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A Novel Concept of Level Graph in Interval-Valued Fuzzy Graphs with Application

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Abstract: Many problems of practical interest can be modeled and solved by using interval-valued fuzzy graph (IVFG) algorithms. An IVFG is a very useful and effective tool for studying various calculations, fields of intelligence, and computer science, such as networking, imaging, and other fields, such as biological sciences. In different applications, they present an appropriate construction means. There were limitations in the definition of fuzzy graphs (FGs), which prompted us to propose a new definition for IVFGs. Some interesting properties related to the new IVFGs are investigated, and enough conditions under which the level graph on IVFGs is equivalent are obtained. Therefore, in this study, we present the properties of a level graph (LG) of an IVFG, and four operations, the Cartesian product (CP), composition (CO), union, and join, are investigated on it. Today, in a treatment system, one of the issues that can be very valuable and important to the quality of service to patients is finding qualified and efficient people in each department, which is not an easy task. But the interval-valued fuzzy graph, as an important fuzzy graph, can help us by considering the ability of each person in the form of intervals of numbers and the effectiveness of each one on the other (according to the relationships between them) in order to find the most worthy people. So, an application of IVFG to find the most effective person in a hospital information system has been introduced.

Keywords: fuzzy set; interval-valued fuzzy graph; level graph; Cartesian product; composition; join

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1. Introduction

Graph theory started its journey with the famous Konigsberg bridge problem. This problem was the birth of graph theory. Euler finally solved this problem with the help of graphs. Though graph theory is a relatively old subject, its growing applications are shown in research. After the presentation of fuzzy sets by Zadeh [1], fuzzy set (FS) theory has become a massive and strong study area in different fields, including life sciences, management science, statistics, graph theory, and decision making. Zadeh [2] presented the notion of an interval-valued fuzzy set (IVFS) as a continuation of FSs. Since the concept of IVFSs is a very useful issue, some researchers have conducted studies in this regard. Roy and Biswas [3] defined IVF relations. Rosenfeld [4] introduced the concept of fuzzy graphs (FGs). FG theory has various applications in medicine, engineering, psychology, and urban planning. Many researchers are trying to use it to optimize and save time. FG theory is finding an increasing number of applications in modeling real-time systems where the level of information in the system is different with various levels of precision. Some remarks on FGs were expressed by Bhattacharya [5]. Mordeson and Peng [6] described several notations on FGs. Akram et al. [7–10] introduced the definition of an IVFG and examined several of its properties. Hongmei and Lianhua [11] explained the new concept of IVFGs. Turksen [12] defined IVFSs based on normal forms. Some root-level modifications in IVFGs were presented by Jan et al. [13]. Nagoorgani and Radha [14] studied isomorphism on FGs. Rashmanlou and Pal [15] defined antipodal IVFGs. Certain types of m-polar IVFGs were proposed by [16]. Zihni et al. [17] introduced IVF soft graphs. IVFGs are of the FG family and have many abilities when involved with issues that FGs and VGs cannot explain. Since the membership value is not known, impartiality is a good advantage that can be well protected with an IVFG. Therefore, in this study, we extend the FG notion to the IVFG and discuss the well-known problems of level graphs, CP, CO, union, and join, on IVFGs. The level graph has a variety of applications in other sciences. They are used to identify the most effective person in an organization. Krishna et al. [18] explained new information in cubic graphs. Dey et al. [19–21] introduced the fuzzy minimum spanning tree with an interval type 2 fuzzy arc length and an interval type 2 FS in a fuzzy shortest path problem. Qiang et al. [22] expressed a novel description of VG structure and also investigated the new results of it. Some types of FGs were studied in [23,24].

The following points influenced us to write this article:

- Due to the enormous applications of LGs in fuzzy graphs, including level graphs for FGs in distinct decision-making problems, it also seems advantageous to expand the notion of LGs in IVFGs.
- There are numerous applications for the operation of LGs in chemistry, computer science, psychology, and other disciplines.
- Moreover, the LG of an IVFG has not yet been discussed and studied in the literature; therefore, we expanded the notion of a LG of an FG to a LG of an IVFG, and four operations, the Cartesian product, composition, union, and join, are investigated.

The intention of our proposed research work is given as follows:

- The aim of this research study is to investigate the notion of the Cartesian product and the composition of interval-valued fuzzy graphs.
- We introduce the concept of union and join of LGs on IVFGs and study their properties with some examples.
- The properties of LGs in IVFGs under important operations, the Cartesian product, composition, union, and join, are clearly shown.
- Finally, we present an application of an IVFG to find the most effective person in a hospital information system.

This paper is organized as follows.

In the first part, we study some theoretical background. In the second part, we explain some essential definitions of IVFGs and give details of new operations of them, such as the CP, CO, union, and join. In the third part, we introduce some operations of level graphs of IVFGs. In the last part, an application of IVFG is given.

2. Preliminaries

In this part, we present some definition which will be used throughout the paper.

A graph $\mathcal{G}^*$ is a ordered pair (X, E), where X is a nonempty set called the vertex set and $E \subseteq X \times X$ is called the edge set.

Definition 1 ([5])**.** *An FG of a graph $G^* = (X, E)$ is a pair $G = (\delta, \rho)$, where δ is an FS on X and ρ is an FS on E, so that*

$$\rho(ef) \leq \min\{\delta(e), \delta(f)\},$$

for all $ef \in E$.

The set of all closed interval on $[0, 1]$ is denoted by $I[0, 1]$.

Definition 2. *Let X be a nonempty set. A mapping $R = [\rho^l_R, \rho^u_R] : X \to I[0,1]$, ($x \to [\rho^l_R(x), \rho^u_R(x)] \in I[0,1]$) is called an IVFS on X. The lower and upper values of the vertex interval in IVFS are defined by ρ^l_R and ρ^u_R, respectively.*

Definition 3 ([7])**.** *A pair $G' = (R, S)$ of a graph $G^* = (X, E)$ is named an IVFG so that $R = [\rho_R^l, \rho_R^u]$ is an IVFS on X and $S = [\rho_S^l, \rho_S^u]$ is an IVFS on E that satisfies the following conditions:*

$$\rho_S^l(ef) \leq \min\{\rho_R^l(e), \rho_R^l(f)\},$$
$$\rho_S^u(ef) \leq \min\{\rho_R^u(e), \rho_R^u(f)\},$$

for all $ef \in E$.

Definition 4. *Assume that $G_1' = (R_1, S_1)$ and $G_2' = (R_2, S_2)$ are two IVFGs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively.*

(i) The union $G_1' \cup G_2'$ is defined as the pair (R, S) of IVFSs on the union of graphs G_1^* and G_2^*, so that

$$(a)\begin{cases}(\rho_{R_1}^l \cup \rho_{R_2}^l)(e) = \rho_{R_1}^l(e) & if\ e \in X_1, e \notin X_2\\ (\rho_{R_1}^l \cup \rho_{R_2}^l)(e) = \rho_{R_2}^l(e) & if\ e \in X_2, e \notin X_1\\ (\rho_{R_1}^l \cup \rho_{R_2}^l)(e) = \max\{\rho_{R_1}^l(e), \rho_{R_2}^l(e)\} & if\ e \in X_1 \cap X_2\end{cases}$$

$$(b)\begin{cases}(\rho_{R_1}^u \cup \rho_{R_2}^u)(e) = \rho_{R_1}^u(e) & if\ e \in X_1, e \notin X_2\\ (\rho_{R_1}^u \cup \rho_{R_2}^u)(e) = \rho_{R_2}^u(e) & if\ e \in X_2, e \notin X_1\\ (\rho_{R_1}^u \cup \rho_{R_2}^u)(e) = \max\{\rho_{R_1}^u(e), \rho_{R_2}^u(e)\} & if\ e \in X_1 \cap X_2\end{cases}$$

$$(c)\begin{cases}(\rho_{S_1}^l \cup \rho_{S_2}^l)(ef) = \rho_{S_1}^l(ef) & if\ ef \in E_1, ef \notin E_2\\ \rho_{S_1}^l \cup \rho_{S_2}^l)(ef) = \rho_{S_2}^l(ef) & if\ ef \in E_2, ef \notin E_1\\ (\rho_{S_1}^l \cup \rho_{S_2}^l)(ef) = \max\{\rho_{S_1}^l(ef), \rho_{S_2}^l(ef)\} & if\ ef \in E_1 \cap E_2\end{cases}$$

$$(d)\begin{cases}(\rho_{S_1}^u \cup \rho_{S_2}^u)(ef) = \rho_{S_1}^u(ef) & if\ ef \in E_1, ef \notin E_2\\ (\rho_{S_1}^u \cup \rho_{S_2}^u)(ef) = \rho_{S_2}^u(ef) & if\ ef \in E_2, ef \notin E_1\\ (\rho_{S_1}^u \cup \rho_{S_2}^u)(ef) = \max\{\rho_{S_1}^u(ef), \rho_{S_2}^u(ef)\} & if\ ef \in E_1 \cap E_2.\end{cases}$$

