

Article

A Minimal Parameterization of Rigid Body Displacement and Motion Using a Higher-Order Cayley Map by Dual Quaternions

Daniel Condurache *  and Ionuț Popa *Department of Theoretical Mechanics, Technical University of Iasi, D. Mangeron Street No. 59,
700050 Iasi, Romania

* Correspondence: daniel.condurache@tuiasi.ro (D.C.); madalin-ionut.popa@student.tuiasi.ro (I.P.)

Abstract: The rigid body displacement mathematical model is a Lie group of the special Euclidean group $SE(3)$. This article is about the Lie algebra $se(3)$ group. The standard exponential map from $se(3)$ onto $SE(3)$ is a natural parameterization of these displacements. In technical applications, a crucial problem is the vector minimal parameterization of manifold $SE(3)$. This paper presents a unitary variant of a general class of such vector parameterizations. In recent years, dual algebra has become a comprehensive framework for analyzing and computing the characteristics of rigid-body movements and displacements. Based on higher-order fractional Cayley transforms for dual quaternions, higher-order Rodrigues dual vectors and multiple vectorial parameters (extended by rotational cases) were computed. For the rigid body movement description, a dual tangent operator (for any vectorial minimal parameterization) was computed. This paper presents a unitary method for the initial value problem of the dual kinematic equation.

Keywords: dual quaternion; orthogonal dual tensor; dual algebra; minimal parameterization

MSC: 34A05



Citation: Condurache, D.; Popa, I. A Minimal Parameterization of Rigid Body Displacement and Motion Using a Higher-Order Cayley Map by Dual Quaternions. *Symmetry* **2023**, *15*, 2011. <https://doi.org/10.3390/sym15112011>

Academic Editor: Haci Mehmet Baskonus

Received: 28 August 2023

Revised: 22 October 2023

Accepted: 29 October 2023

Published: 1 November 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Although classical, studying the motion of rigid solid bodies is still an interesting field in robotics [1–9], computer vision [10–12], kinematic equations and robot manipulation [13], Cosserat media, molecular dynamics, and astrodynamics [14–34]. The representation of the integration of the translational component with that of the rotational motion of a rigid motion is possible if we consider the rigid motion not only as a motion of points but also as a motion of directed lines. Mathematically, this equivalence corresponds to the isomorphism between the Lie group $SE(3)$ and the Lie group of orthogonal dual tensors [3]. Thus, a homogeneous matrix (the generic element of $SE(3)$) uniquely corresponds to a dual orthogonal tensor (the generic element of \overline{SO}_3). The reinvention of unit quaternions for studying rigid rotations (equivalent to the Euler–Rodrigues parameters) suggested the dual version of the notion: unit dual quaternion. It inherits the character of non-singularity and the correspondence with homogeneous matrices dedicated to rigid motion. In the past decades, theoretical bases were reevaluated, and a different technique emerged from the theory of the dual algebra realm [2–5,10–12,14–16,18,35–44]. Numerous applications have utilized dual quaternions, developing multiple algorithms for the kinematic equations associated with robotic manipulators [1–9], hand-eye calibration [10–12], serial and parallel robotic systems control [13], astrodynamics [14–34], etc. [10–12,14–16,18,39,40,42–46].

This paper aims to present a general framework for rigid body displacements in $SE(3)$ through the Lie group parametrization and rigid body motions using dual algebra [47–52].

The novelty of this paper is the approach of higher-order fractional Cayley transforms from the Lie algebra of the Lie group of unit dual quaternions [53]. The closed-form expression of this transform and its inverse are explicitly determined, coordinate-free, and in closed form. By the inverse of this higher-order fractional Cayley map (the inverse of the

modified Cayley transform is a multi-valued function with n -branches), the higher-order Rodrigues dual vector parameter and their shadow are obtained in explicit form. A novel modification of the fractional Cayley transform for vectorial parameterizations of various parameterizations embeds multiple documented attitude parameterization Cayley transforms while expanding their applicability to pose parameterization. Regarding the motion of a rigid body, a unitary method for the dual kinematic equation (the Poisson–Darboux problem) is presented via a dual tangent operator of higher-order fractional Cayley transforms for dual vector parameterization by rigid body motion. According to the authors' knowledge, this paper presents it for the first time.

The paper is structured as follows: in the second section, a mathematical preliminary for dual algebra (dual numbers, dual vectors, dual tensors) is introduced. Using these mathematical results, we investigate in the third section the rigid body motion parameterization through an orthogonal dual tensor, and two equivalent representations are provided. Using these findings, we investigate in the fourth and fifth sections the unit dual quaternions, homomorphism with orthogonal dual tensors, and higher-order Cayley map. In the seventh section, the definition and properties of the higher-order fractional Cayley transform are presented. Moreover, kinematic equations and tangent operators for multiple parameterizations with applications are discussed. The last section presents the conclusions and further works.

2. Mathematical Preliminaries

In the following section, we provide properties of dual numbers, dual vectors, and dual tensors. Additional details can be found in the following references: [35,36,42,45,46].

2.1. Dual Numbers

We denote the set of real dual numbers as $\underline{\mathbb{R}}$:

$$\underline{\mathbb{R}} = \mathbb{R} + \varepsilon\mathbb{R} = \left\{ \underline{a} = a + \varepsilon a_0 \mid a, a_0 \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0 \right\} \quad (1)$$

where $a = Re(\underline{a})$ represents the real component of \underline{a} and $a_0 = Du(\underline{a})$ signifies the dual component. The operations of addition and multiplication among dual numbers establish a ring in $\underline{\mathbb{R}}$ that incorporates a zero divisor structure. This paper emphasizes several properties of dual numbers, with a focus on magnitude and the inverse. The square of a dual number's magnitude adheres to the relationship $|\underline{a}|^2 = \underline{a}^2$, computable using $|\underline{a}| = |a| + \varepsilon \operatorname{sgn}(a)a_0$. Conversely, denoted by $\underline{a}^{-1} \in \underline{\mathbb{R}}$, the inverse of a dual number exists solely when $Re(\underline{a}) \neq 0$, determined through $\underline{a}^{-1} = \frac{1}{a} = \frac{1}{a} - \varepsilon \frac{a_0}{a^2}$. Additionally, a dual number $\underline{a} \in \underline{\mathbb{R}}$ qualifies as a zero divisor if and only if $Re(\underline{a}) = 0$. These properties indicate that the structure $(\underline{\mathbb{R}}, +, \cdot)$ forms a ring that is both commutative and unitary, where each element $\underline{a} \in \underline{\mathbb{R}}$ is either invertible or a zero divisor.

Any differentiable function $f : \mathbb{I} \subset \mathbb{R} \rightarrow \mathbb{R}, f = f(a)$ is completely defined on $\underline{\mathbb{I}} \subset \underline{\mathbb{R}}$ such that:

$$f : \underline{\mathbb{I}} \subset \underline{\mathbb{R}} \rightarrow \underline{\mathbb{R}}, f(\underline{a}) = f(a) + \varepsilon a_0 f'(a) \quad (2)$$

Based on the previous property, we can compute: $\cos \underline{a} = \cos a - \varepsilon a_0 \sin a$; $\sin \underline{a} = \sin a +$

$$\begin{aligned} \varepsilon a_0 \cos a; \sqrt[n]{\underline{a}} &= \sqrt[n]{a} + \varepsilon \frac{a_0}{n \sqrt[n]{a^{n-1}}}; \tan \underline{a} = \tan a + \varepsilon \frac{a_0}{\cos^2 a}; \arctan(\underline{a}) \\ &= \arctan(a) + \varepsilon \frac{a_0}{1+a^2} \end{aligned}$$

2.2. Dual Vectors

In the Euclidean space, the linear space of free vectors with dimension 3 is denoted by V_3 . The ensemble of dual vectors is defined as:

$$\underline{V}_3 = V_3 + \varepsilon V_3 = \left\{ \underline{\mathbf{a}} = \mathbf{a} + \varepsilon \mathbf{a}_0; \mathbf{a}, \mathbf{a}_0 \in V_3, \varepsilon^2 = 0, \varepsilon \neq 0 \right\} \quad (3)$$

where $\mathbf{a} = Re(\underline{\mathbf{a}})$ is the real part of $\underline{\mathbf{a}}$ and $\mathbf{a}_0 = Du(\underline{\mathbf{a}})$ is the dual part. For dual vectors, three products are considered: scalar product (denoted by $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}}$), cross product (denoted by $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$) and triple scalar product (denoted by $\langle \underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}} \rangle = \underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$). Regarding the algebraic structure, $(\underline{V}_3, +, \cdot, \mathbb{R})$ is a free \mathbb{R} -module [42].

For any dual vector $\underline{\mathbf{a}} \in \underline{V}_3$, the magnitude of $\underline{\mathbf{a}}$, denoted by $|\underline{\mathbf{a}}|$, is the dual number which fulfills $|\underline{\mathbf{a}}| \cdot |\underline{\mathbf{a}}| = \underline{\mathbf{a}} \cdot \underline{\mathbf{a}}$ and can be computed using:

$$|\underline{\mathbf{a}}| = \begin{cases} \|\mathbf{a}\| + \varepsilon \frac{\mathbf{a}_0 \cdot \mathbf{a}}{\|\mathbf{a}\|}, Re(\underline{\mathbf{a}}) \neq \mathbf{0} \\ \varepsilon \|\mathbf{a}_0\|, Re(\underline{\mathbf{a}}) = \mathbf{0} \end{cases} \tag{4}$$

where $\|\cdot\|$ is the Euclidean norm. If $|\underline{\mathbf{a}}| = 1$, then $\underline{\mathbf{a}}$ is called the unit dual vector.

Proposition 1 [42]. For any $\underline{\mathbf{a}} \in \underline{V}_3$, a dual number $\underline{\alpha} \in \mathbb{R}$, and a unit dual vector $\underline{\mathbf{u}}_a \in \underline{V}_3$ exist to have:

$$\underline{\mathbf{a}} = \underline{\alpha} \underline{\mathbf{u}}_a \tag{5}$$

The computational formulas for $\underline{\alpha}$ and $\underline{\mathbf{u}}_a$, are $\pm \underline{\alpha} = |\underline{\mathbf{a}}|$

$$\pm \underline{\mathbf{u}}_a = \begin{cases} \frac{\mathbf{a}}{\|\mathbf{a}\|} + \varepsilon \frac{\mathbf{a} \times (\mathbf{a}_0 \times \mathbf{a})}{\|\mathbf{a}\|^3}, Re(\underline{\mathbf{a}}) \neq \mathbf{0} \\ \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|} + \varepsilon \mathbf{v} \times \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|}, \forall \mathbf{v} \in V_3, Re(\underline{\mathbf{a}}) = \mathbf{0} \end{cases} \tag{6}$$

Also, for $Re(\underline{\mathbf{a}}) \neq \mathbf{0}$, $\underline{\alpha}$, and $\underline{\mathbf{u}}_a$ are unique up to a sign change.

The result emphasizes that any dual vector $\underline{\mathbf{a}} \in \underline{V}_3$, with $Re(\underline{\mathbf{a}}) \neq \mathbf{0}$ corresponds with a labeled directed line in the Euclidean three-dimensional space. This directed line has the following parametric equation: $\mathbf{r} = \frac{\mathbf{a} \times \mathbf{a}_0}{\|\mathbf{a}\|^2} + \lambda \frac{\mathbf{a}}{\|\mathbf{a}\|}, \forall \lambda \in \mathbb{R}$. If $Re(\underline{\mathbf{a}}) = \mathbf{0}$ the parametric equation is $\mathbf{r} = \mathbf{v} + \lambda \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|}, \forall \mathbf{v} \in V_3, \forall \lambda \in \mathbb{R}$.

2.3. Dual Tensors

A \mathbb{R} -linear mapping of \underline{V}_3 into \underline{V}_3 is called a Euclidean dual tensor:

$$\underline{\mathbf{T}}(\lambda_1 \underline{\mathbf{v}}_1 + \lambda_2 \underline{\mathbf{v}}_2) = \lambda_1 \underline{\mathbf{T}}(\underline{\mathbf{v}}_1) + \lambda_2 \underline{\mathbf{T}}(\underline{\mathbf{v}}_2), \forall \lambda_1, \lambda_2 \in \mathbb{R}, \forall \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2 \in \underline{V}_3 \tag{7}$$

Any tensor within the Euclidean dual space is referred to succinctly as a dual tensor, and $\mathbf{L}(\underline{V}_3, \underline{V}_3)$ denotes the free \mathbb{R} -module of dual tensors. Each dual tensor $\underline{\mathbf{T}} \in \mathbf{L}(\underline{V}_3, \underline{V}_3)$ is subject to decomposition into $\underline{\mathbf{T}} = \mathbf{T} + \varepsilon \mathbf{T}_0$, with both $\mathbf{T}, \mathbf{T}_0 \in \mathbf{L}(V_3, V_3)$ standing as real tensors. Furthermore, the transposed dual tensor, denoted by $\underline{\mathbf{T}}^T$, is defined by:

$$\underline{\mathbf{v}}_1 \cdot (\underline{\mathbf{T}} \underline{\mathbf{v}}_2) = \underline{\mathbf{v}}_2 \cdot (\underline{\mathbf{T}}^T \underline{\mathbf{v}}_1), \forall \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2 \in \underline{V}_3 \tag{8}$$

while, $\forall \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \in \underline{V}_3, Re(\langle \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \rangle) \neq \mathbf{0}$ the determinant is:

$$\langle \underline{\mathbf{T}} \underline{\mathbf{v}}_1, \underline{\mathbf{T}} \underline{\mathbf{v}}_2, \underline{\mathbf{T}} \underline{\mathbf{v}}_3 \rangle = \det \underline{\mathbf{T}} \langle \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \rangle \tag{9}$$

For any dual vector $\underline{\mathbf{a}} \in \underline{V}_3$ the associated skew-symmetric dual tensor is denoted by $\tilde{\underline{\mathbf{a}}}$ and is defined by:

$$\tilde{\underline{\mathbf{a}}} \underline{\mathbf{b}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}}, \forall \underline{\mathbf{b}} \in \underline{V}_3 \tag{10}$$

The previous definition leads us to the following result: for any skew-symmetric dual tensor $\underline{\mathbf{A}} \in \mathbf{L}(\underline{V}_3, \underline{V}_3), \underline{\mathbf{A}} = -\underline{\mathbf{A}}^T$, there exists a distinct and unique dual vector $\underline{\mathbf{a}} = \text{vect} \underline{\mathbf{A}}, \underline{\mathbf{a}} \in \underline{V}_3$ such that the relationship $\underline{\mathbf{A}} \underline{\mathbf{b}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}}, \forall \underline{\mathbf{b}} \in \underline{V}_3$. The set of skew-symmetric dual tensors forms a structured entity specifically a free \mathbb{R} -module of rank 3 that is isomorphic with \underline{V}_3 .

