 August 2023)). Additionally, we may define the Lucas–Leonardo p-numbers $\mathfrak{K}_{n;p}$ by the recurrence relation

$$\mathfrak{K}_{n;p} = \mathfrak{K}_{n-1;p} + \mathfrak{K}_{n-p-1;p} + p$$

for all $n \geq p+1$ with initial values $\mathfrak{K}_{0;p} = p^2 + p + 1, \mathfrak{K}_{1;p} = \mathfrak{K}_{2;p} = \cdots = \mathfrak{K}_{p;p} = 1$. For $p = 1$, we obtain the Lucas–Leonardo sequence. The nonhomogeneous recurrence relation of Lucas–Leonardo p-numbers can be converted to the following homogeneous recurrence relation, for $n \geq 2p + 1$:

$$\mathfrak{K}_{n;p} = \mathfrak{K}_{n-1;p} + \mathfrak{K}_{n-p;p} - \mathfrak{K}_{n-2p-1;p}.$$

In addition, we have the following proposition.

Proposition 8. *For $n \geq 0$, we have*

$$\mathfrak{K}_{n;p} = (p+1)L_{n;p} - p, \tag{16}$$

where $L_{n;p}$ is the n-th Lucas p-numbers (by letting $h(x,y) = \ell(x,y) = 1$ in (11)).

Proof. For $n = 0$, we obtain $\mathfrak{K}_{0;p} = p^2 + p + 1$ and $(p+1)L_{0;p} - p = (p+1)^2 - p = p^2 + p + 1$. Thus, the relation holds when $n = 0$. For $n = 1, 2, \ldots, p$, it is easy to see that the desired relation holds. Now, we finish the proof by using induction on n. Suppose that the relation holds for some n that is greater than p. For $\mathfrak{K}_{n+1;p}$, we have

$$\begin{aligned}
\mathfrak{K}_{n+1;p} &= \mathfrak{K}_{n;p} + \mathfrak{K}_{n-p;p} + p \\
&= (p+1)L_{n;p} - p + (p+1)L_{n-p;p} - p + p \\
&= (p+1)\big(L_{n;p} + L_{n-p;p}\big) - p \\
&= (p+1)L_{n+1;p} - p.
\end{aligned}$$

Hence, the desired relation holds for $n \geq 0$ by induction. □

By substituting $h(x,y) = \ell(x,y) = 1$ into (12), we obtain $L_{n;p} = F_{n+1;p} + pF_{n-p;p}$. Then, we obtain the following result.

Proposition 9. *For $n \geq 0$, we have*

$$\mathfrak{K}_{n;p} = (p+1)\mathfrak{L}_{n;p} - p\mathfrak{L}_{n-1;p}.$$

In particular, for $p = 1$, we obtain $\mathfrak{K}_n = 2\mathfrak{L}_n - \mathfrak{L}_{n-1}$.

Proof. By using (15) and (14), we obtain

$$\begin{aligned}
\mathfrak{K}_{n;p} &= (p+1)L_{n;p} - p \\
&= (p+1)\left(F_{n+1;p} + pF_{n-p;p}\right) - p \\
&= \mathfrak{L}_{n;p} + p(p+1)F_{n-p;p} \\
&= \mathfrak{L}_{n;p} + p\left(\mathfrak{L}_{n-p-1;p} + p\right) \\
&= \mathfrak{L}_{n;p} + p\left(\mathfrak{L}_{n;p} - \mathfrak{L}_{n-1;p}\right) \\
&= (p+1)\mathfrak{L}_{n;p} - p\mathfrak{L}_{n-1;p}.
\end{aligned}$$

□

Theorem 3. *For any integers $p \geq 1$ and $n \geq 0$ and any real nonzero number r, we have a relation involving the Leonardo p-numbers and the Lucas–Leonardo p-numbers,*

$$r^{n+1}p\mathfrak{L}_{n;p} = \sum_{i=0}^{n} r^i \mathfrak{K}_{i;p} + (rp - p - 1)\sum_{i=0}^{n} r^i \mathfrak{L}_{i;p} - p^2, \tag{17}$$

and an alternating relation involving the Leonardo p-numbers and the Lucas–Leonardo p-numbers,

$$(-1)^n p\mathfrak{L}_{n;p} = \sum_{i=0}^{n}(-1)^{i-1} r^{n-i+1}\mathfrak{K}_{i;p} + (r + p + rp)\sum_{i=0}^{n}(-1)^i r^{n-i}\mathfrak{L}_{i;p} + p^2 r^{n+1}.$$