(ii) The CP of IVFGs G_1' and G_2', denoted by $G_1' \times G_2'$, is the pair (R, S) of IVFSs defined on the CP of graphs G_1^* and G_2^*, so that

$$(a)\begin{cases}(\rho_{R_1}^l \times \rho_{R_2}^l)(e_1, e_2) = \min\{\rho_{R_1}^l(e_1), \rho_{R_2}^l(e_2)\}\\ (\rho_{R_1}^u \times \rho_{R_2}^u)(e_1, e_2) = \min\{\rho_{R_1}^u(e_1), \rho_{R_2}^u(e_2)\}\end{cases} \text{for all } (e_1, e_2) \in X_1 \times X_2$$

$$(b)\begin{cases}(\rho_{S_1}^l \times \rho_{S_2}^l)((e, e_2)(e, f_2)) = \min\{\rho_{R_1}^l(e), \rho_{S_2}^l(e_2 f_2)\}\\ (\rho_{S_1}^u \times \rho_{S_2}^u)((e, e_2)(e, f_2)) = \min\{\rho_{R_1}^u(e), \rho_{S_2}^u(e_2 f_2)\}\end{cases} \text{for all } e \in X_1, e_2 f_2 \in E_2$$

$$(c)\begin{cases}(\rho_{S_1}^l \times \rho_{S_2}^l)((e_1, k)(f_1, k)) = \min\{\rho_{S_1}^l(e_1 f_1), \rho_{R_2}^l(k)\}\\ (\rho_{S_1}^u \times \rho_{S_2}^u)((e_1, k)(f_1, k)) = \min\{\rho_{S_1}^u(e_1 f_1), \rho_{R_2}^u(k)\}\end{cases} \text{for all } k \in X_2, e_1 f_1 \in E_1.$$

(iii) The CO of two IVFGs G_1' and G_2', shown by $G_1'[G_2']$, is the pair (R, S) of IVFSs defined on the CO $G_1'^*[G_2'^*]$, so that

$$(a)\begin{cases}(\rho_{R_1}^l \circ \rho_{R_2}^l)(e_1, e_2) = \min\{\rho_{R_1}^l(e_1), \rho_{R_2}^l(e_2)\}\\ (\rho_{R_1}^u \circ \rho_{R_2}^u)(e_1, e_2) = \min\{\rho_{R_1}^u(e_1), \rho_{R_2}^u(e_2)\}\end{cases} \text{for all } (e_1, e_2) \in X_1 \times X_2$$

$$(b)\begin{cases}(\rho_{S_1}^l \circ \rho_{S_2}^l)((e, e_2)(e, f_2)) = \min\{\rho_{R_1}^l(e), \rho_{S_2}^l(e_2 f_2)\}\\ (\rho_{S_1}^u \circ \rho_{S_2}^u)((e, e_2)(e, f_2)) = \min\{\rho_{R_1}^u(e), \rho_{S_2}^u(e_2 f_2)\}\end{cases} \text{for all } e \in X_1, e_2 f_2 \in E_2$$

$$(c)\begin{cases}(\rho_{S_1}^l \circ \rho_{S_2}^l)((e_1, k)(f_1, k)) = \min\{\rho_{S_1}^l(e_1 f_1), \rho_{R_2}^l(k)\}\\ (\rho_{S_1}^u \circ \rho_{S_2}^u)((e_1, k)(f_1, k)) = \min\{\rho_{S_1}^u(e_1 f_1), \rho_{R_2}^u(k)\}\end{cases} \text{for all } k \in X_2, e_1 f_1 \in E_1.$$

$$(d)\begin{cases}(\rho_{S_1}^l \circ \rho_{S_2}^l)((e_1, e_2)(f_1, f_2)) = \min\{\rho_{S_1}^l(e_1 f_1), \rho_{R_2}^l(e_2), \rho_{R_2}^l(f_2)\}\\ (\rho_{S_1}^u \circ \rho_{S_2}^u)((e_1, e_2)(f_1, f_2)) = \min\{\rho_{S_1}^u(e_1 f_1), \rho_{R_2}^u(e_2), \rho_{R_2}^u(f_2)\}\end{cases} \text{for all } e_2, f_2 \in X_2, e_1 f_1 \in E_1, e_2 \neq f_2.$$

3. Level Graphs of the Interval-Valued Fuzzy Graphs

In this section, we introduce a new concept of $[\psi, \theta]-$ level graph on IVFGs.

Definition 5. *For all $[\psi,\theta] \in I[0,1]$, $[\psi,\theta]$-level set of an IVFS R on X is defined to be $R_{[\psi,\theta]} = \{e \in X : \rho^l_R(e) \geq \psi, \rho^u_R(e) \geq \theta\}$.*

Theorem 1. *Suppose that X is a nonempty set, and $R = [\rho^l_R, \rho^u_R]$ and $S = [\rho^l_S, \rho^u_S]$ are IVFGs on X and E, respectively. Then, $G' = (R,S)$ is an IVFG if and only if $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$, named the $[\psi,\theta]$-level graph of G', is a graph for every $[\psi,\theta] \in I[0,1]$.*

Proof. Let $G' = (R,S)$ be an IVFG. For each $[\psi,\theta] \in I[0,1]$, if $ef \in S_{[\psi,\theta]}$, then, $\rho^l_S(ef) \geq \psi, \rho^u_S(ef) \geq \theta$. Since G' is an IVFG,

$$\psi \leq \rho^l_S(ef) \leq \min\{\rho^l_R(e), \rho^l_R(f)\},$$

$$\theta \leq \rho^u_S(ef) \leq \min\{\rho^u_R(e), \rho^u_R(f)\},$$

and so

$\psi \leq \rho^l_R(e), \psi \leq \rho^l_R(f), \theta \leq \rho^u_R(e), \theta \leq \rho^u_R(f)$, that is, $e, f \in R_{[\psi,\theta]}$.

Therefore, $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is a graph for every $[\psi,\theta] \in I[0,1]$.

Conversely, suppose $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is a graph for all $[\psi,\theta] \in I[0,1]$. For each $ef \in E$, we consider $\rho^l_S(ef) = \psi$ and $\rho^u_S(ef) = \theta$. Then, $ef \in S_{[\psi,\theta]}$. Since $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is a graph; thus, we have $e, f \in R_{[\psi,\theta]}$. Hence, $\psi \leq \rho^l_R(e)$, $\psi \leq \rho^l_R(f)$, $\theta \leq \rho^u_R(e)$, and $\theta \leq \rho^u_R(f)$. Therefore,

$$\rho^l_S(ef) = \psi \leq \min\{\rho^l_R(e), \rho^l_R(f)\},$$

$$\rho^u_S(ef) = \theta \leq \min\{\rho^u_R(e), \rho^u_R(f)\},$$

that is, $G' = (R,S)$ is an IVFG. □

Theorem 2. *Suppose $G'_1 = (R_1, S_1)$ and $G'_2 = (R_2, S_2)$ are the IVFGs of $G^*_1 = (X_1, E_1)$ and $G^*_2 = (X_2, E_2)$, respectively. Then, the pair $G' = (R,S)$ of an IVFG on G^*_1 and G^*_2 is the CP of G'_1 and G'_2 if and only if for every $[\psi,\theta] \in I[0,1]$ the $[\psi,\theta]$-level graph $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is the CP of $((R_1)_{[\psi,\theta]}, (S_1)_{[\psi,\theta]})$ and $((R_2)_{[\psi,\theta]}, (S_2)_{[\psi,\theta]})$.*

Proof. Suppose that $G' = (R,S)$ is the CP of the IVFGs G'_1 and G'_2. For each $[\psi,\theta] \in I[0,1]$, if $(e,f) \in R_{[\psi,\theta]}$, then

$$\psi \leq \rho^l_R(e,f) = \min\{\rho^l_{R_1}(e), \rho^l_{R_2}(f)\},$$

$$\theta \leq \rho^u_R(e,f) = \min\{\rho^u_{R_1}(e), \rho^u_{R_2}(f)\}.$$

Therefore, $e \in (R_1)_{[\psi,\theta]}$ and $f \in (R_2)_{[\psi,\theta]}$; that is, $(e,f) \in (R_1)_{[\psi,\theta]} \times (R_2)_{[\psi,\theta]}$.