Focusing on describing the dual tensor, in this paper we will use the category of invariants called linear invariants, which are denoted by $\text{vect}\underline{\mathbf{I}} = \text{vect}\frac{1}{2}[\underline{\mathbf{I}} - \underline{\mathbf{I}}^T]$, $\text{trace}\underline{\mathbf{I}}$ where:

$$\text{trace}\underline{\mathbf{I}} = \frac{\langle \underline{\mathbf{I}}\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle + \langle \mathbf{v}_1, \underline{\mathbf{I}}\mathbf{v}_2, \mathbf{v}_3 \rangle + \langle \mathbf{v}_1, \mathbf{v}_2, \underline{\mathbf{I}}\mathbf{v}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle} \tag{11}$$

for any $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V_3$ with $\text{Re}(\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle) \neq 0$ [42,45].

Given two dual vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}} \in V_3$, $\underline{\mathbf{a}} \otimes \underline{\mathbf{b}}$ denotes a dual tensor called *tensor (dyadic product)* and is defined by:

$$\underline{\mathbf{a}} \otimes \underline{\mathbf{b}} : V_3 \rightarrow V_3, (\underline{\mathbf{a}} \otimes \underline{\mathbf{b}})\mathbf{v} = (\mathbf{v} \cdot \underline{\mathbf{b}})\underline{\mathbf{a}}, \forall \mathbf{v} \in V_3 \tag{12}$$

An important property of Equation (12) is $(\underline{\mathbf{a}} \otimes \underline{\mathbf{b}})(\underline{\mathbf{c}} \otimes \underline{\mathbf{d}}) = (\underline{\mathbf{b}} \cdot \underline{\mathbf{c}})\underline{\mathbf{a}} \otimes \underline{\mathbf{d}}$. If $\mathbf{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a right handed orthonormal basis of dual vectors, and $\underline{\mathbf{a}} = \sum_{i=1}^3 a^i \mathbf{e}_i$, $\underline{\mathbf{b}} = \sum_{i=1}^3 b^i \mathbf{e}_i$, the dyadic product $\underline{\mathbf{a}} \otimes \underline{\mathbf{b}}$ is linked to a matrix of dual numbers computed as $[\underline{\mathbf{a}} \otimes \underline{\mathbf{b}}] = \underline{\mathbf{a}}[\underline{\mathbf{b}}]^T$, where $\underline{\mathbf{a}} = [a^1, a^2, a^3]^T$ and $\underline{\mathbf{b}} = [b^1, b^2, b^3]^T$. Also, the skew-symmetric tensor is linked to a matrix of dual numbers $\tilde{\underline{\mathbf{a}}} = \begin{bmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{bmatrix}$. More details on relations between dual numbers, dual vectors, and dual matrices can be found in [36,45].

3. Rigid Body Motion Parameterization through Orthogonal Dual Tensors

Let the orthogonal dual tensor set be denoted by:

$$\underline{SO}_3 = \left\{ \underline{\mathbf{R}} \in \mathbf{L}(V_3, V_3) \mid \underline{\mathbf{R}}\underline{\mathbf{R}}^T = \underline{\mathbf{I}}, \det \underline{\mathbf{R}} = 1 \right\} \tag{13}$$

where \underline{SO}_3 represents the set of real special orthogonal dual tensors and $\underline{\mathbf{I}}$ stands as the unit orthogonal dual tensor. The internal structure of any orthogonal dual tensor $\underline{\mathbf{R}} \in \underline{SO}_3$ is illustrated in a series of results that were detailed in our previous work [45].

Theorem 1 [45]. (Structure Theorem). For any $\underline{\mathbf{R}} \in \underline{SO}_3$, a unique decomposition is viable:

$$\underline{\mathbf{R}} = (I + \varepsilon \tilde{\underline{\rho}})Q \tag{14}$$

where $Q \in SO_3$ and $\underline{\rho} \in V_3$ are called structural invariants.

Next, we introduce an isomorphism between the Lie group SE_3 and the Lie group \underline{SO}_3 :

Theorem 2 [45]. (Isomorphism theorem). The special Euclidean group (SE_3, \cdot) and $(\underline{SO}_3, \cdot)$ are connected via the isomorphism:

$$\phi : SE_3 \rightarrow \underline{SO}_3, \phi(g) = (I + \varepsilon \tilde{\underline{\rho}})Q \tag{15}$$

where $g = \begin{bmatrix} Q & \underline{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$.

The inverse of ϕ is:

$$\phi^{-1} : \underline{SO}_3 \leftrightarrow SE_3; \phi^{-1}(\underline{\mathbf{R}}) = \begin{bmatrix} Q & \underline{\rho} \\ \mathbf{0} & 1 \end{bmatrix} \tag{16}$$

where $Q = \text{Re}(\underline{\mathbf{R}})$, $\underline{\rho} = \text{vect}(Du(\underline{\mathbf{R}}) \cdot Q^T)$.

Considering the Lie group structure of $S\mathbb{O}_3$ and the result presented in the previous theorems, we ascertain that any orthogonal dual tensor $\underline{\mathbf{R}} \in S\mathbb{O}_3$ can be used to globally parameterize displacements of rigid bodies, as detailed in references [45,46].

Theorem 3 [45]. (Representation Theorem). For any orthogonal dual tensor $\underline{\mathbf{R}}$ defined as in Equation (14), a dual number $\underline{\alpha} = \alpha + \varepsilon d$ and a dual unit vector $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ can be computed to have:

$$\underline{\mathbf{R}}(\underline{\alpha}, \underline{\mathbf{u}}) = \mathbf{I} + \sin \underline{\alpha} \underline{\mathbf{u}} + (1 - \cos \underline{\alpha}) \underline{\mathbf{u}}^2 = \exp\left(\frac{\underline{\alpha}}{2} \underline{\mathbf{u}}\right) \tag{17}$$

The computational formulas for $\alpha, \mathbf{u}, d, \mathbf{u}_0$, are:

$$\begin{aligned} \alpha &= \text{atan2}\left(\pm \frac{1}{2} \sqrt{(1 + \text{trace}Q)(3 - \text{trace}Q)}; \frac{\text{trace}Q - 1}{2}\right) \\ \mathbf{u} &= \begin{cases} \pm \text{vect}\left(\frac{1}{\sqrt{(1 + \text{trace}Q)(3 - \text{trace}Q)}}(Q - Q^T)\right), & \text{when } \text{trace}Q \in (-1, 3) \\ \frac{Q\mathbf{v} + \mathbf{v}Q}{\|Q\mathbf{v} + \mathbf{v}Q\|}, \forall \mathbf{v} \in V_3, & \text{when } \text{trace}Q = -1 \text{ (} Q \text{ is symmetric)} \\ \frac{\boldsymbol{\rho}}{\|\boldsymbol{\rho}\|}, & \text{when } \text{trace}Q = 3 \text{ (} Q = \mathbf{I}) \end{cases} \\ d &= \boldsymbol{\rho} \cdot \mathbf{u} \\ \mathbf{u}_0 &= \begin{cases} \frac{1}{2}\boldsymbol{\rho} \times \mathbf{u} + \frac{1}{2} \cot \frac{\alpha}{2} \mathbf{u} \times (\boldsymbol{\rho} \times \mathbf{u}), & \alpha \neq 0 \\ \frac{1}{2}\boldsymbol{\rho} \times \mathbf{u}, & \alpha = 0 \end{cases} \end{aligned} \tag{18}$$

The parameters $\underline{\alpha}$ and $\underline{\mathbf{u}}$ are called the natural invariants of $\underline{\mathbf{R}}$. The unit dual vector $\underline{\mathbf{u}}$ gives the Plücker representation of the Mozzi–Chalses axis [35], while the dual angle $\underline{\alpha} = \alpha + \varepsilon d$ contains the rotation angle α and the translation distance d . If $\underline{\alpha} \in \mathbb{R}$, then we have the parameterization of a rotation, while for $\underline{\alpha} \in \varepsilon\mathbb{R}$, the parameterization describes translation.

Theorem 4. The natural invariants $\underline{\alpha} = \alpha + \varepsilon d, \underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ can be used to directly recover the structural invariants Q and $\boldsymbol{\rho}$ from Equation (14):

$$\begin{aligned} Q &= \mathbf{I} + \sin \underline{\alpha} \underline{\mathbf{u}} + (1 - \cos \underline{\alpha}) \underline{\mathbf{u}}^2 \\ \boldsymbol{\rho} &= d\mathbf{u} + \sin \alpha \mathbf{u}_0 + (1 - \cos \alpha) \mathbf{u} \times \mathbf{u}_0 \end{aligned} \tag{19}$$

Proof. To prove Equation (19), we need to use Equations (14) and (17). If these equations are equal, then the structure of their dual parts lead to the result presented in Equation (19). □

4. Dual Quaternions and Orthogonal Dual Tensors

A dual quaternion can be defined as an associated pair of a dual scalar quantity and a free dual vector [17,18,36,41]:

$$\widehat{\mathbf{q}} = (\underline{q}, \underline{\mathbf{q}}), \underline{q} \in \mathbb{R}, \underline{\mathbf{q}} \in V_3 \tag{20}$$

The set of dual quaternions will be denoted as $\underline{\mathbf{Q}}$ and is a \mathbb{R} -module of rank 4, if dual quaternion addition and multiplication with dual numbers are considered.

The product of two dual quaternions $\widehat{\mathbf{q}}_1 = (\underline{q}_1, \underline{\mathbf{q}}_1)$ and $\widehat{\mathbf{q}}_2 = (\underline{q}_2, \underline{\mathbf{q}}_2)$ is defined by:

$$\widehat{\mathbf{q}}_1 \widehat{\mathbf{q}}_2 = (\underline{q}_1 \cdot \underline{q}_2 - \underline{\mathbf{q}}_1 \cdot \underline{\mathbf{q}}_2, \underline{q}_1 \underline{\mathbf{q}}_2 + \underline{q}_2 \underline{\mathbf{q}}_1 + \underline{\mathbf{q}}_1 \times \underline{\mathbf{q}}_2) \tag{21}$$

Given the properties outlined above, it emerges that the \mathbb{R} -module $\underline{\mathbf{Q}}$ evolves into an associative, non-commutative linear dual algebra of the fourth order, grounded in the foundational structure of dual numbers. For every dual quaternion, as characterized by

Equation (20), the following can be computed: the conjugate denoted by $\hat{\mathbf{q}}^* = (q, -\mathbf{q})$, and the norm, denoted by $|\hat{\mathbf{q}}| = \sqrt{\hat{\mathbf{q}}\hat{\mathbf{q}}^*}$. When perceived purely within the context of a free \mathbb{R} -module, \mathbf{Q} contains two remarkable submodules, namely $\mathbf{Q}_{\mathbb{R}}$ and \mathbf{Q}_{V_3} . The initial submodule assembles entities of the form $(q, \mathbf{0})$, $q \in \mathbb{R}$, establishing an isomorphism with \mathbb{R} , and the second one, containing the pairs $(\mathbf{0}, \mathbf{q})$, $\mathbf{q} \in V_3$, achieving an isomorphism with V_3 . Moreover, the expression of any dual quaternion can be written as $\hat{\mathbf{q}} = \underline{q} + \underline{\mathbf{q}}$, where $\underline{q} = (q, \mathbf{0})$ and $\underline{\mathbf{q}} = (\mathbf{0}, \mathbf{q})$, or $\hat{\mathbf{q}} = \hat{\mathbf{q}} + \varepsilon\hat{\mathbf{q}}_0$, where $\hat{\mathbf{q}}, \hat{\mathbf{q}}_0$ are real quaternions.

Denoting \mathbb{U} the set of real unit quaternions ($|\hat{\mathbf{q}}| = 1$) and $\underline{\mathbb{U}}$ the set of dual unit quaternions ($|\hat{\mathbf{q}}| = 1$) paves the way for understanding the components of a unit dual quaternion, commonly referred to as dual Euler parameters in their scalar and vector forms.

Theorem 5. For any $\hat{\mathbf{q}} \in \underline{\mathbb{U}}$, the following representation is valid:

$$\hat{\mathbf{q}} = \left(1 + \varepsilon \frac{1}{2} \hat{\rho}\right) \hat{\mathbf{q}} \quad (22)$$

where $\rho \in V_3$ and $\hat{\mathbf{q}} \in \mathbb{U}$.