Proof. We use the result in Proposition 9 to compute the summation:

$$\begin{aligned}
\sum_{i=0}^{n} r^i \mathfrak{K}_{i;p} + (rp - p - 1)r^i \mathfrak{L}_{i;p} &= p^2 + rp + \sum_{i=1}^{n} r^i \mathfrak{K}_{i;p} + (rp - p - 1)r^i \mathfrak{L}_{i;p} \\
&= p^2 + rp + p\sum_{i=1}^{n}\left[r^{i+1}\mathfrak{L}_{i;p} - r^i \mathfrak{L}_{i-1;p}\right] \\
&= p^2 + rp + p\left(r^{n+1}\mathfrak{L}_{n;p} - r\mathfrak{L}_{0;p}\right) \\
&= p^2 + r^{n+1}p\mathfrak{L}_{n;p}.
\end{aligned}$$

Then the second assertion follows from the first assertion by substituting r with $-1/r$. □

We note that relation (17) implies

$$p \mid \left(\mathfrak{K}_{n;p} - \mathfrak{L}_{n;p}\right),$$

for any integers $n \geq 0$ and $p \geq 1$. This follows also from Proposition 9.

Corollary 6. *For any integer $n \geq 0$ and any real nonzero number r, we have*

$$r^{n+1}\mathfrak{L}_n = \sum_{i=0}^{n} r^i \mathfrak{K}_i + (r-2)\sum_{i=0}^{n} r^i \mathfrak{L}_i - 1.$$

In particular, we have

$$2^{n+1}\mathfrak{L}_n = \sum_{i=0}^{n} 2^i \mathfrak{K}_i - 1.$$

Recently, Tan and Leung [7] investigated incomplete Leonardo p-numbers and gave some properties of these numbers. Indeed, in their paper [7], they defined the incomplete Leonardo p-numbers as

$$\mathfrak{L}_{n;p}(s) = (p+1)\sum_{j=0}^{s}\binom{n-jp}{j} - p,$$

where s is an integer with $0 \leq s \leq \lfloor \frac{n}{p+1} \rfloor$. From this definition and (15), it is clear to see that $\mathfrak{L}_{n;p}(0) = 1, \mathfrak{L}_{n;p}(1) = (p+1)(n-p)+1$, and $\mathfrak{L}_{n;p}(\lfloor \frac{n}{p+1} \rfloor) = \mathfrak{L}_{n;p}$. The incomplete Leonardo p-numbers $\mathfrak{L}_{n;p}(s)$ satisfy the recurrence relation

$$\mathfrak{L}_{n;p}(s+1) = \mathfrak{L}_{n-1;p}(s+1) + \mathfrak{L}_{n-p-1;p}(s) + p,$$

for $0 \leq s \leq \frac{n-p-2}{p+1}$. One can find proof in [7].

Similarly, we consider relation (16) and may define the incomplete Lucas–Leonardo p-numbers as below. For integers $n \geq 0$ and $p \geq 1$ and an integer s with $0 \leq s \leq \lfloor \frac{n}{p+1} \rfloor$, we define

$$\mathfrak{K}_{n;p}(s) = (p+1)\sum_{j=0}^{s}\frac{n}{n-jp}\binom{n-jp}{j} - p.$$

Some special cases of the above definition are

1. $\mathfrak{K}_{n;p}(0) = 1$,
2. $\mathfrak{K}_{n;p}(1) = (p+1)(n+1) - p$,
3. $\mathfrak{K}_{n;p}(\lfloor \frac{n}{p+1} \rfloor) = (p+1)L_{n;p} - p = \mathfrak{K}_{n;p}$.

Furthermore, we have the following proposition.

Proposition 10. *The recurrence relation of the incomplete Lucas–Leonardo p-numbers* $\mathfrak{K}_{n;p}(s)$ *is*

$$\mathfrak{K}_{n;p}(s+1) = \mathfrak{K}_{n-1;p}(s+1) + \mathfrak{K}_{n-p-1;p}(s) + p, \tag{18}$$

for $0 \leq s \leq \lfloor \frac{n-p-1}{p+1} \rfloor$.