Therefore, $R_{[\psi,\theta]} \subseteq (R_1)_{[\psi,\theta]} \times (R_2)_{[\psi,\theta]}$. If $(e,f) \in (R_1)_{[\psi,\theta]} \times (R_2)_{[\psi,\theta]}$, then $e \in (R_1)_{[\psi,\theta]}$ and $f \in (R_2)_{[\psi,\theta]}$. Here, we have

$$\psi \leq \min\{\rho^l_{R_1}(e), \rho^l_{R_2}(f)\},$$

$$\theta \leq \min\{\rho^u_{R_1}(e), \rho^u_{R_2}(f)\}.$$

Because (R,S) is the CP of G'_1 and G'_2, $\psi \leq \rho^l_R(e,f)$ and $\theta \leq \rho^u_R(e,f)$, so we have $(e,f) \in R_{[\psi,\theta]}$. Thus,

$$(R_1)_{[\psi,\theta]} \times (R_2)_{[\psi,\theta]} \subseteq R_{[\psi,\theta]}$$

and so

$$(R_1)_{[\psi,\theta]} \times (R_2)_{[\psi,\theta]} = R_{[\psi,\theta]}.$$

Now, we prove $S_{[\psi,\theta]} = E$, such that E is the edge set of the CP $(G'_1)_{[\psi,\theta]} \times (G'_2)_{[\psi,\theta]}$ for every $[\psi,\theta] \in I[0,1]$.

Suppose $((e_1, e_2)(f_1, f_2)) \in S_{[\psi,\theta]}$. Therefore,

$$\rho_S^l(e_1, e_2)(f_1, f_2) \geq \psi, \quad \rho_S^u(e_1, e_2)(f_1, f_2) \geq \theta.$$

Since (R, S) is the CP of G_1' and G_2', it satisfies one of the below conditions:

(i) $e_1 = f_1$ and $e_2 f_2 \in E_2$,
(ii) $e_2 = f_2$ and $e_1 f_1 \in E_1$.

In the first condition (i), we have:

$$\rho_S^l((e_1, e_2)(f_1, f_2)) = \min\{\rho_{R_1}^l(e_1), \rho_{S_2}^l(e_2 f_2)\} \geq \psi,$$

$$\rho_S^u((e_1, e_2)(f_1, f_2)) = \min\{\rho_{R_1}^u(e_1), \rho_{S_2}^u(e_2 f_2)\} \geq \theta,$$

and so $\rho_{R_1}^l(e_1) \geq \psi$, $\rho_{R_1}^u(e_1) \geq \theta$, $\rho_{S_2}^l(e_2 f_2) \geq \psi$, $\rho_{S_2}^u(e_2 f_2) \geq \theta$.

It follows that $e_1 = f_1 \in (R_1)_{[\psi,\theta]}$ and $e_2 f_2 \in (S_2)_{[\psi,\theta]}$; that is, $((e_1, e_2)(f_1, f_2)) \in E$.

Similarly, for condition (ii), we conclude $((e_1, e_2)(f_1, f_2)) \in E$. Thus, $S_{[\psi,\theta]} \subseteq E$. For each $((e, e_2)(e, f_2)) \in E$, $\rho_{R_1}^l(e) \geq \psi$, $\rho_{R_1}^u(e) \geq \theta$, $\rho_{S_2}^l(e_2 f_2) \geq \psi$, $\rho_{S_2}^u(e_2 f_2) \geq \theta$.

Because (R, S) is the CP of G_1' and G_2', we have:

$$\rho_S^l((e, e_2)(e, f_2)) = \min\{\rho_{R_1}^l(e), \rho_{S_2}^l(e_2 f_2)\} \geq \psi,$$

$$\rho_S^u((e, e_2)(e, f_2)) = \min\{\rho_{R_1}^u(e), \rho_{S_2}^u(e_2 f_2)\} \geq \psi.$$

Therefore, $((e, e_2)(e, f_2)) \in S_{[\psi,\theta]}$. Similarly, for each $(e_1, k)(f_1, k) \in E$, we have $((e_1, k)(f_1, k)) \in S_{[\psi,\theta]}$. Thus, $E \subseteq S_{[\psi,\theta]}$, and so $E = S_{[\psi,\theta]}$.

Conversely, suppose the $[\psi, \theta]$-level graph $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is the CP of $((R_1)_{[\psi,\theta]}, (S_1)_{[\psi,\theta]})$ and $((R_2)_{[\psi,\theta]}, (S_2)_{[\psi,\theta]})$, for all $[\psi, \theta] \in I[0, 1]$.

Suppose $\min\{\rho_{R_1}^l(e_1), \rho_{R_2}^l(e_2)\} = \psi$ and $\min\{\rho_{R_1}^u(e_1), \rho_{R_2}^u(e_2)\} = \theta$, for $(e_1, e_2) \in X_1 \times X_2$. Then, $e_1 \in R_{1[\psi,\theta]}$ and $e_2 \in R_{2[\psi,\theta]}$. By the hypothesis, $(e_1, e_2) \in R_{[\psi,\theta]}$. Hence,

$$\rho_R^l(e_1, e_2) \geq \psi = \min\{\rho_{R_1}^l(e_1), \rho_{R_2}^l(e_2)\},$$

$$\rho_R^u(e_1, e_2) \geq \theta = \min\{\rho_{R_1}^u(e_1), \rho_{R_2}^u(e_2)\}.$$

Consider $\rho_R^l(e_1, e_2) = \psi'$ and $\rho_R^u(e_1, e_2) = \theta'$. Then, we have $(e_1, e_2) \in R_{[\psi',\theta']}$. Since $(R_{[\psi',\theta']}, S_{[\psi',\theta']})$ is the CP of $(R_{1[\psi',\theta']}, S_{1[\psi',\theta']})$ and $(R_{2[\psi',\theta']}, S_{2[\psi',\theta']})$, then $e_1 \in R_{1[\psi',\theta']}$ and $e_2 \in R_{2[\psi',\theta']}$. Hence,

$$\rho_{R_1}^l(e_1) \geq \psi' = \rho_R^l(e_1, e_2), \quad \rho_{R_1}^u(e_1) \geq \theta' = \rho_R^u(e_1, e_2),$$

$$\rho_{R_2}^l(e_2) \geq \psi' = \rho_R^l(e_1, e_2), \quad \rho_{R_2}^u(e_2) \geq \theta' = \rho_R^u(e_1, e_2).$$

It follows that

$$\rho_R^l(e_1, e_2) \leq \min\{\rho_{R_1}^l(e_1), \rho_{R_2}^l(e_2)\},$$
$$\rho_R^u(e_1, e_2) \leq \min\{\rho_{R_1}^u(e_1), \rho_{R_2}^u(e_2)\}.$$

Therefore,

$$\rho_R^l(e_1, e_2) = \min\{\rho_{R_1}^l(e_1), \rho_{R_2}^l(e_2)\},$$
$$\rho_R^u(e_1, e_2) = \min\{\rho_{R_1}^u(e_1), \rho_{R_2}^u(e_2)\},$$

for all $(e_1, e_2) \in X_1 \times X_2$. That is, $\rho_R^l = \rho_{R_1}^l \times \rho_{R_2}^l$, $\rho_R^u = \rho_{R_1}^u \times \rho_{R_2}^u$.

For each $e \in X_1$ and $e_2 f_2 \in E_2$, suppose

$$\min\{\rho_{R_1}^l(e), \rho_{S_2}^l(e_2 f_2)\} = \psi, \quad \min\{\rho_{R_1}^u(e), \rho_{S_2}^u(e_2 f_2)\} = \theta,$$

$$\rho^l_S((e,e_2)(e,f_2)) = \psi', \quad \rho^u_S((e,e_2)(e,f_2)) = \theta'.$$

Then, $\rho^l_{R_1}(e) \geq \psi$, $\rho^u_{R_1}(e) \geq \theta$, $\rho^l_{S_2}(e_2f_2) \geq \psi$, $\rho^u_{S_2}(e_2f_2) \geq \theta$ and $((e,e_2)(e,f_2)) \in S_{[\psi',\theta']}$, i.e., $e \in R_1[\psi,\theta]$, $e_2f_2 \in S_2[\psi,\theta]$, and $(e,e_2)(e,f_2) \in S_{[\psi',\theta']}$.

Since $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is the CP of $((R_1)_{[\psi,\theta]}, (S_1)_{[\psi,\theta]})$ and $((R_2)_{[\psi,\theta]}, (S_2)_{[\psi,\theta]})$, and also $(R_{[\psi',\theta']}, S_{[\psi',\theta']})$ is the CP of $((R_1)_{[\psi',\theta']}, (S_1)_{[\psi',\theta']})$ and $((R_2)_{[\psi',\theta']}, (S_2)_{[\psi',\theta']})$, we have $(e,e_2)(e,f_2) \in S_{[\psi,\theta]}$, $e \in R_{1[\psi',\theta']}$, and $e_2f_2 \in S_{2[\psi',\theta']}$, which implies $((e,e_2)(e,f_2)) \in S_{[\psi,\theta]}$, $\rho^l_{R1}(e) \geq \psi', \rho^u_{R1}(e) \geq \theta', \rho^l_{S2}(e_2f_2) \geq \psi', \rho^u_{S2}(e_2f_2) \geq \theta'$.