Proof. $\hat{\mathbf{q}} \in \underline{\mathbb{U}} \iff \hat{\mathbf{q}}\hat{\mathbf{q}}^* = \mathbf{1} \iff (\hat{\mathbf{q}} + \varepsilon\hat{\mathbf{q}}_0)(\hat{\mathbf{q}}^* + \varepsilon\hat{\mathbf{q}}_0^*) = \mathbf{1} \iff \hat{\mathbf{q}}\hat{\mathbf{q}}^* + \varepsilon(\hat{\mathbf{q}}_0\hat{\mathbf{q}}^* + \hat{\mathbf{q}}\hat{\mathbf{q}}_0^*) = \mathbf{1}$. From previous equation it follows: $\hat{\mathbf{q}}\hat{\mathbf{q}}^* = \mathbf{1} \iff \hat{\mathbf{q}} \in \mathbb{U}$ and $\hat{\mathbf{q}}_0\hat{\mathbf{q}}^* + \hat{\mathbf{q}}\hat{\mathbf{q}}_0^* = \mathbf{0} \iff \hat{\mathbf{q}}_0\hat{\mathbf{q}}^* \in V_3$. By denotation $\hat{\rho} = 2\hat{\mathbf{q}}_0\hat{\mathbf{q}}^*$, Theorem 5 is proved. \square

This representation is the quaternion counterpart to Equation (14). Based on Theorem 5, a dual number $\underline{\alpha}$ and a dual vector $\underline{\mathbf{u}}$ exist so that:

$$\hat{\mathbf{q}} = \cos \frac{\underline{\alpha}}{2} + \underline{\mathbf{u}} \sin \frac{\underline{\alpha}}{2} = \exp\left(\frac{\underline{\alpha}}{2} \underline{\mathbf{u}}\right), \forall \hat{\mathbf{q}} \in \underline{\mathbb{U}} \quad (23)$$

Lemma 1. The mapping denoted $\exp : V_3 \rightarrow \underline{\mathbb{U}}$, $\hat{\mathbf{q}} = \exp \frac{\underline{\alpha}}{2}$ is well defined and onto.

Given the way $\underline{\mathbb{U}}$ is constructed, along with the rules governing the multiplication of dual quaternions, we can conclude that the Lie group structure (V_3 being the associated Lie algebra, where the cross-product operation between dual vectors is intrinsic), allows for the global parameterization of all possible rigid body motions.

Using the internal structure of any element from \underline{SO}_3 , the following theorem is valid:

Theorem 6. The Lie groups $\underline{\mathbb{U}}$ and \underline{SO}_3 are linked by an onto homomorphism:

$$\Delta : \underline{\mathbb{U}} \rightarrow \underline{SO}_3, \Delta(\underline{q} + \underline{\mathbf{q}}) = \mathbf{I} + 2\underline{q}\tilde{\mathbf{q}} + 2\tilde{\mathbf{q}}^2 \quad (24)$$

Proof. Considering that any dual quaternion $\hat{\mathbf{q}} \in \underline{\mathbb{U}}$ can undergo decomposition as indicated in Equation (23), it follows that the expression $\Delta(\hat{\mathbf{q}}) = \exp(\underline{\alpha} \tilde{\mathbf{u}}) \in \underline{SO}_3$. This validates that relation Equation (24) is well defined. By straightforward computation, one can confirm that $\Delta(\hat{\mathbf{q}}_2\hat{\mathbf{q}}_1) = \Delta(\hat{\mathbf{q}}_2)\Delta(\hat{\mathbf{q}}_1)$.

Furthermore, any orthogonal dual tensor $\mathbf{R} \in \underline{SO}_3$ can be expressed as per Theorem 3, specifically, $\mathbf{R} = \exp(\underline{\alpha} \tilde{\mathbf{u}})$. Consequently, it is possible to identify a dual quaternion $\hat{\mathbf{q}} = \exp(\frac{\underline{\alpha}}{2} \underline{\mathbf{u}})$ such that $\Delta(\hat{\mathbf{q}}) = \mathbf{R}$, substantiating that Δ operates as a surjective homomorphism. The proof is thus complete. \square

An important property of the previous homomorphism is that for $\hat{\mathbf{q}}$ and $-\hat{\mathbf{q}}$ we can associate the same orthogonal dual tensor, which shows that Equation (24) is not injective and $\underline{\mathbb{U}}$ is a double cover of $\underline{S\mathbb{O}}_3$.

5. Cayley Transform for Dual Orthogonal Tensors

The Lie algebra of $\underline{S\mathbb{O}}_3$ is the skew-symmetric dual tensors set denoted by $\underline{so}_3 = \left\{ \tilde{\underline{\alpha}} \in \mathbf{L}(\mathbf{V}_3, \mathbf{V}_3) \mid \tilde{\underline{\alpha}} = -\tilde{\underline{\alpha}}^T \right\}$, where the internal mapping is $\langle \tilde{\underline{\alpha}}_1, \tilde{\underline{\alpha}}_2 \rangle = \widetilde{\underline{\alpha}}_1 \underline{\alpha}_2$ [45]. The link between the Lie algebra \underline{so}_3 , the Lie group $\underline{S\mathbb{O}}_3$, and the exponential map is given by:

Theorem 7. [45]. *The mapping*

$$\exp : \underline{so}_3 \rightarrow \underline{S\mathbb{O}}_3, \exp(\tilde{\underline{\alpha}}) = \mathbf{e}^{\tilde{\underline{\alpha}}} = \sum_{k=0}^{\infty} \frac{\tilde{\underline{\alpha}}^k}{k!} \quad (25)$$

is well defined and onto.

A screw axis is representable through a unit dual vector, while the screw parameters (the angle of rotation about the screw and the translation along the screw axis) can be combined into a dual angle. The computation of the screw axis intertwines with the task of calculating the logarithm of an orthogonal dual tensor, represented as $\underline{\mathbf{R}}$, as it involves a multifunction defined by

$$\log : \underline{S\mathbb{O}}_3 \rightarrow \underline{so}_3, \log \underline{\mathbf{R}} = \left\{ \tilde{\underline{\alpha}} \in \underline{so}_3 \mid \exp(\tilde{\underline{\alpha}}) = \underline{\mathbf{R}} \right\} \quad (26)$$

and is the inverse of Equation (25).

Based on Theorem 3, for any orthogonal dual tensor $\underline{\mathbf{R}}$, a Euler dual vector $\underline{\alpha} = \underline{\alpha} \mathbf{u} = \underline{\alpha} + \varepsilon \alpha_0$ can be computed and represents the screw dual vector which embeds the screw axis and screw parameters. The form of $\underline{\alpha}$ implies that $\tilde{\underline{\alpha}} \in \log \underline{\mathbf{R}}$. The types of rigid displacements that can be parameterized by $\underline{\alpha}$ are: (i) general screw displacement (if $\underline{\alpha} \neq 0$, $\alpha_0 \neq 0$ and $\underline{\alpha} \cdot \alpha_0 \neq 0$); (ii) pure translation (if $\underline{\alpha} = 0$ and $\alpha_0 \neq 0$); and (iii) pure rotation ($\underline{\alpha} \neq 0$ and $\underline{\alpha} \cdot \alpha_0 = 0$). Also, if $|\underline{\alpha}| < \pi$, Theorem 3 can be used to uniquely recover the Euler dual vector $\underline{\alpha}$ which is equivalent to computing $\log \underline{\mathbf{R}}$.

The first-order Cayley transform is a map between the Lie algebra \underline{so}_3 and the Lie group $\underline{S\mathbb{O}}_3$.

Theorem 8. *The map*

$$\text{cay}(\cdot) : \underline{so}_3 \rightarrow \underline{S\mathbb{O}}_3, \text{cay}(\tilde{\underline{\mathbf{v}}}) = (\mathbf{I} + \tilde{\underline{\mathbf{v}}})(\mathbf{I} - \tilde{\underline{\mathbf{v}}})^{-1} \quad (27)$$

is well defined and onto.

Proof. For any $\tilde{\underline{\mathbf{v}}} \in \underline{S\mathbb{O}}_3$, the value of $\det(\text{cay}(\tilde{\underline{\mathbf{v}}}))$ is:

$$\det(\text{cay}(\tilde{\underline{\mathbf{v}}})) = \det\left[(\mathbf{I} + \tilde{\underline{\mathbf{v}}})(\mathbf{I} - \tilde{\underline{\mathbf{v}}})^{-1}\right] = \frac{1 + |\underline{\mathbf{v}}|^2}{1 - |\underline{\mathbf{v}}|^2} = 1 \quad (28)$$

while

$$\text{cay}(\tilde{\underline{\mathbf{v}}}) [\text{cay}(\tilde{\underline{\mathbf{v}}})]^T = (\mathbf{I} + \tilde{\underline{\mathbf{v}}})(\mathbf{I} - \tilde{\underline{\mathbf{v}}})^{-1} (\mathbf{I} - \tilde{\underline{\mathbf{v}}})(\mathbf{I} + \tilde{\underline{\mathbf{v}}})^{-1} = \mathbf{I} \quad (29)$$

Equations (28) and (29) prove that the first-order Cayley transform is well defined.

The computation for the inverse of tensor $(\mathbf{I} - \tilde{\underline{\mathbf{v}}})$, is:

$$(\mathbf{I} - \tilde{\underline{\mathbf{v}}})^{-1} = \frac{1}{1 + |\underline{\mathbf{v}}|^2} (\tilde{\underline{\mathbf{v}}}^2 + \tilde{\underline{\mathbf{v}}}) + \mathbf{I} \quad (30)$$

Equations (27) and (30) leads to:

$$cay(\tilde{\mathbf{v}}) = \mathbf{I} + 2 \frac{|\mathbf{v}|}{1 + |\mathbf{v}|^2} \frac{\tilde{\mathbf{v}}}{|\mathbf{v}|} + \left(1 - \frac{1 - |\mathbf{v}|^2}{1 + |\mathbf{v}|^2}\right) \frac{\tilde{\mathbf{v}}^2}{|\mathbf{v}|^2} \tag{31}$$

Considering

$$\sin \underline{\alpha} = \frac{2 \tan \frac{\underline{\alpha}}{2}}{1 + \tan^2 \frac{\underline{\alpha}}{2}}; \cos \underline{\alpha} = \frac{1 - \tan^2 \frac{\underline{\alpha}}{2}}{1 + \tan^2 \frac{\underline{\alpha}}{2}} \tag{32}$$

result:

$$cay(\tilde{\mathbf{v}}) = \mathbf{I} + \sin \underline{\alpha} \tilde{\mathbf{u}} + (1 - \cos \underline{\alpha}) \tilde{\mathbf{u}}^2 \tag{33}$$

where $\underline{\alpha} = 2 \arctan |\mathbf{v}|$. Therefore, the first-order Cayley transform is onto. \square

The previous theorem leads to the following result:

Corollary 1. *The parametrization $cay(\tilde{\mathbf{v}})$ is associated with pure rotation if and only if $|\mathbf{v}| \in \mathbb{R}$. Conversely, when $|\mathbf{v}| \in \varepsilon \mathbb{R}$, $cay(\tilde{\mathbf{v}})$ represents pure translation. In all other cases, the mapping $cay(\tilde{\mathbf{v}})$ describes a general rigid body displacement.*

A dual vector $\underline{\mathbf{b}} = (\tan \frac{\underline{\alpha}}{2}) \underline{\mathbf{u}}$ (named Rodrigues dual vector) exists for the purpose of having $cay(\tilde{\mathbf{b}}) = \mathbf{R}, \forall \mathbf{R} \in \underline{SO}_3, \mathbf{R} = \mathbf{R}(\underline{\alpha}, \underline{\mathbf{u}})$ and $Re(\underline{\alpha}) \neq k\pi, k \in \mathbb{Z}$.

6. Cayley Transforms for Unit Dual Quaternions

In this section we studied the Cayley transform from Lie algebra to dual vectors \underline{V}_3 by Lie group of unit dual quaternions \underline{U} .

Theorem 9. *The Cayley map $cay : \underline{V}_3 \rightarrow \underline{U}$*

$$cay(\underline{\mathbf{v}}) = (1 + \underline{\mathbf{v}})(1 - \underline{\mathbf{v}})^{-1} \tag{34}$$

is well defined and onto.

Proof. Based on the quaternion product results, we can see that:

$$cay(\underline{\mathbf{v}}) = \frac{1 + \underline{\mathbf{v}}}{1 - \underline{\mathbf{v}}} = \frac{(1 + \underline{\mathbf{v}})^2}{1 + |\mathbf{v}|^2} \tag{35}$$

If we compute the norm of the previous equation, using quaternionic calculus:

$$|cay(\underline{\mathbf{v}})| = \frac{|1 + \underline{\mathbf{v}}|}{|1 - \underline{\mathbf{v}}|} = \frac{\sqrt{1 + |\mathbf{v}|^2}}{\sqrt{1 + |\mathbf{v}|^2}} = 1 \tag{36}$$

which proves that the map is well defined. In Equation (36) on using the definition: $|\hat{\mathbf{q}}| = \sqrt{\hat{\mathbf{q}}\hat{\mathbf{q}}^*} \iff |1 + \underline{\mathbf{v}}| = \sqrt{(1 + \underline{\mathbf{v}})(1 - \underline{\mathbf{v}})} = \sqrt{(1 - \underline{\mathbf{v}}^2)} = \sqrt{1 + |\mathbf{v}|^2}$. $\forall \hat{\mathbf{q}} \in \underline{U}$ we can find by Equation (35) a dual vector $\underline{\mathbf{v}} \in \underline{V}_3, \underline{\mathbf{v}} = \frac{\hat{\mathbf{q}} - 1}{\hat{\mathbf{q}} + 1}$ to have $cay(\underline{\mathbf{v}}) = \hat{\mathbf{q}}$, which proves that the map is onto. \square

Corollary 2. *$cay(\underline{\mathbf{v}})$ is the parameterization of a pure rotation if and only if $|\mathbf{v}| \in \mathbb{R}$. Meanwhile, if $|\mathbf{v}| \in \varepsilon \mathbb{R}$, the mapping $cay(\underline{\mathbf{v}})$ is the parameterization of a pure translation. Otherwise, $cay(\underline{\mathbf{v}})$ is the parameterization of general rigid body displacement.*

A dual number $\underline{\alpha}$ and a dual vector $\underline{\mathbf{u}}$ can be considered to have:

$$\widehat{\underline{\mathbf{q}}} = \cos \frac{\alpha}{2} + \underline{\mathbf{u}} \sin \frac{\alpha}{2} \tag{37}$$

The results for $Re(\underline{\alpha}) \neq 2k\pi$

$$\underline{\sigma} = \tan \frac{\alpha}{4} \underline{\mathbf{u}} \tag{38}$$

which can be interpreted as a modified Rodrigues dual vector [54] (a.k.a Wiener–Milenković dual vector [55]).