Proof. By definition, for $0 \leq s \leq \lfloor \frac{n-p-1}{p+1} \rfloor$, we have

$$\begin{aligned}
&\mathfrak{K}_{n-1;p}(s+1) + \mathfrak{K}_{n-p-1;p}(s) + p \\
&= (p+1)\sum_{j=0}^{s+1}\frac{n-1}{n-jp-1}\binom{n-jp-1}{j} \\
&\qquad + (p+1)\sum_{j=0}^{s}\frac{n-p-1}{n-(j+1)p-1}\binom{n-(j+1)p-1}{j} - p \\
&= (p+1) + (p+1)\sum_{j=1}^{s+1}\frac{n-1}{n-jp-1}\binom{n-jp-1}{j} + \frac{n-p-1}{n-jp-1}\binom{n-jp-1}{j-1} - p \\
&= (p+1)\sum_{j=1}^{s+1}\frac{n}{n-jp}\binom{n-jp}{j} + 1 \\
&= (p+1)\sum_{j=0}^{s+1}\frac{n}{n-jp}\binom{n-jp}{j} - p \\
&= \mathfrak{K}_{n;p}(s+1).
\end{aligned}$$

□

It is easy to see that relation (18) can be transformed into the nonhomogeneous recurrence relation:

$$\mathfrak{K}_{n;p}(s) = \mathfrak{K}_{n-1;p}(s) + \mathfrak{K}_{n-p-1;p}(s) + p - (p+1)\frac{n-p-1}{n-(s+1)p-1}\binom{n-(s+1)p-1}{s},$$

for $0 \le s \le \lfloor \frac{n-p-1}{p+1} \rfloor$.

The following result gives a link between the incomplete Leonardo p-numbers and the incomplete Lucas–Leonardo p-numbers.

Lemma 4. *Notations as above, we have*

$$\mathfrak{K}_{n;p}(s) = (p+1)\mathfrak{L}_{n;p}(s) - p\mathfrak{L}_{n-1;p}(s), \tag{19}$$

for $0 \le s \le \lfloor \frac{n-1}{p+1} \rfloor$.

Proof.

$$\begin{aligned}
&\mathfrak{K}_{n;p}(s) + p\mathfrak{L}_{n-1;p}(s) \\
&= (p+1)\sum_{j=0}^{s}\left[\frac{n}{n-jp}\binom{n-jp}{j} + p\binom{n-jp-1}{j}\right] - p - p^2 \\
&= (p+1)\sum_{j=0}^{s}(p+1)\binom{n-jp}{j} - p(p+1) \\
&= (p+1)\mathfrak{L}_{n;p}(s).
\end{aligned}$$

□

Our next goal is to transform our Theorem 3 into an incomplete version. For this purpose, note that $\mathfrak{L}_{n;p}(\lfloor \frac{n}{p+1} \rfloor) = \mathfrak{L}_{n;p}$ and $\mathfrak{K}_{n;p}(\lfloor \frac{n}{p+1} \rfloor) = \mathfrak{K}_{n;p}$, and then we write relation (17) in an equivalent form:

$$r^{n+1}p\mathfrak{L}_{n;p}(\lfloor \frac{n}{p+1} \rfloor) = \sum_{i=0}^{n} r^i \mathfrak{K}_{i;p}(\lfloor \frac{i}{p+1} \rfloor) + (rp-p-1)\sum_{i=0}^{n} r^i \mathfrak{L}_{i;p}(\lfloor \frac{i}{p+1} \rfloor) - p^2.$$

Theorem 4. *For any integers* $p \ge 1$ *and* $n \ge 0$ *and any real nonzero number* r*, we have a relation involving the incomplete Leonardo* p*-numbers and the incomplete Lucas–Leonardo* p*-numbers,*

$$\begin{aligned}
r^{n+1}p\mathfrak{L}_{n;p}(s) = r^{s(p+1)+1}p\mathfrak{L}_{s(p+1);p}(s) + \sum_{i=s(p+1)+1}^{n} r^i \mathfrak{K}_{i;p}(s) \\
+ (rp-p-1)\sum_{i=s(p+1)+1}^{n} r^i \mathfrak{L}_{i;p}(s) - p^2,
\end{aligned} \tag{20}$$

or any integer s *with* $0 \le s \le \lfloor \frac{n}{p+1} \rfloor$.