Therefore,

$$\rho^l_S((e,e_2)(e,f_2)) \geq \psi = \min\{\rho^l_{R_1}(e), \rho^l_{S_2}(e_2f_2)\},$$
$$\rho^u_S((e,e_2)(e,f_2)) \geq \theta = \min\{\rho^u_{R_1}(e), \rho^u_{S_2}(e_2f_2)\},$$

and also

$$\min\{\rho^l_{R_1}(e), \rho^l_{S_2}(e_2f_2)\} \geq \psi' = \rho^l_S((e,e_2)(e,f_2)),$$
$$\min\{\rho^u_{R_1}(e), \rho^u_{S_2}(e_2f_2)\} \geq \theta' = \rho^u_S((e,e_2)(e,f_2)).$$

Thus,

$$\min\{\rho^l_{R_1}(e), \rho^l_{S_2}(e_2f_2)\} = \rho^l_S((e,e_2)(e,f_2)),$$
$$\min\{\rho^u_{R_1}(e), \rho^u_{S_2}(e_2f_2)\} = \rho^u_S((e,e_2)(e,f_2)),$$

for all $e \in X_1$ and $e_2f_2 \in E_2$. Also, similarly, we obtain

$$\min\{\rho^l_{S_1}(e_1f_1), \rho^l_{R_2}(k)\} = \rho^l_S((e_1,k)(f_1,k)),$$

$$\min\{\rho^u_{S_1}(e_1f_1), \rho^u_{R_2}(k)\} = \rho^l_S((e_1,k)(f_1,k)),$$

for all $k \in X_2$ and $e_1f_1 \in E_1$. □

Example 1. *Suppose $G_1' = (R_1, S_1)$ and $G_2' = (R_2, S_2)$ are two IVFGs of the graphs $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively, so that $X_1 = \{a,b,c,d\}$, $X_2 = \{g,h\}$, $E_1 = \{ab, bd, , cd, ac, bc\}$, and $E_2 = \{gh\}$, as is shown in Figure 1.*

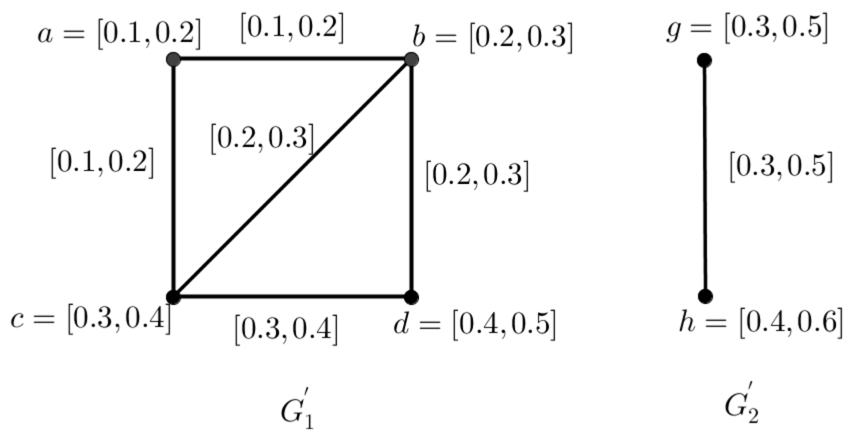

Figure 1. IVFGs G_1' and G_2'.

$$(\rho^l_{S_1} \times \rho^l_{S_2})((a,g)(b,g)) = 0.1, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((a,g)(b,g)) = 0.2$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((a,g)(a,h)) = 0.1, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((a,g)(a,h)) = 0.2$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((a,g)(c,g)) = 0.1, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((a,g)(c,g)) = 0.2$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,g)(b,h)) = 0.2, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((b,g)(b,h)) = 0.3$$

$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,g)(d,g)) = 0.2, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((b,g)(d,g)) = 0.3$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,g)(c,g)) = 0.2, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((b,g)(c,g)) = 0.3$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,h)(d,h)) = 0.2, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((b,h)(d,h)) = 0.3$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,h)(a,h)) = 0.1, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((b,h)(a,h)) = 0.2$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,h)(c,h)) = 0.2, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((b,h)(c,h)) = 0.3$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((d,h)(d,g)) = 0.3, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((d,h)(d,g)) = 0.5$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((d,h)(c,h)) = 0.3, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((d,h)(c,h)) = 0.4$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((c,h)(c,g)) = 0.3, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((c,h)(c,g)) = 0.4$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((c,h)(a,h)) = 0.1, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((c,h)(a,h)) = 0.2$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((c,g)(d,g)) = 0.3, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((c,g)(d,g)) = 0.4$$

We obtained the CP of the IVFGs $G_1^{'}$ and $G_2^{'}$, which are shown in Figure 2.

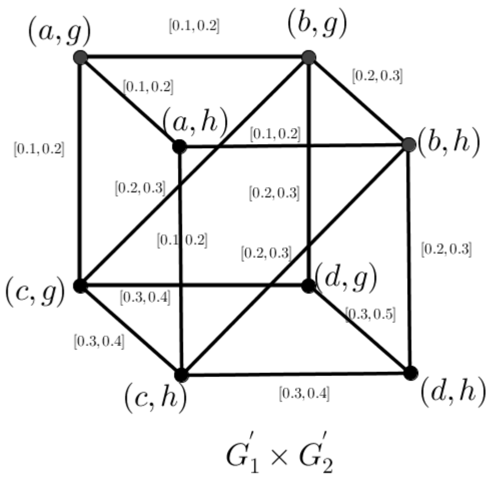

Figure 2. The CP of the IVFGs $G_1^{'}$ and $G_2^{'}$.

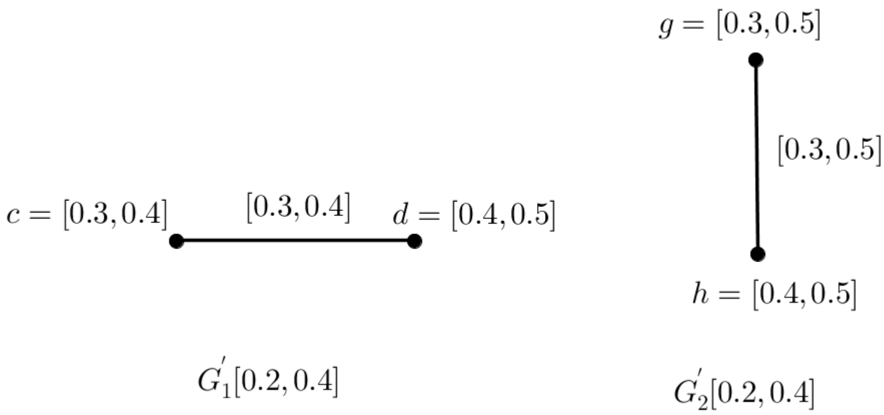

Figure 3. The $[0.2, 0.4]$-level graphs of the IVFGs $G_1^{'}$ and $G_2^{'}$.

Assume $[\psi, \theta] = [0.2, 0.4]$. *Then,* $R_{1[0.2,0.4]} = \{c, d\}$, $R_{2[0.2,0.4]} = \{g, h\}$, $E_{1[0.2,0.4]} = \{cd\}$, *and* $E_{2[0.2,0.4]} = \{gh\}$, *as is shown in Figure 3.*

We obtain the CP of the IVFGs $G'_{1[0.2,0.4]}$ and $G'_{2[0.2,0.4]}$. The CP of the IVFGs $G'_{1[0.2,0.4]}$ and $G'_{2[0.2,0.4]}$ is shown in Figure 4.

$$(\rho^l_{S_1} \times \rho^l_{S_2})((c,g)(c,h)) = 0.3, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((c,g)(c,h)) = 0.4$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((d,g)(d,h)) = 0.3, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((d,g)(d,h)) = 0.5$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((c,h)(d,h)) = 0.3, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((c,h)(d,h)) = 0.4$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((c,g)(d,g)) = 0.3, \quad (\rho^u_{S_1} \times \rho^u_{S_2})((c,g)(d,g)) = 0.4$$

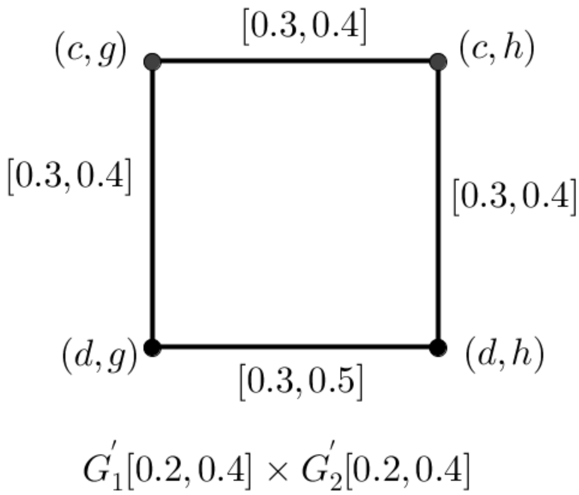

Figure 4. The CP of graphs $G'_{1[0.2,0.4]}$ and $G'_{2[0.2,0.4]}$.