The representation given by Equation (34) plays an important role in addressing the complexities of successive rigid body displacements. Consider dual vectors $\underline{\sigma}_1 = \tan \frac{\alpha_1}{4} \underline{\mathbf{u}}_1$ and $\underline{\sigma}_2 = \tan \frac{\alpha_2}{4} \underline{\mathbf{u}}_2$ to describe two separate rigid displacements.

Consider $\widehat{\underline{\mathbf{q}}}$ as the dual quaternion representing the composition of two rigid body displacements. Let $\underline{\sigma} = \tan \frac{\alpha}{4} \underline{\mathbf{u}}$ be the modified Rodrigues dual vector of this rigid body composition. The dual vector $\underline{\sigma}$ is expressed by the equation:

$$\underline{\sigma} = \frac{\widehat{\underline{\mathbf{q}}} - 1}{\widehat{\underline{\mathbf{q}}} + 1} \tag{39}$$

Considering that $\widehat{\underline{\mathbf{q}}} = cay(\underline{\sigma}_2)cay(\underline{\sigma}_1)$, we obtain:

$$\underline{\sigma} = \frac{\left(\frac{1+\underline{\sigma}_2}{1-\underline{\sigma}_2}\right)\left(\frac{1+\underline{\sigma}_1}{1-\underline{\sigma}_1}\right) - 1}{\left(\frac{1+\underline{\sigma}_2}{1-\underline{\sigma}_2}\right)\left(\frac{1+\underline{\sigma}_1}{1-\underline{\sigma}_1}\right) + 1} \tag{40}$$

The dual vector associated with Equation (40) is, after some quaternionic product calculations:

$$\underline{\sigma} = \frac{(1 - |\underline{\sigma}_2|^2)\underline{\sigma}_1 + (1 - |\underline{\sigma}_1|^2)\underline{\sigma}_2 + 2\underline{\sigma}_2 \times \underline{\sigma}_1}{(1 - \underline{\sigma}_2 \cdot \underline{\sigma}_1)^2 + |\underline{\sigma}_2 \times \underline{\sigma}_1|^2} \tag{41}$$

Next, we present a Cayley-like fractional map:

Theorem 10. *The fractional order Cayley-like map denoted as $cay_{\frac{1}{2}} : \underline{V}_3 \rightarrow \underline{U}$ is established as follows:*

$$cay_{\frac{1}{2}}(\underline{\mathbf{v}}) = (1 + \underline{\mathbf{v}})^{\frac{1}{2}}(1 - \underline{\mathbf{v}})^{-\frac{1}{2}} \tag{42}$$

It is well defined and onto.

Proof. Using quaternionic calculus the chosen Cayley transform can also be expressed as:

$$cay_{\frac{1}{2}}(\underline{\mathbf{v}}) = \sqrt{\frac{1 + \underline{\mathbf{v}}}{1 - \underline{\mathbf{v}}}} = \pm \frac{1 + \underline{\mathbf{v}}}{\sqrt{1 + |\underline{\mathbf{v}}|^2}} \tag{43}$$

It is clear that $cay_{\frac{1}{2}}(\underline{\mathbf{v}})$ is well defined and $|cay_{\frac{1}{2}}(\underline{\mathbf{v}})| = 1$.

Now, for any $\widehat{\underline{\mathbf{q}}} \in \underline{U}$, $\widehat{\underline{\mathbf{q}}} = \cos \frac{\alpha}{2} + \underline{\mathbf{u}} \sin \frac{\alpha}{2}$, a dual vector $\underline{\mathbf{v}} \in \underline{V}_3$ exists to have, by Equation (42):

$$\underline{\mathbf{v}} = \frac{\widehat{\underline{\mathbf{q}}}^2 - 1}{\widehat{\underline{\mathbf{q}}}^2 + 1} \tag{44}$$

This dual vector is $\underline{\mathbf{v}} = \tan \frac{\alpha}{2} \underline{\mathbf{u}}$, $Re(\underline{\alpha}) \neq k\pi$, which proves that the map $cay_{\frac{1}{2}}(\underline{\mathbf{v}})$ is onto. \square

This also shows that the Rodrigues dual vector can be recovered through the inverse of the Cayley map given in Equation (44).

Corollary 3. *The parametrization $cay_{\frac{1}{2}}(\underline{\mathbf{v}})$ is associated with pure rotation if and only if $|\underline{\mathbf{v}}| \in \mathbb{R}$. Meanwhile, if $|\underline{\mathbf{v}}| \in \epsilon\mathbb{R}$, the mapping $cay_{\frac{1}{2}}(\underline{\mathbf{v}})$ is the parameterization of a pure translation. Otherwise, $cay_{\frac{1}{2}}(\underline{\mathbf{v}})$ describes a general rigid body displacement.*

Consider $\underline{\mathbf{b}}_1 = \tan \frac{\alpha_1}{2} \underline{\mathbf{u}}_1$ and $\underline{\mathbf{b}}_2 = \tan \frac{\alpha_2}{2} \underline{\mathbf{u}}_2$, the Rodrigues dual vectors that parameterize two rigid displacements. The Rodrigues dual vector for the composition of these two rigid body displacements is given by:

$$\underline{\mathbf{b}} = \frac{\underline{\mathbf{b}}_1 + \underline{\mathbf{b}}_2 + \underline{\mathbf{b}}_2 \times \underline{\mathbf{b}}_1}{1 - \underline{\mathbf{b}}_1 \cdot \underline{\mathbf{b}}_2} \tag{45}$$

This result is a clear indication that we need to explore the properties of higher-order fractional Cayley transforms.

7. Higher-Order Fractional Modified Cayley Transform

Up next, we introduce a range of findings derived from utilizing a set of fractional Cayley transforms. Notably, these fractional Cayley transforms differ from those previously documented in [20,21,56–58].

Theorem 11. *The fractional order Cayley transform:*

$$\begin{aligned} cay_{\frac{n}{2}} : \mathbf{V}_3 &\rightarrow \mathbb{U} \\ cay_{\frac{n}{2}}(\underline{\mathbf{v}}) &= (1 + \underline{\mathbf{v}})^{\frac{n}{2}}(1 - \underline{\mathbf{v}})^{-\frac{n}{2}}, n \in \mathbb{N}^* \end{aligned} \tag{46}$$

are well defined and onto.

Proof. Using direct calculus we find that $cay_{\frac{n}{2}}(\underline{\mathbf{v}})cay_{\frac{n}{2}}^*(\underline{\mathbf{v}}) = 1$ and $|cay_{\frac{n}{2}}(\underline{\mathbf{v}})| = 1$. □

The surjectivity is proved by the following theorem:

Theorem 12. *The inverse of the previous fractional order Cayley map is a multi-valued function with n branches $cay_{\frac{n}{2}}(\underline{\mathbf{v}})^{-1} : \mathbb{U} \rightarrow \mathbf{V}_3$ given by:*

$$\underline{\mathbf{v}} = \frac{\sqrt[n]{\hat{\mathbf{q}}^2} - 1}{\sqrt[n]{\hat{\mathbf{q}}^2} + 1} \tag{47}$$

Proof. By Equation (47): $\hat{\mathbf{q}} = (1 + \underline{\mathbf{v}})^{\frac{n}{2}}(1 - \underline{\mathbf{v}})^{-\frac{n}{2}} = \left(\frac{1+\underline{\mathbf{v}}}{1-\underline{\mathbf{v}}}\right)^{\frac{n}{2}} \iff \frac{1+\underline{\mathbf{v}}}{1-\underline{\mathbf{v}}} = \sqrt[n]{\hat{\mathbf{q}}^2} \iff \underline{\mathbf{v}} = \frac{\sqrt[n]{\hat{\mathbf{q}}^2}-1}{\sqrt[n]{\hat{\mathbf{q}}^2}+1}$. □

Corollary 4. *The parameterization $cay_{\frac{n}{2}}(\underline{\mathbf{v}})$ is associated with a pure rotation if and only if $|\underline{\mathbf{v}}| \in \mathbb{R}$. Meanwhile, if $|\underline{\mathbf{v}}| \in \epsilon\mathbb{R}$ the mapping $cay_{\frac{n}{2}}(\underline{\mathbf{v}})$ is the parameterization of a pure translation. Otherwise, the parameterization $cay_{\frac{n}{2}}(\underline{\mathbf{v}})$ describes a general screw displacement.*

Considering that a dual number $\underline{\alpha}$ and a dual vector $\underline{\mathbf{u}}$ exist to have:

$$\hat{\mathbf{q}} = \cos \frac{\alpha}{2} + \underline{\mathbf{u}} \sin \frac{\alpha}{2} \tag{48}$$

we find that by Equation (48):

$$\underline{\mathbf{v}} = \tan \frac{\alpha + 2k\pi}{2n} \underline{\mathbf{u}}, k = \{0, 1, \dots, n - 1\} \tag{49}$$

From Equation (49), $k = 0$, and the principal parameterization $\mathbf{v}_0 = \tan \frac{\alpha}{2n} \mathbf{u}$, which is the higher-order Rodrigues dual vector.

For $k = \{1, \dots, n - 1\}$, the dual vectors $\mathbf{v}_k = \tan \frac{\alpha + 2k\pi}{2n} \mathbf{u}$ are the shadow parameterizations that can be used to describe the same pose. Based on $|\mathbf{v}_0| = \tan \frac{\alpha}{2n}$ and $|\mathbf{v}_k| = \tan \frac{\alpha + 2k\pi}{2n}$, shows that $|\mathbf{v}_k| = \frac{|\mathbf{v}_0| + \tan \frac{k\pi}{n}}{1 - |\mathbf{v}_0| \tan \frac{k\pi}{n}}$. If $Re(|\mathbf{v}_0|) \rightarrow \infty$, then $Re(|\mathbf{v}_k|) \rightarrow -\cot \frac{k\pi}{n}$, which allows the avoidance of any singularity of type $Re(\frac{\alpha}{2n}) = \frac{\pi}{2} + k\pi$.

Theorem 13. If $\mathbf{v} \in \mathbf{V}_3$ is the parameterization of a displacement obtained from Equation (41),

$$\pm \hat{\mathbf{q}} = \frac{1}{\sqrt{(1 + |\mathbf{v}|^2)^n}} [p_n(|\mathbf{v}|) + q_n(|\mathbf{v}|)\mathbf{v}] \tag{50}$$

where:

$$p_n(X) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} X^{2k} \tag{51}$$

$$q_n(X) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} X^{2k} \tag{52}$$

In Equation (51), $\binom{n}{k}$ are binomial coefficients and $\lfloor \cdot \rfloor$ represents the floor of a number.

Proof. Consider $\pm \hat{\mathbf{q}} = (\mathbf{1} + \mathbf{v})^{\frac{n}{2}} (\mathbf{1} - \mathbf{v})^{\frac{n}{2}}$. Based on the properties of the dual quaternions product results:

$$\pm \hat{\mathbf{q}} = \frac{(\mathbf{1} + \mathbf{v})^n}{\left(\sqrt{\mathbf{1} + |\mathbf{v}|^2}\right)^n} = \frac{1}{\left(\sqrt{\mathbf{1} + |\mathbf{v}|^2}\right)^n} \sum_{k=0}^n \binom{n}{k} \mathbf{v}^k. \tag{53}$$

Considering that

$$\mathbf{v}^k = \begin{cases} |\mathbf{v}|^{4p}, & k = 4p \\ |\mathbf{v}|^{4p} \mathbf{v}, & k = 4p + 1 \\ -|\mathbf{v}|^{4p+2}, & k = 4p + 2 \\ -|\mathbf{v}|^{4p+2} \mathbf{v}, & k = 4p + 3 \end{cases}, p \in \mathbb{N} \tag{54}$$

the dual quaternion $\pm \hat{\mathbf{q}}$ can be expressed by:

$$\pm \hat{\mathbf{q}} = \frac{1}{\left(\sqrt{\mathbf{1} + |\mathbf{v}|^2}\right)^n} [p_n(\mathbf{v}) + q_n(\mathbf{v}) \cdot \mathbf{v}] \tag{55}$$

which proves the theorem. \square

Lemma 2. The structure of the polynomials $p_n(X)$ and $q_n(X)$, given by Equations (51) and (52), can be used to obtain the following iterative expressions:

$$\begin{aligned} p_{n+1}(X) &= p_n(X) - X^2 q_n(X) \\ q_{n+1}(X) &= q_n(X) + q_n(X) \\ p_1(X) &= 1, q_1(X) = 1 \end{aligned} \tag{56}$$

To assess the practicality of the iterative formulas, we present polynomials ranging from the second to the fifth order, along with the corresponding quaternions that emerge:

- $p_1(X) = 1; q_1(X) = 1;$
 $\bullet \quad \pm \hat{\mathbf{q}} = \frac{1}{\sqrt{1+|\mathbf{v}|^2}} [1 + \mathbf{v}]; \mathbf{v} = \tan \frac{\alpha}{2} \mathbf{u}$
- $p_2(X) = 1 - X^2; q_2(X) = 2;$
 $\bullet \quad \pm \hat{\mathbf{q}} = \frac{1}{1+|\mathbf{v}|^2} [1 - |\mathbf{v}|^2 + 2\mathbf{v}]; \mathbf{v} = \tan \frac{\alpha}{4} \mathbf{u}$
- $p_3(X) = 1 - 3X^2; q_3(X) = 3 - X^2;$
 $\bullet \quad \pm \hat{\mathbf{q}} = \frac{1}{\sqrt{(1+|\mathbf{v}|^2)^3}} [1 - 3|\mathbf{v}|^2 + (3 - |\mathbf{v}|^2)\mathbf{v}]; \mathbf{v} = \tan \frac{\alpha}{6} \mathbf{u}$
- $p_4(X) = 1 - 6X^2 + X^4; q_4(X) = 4 - 4X^2;$
 $\bullet \quad \pm \hat{\mathbf{q}} = \frac{1}{(1+|\mathbf{v}|^2)^2} [1 - 6|\mathbf{v}|^2 + |\mathbf{v}|^4 + 4(1 - |\mathbf{v}|^2)\mathbf{v}]; \mathbf{v} = \tan \frac{\alpha}{8} \mathbf{u}$
- $p_5(X) = 1 - 10X^2 + 5X^4; q_5(X) = 5 - 10X^2 + X^4;$
 $\bullet \quad \pm \hat{\mathbf{q}} = \frac{1}{\sqrt{(1+|\mathbf{v}|^2)^5}} [1 - 10|\mathbf{v}|^2 + 5|\mathbf{v}|^4 + (5 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)\mathbf{v}]; \mathbf{v} = \tan \frac{\alpha}{10} \mathbf{u}$