Proof. By Theorem 3, we have the sum

$$\sum_{i=0}^{s(p+1)} r^i \mathfrak{K}_{i;p}(\lfloor \frac{i}{p+1} \rfloor) + (rp-p-1)\sum_{i=0}^{s(p+1)} r^i \mathfrak{L}_{i;p}(\lfloor \frac{i}{p+1} \rfloor)$$

must be equal to $r^{s(p+1)+1}p\mathfrak{L}_{s(p+1);p}(s)+p^2$. Hence, in order to prove the assertion, it is suffice to show

$$r^{n+1}p\mathfrak{L}_{n;p}(s)+p^2=\sum_{i=0}^{s(p+1)}\Big[r^i\mathfrak{K}_{i;p}+(rp-p-1)r^i\mathfrak{L}_{i;p}\Big]$$
$$+\sum_{i=s(p+1)+1}^{n}\Big[r^i\mathfrak{K}_{i;p}(s)+(rp-p-1)r^i\mathfrak{L}_{i;p}(s)\Big],$$

for all integer s with $0\le s\le\lfloor\frac{n}{p+1}\rfloor$. The first summation of the right-hand side deals with a complete form, and the second deals with an incomplete form. For $0\le s\le\lfloor\frac{n-1}{p+1}\rfloor$, we use (19) of Lemma 4 to compute

$$\begin{aligned}&\sum_{i=s(p+1)+1}^{n}\Big[r^i\mathfrak{K}_{i;p}(s)+(rp-p-1)r^i\mathfrak{L}_{i;p}(s)\Big]\\&=p\sum_{i=s(p+1)+1}^{n}\Big[r^{i+1}\mathfrak{L}_{i;p}(s)-r^i\mathfrak{L}_{i-1;p}(s)\Big]\\&=r^{n+1}p\mathfrak{L}_{n;p}(s)-r^{s(p+1)+1}p\mathfrak{L}_{s(p+1);p}(s)\\&=r^{n+1}p\mathfrak{L}_{n;p}(s)-\sum_{i=0}^{s(p+1)}\Big[r^i\mathfrak{K}_{i;p}+(rp-p-1)r^i\mathfrak{L}_{i;p}\Big]+p^2.\end{aligned}$$

Hence, the proof finishes if $0\le s\le\lfloor\frac{n-1}{p+1}\rfloor$. For n is a multiplier of $p+1$ and $s=\lfloor\frac{n}{p+1}\rfloor$, relation (20) holds obviously. □

We can replace r with $-1/r$ in (20) to obtain the following alternating relation involving the incomplete Leonardo p-numbers and the incomplete Lucas–Leonardo p-numbers:

$$\begin{aligned}(-1)^np\mathfrak{L}_{n;p}(s)=&(-1)^{s(p+1)}r^{n-s(p+1)}p\mathfrak{L}_{s(p+1);p}(s)+\sum_{i=s(p+1)}^{n}(-1)^{i+1}r^{n+1-i}\mathfrak{K}_{i;p}(s)\\&+(rp-p-1)\sum_{i=s(p+1)}^{n}(-1)^{i+1}r^{n+1-i}\mathfrak{L}_{i;p}(s)+r^{n+1}p^2.\end{aligned}$$

4. Conclusions

The goal of this note is to establish some generalized relations of the well-known Fibonacci–Lucas relation. The regular (complete) Fibonacci–Lucas-type relation can be obtained by the crucial inter-relationship (such as Lemmas 1–4) between these numbers (polynomials). Such a relation can be also obtained based on their generating functions. We present the whole procedure for how to deduce an incomplete Fibonacci–Lucas-type relation from the known complete one. We provide a Fibonacci–Lucas-type relation involving the incomplete generalized bivariate Fibonacci p-polynomials and the incomplete generalized bivariate Lucas p-polynomials and a relation involving the incomplete Leonardo p-numbers and the incomplete Lucas–Leonardo p-numbers.

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References

1. Filipponi, P. Incomplete Fibonacci and Lucas numbers. *Rend. Del Circ. Mat. Palermo* **1996**, *45*, 37–56. [CrossRef]
2. Sury, B. A polynomial parent to a Fibonacci-Lucas relation. *Am. Math. Mon.* **2014**, *3*, 236. [CrossRef]
3. Chung, C.-L. Some polynomial sequence relations. *Mathematics* **2019**, *7*, 750. [CrossRef]
4. Chung, C.-L.; Yao, J.; Zhou, K. Extensions of Sury's relation involving Fibonacci k-step and Lucas k-step polynomials. *Integers* **2023**, *23*, A38.
5. Tasci, D.; Cetin Firengiz, M.; Tuglu, N. Incomplete bivariate Fibonacci and Lucas p-polynomials. *Discret. Dyn. Nat. Soc.* **2012**, *2012*, 840345. [CrossRef]
6. Ramírez, J.L. Incomplete generalized Fibonacci and Lucas polynomials. *Hacet. J. Math. Stat.* **2015**, *44*, 363–373. [CrossRef]
7. Tan, E.; Leung, H.-H. On Leonardo p-numbers. *Integers* **2023**, *23*, A7.