Here, we investigated the properties of the CP IVFGs G'_1 and G'_2 from their $[0.2,0.4]$-level graphs of G'_1 and G'_2.

We concluded that the pair $G' = (R,S)$ is the CP of G'_1 and G'_2 if and only if the $[0.2,0.4]$-level graph $(R_{[0.2,0.4]}, S_{[0.2,0.4]})$ is the CP of the two graphs $G'_{1[0.2,0.4]}$ and $G'_{2[0.2,0.4]}$.

Also, we consider $[\psi,\theta] = [0.2,0.3]$. Then, $R_{1[0.2,0.3]} = \{c,d\}$, $R_{2[0.2,0.3]} = \{g,h\}$, $E_{1[0.2,0.3]} = \{bc,bd,cd\}$, and $E_{2[0.2,0.3]} = \{gh\}$, is shown in Figure 5.

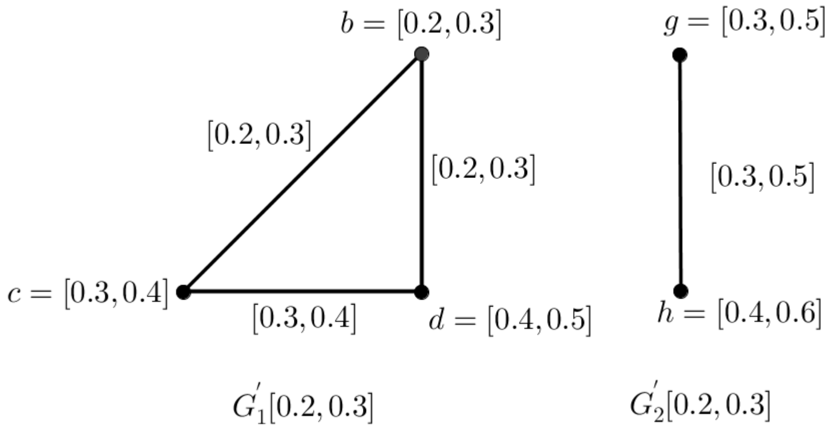

Figure 5. The $[0.2,0.3]$-level graphs of the IVFGs G'_1 and G'_2.

We obtain the CP of the IVFGs $G'_{1[0.2,0.3]}$ and $G'_{2[0.2,0.3]}$. The CP of the IVFGs $G'_{1[0.2,0.3]}$ and $G'_{2[0.2,0.3]}$ is shown in Figure 6.

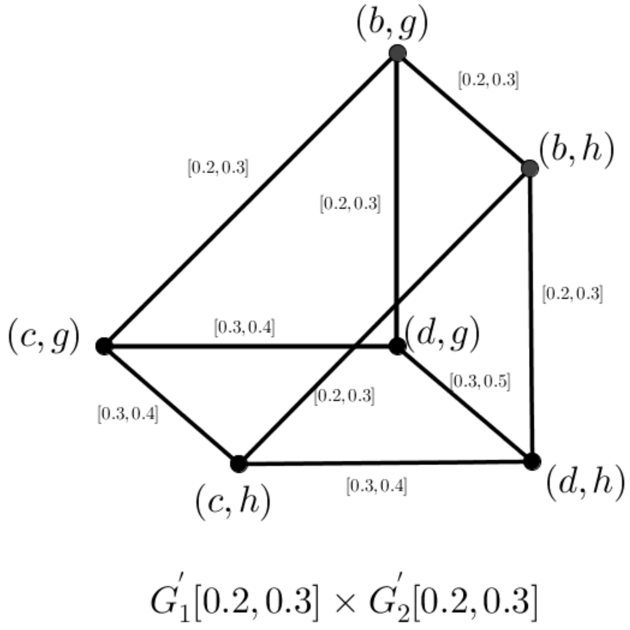

Figure 6. The CP of graphs $G^{'}_{1[0.2,0.3]}$ and $G^{'}_{2[0.2,0.3]}$

$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,g)(b,h)) = 0.2 \quad , (\rho^u_{S_1} \times \rho^u_{S_2})((b,g)(b,h)) = 0.3$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((d,g)(d,h)) = 0.3 \quad , (\rho^u_{S_1} \times \rho^u_{S_2})((d,g)(d,h)) = 0.5$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((c,g)(c,h)) = 0.3 \quad , (\rho^u_{S_1} \times \rho^u_{S_2})((c,g)(c,h)) = 0.4$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,g)(d,g)) = 0.2 \quad , (\rho^u_{S_1} \times \rho^u_{S_2})((b,g)(d,g)) = 0.3$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,g)(c,g)) = 0.2 \quad , (\rho^u_{S_1} \times \rho^u_{S_2})((b,g)(c,g)) = 0.3$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((c,g)(d,g)) = 0.3 \quad , (\rho^u_{S_1} \times \rho^u_{S_2})((c,g)(d,g)) = 0.4$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,h)(d,h)) = 0.2 \quad , (\rho^u_{S_1} \times \rho^u_{S_2})((b,h)(d,h)) = 0.3$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((b,h)(c,h)) = 0.2 \quad , (\rho^u_{S_1} \times \rho^u_{S_2})((b,h)(c,h)) = 0.3$$
$$(\rho^l_{S_1} \times \rho^l_{S_2})((d,h)(c,h)) = 0.3 \quad , (\rho^u_{S_1} \times \rho^u_{S_2})((d,h)(c,h)) = 0.4$$

In this example, we investigated the properties of the CP IVFGs $G_1^{'}$ and $G_2^{'}$ from their $[0.2, 0.3]$*-level graphs of $G_1^{'}$ and $G_2^{'}$.*

We concluded that the pair $G^{'} = (R, S)$ is the CP of $G_1^{'}$ and $G_2^{'}$ if and only if the $[0.2, 0.4]$*-level graph $(R_{[0.2,0.3]}, S_{[0.2,0.3]})$ is the CP of two graphs $G^{'}_{1[0.2,0.3]}$ and $G^{'}_{2[0.2,0.3]}$.*

Corollary 1. *Let $G_1^{'} = (R_1, S_1)$ and $G_2^{'} = (R_2, S_2)$ be IVFGs. Then, the CP $G_1^{'} \times G_2^{'}$ is an IVFG.*

Theorem 3. *Suppose $G_1^{'} = (R_1, S_1)$ and $G_2^{'} = (R_2, S_2)$ are two IVFGs of G_1^* and G_2^*, respectively. Then, the pair $G^{'} = (R, S)$ of the IVFGs on $G_1^*[G_2^*]$ is the CO of $G_1^{'}$ and $G_2^{'}$ if and only if for every $[\psi, \theta] \in I[0, 1]$, the $[\psi, \theta]$-level graph $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is the CO of $((R_1)_{[\psi,\theta]}, (S_1)_{[\psi,\theta]})$ and $((R_2)_{[\psi,\theta]}, (S_2)_{[\psi,\theta]})$.*

Proof. Suppose that $G' = (R, S)$ is the CO of the IVFGs G_1' and G_2'. By Theorem 2, we have $R_{[\psi,\theta]} = (R_1)_{[\psi,\theta]} \times (R_2)_{[\psi,\theta]}$. We prove $S_{[\psi,\theta]} = E$, where E is the edge set of the CO $(G_1')_{[\psi,\theta]}[(G_2')_{[\psi,\theta]}]$, for all $[\psi,\theta] \in I[0,1]$.

Suppose $(e_1,e_2)(f_1,f_2) \in S_{[\psi,\theta]}$. Then, $\rho_S^l((e_1,e_2)(f_1,f_2)) \geq \psi$ and $\rho_S^u((e_1,e_2)(f_1,f_2)) \geq \theta$.

Since $G' = (R, S)$ is the CO $G_1'[G_2']$, one of the below conditions holds:

(i) $e_1 = f_1$ and $e_2 f_2 \in E_2$,
(ii) $e_2 = f_2$ and $e_1 f_1 \in E_1$,
(iii) $e_2 \neq f_2$ and $e_1 f_1 \in E_1$.