A direct consequence is that previous equations hold true for both the primary parameterization and its corresponding shadows. Another important result is the following:

Theorem 14. *The orthogonal dual tensor associated with Equation (55) is (according to Theorem 6):*

$$\mathbf{R} = \mathbf{I} + \frac{2p_n(|\mathbf{v}|)q_n(|\mathbf{v}|)\tilde{\mathbf{v}}}{(1 + |\mathbf{v}|^2)^n} + \frac{2q_n^2(|\mathbf{v}|)\tilde{\mathbf{v}}^2}{(1 + |\mathbf{v}|^2)^n} \tag{57}$$

To demonstrate the efficacy of the iterative terms, we supply orthogonal dual tensors from the second through the fifth order:

- $\bullet \quad \mathbf{R} = \mathbf{I} + \frac{2}{1+|\mathbf{v}|^2} [\tilde{\mathbf{v}} + \tilde{\mathbf{v}}^2]; \mathbf{v} = \tan \frac{\alpha}{2} \mathbf{u};$
- $\bullet \quad \mathbf{R} = \mathbf{I} + \frac{4}{(1+|\mathbf{v}|^2)^2} [(1 - |\mathbf{v}|^2)\tilde{\mathbf{v}} + 2\tilde{\mathbf{v}}^2]; \mathbf{v} = \tan \frac{\alpha}{4} \mathbf{u};$
- $\bullet \quad \mathbf{R} = \mathbf{I} + \frac{2(3-|\mathbf{v}|^2)}{(1+|\mathbf{v}|^2)^3} [(1 - 3|\mathbf{v}|^2)\tilde{\mathbf{v}} + (3 - |\mathbf{v}|^2)\tilde{\mathbf{v}}^2]; \mathbf{v} = \tan \frac{\alpha}{6} \mathbf{u};$
- $\bullet \quad \mathbf{R} = \mathbf{I} + \frac{8(1-|\mathbf{v}|^2)}{(1+|\mathbf{v}|^2)^4} [(1 - 6|\mathbf{v}|^2 + |\mathbf{v}|^4)\tilde{\mathbf{v}} + 4(1 - |\mathbf{v}|^2)\tilde{\mathbf{v}}^2]; \mathbf{v} = \tan \frac{\alpha}{8} \mathbf{u};$
- $\bullet \quad \mathbf{R} = \mathbf{I} + \frac{2(5-10|\mathbf{v}|^2+|\mathbf{v}|^4)}{(1+|\mathbf{v}|^2)^5} [(1 - 10|\mathbf{v}|^2 + 5|\mathbf{v}|^4)\tilde{\mathbf{v}} + (5 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)\tilde{\mathbf{v}}^2]; \mathbf{v} = \tan \frac{\alpha}{10} \mathbf{u}.$

The composition of two rigid displacements parameterized by \mathbf{v}_1 and \mathbf{v}_2 can be modeled by:

$$\mathbf{v} = \frac{\sqrt[n]{\left[\left(\frac{1+\mathbf{v}_2}{1-\mathbf{v}_2}\right)^{\frac{n}{2}} \left(\frac{1+\mathbf{v}_1}{1-\mathbf{v}_1}\right)^{\frac{n}{2}}\right]^2} - 1}{\sqrt[n]{\left[\left(\frac{1+\mathbf{v}_2}{1-\mathbf{v}_2}\right)^{\frac{n}{2}} \left(\frac{1+\mathbf{v}_1}{1-\mathbf{v}_1}\right)^{\frac{n}{2}}\right]^2} + 1} \tag{58}$$

Until now we have discussed about the fractional Cayley transform for vectorial parameterizations of type $\tan \frac{\alpha}{2n} \mathbf{u}$ and their shadows.

Next, we uncover a new type of fractional Cayley transform for vectorial parameterizations of type $\underline{\mathbf{v}} = \varphi(\underline{\alpha})\underline{\mathbf{u}}$, where $\underline{\alpha}$ and $\underline{\mathbf{u}}$ are the natural invariants and $\varphi(\underline{\alpha}) : \underline{\mathbb{R}} \rightarrow \underline{\mathbb{R}}$ is smooth mapping.

Theorem 15. Consider $a : \underline{V}_3 \rightarrow \underline{\mathbb{R}}, a = a(\underline{\mathbf{v}})$, with $Re(a(\underline{\mathbf{v}})) \neq 0$. The mapping $caym_{\frac{n}{2}}(\underline{\mathbf{v}}) : \underline{V}_3 \rightarrow \underline{\mathbb{U}}$,

$$caym_{\frac{n}{2}}(\underline{\mathbf{v}}) = (a(\underline{\mathbf{v}}) + \underline{\mathbf{v}})^{\frac{n}{2}}(a(\underline{\mathbf{v}}) - \underline{\mathbf{v}})^{-\frac{n}{2}} \tag{59}$$

are well defined.

Proof. For any dual vector $\underline{\mathbf{v}} \in \underline{V}_3$ we have $caym_{\frac{n}{2}}(\underline{\mathbf{v}})caym_{\frac{n}{2}}(\underline{\mathbf{v}})^*(\underline{\mathbf{v}}) = 1$ and $caym_{\frac{n}{2}}(\underline{\mathbf{v}}) = \pm \frac{[a(\underline{\mathbf{v}}) + \underline{\mathbf{v}}]^n}{\sqrt{(|a(\underline{\mathbf{v}})|^2 + |\underline{\mathbf{v}}|^2)^n}}$.

Now consider $\hat{\underline{\mathbf{q}}} = \cos \frac{\alpha}{2} + \underline{\mathbf{u}} \sin \frac{\alpha}{2} \in \underline{\mathbb{U}}$, the inverse of the mapping Equation (59) contains n branches to have:

$$\underline{\mathbf{v}}(\hat{\underline{\mathbf{q}}}) = a(\underline{\mathbf{v}}) \tan \frac{\alpha + 2k\pi}{2n} \underline{\mathbf{u}}, k = \{0, 1, \dots, n - 1\} \tag{60}$$

□

Theorem 16. If $\underline{\mathbf{v}} = \varphi(\underline{\alpha})\underline{\mathbf{u}}$ is a vectorial parameterization of displacement, with $\varphi : \underline{\mathbb{R}} \rightarrow \underline{\mathbb{R}}$ being invertible (denoted by $\varphi^{-1}(|\underline{\mathbf{v}}|)$), then the corresponding dual quaternion is:

$$\pm \hat{\underline{\mathbf{q}}}(\underline{\mathbf{v}}) = \frac{(a(\underline{\mathbf{v}}) + \underline{\mathbf{v}})^n}{\sqrt{(|a(\underline{\mathbf{v}})|^2 + |\underline{\mathbf{v}}|^2)^n}} \tag{61}$$

where $a(\underline{\mathbf{v}}) = |\underline{\mathbf{v}}| \cot \frac{\varphi^{-1}(|\underline{\mathbf{v}}|) + 2k\pi}{2n}, k \in \{0, 1, \dots, n - 1\}$.

Proof. According to Equation (61) the following expression is true: $a(\underline{\mathbf{v}}) = |\underline{\mathbf{v}}| \cot \frac{\alpha + 2k\pi}{2n}$. This expression combined with $\underline{\alpha} = \varphi^{-1}(|\underline{\mathbf{v}}|)$ proves the theorem. □

Theorem 17. If $\underline{\mathbf{v}} = \varphi(\underline{\alpha})\underline{\mathbf{u}}$ represents a vectorial parameterization of displacement, then the associated dual quaternion is:

$$\pm \hat{\underline{\mathbf{q}}}(\underline{\mathbf{v}}) = \frac{1}{\sqrt{(1 + \underline{w}^2)^n}} [p_n(\underline{w}) + q_n(\underline{w})\underline{\mathbf{v}}] \tag{62}$$

where $\underline{w} = \frac{|\underline{\mathbf{v}}|}{a(\underline{\mathbf{v}})}$ and

$$p_n(X) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} X^{2k} \tag{63}$$

$$q_n(X) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} X^{2k} \tag{64}$$

and $\lfloor . \rfloor$ represents the floor of a number and $\binom{n}{k}$ are binomial coefficients.

Corollary 5. The orthogonal dual tensor associated with Equation (63) is:

$$\underline{\mathbf{R}} = \underline{\mathbf{I}} + \frac{2p_n(\underline{w})q_n(\underline{w})}{(1 + \underline{w}^2)^n} \underline{\mathbf{v}} + \frac{2q_n^2(\underline{w})}{(1 + \underline{w}^2)^n} \underline{\mathbf{v}}^2 \tag{65}$$

where $\underline{w} = \frac{|\underline{\mathbf{v}}|}{a(\underline{\mathbf{v}})}$.

Selections for the dual function $a(\underline{\mathbf{v}})$ enable the possibility of recovering various vectorial parameterizations, including azimuthal projections [7], and have the potential for extension to encompass pose parameterizations as well.

1 Let the normal higher-order dual Rodrigues vector $\underline{\mathbf{v}} = 2n \tan \frac{\alpha}{2n} \underline{\mathbf{u}}$, $a(\underline{\mathbf{v}}) = 2n$

- $\pm \hat{\underline{\mathbf{q}}} = \frac{1}{\sqrt{(1+w^2)^n}} [p_n(w) + q_n(w)\underline{\mathbf{v}}]$, where $w = \frac{|\underline{\mathbf{v}}|}{2n}$
- $\underline{\mathbf{R}} = \underline{\mathbf{I}} + \frac{2p_n(w)q_n(w)}{(1+w^2)^n} \underline{\tilde{\mathbf{v}}} + \frac{2q_n^2(w)}{(1+w^2)^n} \underline{\tilde{\mathbf{v}}}^2$, where $w = \frac{|\underline{\mathbf{v}}|}{2n}$

These parameterizations have the following asymptotical behaviors:

- $\lim_{n \rightarrow \infty} \frac{(2n+\underline{\mathbf{v}})^n}{(\sqrt{4n^2+|\underline{\mathbf{v}}|^2})^n} = \exp\left(\frac{\alpha}{2}\right)$
- $\lim_{n \rightarrow \infty} \left[\underline{\mathbf{I}} + \frac{2p_n(w)q_n(w)}{(1+w^2)^n} \underline{\tilde{\mathbf{v}}} + \frac{2q_n^2(w)}{(1+w^2)^n} \underline{\tilde{\mathbf{v}}}^2 \right] = \exp\left(\underline{\tilde{\mathbf{v}}}\right)$.

2 Let the dual orthographic projection $\underline{\mathbf{v}} = \sin \frac{\alpha}{2} \underline{\mathbf{u}}$, $a(\underline{\mathbf{v}}) = \sqrt{1 - |\underline{\mathbf{v}}|^2}$ and $n = 1$:

- $\pm \hat{\underline{\mathbf{q}}}(\underline{\mathbf{v}}) = \sqrt{1 - |\underline{\mathbf{v}}|^2} + \underline{\mathbf{v}} \in \mathbb{U}$
- $\underline{\mathbf{R}}(\underline{\mathbf{v}}) = \underline{\mathbf{I}} + 2\sqrt{1 - |\underline{\mathbf{v}}|^2} \underline{\tilde{\mathbf{v}}} + 2\underline{\tilde{\mathbf{v}}}^2$.

3 Let the dual Lambert parameters $\underline{\mathbf{v}} = \sin \frac{\alpha}{4} \underline{\mathbf{u}}$, $a(\underline{\mathbf{v}}) = \sqrt{1 - |\underline{\mathbf{v}}|^2}$ and $n = 2$:

- $\pm \hat{\underline{\mathbf{q}}}(\underline{\mathbf{v}}) = 1 - 2|\underline{\mathbf{v}}|^2 + 2\sqrt{1 - |\underline{\mathbf{v}}|^2} \underline{\mathbf{v}} \in \mathbb{U}$
- $\underline{\mathbf{R}}(\underline{\mathbf{v}}) = \underline{\mathbf{I}} + 4(1 - |\underline{\mathbf{v}}|^2) \sqrt{1 - |\underline{\mathbf{v}}|^2} \underline{\tilde{\mathbf{v}}} + 8(1 - |\underline{\mathbf{v}}|^2) \underline{\tilde{\mathbf{v}}}^2$.

4 Let the dual Breusing parameters $\underline{\mathbf{v}} = \tan \frac{\alpha}{4} \sqrt{\cot \frac{\alpha}{4}} \underline{\mathbf{u}}$, $a(\underline{\mathbf{v}}) = \sqrt{\frac{\sqrt{4+|\underline{\mathbf{v}}|^4} - |\underline{\mathbf{v}}|^2}{2}}$ and $n = 2$:

- $\pm \hat{\underline{\mathbf{q}}}(\underline{\mathbf{v}}) = \frac{a^2(\underline{\mathbf{v}}) - |\underline{\mathbf{v}}|^2}{|a(\underline{\mathbf{v}})|^2 + |\underline{\mathbf{v}}|^2} + \frac{2a(\underline{\mathbf{v}})}{|a(\underline{\mathbf{v}})|^2 + |\underline{\mathbf{v}}|^2} \underline{\mathbf{v}}$
- $\underline{\mathbf{R}}(\underline{\mathbf{v}}) = \underline{\mathbf{I}} + \frac{4a(\underline{\mathbf{v}})[a^2(\underline{\mathbf{v}}) - |\underline{\mathbf{v}}|^2]}{(|a(\underline{\mathbf{v}})|^2 + |\underline{\mathbf{v}}|^2)^2} \underline{\tilde{\mathbf{v}}} + \frac{8a^2(\underline{\mathbf{v}})}{(|a(\underline{\mathbf{v}})|^2 + |\underline{\mathbf{v}}|^2)^2} \underline{\tilde{\mathbf{v}}}^2$.