For conditions (i) and (ii) in the proof of Theorem 2, we obtain $((e_1,e_2)(f_1,f_2)) \in E$. For case (iii), we have

$$\rho_S^l((e_1,e_2)(f_1,f_2)) = \min\{\rho_{R_2}^l(e_2), \rho_{R_2}^l(f_2), \rho_{S_1}^l(e_1 f_1)\} \geq \psi,$$

$$\rho_S^u((e_1,e_2)(f_1,f_2)) = \min\{\rho_{R_2}^u(e_2), \rho_{R_2}^u(f_2), \rho_{S_1}^u(e_1 f_1)\} \geq \theta.$$

Therefore,

$$\rho_{R_2}^l(e_2) \geq \psi, \quad \rho_{R_2}^l(f_2) \geq \psi, \quad \rho_{S_1}^l(e_1 f_1) \geq \psi,$$
$$\rho_{R_2}^u(e_2) \geq \theta, \quad \rho_{R_2}^u(f_2) \geq \theta, \quad \rho_{S_1}^u(e_1 f_1) \geq \theta.$$

It follows that $e_2, f_2 \in (R_2)_{[\psi,\theta]}$ and $e_1 f_1 \in (S_1)_{[\psi,\theta]}$; that is, $((e_1,e_2)(f_1,f_2)) \in E$. Thus, $S_{[\psi,\theta]} \subseteq E$. For each $((e,e_2)(e,f_2)) \in E$,

$$\rho_{R_1}^l(e) \geq \psi, \quad \rho_{R_1}^u(e) \geq \theta, \quad \rho_{S_2}^l(e_2 f_2) \geq \psi, \quad \rho_{S_2}^u(e_2 f_2) \geq \theta.$$

Since $G' = (R, S)$ is the CO $G_1'[G_2']$, we have

$$\rho_S^l((e,e_2)(e,f_2)) = \min\{\rho_{R_1}^l(e), \rho_{S_2}^l(e_2 f_2)\} \geq \psi$$

$$\rho_S^u((e,e_2)(e,f_2)) = \min\{\rho_{R_1}^u(e), \rho_{S_2}^u(e_2 f_2)\} \geq \theta.$$

Thus, $(e,e_1)(e,f_2) \in S_{[\psi,\theta]}$. Similarly, for each $((e_1,k)(f_1,k)) \in E$, we have $(e,e_2)(e,f_2) \in S_{[\psi,\theta]}$. For each $((e_1,e_2)(f_1,f_2)) \in E$, $e_2 \neq f_2$.

$$\rho_{S_1}^l(e_1 f_1) \geq \psi, \quad \rho_{S_1}^u(e_1 f_1) \geq \theta,$$

$$\rho_{R_2}^l(f_2) \geq \psi, \quad \rho_{R_2}^u(f_2) \geq \theta,$$
$$\rho_{R_2}^l(e_2) \geq \psi, \quad \rho_{R_2}^u(e_2) \geq \theta.$$

Since $G' = (R, S)$ is the CO $G_1'[G_2']$, we have:

$$\rho_S^l((e_1,e_2)(f_1,f_2)) = \min\{\rho_{R_2}^l(e_2), \rho_{R_2}^l(f_2), \rho_{S_1}^l(e_1 f_1)\} \geq \psi,$$

$$\rho_S^u((e_1,e_2)(f_1,f_2)) = \min\{\rho_{R_2}^u(e_2), \rho_{R_2}^u(f_2), \rho_{S_1}^u(e_1 f_1)\} \geq \theta.$$

Hence, $(e_1,e_2)(f_1,f_2) \in S_{[\psi,\theta]}$. Therefore, $E \in S_{[\psi,\theta]}$ and so $E = S_{[\psi,\theta]}$.

Conversely, let $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$, $[\psi,\theta] \in I[0,1]$ be the CO of $((R_1)_{[\psi,\theta]}, (S_1)_{[\psi,\theta]})$, and $((R_2)_{[\psi,\theta]}, (S_2)_{[\psi,\theta]})$. By Theorem 2, we have

(i) $\begin{cases} \rho_R^l(e_1,e_2) = \min\{\rho_{R_1}^l(e_1), \rho_{R_2}^l(e_2)\} \\ \rho_R^u(e_1,e_2) = \min\{\rho_{R_1}^u(e_1), \rho_{R_2}^u(e_2)\} \end{cases}$ for all $(e_1,e_2) \in X_1 \times X_2$.

(ii) $\begin{cases} \rho_S^l((e,e_2)(e,f_2)) = \min\{\rho_{R_1}^l(e), \rho_{S_2}^l(e_2 f_2)\} \\ \rho_S^u((e,e_2)(e,f_2)) = \min\{\rho_{R_1}^u(e), \rho_{S_2}^u(e_2 f_2)\} \end{cases}$ for all $e \in X_1$ and $e_2 f_2 \in E_2$.

(iii) $\begin{cases} \rho_S^l((e_1,k)(f_1,k)) = \min\{\rho_{S_1}^l(e_1 f_1), \rho_{R_2}^l(k)\} \\ \rho_S^u((e_1,k)(f_1,k)) = \min\{\rho_{S_1}^u(e_1 f_1), \rho_{R_2}^u(k)\} \end{cases}$ for all $k \in X_2$ and $e_1 f_1 \in E_1$.

Also, we obtain:

$$\rho^l_S((e_1,e_2)(f_1,f_2)) = \min\{\rho^l_{R_2}(e_2), \rho^l_{R_2}(f_2), \rho^l_{S_1}(e_1f_1)\},$$

$$\rho^u_S((e_1,e_2)(f_1,f_2)) = \min\{\rho^u_{R_2}(e_2), \rho^u_{R_2}(f_2), \rho^u_{S_1}(e_1f_1)\},$$

for all $e_2, f_2 \in X_2$, $e_2 \neq f_2$ and $e_1f_1 \in E_1$. $\square$

Corollary 2. *Let $G_1^{'} = (R_1, S_1)$ and $G_2^{'} = (R_2, S_2)$ be IVFGs. Then, the CO $G_1^{'}[G_2^{'}]$ is an IVFG.*

Theorem 4. *Assume $G_1^{'} = (R_1, S_1)$ and $G_2^{'} = (R_2, S_2)$ are IVFGs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively, so that $X_1 \cap X_2 = \emptyset$. Then, $G^{'} = (R, S)$ is the union of $G_1^{'}$ and $G_2^{'}$ if and only if for every $[\psi, \theta] \in I[0, 1]$, the level graph $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is the union of $((R_1)_{[\psi,\theta]}, (S_1)_{[\psi,\theta]})$ and $((R_2)_{[\psi,\theta]}, (S_2)_{[\psi,\theta]})$.*

Proof. Suppose that $G^{'} = (R, S)$ is the union of the IVFGs $G_1^{'}$ and $G_2^{'}$. We show that $R_{[\psi,\theta]} = (R_1)_{[\psi,\theta]} \cup (R_2)_{[\psi,\theta]}$ for every $[\psi, \theta] \in I[0, 1]$.

Assume that $e \in R_{[\psi,\theta]}$, then $e \in X_1 \backslash X_2$ or $e \in X_2 \backslash X_1$. If $e \in X_1 \backslash X_2$, then $\rho^l_{R_1}(e) = \rho^l_R(e) \geq \psi$ and $\rho^u_{R_1}(e) = \rho^u_R(e) \geq \theta$, which implies $e \in (R_1)_{[\psi,\theta]}$. If $e \in X_2 \backslash X_1$, we have $e \in (R_2)_{[\psi,\theta]}$. Therefore, $e \in (R_1)_{[\psi,\theta]} \cup (R_2)_{[\psi,\theta]}$, and so $R_{[\psi,\theta]} \subseteq (R_1)_{[\psi,\theta]} \cup (R_2)_{[\psi,\theta]}$.

Let $e \in (R_1)_{[\psi,\theta]} \cup (R_2)_{[\psi,\theta]}$, then $e \in (R_1)_{[\psi,\theta]}$, $e \notin (R_2)_{[\psi,\theta]}$ or $e \in (R_2)_{[\psi,\theta]}$ and $e \notin (R_1)_{[\psi,\theta]}$. In the first situation, $\rho^l_R(e) = \rho^l_{R1}(e) \geq \psi$ and $\rho^u_R(e) = \rho^u_{R1}(e) \geq \theta$, which implies $e \in R_{[\psi,\theta]}$. In the second condition, $\rho^l_R(e) = \rho^l_{R2}(e) \geq \psi$ and $\rho^u_R(e) = \rho^u_{R2}(e) \geq \theta$, hence $e \in R_{[\psi,\theta]}$.

Consequently, $(R_1)_{[\psi,\theta]} \cup (R_2)_{[\psi,\theta]} \subseteq R_{[\psi,\theta]}$. We prove that $(S_1)_{[\psi,\theta]} \cup (S_2)_{[\psi,\theta]} = S_{[\psi,\theta]}$ for all $[\psi, \theta] \in I[0, 1]$. Assume that $ef \in S_{[\psi,\theta]}$, then $ef \in E_1 \backslash E_2$ or $ef \in E_2 \backslash E_1$.

For $ef \in E_1 \backslash E_2$, we have $\rho^l_{S_1}(ef) = \rho^l_S(ef) \geq \psi$ and $\rho^u_{S_1}(ef) = \rho^u_S(ef) \geq \theta$. Therefore, $ef \in S_{1[\psi,\theta]}$. Also, if $ef \in E_2 \backslash E_1$, then $ef \in S_{2[\psi,\theta]}$. Thus, $S_{[\psi,\theta]} \subseteq (S_1)_{[\psi,\theta]} \cup (S_2)_{[\psi,\theta]}$.

If $ef \in (S_1)_{[\psi,\theta]} \cup (S_2)_{[\psi,\theta]}$, then $ef \in (S_1)_{[\psi,\theta]} \backslash (S_2)_{[\psi,\theta]}$ or $ef \in (S_2)_{[\psi,\theta]} \backslash (S_1)_{[\psi,\theta]}$.

In the first situation, $\rho^l_S(ef) = \rho^l_{S1}(ef) \geq \psi$ and $\rho^u_S(ef) = \rho^u_{S1}(ef) \geq \theta$, which implies $e \in S_{[\psi,\theta]}$.

In the second condition, there is also $ef \in S_{[\psi,\theta]}$. Thus, $(S_1)_{[\psi,\theta]} \cup (S_2)_{[\psi,\theta]} \subseteq S_{[\psi,\theta]}$.

Conversely, suppose for all $[\psi, \theta] \in I[0, 1]$, $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is the union of $((R_{1[\psi,\theta]}),$ $(S_{1[\psi,\theta]}))$ and $((R_{2[\psi,\theta]}), (S_{2[\psi,\theta]}))$. Let $e \in X_1$ and $\rho^l_{R_1}(e) = \psi$, $\rho^u_{R_1}(e) = \theta$, $\rho^l_R(e) = \psi^{'}$, $\rho^u_R(e) = \theta^{'}$, for $\psi, \theta, \psi^{'}, \theta^{'} \in [0, 1]$.

Then, $e \in (R_1)_{[\psi,\theta]}$ and $e \in R_{[\psi^{'},\theta^{'}]}$. By the hypothesis, $e \in R_{[\psi,\theta]}$ and $e \in (R_1)_{[\psi^{'},\theta^{'}]}$.

Therefore,

$$\rho^l_R(e) \geq \psi, \quad \rho^u_R(e) \geq \theta, \quad \rho^l_{R_1}(e) \geq \psi^{'}, \quad \rho^u_{R_1}(e) \geq \theta^{'}.$$

It follows, $\rho^l_{R_1}(e) \leq \rho^l_R(e)$, $\rho^u_{R_1}(e) \leq \rho^u_R(e)$, $\rho^l_{R_1}(e) \geq \rho^l_R(e)$ $\rho^u_{R_1}(e) \geq \rho^u_R(e)$. Hence, $\rho^l_{R_1}(e) = \rho^l_R(e)$, $\rho^u_{R_1}(e) = \rho^u_R(e)$.

Similarly, for every $e \in X_2$, we obtain $\rho^l_{R_2}(e) = \rho^l_R(e)$, $\rho^u_{R_2}(e) = \rho^u_R(e)$.

Also, we obtain:

(i) $\begin{cases} \rho^l_S(ef) = \rho^l_{S_1}(ef) & if\ ef \in E_1 \\ \rho^l_S(ef) = \rho^l_{S_2}(ef) & if\ ef \in E_2. \end{cases}$

(ii) $\begin{cases} \rho^u_S(ef) = \rho^u_{S_1}(ef) & if\ ef \in E_1 \\ \rho^u_S(ef) = \rho^u_{S_2}(ef) & if\ ef \in E_2 \end{cases}$ $\square$

Corollary 3. *Let $G_1^{'} = (R_1, S_1)$ and $G_2^{'} = (R_2, S_2)$ be IVFGs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively, and $X_1 \cap X_2 = \emptyset$, then $G_1^{'} \cup G_2^{'}$ is an IVFG.*

Theorem 5. *Suppose $G_1' = (R_1, S_1)$ and $G_2' = (R_2, S_2)$ are IVFGs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively, and $X_1 \cap X_2 = \emptyset$. Then, the pair $G' = (R, S)$ of the IVFGd on $G_1^* + G_2^*$ is the join of G_1' and G_2' if and only if for every $[\psi, \theta] \in I[0, 1]$, $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is the join of $((R_1)_{[\psi,\theta]}, (S_1)_{[\psi,\theta]})$ and $((R_2)_{[\psi,\theta]}, (S_2)_{[\psi,\theta]})$.*

Proof. Suppose that $G' = (R, S)$ is the join of the IVFGs G_1' and G_2'. We show that $R_{[\psi,\theta]} = (R_1)_{[\psi,\theta]} \cup (R_2)_{[\psi,\theta]}$ and $S_{[\psi,\theta]} = (S_1)_{[\psi,\theta]} \cup (S_2)_{[\psi,\theta]} \cup E^*_{[\psi,\theta]}$ for all $[\psi, \theta] \in I[0, 1]$, where $E^*_{[\psi,\theta]}$ is the set of all edges joining the nodes $(R_1)_{[\psi,\theta]}$ and $(R_2)_{[\psi,\theta]}$.

According to the proof of Theorem 4, $(S_1)_{[\psi,\theta]} \cup (S_2)_{[\psi,\theta]} \subseteq S_{[\psi,\theta]}$. If $ef \in E^*_{[\psi,\theta]}$, then $\rho^l_{R1}(e) \geq \psi, \quad \rho^u_{R1}(e) \geq \theta, \quad \rho^l_{R2}(f) \geq \psi, \quad \rho^u_{R2}(f) \geq \theta$. Hence,

$$\rho^l_S(ef) = \min\{\rho^l_{R_1}(e), \rho^l_{R_2}(f)\} \geq \psi,$$

and

$$\rho^u_S(ef) = \min\{\rho^u_{R_1}(e), \rho^u_{R_2}(f)\} \geq \theta.$$

Therefore, $ef \in S_{[\psi,\theta]}$. It follows, $(S_1)_{[\psi,\theta]} \cup (S_2)_{[\psi,\theta]} \cup E^*_{[\psi,\theta]} \subseteq S_{[\psi,\theta]}$.

For each $ef \in S_{[\psi,\theta]}$, if $ef \in E_1 \cup E_2$, then $ef \in (S_1)_{[\psi,\theta]} \cup (S_2)_{[\psi,\theta]}$. If $e \in X_1$ and $f \in X_2$, then

$$\min\{\rho^l_{R_1}(e), \rho^l_{R_2}(f)\} = \rho^l_S(ef) \geq \psi,$$

and

$$\min\{\rho^u_{R_1}(e), \rho^u_{R_2}(f)\} = \rho^u_S(ef) \geq \theta.$$

Therefore, $e \in (R_1)_{[\psi,\theta]}$, $f \in (R_2)_{[\psi,\theta]}$, and $ef \in E^*_{[\psi,\theta]}$.

Thus, $S_{[\psi,\theta]} \subseteq (S_1)_{[\psi,\theta]} \cup (S_2)_{[\psi,\theta]} \cup E^*_{[\psi,\theta]}$.

Conversely, suppose $(R_{[\psi,\theta]}, S_{[\psi,\theta]})$ is the join of $((R_1)_{[\psi,\theta]}, (S_1)_{[\psi,\theta]})$ and $((R_2)_{[\psi,\theta]}, (S_2)_{[\psi,\theta]})$ for $[\psi, \theta] \in I[0, 1]$.

We have

(i) $\begin{cases} \rho^l_R(e) = \rho^l_{R_1}(e) & if\ e \in X_1 \\ \rho^l_R(e) = \rho^l_{R_2}(e) & if\ e \in X_2 \end{cases}$

(ii) $\begin{cases} \rho^u_R(e) = \rho^u_{R_1}(e) & if\ e \in X_1 \\ \rho^u_R(e) = \rho^u_{R_2}(e) & if\ e \in X_2. \end{cases}$

Also, we obtain

(iii) $\begin{cases} \rho^l_S(ef) = \rho^l_{S_1}(ef) & if\ ef \in E_1 \\ \rho^l_S(ef) = \rho^l_{S_2}(ef) & if\ ef \in E_2 \end{cases}$

(iv) $\begin{cases} \rho^u_S(ef) = \rho^u_{S_1}(ef) & if\ ef \in E_1 \\ \rho^u_S(ef) = \rho^u_{S_2}(ef) & if\ ef \in E_2. \end{cases}$

Suppose $e \in X_1$, $f \in X_2$ and consider $min\{\rho^l_{R_1}(e), \rho^l_{R_2}(f)\} = \psi'$, $min\{\rho^u_{R_1}(e), \rho^u_{R_2}(f)\} = \tau'$, $\rho^l_S(ef) = \psi'$, $\rho^u_S(ef) = \theta'$.

Then, $e \in (R_1)_{[\psi,\theta]}$, $f \in (R_2)_{[\psi,\theta]}$ and $ef \in S_{[\psi',\theta']}$.