5 Let the dual sin family parameters $\underline{\mathbf{v}} = \sin \frac{\alpha}{2n} \underline{\mathbf{u}}$; $a(\underline{\mathbf{v}}) = \sqrt{1 - |\underline{\mathbf{v}}|^2}$, $n \in \mathbb{N}^*$:

- $\pm \hat{\underline{\mathbf{q}}}(\underline{\mathbf{v}}) = \left(\sqrt{1 - |\underline{\mathbf{v}}|^2} + \underline{\mathbf{v}} \right)^n = P_n(|\underline{\mathbf{v}}|) + Q_n(|\underline{\mathbf{v}}|)\underline{\mathbf{v}}$
- $\underline{\mathbf{R}}(\underline{\mathbf{v}}) = \underline{\mathbf{I}} + 2P_n(|\underline{\mathbf{v}}|)Q_n(|\underline{\mathbf{v}}|)\underline{\tilde{\mathbf{v}}} + 2Q_n^2(|\underline{\mathbf{v}}|)\underline{\tilde{\mathbf{v}}}^2$.

where $P_n(|\underline{\mathbf{v}}|) = \left(\sqrt{1 - |\underline{\mathbf{v}}|^2} \right)^n p_n\left(\frac{|\underline{\mathbf{v}}|}{\sqrt{1 - |\underline{\mathbf{v}}|^2}}\right)$ and $Q_n(|\underline{\mathbf{v}}|) = \left(\sqrt{1 - |\underline{\mathbf{v}}|^2} \right)^{n-1} q_n\left(\frac{|\underline{\mathbf{v}}|}{\sqrt{1 - |\underline{\mathbf{v}}|^2}}\right)$, $p_n(|\underline{\mathbf{v}}|)$ and $q_n(|\underline{\mathbf{v}}|)$ being computed from Equations (63) and (64).

8. Kinematic Equation and Tangent Operator by Multiple Parameterizations of Motion

Let $\mathbf{Q} = \mathbf{Q}(t) \in S\mathbb{O}_3^{\mathbb{R}}$ and $\boldsymbol{\rho} = \boldsymbol{\rho}(t) \in V_3^{\mathbb{R}}$ represent the parametric expressions for any rigid body motion.

The dual tensor function $\underline{\mathbf{R}}(t) = \left(\mathbf{I} + \varepsilon \tilde{\boldsymbol{\rho}}(t) \right) \mathbf{Q}(t)$, whererepresents the time variable, which can be parameterized by a curve in $S\mathbb{O}_3$. Consider $\underline{\mathbf{l}}_0$ which contains the Plücker coordinates of a line feature at the time $t = t_0$. At a specific time instant t , the line undergoes a transformation to become:

$$\underline{\mathbf{l}}(t) = \underline{\mathbf{R}}(t)\underline{\mathbf{l}}_0 \quad (66)$$

Theorem 18. The velocity dual tensor function $\underline{\Phi}$ is defined by the equation:

$$\dot{\underline{\mathbf{l}}} = \underline{\Phi} \underline{\mathbf{l}}, \forall \underline{\mathbf{l}} \in V_3^{\mathbb{R}} \quad (67)$$

which gives:

$$\underline{\Phi} = \dot{\underline{\mathbf{R}}} \underline{\mathbf{R}}^T \quad (68)$$

Consider $\underline{\Phi} = \dot{\underline{\mathbf{R}}} \underline{\mathbf{R}}^T$, which leads to $\dot{\underline{\mathbf{R}}} \underline{\mathbf{R}}^T + \underline{\mathbf{R}} \dot{\underline{\mathbf{R}}}^T = \underline{\mathbf{0}}$, equivalent to $\underline{\Phi} = -\underline{\Phi}^T$. This equivalence demonstrates that $\underline{\Phi} \in \underline{S\mathbb{O}}_3^{\mathbb{R}}$.

The dual vector $\underline{\omega} = \text{vect} \underline{\mathbf{R}} \underline{\mathbf{R}}^T$, also known as dual angular velocity of the rigid body or dual spatial twist, can be expressed as:

$$\underline{\omega} = \underline{\omega} + \underline{\varepsilon} \mathbf{v} \quad (69)$$

The pair $(\underline{\omega}, \mathbf{v})$ is commonly referred to as the spatial twist of the rigid body. Here, $\underline{\omega}$ represents the instantaneous angular velocity of the rigid body and the velocity of spatial twist; $\mathbf{v} = \dot{\underline{\rho}} - \underline{\omega} \times \underline{\rho}$ represents the linear velocity of the point of the body that coincides at any given time with the origin of the reference frame.

The following theorem enables the reconstruction of the rigid body motion based on the knowledge of the spatial twist of the rigid body at any given time, which is equivalent to having knowledge of the time-dependent dual angular velocity function.

Theorem 19. For a continuous dual function $\underline{\omega} \in V_3^{\mathbb{R}}$, there exists a unique dual tensor $\underline{\mathbf{R}} \in \underline{S\mathbb{O}}_3^{\mathbb{R}}$ such that:

$$\begin{cases} \dot{\underline{\mathbf{R}}} = \underline{\tilde{\omega}} \underline{\mathbf{R}} \\ \underline{\mathbf{R}}(t_0) = \underline{\mathbf{R}}_0, \underline{\mathbf{R}}_0 \in \underline{S\mathbb{O}}_3 \end{cases} \quad (70)$$

Proof. The initial value problem Equation (70) provides a unique solution if $\underline{\tilde{\omega}} = \underline{\tilde{\omega}}(t)$ is a continuous function.

Consider $\underline{\mathbf{R}}^T$, the transpose of dual tensor $\underline{\mathbf{R}}$. Computing

$$\frac{d}{dt} (\underline{\mathbf{R}} \underline{\mathbf{R}}^T) = \dot{\underline{\mathbf{R}}} \underline{\mathbf{R}}^T + \underline{\mathbf{R}} \dot{\underline{\mathbf{R}}}^T = \underline{\tilde{\omega}} \underline{\mathbf{R}} \underline{\mathbf{R}}^T - \underline{\mathbf{R}} \underline{\mathbf{R}}^T \underline{\tilde{\omega}} = \underline{\mathbf{0}} \quad (71)$$

shows that:

$$\underline{\mathbf{R}} \underline{\mathbf{R}}^T = \underline{\mathbf{R}} \underline{\mathbf{R}}^T(t_0) = \underline{\mathbf{I}} \quad (72)$$

From Equation (72) it can be seen that $\det(\underline{\mathbf{R}}) \in \{-1, 1\}$. As $\det(\underline{\mathbf{R}}(t_0)) = \det \underline{\mathbf{R}}_0 = 1$, we see that:

$$\begin{cases} \underline{\mathbf{R}} \underline{\mathbf{R}}^T = \underline{\mathbf{I}} \\ \det(\underline{\mathbf{R}}) = 1 \end{cases} \quad (73)$$

Hence, tensor $\underline{\mathbf{R}} \in \underline{S\mathbb{O}}_3^{\mathbb{R}}$ qualifies as a dual orthogonal tensor map.

The orthogonal dual tensor $\underline{\mathbf{R}}$ fully models the six-degree-of-freedom rigid body motion, Theorem 19 represents the dual form of kinematic equation. \square

Corollary 6. If we consider the dual angular velocity in the body frame, $\underline{\omega}^B = \underline{\mathbf{R}}^T \underline{\omega}$, thus transforming Equation (70) into:

$$\begin{cases} \dot{\underline{\mathbf{R}}} = \underline{\mathbf{R}} \underline{\tilde{\omega}}^B \\ \underline{\mathbf{R}}(t_0) = \underline{\mathbf{R}}_0, \underline{\mathbf{R}}_0 \in \underline{S\mathbb{O}}_3 \end{cases} \quad (74)$$

The dual tensor initial value problems Equation (70) and Equation (74) can be expressed as 18 separate real differential equations.

Corollary 7. Let $\hat{\mathbf{q}} \in \mathbb{U}^{\mathbb{R}}$ be a dual quaternion parameterization of rigid body motion, such that $\Delta(\hat{\mathbf{q}}) = \mathbf{R}$. The dual Poisson–Darboux problems Equations (70) and (74) can be considered as equivalent to:

$$\begin{cases} \dot{\hat{\mathbf{q}}} = \frac{1}{2}\boldsymbol{\omega}\hat{\mathbf{q}} \\ \hat{\mathbf{q}}(t_0) = \hat{\mathbf{q}}_0 \end{cases} \tag{75}$$

and

$$\begin{cases} \dot{\hat{\mathbf{q}}} = \frac{1}{2}\hat{\mathbf{q}}\boldsymbol{\omega}^B \\ \hat{\mathbf{q}}(t_0) = \hat{\mathbf{q}}_0 \end{cases} \tag{76}$$

where $\Delta(\hat{\mathbf{q}}_0) = \mathbf{R}_0$.

Initial value problems Equations (75) and (76) are equivalent to eight real differential equations.

If the **natural** invariant of the orthogonal dual tensor map $\mathbf{R} = \mathbf{R}(t)$ is denoted by $\underline{\alpha} = \underline{\alpha}(t)$ and $\underline{\mathbf{u}} = \underline{\mathbf{u}}(t)$, the following equations result, by Equations (23), (75), and (76):

$$\dot{\underline{\alpha}} = \underline{\boldsymbol{\omega}} \cdot \underline{\mathbf{u}} = \underline{\boldsymbol{\omega}}^B \cdot \underline{\mathbf{u}}, \tag{77}$$

$$\dot{\underline{\mathbf{u}}} = \underline{\boldsymbol{\Theta}} \underline{\boldsymbol{\omega}} = \underline{\boldsymbol{\Theta}}^T \underline{\boldsymbol{\omega}}^B, \tag{78}$$

where the dual tensor $\underline{\boldsymbol{\Theta}} = -\frac{1}{2} [\tilde{\underline{\mathbf{u}}} + \cot(\frac{1}{2}\alpha)\tilde{\underline{\mathbf{u}}}^2]$.

The dual twist of the rigid body in space and in the body frame is expended, respectively, using Equations (23), (75), and (76) by equations:

$$\underline{\boldsymbol{\omega}} = \dot{\underline{\alpha}} \underline{\mathbf{u}} + \sin \alpha \dot{\underline{\mathbf{u}}} + (1 - \cos \alpha) \underline{\mathbf{u}} \times \dot{\underline{\mathbf{u}}} \tag{79}$$

$$\underline{\boldsymbol{\omega}}^B = \dot{\underline{\alpha}} \underline{\mathbf{u}} + \sin \alpha \dot{\underline{\mathbf{u}}} - (1 - \cos \alpha) \underline{\mathbf{u}} \times \dot{\underline{\mathbf{u}}} \tag{80}$$

Consider $\underline{\alpha} \in V_3^{\mathbb{R}}$ a Euler dual vector such that $\mathbf{R} = \exp(\tilde{\underline{\alpha}})$.

According to the Equations (23), (77), and (78) we find:

$$\underline{\boldsymbol{\omega}} = dexp_{\underline{\alpha}} \dot{\underline{\alpha}}, \tag{81}$$

where the $dexp_{\underline{\alpha}}$ is a tangent tensor:

$$dexp_{\underline{\alpha}} = \mathbf{I} + \frac{1}{2} \text{sinc}^2 \frac{|\underline{\alpha}|}{2} \tilde{\underline{\alpha}} + (1 - \text{sinc}|\underline{\alpha}|) \frac{\tilde{\underline{\alpha}}^2}{|\underline{\alpha}|^2} \tag{82}$$

In Equation (82) $\text{sinc}|\underline{\alpha}| = \begin{cases} \frac{\sin|\underline{\alpha}|}{|\underline{\alpha}|} \text{ if } \text{Re}\underline{\alpha} \neq \mathbf{0} \\ 1 \text{ if } \text{Re}\underline{\alpha} = \mathbf{0} \end{cases}$.

The Poisson–Darboux problem Equation (70) corresponds to:

$$\begin{cases} \dot{\underline{\alpha}} = dexp_{\underline{\alpha}}^{-1} \underline{\boldsymbol{\omega}} \\ \underline{\alpha}(t_0) = \underline{\alpha}_0 \end{cases} \tag{83}$$

where $\exp \tilde{\underline{\alpha}}_0 = \mathbf{R}_0$, and tensor $dexp_{\underline{\alpha}}^{-1}$, if the real part of $\underline{\alpha}$ is different by 2π , we find that:

$$dexp_{\underline{\alpha}}^{-1} = \mathbf{I} - \frac{1}{2} \tilde{\underline{\alpha}} + \left(1 - \frac{|\underline{\alpha}|}{2} \cot \frac{|\underline{\alpha}|}{2}\right) \frac{\tilde{\underline{\alpha}}^2}{|\underline{\alpha}|^2} \tag{84}$$

The kinematic equation problem from Equation (74) is equivalent to:

$$\begin{cases} \dot{\underline{\alpha}} = dexp_{\underline{\alpha}}^{-T} \underline{\omega}^B \\ \underline{\alpha}(t_0) = \underline{\alpha}_0 \end{cases} \quad (85)$$

Theorem 20. If \underline{v} is a higher-order Rodrigues dual vector, there are two dual tensors so that:

$$\begin{cases} \underline{\omega} = dcay_n \dot{\underline{v}} \\ \underline{\omega}^B = dcay_n^T \dot{\underline{v}} \end{cases} \quad (86)$$

$$\begin{cases} \dot{\underline{v}} = dcay_n^{-1} \underline{\omega} \\ \dot{\underline{v}} = dcay_n^{-T} \underline{\omega}^B \end{cases} \quad (87)$$

The tangent operator $dcay_n$ and the kinematic tensor $dcay_n^{-1}$ are given by the following closed form equations:

$$\begin{aligned} dcay_n &= \frac{2q_n(|\underline{v}|)p_n(|\underline{v}|)}{(1+|\underline{v}|^2)^n} \mathbf{I} + \frac{2q_n^2(|\underline{v}|)}{(1+|\underline{v}|^2)^n} \tilde{\underline{v}} \\ &+ 2 \frac{n(1+|\underline{v}|^2)^{n-1} - q_n(|\underline{v}|)p_n(|\underline{v}|)}{|\underline{v}|^2(1+|\underline{v}|^2)^n} \underline{v} \otimes \underline{v} \end{aligned} \quad (88)$$

$$dcay_n^{-1} = \frac{p_n(|\underline{v}|)}{2q_n(|\underline{v}|)} \mathbf{I} - \frac{1}{2} \tilde{\underline{v}} + \frac{(1+|\underline{v}|^2)q_n(|\underline{v}|) - np_n(|\underline{v}|)}{2n|\underline{v}|^2q_n(|\underline{v}|)} \underline{v} \otimes \underline{v} \quad (89)$$

where $cay_n \underline{v} = \underline{R}$, and the polynomials p_n, q_n are given by the Equations (51) and (52).