It follows that $ef \in S_{[\psi,\theta]}$, $e \in (R_1)_{[\psi',\theta']}$ and $f \in (R_2)_{[\psi',\theta']}$. So, $\rho^l_S(ef) \geq \psi, \quad \rho^u_S(ef) \geq \theta, \quad \rho^l_{R_1}(e) \geq \psi', \quad \rho^u_{R_1}(e) \geq \theta', \rho^l_{R_2}(f) \geq \psi', \quad \rho^u_{R_2}(f) \geq \theta'$. Therefore,

$$\rho^l_S(ef) \geq \psi = \min\{\rho^l_{R_1}(e), \rho^l_{R_2}(f)\} \geq \psi' = \rho^l_S(ef),$$

$$\rho^u_S(ef) \geq \theta = \min\{\rho^u_{R_1}(e), \rho^u_{R_2}(f)\} \geq \theta' = \rho^u_S(ef),$$

and then we have

$$\rho^l_S(ef) = \min\{\rho^l_{R_1}(e), \rho^l_{R_2}(f)\}, \quad \rho^u_S(ef) = \min\{\rho^u_{R_1}(e), \rho^u_{R_2}(f)\}.$$

□

4. Application of Interval-Valued Fuzzy Graph to Find the Most Effective Person in a Hospital Information System

The growth of the communication, telecommunication, and informatics industries is facing a new revolution every day. The information and communication technology revolution has had a profound effect on all economic, social, political, and security sectors of the country. One of the most important areas of information technology applications is in the field of health and treatment. A hospital information system is the first and most basic system for providing health care. Hospital information systems are computer systems designed to easily manage medical and hospital information and to improve the quality of health care. Research has shown that the use of hospital information systems has improved the quality of health services and increased customer satisfaction. Some of the problems of health care systems are as follows: the dispersion of patients' information and lack of access to their records, poor cooperation between physicians and health care workers, and poor access to required medical information. These problems can be solved through the development of information technology, especially hospital (health) information systems. Hospital information systems are designed to automate the affairs of hospitals, such as reporting test results, entering doctor's instructions, prescribing medication, controlling pharmacy inventory, central warehouses, feeding units, etc. In the hospital information system, an electronic file is created for each patient that covers all hospital activities (including treatment, diagnosis, finance, etc.) from admission to discharge. In this system, all medical procedures, medication orders, and diagnostic services are sent through the system to clinics and paraclinics and even administrative centers, such as accounting, pharmacies, warehouses, and other units, and their answers are received. Therefore, the start and end times of all actions in the system are clear and traceable. Hence, a hospital information system is an information system in which information is stored in a comprehensive database and is available to consumers in special forms at the time and place of need. Therefore, considering the importance of the hospital information system and its role in improving medical and health services, we intend to specify the most effective employee in the field of technology and information of a hospital in terms of registering information about patients, medicines, finances, laboratories, etc. Suppose the vertices of the IVFG are an information register building and the edges of this graph are the degree of interaction in between. The set of staff is

A = {Moradi, Kamali, Ahmadi, Yegane, Bahmani, Nazari}

(a) Moradi and Bahmani have been co-workers for 16 years.
(b) Yegane is very accurate in calculating and recording information about the hospital's finances, and all employees are satisfied with him.
(c) In recording information about nurses, it is very important to record their expertise, work experience, and the duration of each activity during a day with great care to have the most active nurses serve patients. Kamali is the best choice to undertake this accountability process.
(d) Nazari and Ahmadi have a long history of conflict.
(e) According to Ahmadi's experiences recognizing drugs and their effectiveness, he is the best option for recording information about drugs.

According to the above values, we consider an IVFG. The vertices show each of the department staff members. The edges indicate the level of friendship and fondness between staff. For the weight of the vertices, the lower bound and the upper bound mean the level of staff capability. The lower bound and the upper bound for the weight of the edges mean the amount of friendship and conflict, respectively. The name of the staff and level of staff capability are indicated in Tables 1 and 2. The adjacency matrix corresponding to Figure 7 is indicated in Table 3.

Table 1. The name of employees in hospital and their services.

Name	**Services**
Moradi	Responsible for recording laboratory information
Kamali	Responsible for registering nurses' information
Ahmadi	Responsible for registering drug information
Yegane	Responsible for registering financial information
Bahmani	Responsible for registering radiological information
Nazari	Responsible for registering clinical information

Table 2. The level of staff capability.

Name	$[\rho_S^l, \rho_S^u]$
Moradi	[0.3, 0.4]
Kamali	[0.4, 0.5]
Ahmadi	[0.2, 0.3]
Yegane	[0.3, 0.4]
Bahmani	[0.2, 0.4]
Nazari	[0.1, 0.2]

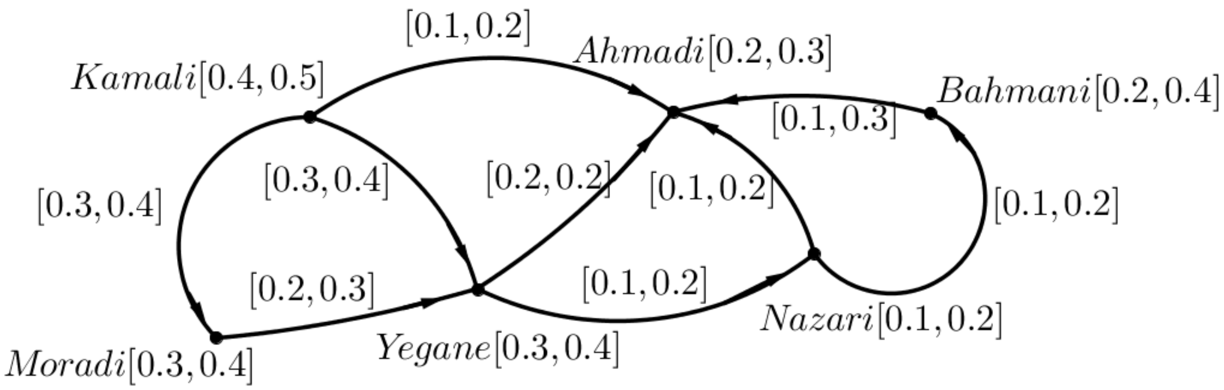

Figure 7. Interval-valued fuzzy digraph.

Table 3. Adjacency matrix corresponding to Figure 7.

Name	**Moradi**	**Kamali**	**Ahmadi**	**Yegane**	**Bahmani**	**Nazari**
Moradi	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
Kamali	$[0.3,0.4]$	$[0,0]$	$[0.1,0.2]$	$[0.3,0.4]$	$[0,0]$	$[0,0]$
Ahmadi	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
Yegane	$[0,0]$	$[0,0]$	$[0.2,0.2]$	$[0,0]$	$[0,0]$	$[0.1,0.2]$
Bahmani	$[0,0]$	$[0,0]$	$[0.1,0.3]$	$[0,0]$	$[0,0]$	$[0,0]$
Nazari	$[0,0]$	$[0,0]$	$[0.1,0.2]$	$[0,0]$	$[0.1,0.2]$	$[0,0]$

Figure 7 shows that Ahmadi has 20% of the necessary ability to register drug information in the system, but unfortunately, he does not have the necessary 30% authority to do so. The directional edge Yegane–Nazari indicates that there is only 10% comradeship among these two staff members, and unfortunately they have 20% conflict. Clearly, Kamali has dominion over both Moradi and Yegane, with his dominance over both at 30%. It is clear that Kamali is the most influential staff member of the hospital information system because he inspects both the laboratory information officer and the pecuniary information officer and has 30% of the authority in the hospital.

As we saw in the above example, fuzzy influence graphs play a significant role in determining worthy and effective people in an organization and social institutions, and they are used in the fields of medical and psychological sciences to diagnose diseases. Interval-valued fuzzy influence graphs are useful tools for chemical engineers to model various relationships in a process. These graphs are used to systematically map an entire chain of processes and controllers to describe the effect. The interval-valued fuzzy influence graph theory is growing as a dominant field of research in mathematics because of its application to a variety of problems, including clustering, data mining, decision-making, communication, etc. It has major contribution potential in modeling, preserving, and performing different types of physical problems in networking and trafficking. Several interval-valued fuzzy influence graph parameters are real indicators of network performance and efficiency. They are very useful to handle networks with extraneous support and flows. Especially, the modeling of the ramping system of highways can be performed using interval-valued fuzzy influence graphs in order to control the unpredicted flow between cities and highways.

5. Conclusions

IVFGs are very useful tools for studying different computational intelligence and computer science domains. They have many applications in different sciences, such as optimization, topology, neural networks, and operations research. In graph theory, operations are easily used in many hybrid applications. In different situations, they present appropriate construction means. Hence, in this paper, we introduced four important operations on an IVFG level graph: the Cartesian product, composition, union, and join. Some properties of level graphs of IVFGs were discussed. Finally, an application of IVFG was presented. Our upcoming investigation involves an in-depth exploration of diverse forms of domination on interval-valued fuzzy influence graphs, covering global domination, restrained domination, semi-global domination, and K-domination. We aim to provide a clear understanding of the properties and characteristics of each of these domination types and their practical implications for real-world applications.

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