Consider $\underline{v} \in \mathbb{V}_3^{\mathbb{R}}$ such that $\underline{R} = cay_n \underline{v}$. Taking into account Equation (86), the problems (70) and (74) are equivalent to :

$$\begin{cases} \dot{\underline{v}} = dcay_n^{-1} \underline{\omega} \\ \underline{v}(t_0) = \underline{v}_0 \end{cases}, \quad (90)$$

$$\begin{cases} \dot{\underline{v}} = dcay_n^{-T} \underline{\omega}^B \\ \underline{v}(t_0) = \underline{v}_0 \end{cases} \quad (91)$$

where $cay_n \underline{v}_0 = \underline{R}_0$.

The initial value problems (83), (85), (90), and (91) are equivalent to six real differential equations and are the minimal parameterizations of dual Poisson–Darboux problems (70) and (74). According to the authors' knowledge, this paper presents it for the first time.

9. Application of Minimal Parameterization of Rigid Body Displacement and Motion Using Higher-Order Rodrigues Dual Vector

To assess the practicality of the iterative formulations, we present polynomials from the second to fifth order, alongside the corresponding unit dual quaternion, orthogonal dual tensor, tangent, and kinematic dual tensor:

- First order:

$$\underline{v} = \tan \frac{\alpha}{2} \underline{u}; \quad (92)$$

$$p_1(X) = 1; \quad (93)$$

$$q_1(X) = 1; \quad (94)$$

$$\pm \hat{\underline{q}} = \frac{1}{\sqrt{1+|\underline{v}|^2}} [1 + \underline{v}]; \quad (95)$$

$$\underline{\mathbf{R}} = \underline{\mathbf{I}} + \frac{2}{1 + |\underline{\mathbf{v}}|^2} \left[\underline{\tilde{\mathbf{v}}} + \underline{\tilde{\mathbf{v}}}^2 \right]; \tag{96}$$

$$\text{dcay}_1 = \frac{2}{1 + |\underline{\mathbf{v}}|^2} \left[\underline{\mathbf{I}} + \underline{\tilde{\mathbf{v}}} \right]; \tag{97}$$

$$\text{dcay}_1^{-1} = \frac{1}{2} \underline{\mathbf{I}} - \frac{1}{2} \underline{\tilde{\mathbf{v}}} + \frac{1}{2} \underline{\mathbf{v}} \otimes \underline{\mathbf{v}} \tag{98}$$

- Second order:

$$\underline{\mathbf{v}} = \tan \frac{\alpha + 2k\pi}{4} \underline{\mathbf{u}}; k = \overline{0, 1}; \tag{99}$$

$$p_2(X) = 1 - X^2; \tag{100}$$

$$q_2(X) = 2 \tag{101}$$

$$\pm \underline{\hat{\mathbf{q}}} = \frac{1}{1 + |\underline{\mathbf{v}}|^2} \left[1 - |\underline{\mathbf{v}}|^2 + 2\underline{\mathbf{v}} \right] \tag{102}$$

$$\underline{\mathbf{R}} = \underline{\mathbf{I}} + \frac{4}{(1 + |\underline{\mathbf{v}}|^2)^2} \left[(1 - |\underline{\mathbf{v}}|^2) \underline{\tilde{\mathbf{v}}} + 2\underline{\tilde{\mathbf{v}}}^2 \right] \tag{103}$$

$$\text{dcay}_2 = \frac{4}{(1 + |\underline{\mathbf{v}}|^2)^2} \left[(1 - |\underline{\mathbf{v}}|^2) \underline{\mathbf{I}} + 2\underline{\tilde{\mathbf{v}}} + 2\underline{\mathbf{v}} \otimes \underline{\mathbf{v}} \right] \tag{104}$$

$$\text{dcay}_2^{-1} = \frac{1 - |\underline{\mathbf{v}}|^2}{2} \underline{\mathbf{I}} - \frac{1}{2} \underline{\tilde{\mathbf{v}}} + \frac{1}{2} \underline{\mathbf{v}} \otimes \underline{\mathbf{v}} \tag{105}$$

- Third order:

$$\underline{\mathbf{v}} = \tan \frac{\alpha + 2k\pi}{6} \underline{\mathbf{u}}; k = \overline{0, 2} \tag{106}$$

$$p_3(X) = 1 - 3X^2; \tag{107}$$

$$q_3(X) = 3 - X^2; \tag{108}$$

$$\pm \underline{\hat{\mathbf{q}}} = \frac{1}{\sqrt{(1 + |\underline{\mathbf{v}}|^2)^3}} \left[1 - 3|\underline{\mathbf{v}}|^2 + (3 - |\underline{\mathbf{v}}|^2) \underline{\mathbf{v}} \right] \tag{109}$$

$$\underline{\mathbf{R}} = \underline{\mathbf{I}} + \frac{2(3 - |\underline{\mathbf{v}}|^2)}{(1 + |\underline{\mathbf{v}}|^2)^3} \left[(1 - 3|\underline{\mathbf{v}}|^2) \underline{\tilde{\mathbf{v}}} + (3 - |\underline{\mathbf{v}}|^2) \underline{\tilde{\mathbf{v}}}^2 \right] \tag{110}$$

$$\begin{aligned} \text{dcay}_3 = & \frac{2}{(1 + |\underline{\mathbf{v}}|^2)^3} \left[(1 - 3|\underline{\mathbf{v}}|^2) (3 - |\underline{\mathbf{v}}|^2) \underline{\mathbf{I}} \right. \\ & \left. + (3 - |\underline{\mathbf{v}}|^2)^2 \underline{\tilde{\mathbf{v}}} + (16 + 9|\underline{\mathbf{v}}|^2) \underline{\mathbf{v}} \otimes \underline{\mathbf{v}} \right] \end{aligned} \tag{111}$$

$$\text{dcay}_3^{-1} = \frac{1 - 3|\underline{\mathbf{v}}|^2}{3 - |\underline{\mathbf{v}}|^2} \underline{\mathbf{I}} - \frac{1}{2} \underline{\tilde{\mathbf{v}}} + \frac{11 - |\underline{\mathbf{v}}|^2}{6(3 - |\underline{\mathbf{v}}|^2)} \underline{\mathbf{v}} \otimes \underline{\mathbf{v}} \tag{112}$$

- Fourth order:

$$\mathbf{v} = \tan \frac{\alpha + 2k\pi}{8} \mathbf{u}; k = \overline{0,3}; \tag{113}$$

$$p_4(X) = 1 - 6X^2 + X^4; \tag{114}$$

$$q_4(X) = 4 - 4X^2; \tag{115}$$

$$\pm \hat{\mathbf{q}} = \frac{1}{(1 + |\mathbf{v}|^2)^2} [1 - 6|\mathbf{v}|^2 + |\mathbf{v}|^4 + 4(1 - |\mathbf{v}|^2)\mathbf{v}]; \tag{116}$$

$$\mathbf{R} = \mathbf{I} + \frac{8(1 - |\mathbf{v}|^2)}{(1 + |\mathbf{v}|^2)^4} [(1 - 6|\mathbf{v}|^2 + |\mathbf{v}|^4)\tilde{\mathbf{v}} + 4(1 - |\mathbf{v}|^2)\tilde{\mathbf{v}}^2] \tag{117}$$

$$\text{dcay}_4 = \frac{8}{(1 + |\mathbf{v}|^2)^4} [(1 - 6|\mathbf{v}|^2 + |\mathbf{v}|^4)(1 - |\mathbf{v}|^2)\mathbf{I} + 4(1 - |\mathbf{v}|^2)\tilde{\mathbf{v}} + (|\mathbf{v}|^4 - 3|\mathbf{v}|^2 + 10)\mathbf{v} \otimes \mathbf{v}] \tag{118}$$

$$\text{dcay}_4^{-1} = \frac{1 - 6|\mathbf{v}|^2 + |\mathbf{v}|^4}{8(1 - |\mathbf{v}|^2)} \mathbf{I} - \frac{1}{2} \tilde{\mathbf{v}} - \frac{8|\mathbf{v}|^2 + 3}{8(1 - |\mathbf{v}|^2)} \mathbf{v} \otimes \mathbf{v} \tag{119}$$

- Fifth order:

$$\mathbf{v} = \tan \frac{\alpha + 2k\pi}{10} \mathbf{u}; k = \overline{0,4}; \tag{120}$$

$$p_5(X) = 1 - 10X^2 + 5X^4$$

$$q_5(X) = 5 - 10X^2 + X^4; \tag{121}$$

$$\pm \hat{\mathbf{q}} = \frac{1}{\sqrt{(1 + |\mathbf{v}|^2)^5}} [1 - 10|\mathbf{v}|^2 + 5|\mathbf{v}|^4 + (5 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)\mathbf{v}]; \tag{122}$$

$$\mathbf{R} = \mathbf{I} + \frac{2(5 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)}{(1 + |\mathbf{v}|^2)^5} [(1 - 10|\mathbf{v}|^2 + 5|\mathbf{v}|^4)\tilde{\mathbf{v}} + (5 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)\tilde{\mathbf{v}}^2]; \tag{123}$$

$$\text{dcay}_5 = \frac{2}{(1 + |\mathbf{v}|^2)^5} [(1 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)(5 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)\mathbf{I} + (5 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)\tilde{\mathbf{v}} + (40 - 48|\mathbf{v}|^2 + 40|\mathbf{v}|^4)\mathbf{v} \otimes \mathbf{v}] \tag{124}$$

$$\text{dcay}_5^{-1} = \frac{1 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4}{2(5 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)} \mathbf{I} - \frac{1}{2} \tilde{\mathbf{v}} + \frac{45 - 16|\mathbf{v}|^2 + |\mathbf{v}|^4}{10(5 - 10|\mathbf{v}|^2 + |\mathbf{v}|^4)} \mathbf{v} \otimes \mathbf{v} \tag{125}$$

Equations (92)–(125) completely solve in closed form and a coordinate-free way the minimal parameterization of rigid body displacement and motion, employing a higher-order Rodrigues dual vector for $n \in \{1, 2, 3, 4, 5\}$.

For example, in [54], for the Rodrigues dual vector ($n = 1$), the real and dual part components are obtained (dual Rodrigues parameters):

$$\mathbf{b}_1 = \tan \frac{\alpha}{2} \mathbf{u}_1 \tag{126}$$

$$\mathbf{b}_2 = \tan \frac{\alpha}{2} \mathbf{u}_2 \quad (127)$$

$$\mathbf{b}_3 = \tan \frac{\alpha}{2} \mathbf{u}_3 \quad (128)$$

$$\mathbf{b}_1^0 = d[1 + (\tan \frac{\alpha}{2})^2] \mathbf{u}_1^0 \quad (129)$$

$$\mathbf{b}_2^0 = d[1 + (\tan \frac{\alpha}{2})^2] \mathbf{u}_2^0 \quad (130)$$

$$\mathbf{b}_3^0 = d[1 + (\tan \frac{\alpha}{2})^2] \mathbf{u}_3^0, \quad (131)$$

which were obtained after rather laborious reasoning. Equation (92): $\underline{\mathbf{b}} = (\tan \frac{\alpha}{2}) \underline{\mathbf{u}}$, for $\underline{\alpha} = \alpha + \varepsilon d$ and a dual unit vector $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$, compacted succinctly includes all six equations presented in [54]. Equations (95)–(98) present, respectively, the dual orthogonal tensor, dual quaternion, dual tangent tensor, and dual kinematic tensor.

10. Conclusions

This paper proposes a novel approach to the minimal pose parameterization technique using higher-order modified Cayley transforms. Our research is grounded on the properties of maps connecting dual vectors to orthogonal dual tensors and unitary dual quaternions. The evolved parameterization approach is comprehensive, coordinate-free, presented in closed form, and incorporates various previously documented attitude parameterization Cayley maps while extending them toward pose parameterization. The expressions rely solely on dual vector algebra and have no transcendent functions. We also provide a unitary method for dual kinematic equations (the Poisson–Darboux problem) via a dual tangent operator of higher-order fractional Cayley transforms for dual vector parameterization by rigid body motion. A general minimal parameterization of the kinematic equation of rigid body motion is presented, according to the authors' knowledge, for the first time in this paper. In further work, we will research what can be changed if we replace dual Lie algebra with dual mock Lie algebra [59,60] and how to define dual triple systems [61,62].

Author Contributions: The individual contributions of the authors are the following: conceptualization, D.C.; methodology, D.C.; data curation, D.C. and I.P.; writing—original draft preparation, I.P.; writing—review and editing, I.P.; visualization, I.P.; supervision, D.C.; project administration, D.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Nomenclature

$\underline{\mathbb{R}}(\mathbb{R})$	dual (real) numbers set
$\underline{a}(a)$	dual (real) number
$\underline{V}_3(V_3)$	dual (real) vectors set
$\underline{\mathbf{a}}(\mathbf{a})$	dual (real) vector
$L(\underline{V}_3, \underline{V}_3)$	Euclidean dual tensor set
$\underline{\mathbf{A}}(A)$	dual (real) tensor

$\tilde{\mathbf{a}}$	skew-symmetric dual tensor corresponding to the dual vector \mathbf{a}
$\underline{SO}_3(SO_3)$	orthogonal Euclidean dual (real) orthogonal tensor set
$\underline{V}_3^{\mathbb{R}}(V_3^{\mathbb{R}})$	Time-depending dual (real) vectorial functions
$\underline{SO}_3^{\mathbb{R}}(SO_3^{\mathbb{R}})$	Time-depending Euclidean dual (real) orthogonal tensorial functions

References

- Davidson, J.; Hunt, K. Robots and Screw Theory applications of kinematics and statistics to robotics. *J. Mech. Des.* **2004**, *126*, 763–764. [\[CrossRef\]](#)
- Cui, L.; Dai, J. A coordinate-free approach to instantaneous kinematics of two rigid objects with rolling contact and its implications for trajectory planning. In Proceedings of the IEEE International Conference on Robotics and Automation, Kobe, Japan, 12–17 May 2009; pp. 612–617.
- Sariyildiz, E.; Temeltas, H. A new formulation method for solving kinematic problems of multiarm robot systems using quaternion algebra in the screw theory framework. *Turk. J. Electr. Eng. Comput. Sci.* **2012**, *20*, 607–628. [\[CrossRef\]](#)
- Dahmouche, R.; Andreff, N.; Mezouar, Y.; Ait-Aider, O.; Martinet, P. Dynamic visual servoing from sequential regions of interest acquisition. *Int. J. Robot. Res.* **2012**, *31*, 520–537. [\[CrossRef\]](#)
- Han, D.-P.; Wei, Q.; Li, Z.-X. Kinematic control of free rigid bodies using dual quaternions. *Int. J. Autom. Comput.* **2008**, *5*, 319–324.
- Muller, A. Group theoretical approaches to vector parameterization of rotations. *J. Geom. Symmetry Phys.* **2010**, *19*, 43–72.
- Tanygin, S. Attitude Parameterizations as Higher-Dimensional Map Projections. *J. Guid. Control Dyn.* **2012**, *35*, 13–24.
- Angeles, J. *Fundamentals of Robotic Mechanical Systems Theory, Methods, and Algorithms*, 5th ed.; Springer: Berlin/Heidelberg, Germany, 2014.
- Segal, S.; Gurfil, P. Effect of Kinematic Rotation-Translation Coupling on Relative Spacecraft Translational Dynamics. *J. Guid. Control Dyn.* **2009**, *32*, 1045–1050. [\[CrossRef\]](#)
- Daniilidis, K. Hand-Eye Calibration Using Dual Quaternions. *Int. J. Robot. Res.* **1999**, *18*, 286–298. [\[CrossRef\]](#)
- Zhao, Z.; Liu, Y. A hand—eye calibration algorithm based on screw motions. *Robotica* **2009**, *27*, 217–223.
- Condurache, D.; Ciureanu, I. A novel solution for AX=YB sensor calibration problem using dual Lie algebra. In Proceedings of the 2019 6th International Conference on Control, Decision and Information Technologies (CoDIT), Paris, France, 23–26 April 2019; pp. 302–307.
- Yang, X.; Wu, H.; Li, Y.; Kang, S.; Chen, B. Computationally Efficient Inverse Dynamics of a Class of Six-DOF Parallel Robots: Dual Quaternion Approach. *J. Intell. Robot. Syst. Theory Appl.* **2019**, *94*, 101–113.
- Wang, J.; Liang, H.; Sun, Z.; Zhang, S.; Liu, M. Finite-time control for spacecraft formation with dual number-based description. *J. Guid. Control Dyn.* **2012**, *35*, 950–962. [\[CrossRef\]](#)
- Condurache, D.; Burlacu, A. On six D.O.F relative orbital motion parameterization using rigid bases of dual vectors. *Adv. Astronaut. Sci.* **2013**, *150*, 2293–2312.
- O’Keefe, S.; Schaub, H. Shadow set considerations for modified Rodrigues parameter attitude filtering. In Proceedings of the AAS/AIAA Astrodynamics Specialist Conference, Hilton Head, SC, USA, 11–15 August 2013; pp. 1–11.
- Filipe, N.; Tsiotras, P. Adaptive model-independent tracking of rigid body position and attitude motion with mass and inertia matrix identification using dual quaternions. In Proceedings of the AIAA Guidance, Navigation, and Control Conference, Boston, MA, USA, 19–22 August 2013; pp. 1–15.
- Valverde, A.; Tsiotras, P. Spacecraft Trajectory Tracking with Identification of Mass Properties Using Dual Quaternions. In Proceedings of the 2018 AIAA Guidance, Navigation, and Control Conference, Kissimmee, FL, USA, 8–12 January 2018.
- Condurache, D.; Burlacu, A. Onboard Exact Solution to the Full-Body Relative Orbital Motion Problem. *J. Guid. Control Dyn.* **2016**, *39*, 2638–2648.
- Deng, Y.; Wang, Z. Modeling and Control for Spacecraft Relative Pose Motion by Representation. *J. Guid. Control Dyn.* **2016**, *39*, 1147–1154. [\[CrossRef\]](#)
- Condurache, D. Singularity-Free Extraction of a Dual Quaternion from Feature-Based Representation of Motion. *Adv. Astronaut. Sci.* **2019**, *171*, 4005–4024.
- Deng, Y.; Wang, Z.; Liu, L. Unscented Kalman filter for spacecraft pose estimation using twistors. *J. Guid. Control Dyn.* **2016**, *39*, 1844–1856.
- Wu, D.; Wang, Z. Strapdown Inertial Navigation System Algorithms Based on Geometric Algebra. *Adv. Appl. Clifford Algebras* **2012**, *22*, 1151–1167. [\[CrossRef\]](#)
- Gui, H.; Vukovich, G. Dual-quaternion-based adaptive motion tracking of spacecraft with reduced control effort. *Nonlinear Dyn.* **2016**, *83*, 597–614.
- Yang, J.; Stoll, E. Adaptive Sliding Mode Control for Spacecraft Proximity Operations Based on Dual Quaternions. *J. Guid. Control Dyn.* **2019**, *42*, 2356–2368.
- Zhang, F.; Duan, G. Robust Integrated Translation and Rotation Finite-Time Maneuver of a Rigid Spacecraft Based on Dual Quaternion. In Proceedings of the AIAA Guidance, Navigation, and Control Conference 2011, Portland, OR, USA, 8–11 August 2011.

27. Razgus, B.; Mooij, E.; Choukroun, D. Relative Navigation in Asteroid Missions Using Dual Quaternion Filtering. *J. Guid. Control Dyn.* **2017**, *40*, 2151–2166. [[CrossRef](#)]
28. Zivan, Y.; Choukroun, D. Dual Quaternion Kalman Filters for Spacecraft Relative Navigation. In Proceedings of the 2018 AIAA Guidance, Navigation, and Control Conference, AIAA Paper 2018-1347, Kissimmee, FL, USA, 8–12 January 2018.
29. Lee, U.; Mesbahi, M. Optimal Power Descent Guidance with 6-DOF Line of Sight Constraints via Unit Dual Quaternions. In Proceedings of the AIAA Guidance, Navigation, and Control Conference, Kissimmee, FL, USA, 5–9 January 2015.
30. Kwon, J.-W.; Lee, D.; Bang, H. Virtual Trajectory Augmented Landing Control Based on Dual Quaternion for Lunar Lander. *J. Guid. Control Dyn.* **2016**, *39*, 2044–2057. [[CrossRef](#)]
31. Dong, H.; Hu, Q.; Akella, M.R. Dual-Quaternion-Based Spacecraft Autonomous Rendezvous and Docking Under Six-Degree-of-Freedom Motion Constraints. *J. Guid. Control Dyn.* **2018**, *41*, 1150–1162. [[CrossRef](#)]
32. Antonello, A.; Valverde, A.; Tsiotras, P. Dynamics and Control of Spacecraft Manipulators with Thrusters and Momentum Exchange Devices. *J. Guid. Control Dyn.* **2019**, *42*, 15–29. [[CrossRef](#)]
33. Antonello, A.; Valverde, A.; Tsiotras, P. Free-Flying Spacecraft-Mounted Manipulators: A Tool for Simulating Dynamics and Control. In Proceedings of the AIAA Scitech 2019 Forum, San Diego, CA, USA, 7–11 January 2019.
34. Filipe, N.; Tsiotras, P. Adaptive Position and Attitude-Tracking Controller for Satellite Proximity Operations Using Dual Quaternions. *J. Guid. Control Dyn.* **2015**, *38*, 566–577. [[CrossRef](#)]
35. Angeles, J. The application of dual algebra to kinematic analysis. *Comput. Methods Mech. Syst.* **1998**, *161*, 3–31.
36. Pennestri, E.; Valentini, P. Linear dual algebra algorithms and their application to kinematics. *Multibody Dyn. Comput. Methods Appl.* **2009**, *12*, 207–229.
37. Condurache, D.; Ciureanu, I.-A. Baker–Campbell–Hausdorff–Dynkin Formula for the Lie Algebra of Rigid Body Displacements. *Mathematics* **2020**, *8*, 1185. [[CrossRef](#)]
38. Condurache, D. A Davenport dual angles approach for minimal parameterization of the rigid body displacement and motion. *Mech. Mach. Theory* **2019**, *140*, 104–122. [[CrossRef](#)]
39. Pennestri, E.; Valentini, P. Dual Quaternions as a Tool for Rigid Body Motion Analysis: A Tutorial with an Application to Biomechanics. *Arch. Mech. Eng.* **2010**, *57*, 187–205. [[CrossRef](#)]
40. Leclercq, G.; Lefèvre, P.; Blohm, G. 3D kinematics using dual quaternions: Theory and applications in neuroscience. *Front. Behav. Neurosci.* **2013**, *7*, 1–25. [[CrossRef](#)]
41. Perez, A.; McCarthy, J.M. Dual Quaternion Synthesis of Constrained Robotic Systems. *J. Mech. Des.* **2004**, *126*, 425–435. [[CrossRef](#)]
42. Condurache, D.; Burlacu, A. Orthogonal dual tensor method for solving the $AX = XB$ sensor calibration problem. *Mech. Mach. Theory* **2016**, *104*, 382–404. [[CrossRef](#)]
43. Bauchau, O.A.; Choi, J.-Y. The Vector Parameterization of Motion. *Nonlinear Dyn.* **2003**, *33*, 165–188. [[CrossRef](#)]
44. Bauchau, O.A.; Li, L. Tensorial Parameterization of Rotation and Motion. *J. Comput. Nonlinear Dyn.* **2011**, *6*, 31007. [[CrossRef](#)]
45. Condurache, D.; Burlacu, A. Dual tensors based solutions for rigid body motion parameterization. *Mech. Mach. Theory* **2014**, *74*, 390–412. [[CrossRef](#)]
46. Condurache, D.; Burlacu, A. Dual Lie algebra representations of the rigid body motion. In Proceedings of the SPACE 2014—Astrodynamics Specialist Conference, San-Diego, CA, USA, 4–7 August 2014; pp. 1–20.
47. Fischer, I. *Dual-Number Methods in Kinematics, Statics and Dynamics*; CRC Press: Boca Raton, FL, USA, 1999.
48. Yang, A.T.; Freudenstein, F. Application of Dual-Number Quaternion Algebra to the Analysis of Spatial Mechanisms. *J. Appl. Mech. Trans. ASME* **1964**, *31*, 300–308. [[CrossRef](#)]
49. Kavan, L.; Collins, S.; Žára, J.; O’Sullivan, C. Geometric skinning with approximate dual quaternion blending. *ACM Trans. Graph.* **2008**, *27*, 1–23. [[CrossRef](#)]
50. Valverde, A.; Tsiotras, P. Dual Quaternion Framework for Modeling of Spacecraft-Mounted Multibody Robotic Systems. *Front. Robot. AI* **2018**, *5*, 128. [[CrossRef](#)]
51. Reynolds, T.P.; Szmuk, M.; Malyuta, D.; Mesbahi, M.; Açıkmeşe, B.; Carson, J.M. Dual Quaternion-Based Powered Descent Guidance with State-Triggered Constraints. *J. Guid. Control Dyn.* **2020**, *43*, 1584–1599. [[CrossRef](#)]
52. Vasilescu, F. Quaternionic Cayley transform. *J. Funct. Anal.* **1999**, *164*, 134–162. [[CrossRef](#)]
53. Condurache, D.; Burlacu, A. Fractional order Cayley transforms for dual quaternions-based pose representation. *Adv. Astronaut. Sci.* **2016**, *156*, 1317–1339.
54. Karakılıç, İ. The Dual Rodrigues Parameters. *Int. J. Eng. Appl. Sci.* **2010**, *2*, 23–32.
55. Wang, Q.; Yu, W. Geometrically nonlinear analysis of composite beams using Wiener-Milenković parameters. *J. Renew. Sustain. Energy* **2017**, *9*, 33306. [[CrossRef](#)]
56. Selig, J. Cayley maps for SE(3). In Proceedings of the 12th IFToMM World Congress, Besançon, France, 17–20 June 2007; pp. 1–6.
57. Hurtado, J. Interior parameters, exterior parameters, and a Cayley-like transform. *J. Guid. Control Dyn.* **2009**, *32*, 653–657. [[CrossRef](#)]
58. Tsiotras, P.; Junkins, J.L.; Schaub, H. Higher-Order Cayley Transforms with Applications to Attitude Representations. *J. Guid. Control Dyn.* **1997**, *20*, 528–534. [[CrossRef](#)]
59. Baklouti, A.; Mansour, S. Unified products and matched pairs analysis of dual mock-Lie algebras. *Comput. Sci.* **2023**, *18*, 539–555.
60. Camacho, L.M.; Kaygorodov, I.; Lopatkin, V.; Salim, M.A. The variety of dual mock-Lie algebras. *Commun. Math.* **2020**, *28*, 161–178. [[CrossRef](#)]

61. Guo, S.; Wang, S.; Zhang, X. 3-Hom Lie Yang Baxter Equation and 3-Hom Lie Bialgebras. *Mathematics* **2022**, *10*, 2485. [[CrossRef](#)]
62. Baklouti, A. Quadratic Hom-Lie triple systems. *J. Geom. Phys.* **2017**, *121*, 166–175. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.