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Exploring Fuzzy Triple Controlled Metric Spaces: Applications in Integral Equations

Fatima M. Azmi

Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia; fazmi@psu.edu.sa

Abstract: In this article, we delve into the study of fuzzy triple controlled metric spaces, investigating their properties and presenting a range of illustrative examples. We emphasize the broader applicability of this concept in comparison to fuzzy rectangular metric spaces and fuzzy rectangular b-metric spaces. By introducing the novel concept of (α-ψ)-fuzzy contractive mappings, we derive fixed point results specifically designed for complete fuzzy triple controlled metric spaces. Our theorems extend and enrich previous findings in this field. Additionally, we demonstrate the practical significance of our study by applying our findings to the solution of an integral equation and providing an example of its application. Furthermore, we propose potential avenues for future research endeavors.

Keywords: fixed point; fuzzy rectangular metric space; controlled fuzzy metric space; α-admissible mapping; (α-ψ)-contraction; fuzzy triple controlled metric space; integral equations

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1. Introduction

The contraction principle, introduced by Banach in 1922 [1], has become a fundamental tool for proving fixed point results in metric spaces. This has spurred rapid growth and excitement in the field of fixed point theory, with its wide-ranging applications across various areas of mathematics. To extend the concept of classical metric spaces, Bakhtin [2] introduced b-metric spaces, which was followed by the proposal of extended b-metric spaces by Kamran et al. [3]. The notion of controlled metric type spaces was then introduced by Mlaiki et al. [4]. Abdeljawad et al. [5] expanded upon this concept, evolving it into double controlled metric type spaces. Building upon this foundation, Azmi [6] established fixed point results in double controlled metric type spaces by utilizing (α-ψ)-contractive mappings. More recently, Tasneem et al. [7] introduced triple controlled metric type spaces and derived their own fixed point results. Related works can be found in [8,9].

Everyday life presents challenges that often involve uncertain information beyond the scope of traditional mathematics. Two mathematical frameworks, fuzzy set theory pioneered by Zadeh in 1965 [10] and the theory of soft sets introduced by Molodstov in 1999 [11], offer specialized approaches for addressing such uncertainties. In the first framework, Zadeh introduced the theory of fuzzy sets [10] as an extension of traditional crisp sets, gaining significant interest due to its ability to address uncertainty and provide more accurate results than traditional crisp sets. This has led to the exploration of fuzzy sets in various mathematical disciplines, including topology, logic, analysis, algebra, and even artificial intelligence. In 1975, Kramosil and Michalek introduced fuzzy metric spaces [12], sparking further developments and extensions by subsequent researchers. Grabiec established fixed point results in fuzzy metric spaces, recognizing the importance of topological properties in metric spaces [13]. George and Veeramani then generalized and modified the concept of fuzzy metric spaces, illustrating the connection between fuzzy metrics and Hausdorff topologies [14]. Chugh and Kumar introduced fuzzy rectangular metric spaces [15], while Nadaban introduced fuzzy b-metric spaces [16], with fixed point results in fuzzy b-metric spaces provided by Kim et al. [17]. In 2017, Mehmood et al. introduced the concept of extended fuzzy b-metric spaces [18] and laid the groundwork for the contraction principle. Subsequently, in 2019, Mehmood et al. proceeded to introduce the concept of fuzzy rectangular b-metric spaces [19]. Expanding upon these concepts, Saleem et al. introduced the notion of extended rectangular fuzzy b-metric spaces and established fixed point results in their work [20]. Additionally, in [21], Saleem et al. introduced the concept of fuzzy double controlled metric spaces and demonstrated the application of the Banach contraction mapping principle in this context. Furthermore, Furqan et al. introduced the concept of fuzzy triple controlled metric spaces and established fixed point results within this framework [22].

Regarding the second framework, which deals with the theory of soft sets, Das and Samanta introduced the concepts of soft metric spaces, contributing significantly to this area [23–25]. Later, Beaula and Raja innovatively combined the notions of soft metric spaces and fuzzy metric spaces, resulting in the novel concept of a fuzzy soft metric space. They proceeded to formulate several concepts, leveraging the foundational principles of fuzzy soft sets [26]. Given that the Banach contraction principle is the cornerstone of fixed point theory, Sonam et al. introduced the notion of soft fuzzy contraction mappings and established fixed point results in soft fuzzy metric spaces [27] as well as in soft rectangular b-metric spaces [28].

Building upon the work of Samet et al. [29], who introduced the notion of α-admissible mappings and (α-ψ)-contractive mappings in metric spaces, Gopal and Vetro [30] extended the concept of (α-ψ)-contractive mappings to (α-ψ)-fuzzy contractive mappings in complete metric spaces, leading to the development of various fixed point theorems (see also [31]). Additionally, Azmi [32] used (α-ψ)-contractive mappings to establish fixed point results in fuzzy double controlled metric spaces.

In this article, we introduce the (α-ψ)-fuzzy contractive mappings in the context of fuzzy triple controlled metric spaces and derive corresponding fixed point theorems. We illustrate the effectiveness of our approach with examples and demonstrate its practical use in solving integral equations, providing a concrete example. Lastly, we offer recommendations for potential future research directions.

2. Preliminaries

In the subsequent sections, we will revisit certain concepts and definitions that are essential for our main results.

Definition 1 ([33])**.** *Consider a binary operation denoted as $* : [0,1]^2 \to [0,1]$. We designate $*$ as a continuous t-norm when it adheres to the following criteria:*

1. *$*$ is commutative and associative.*
2. *The binary operation $*$ is continuous.*
3. *$\eta * 1 = \eta$ holds for all $\eta \in [0,1]$.*
4. *$\eta_1 * \xi_1 \leq \eta_2 * \xi_2$, if $\eta_1 \leq \eta_2$ and $\xi_1 \leq \xi_2$, for all $\eta_1, \eta_2, \xi_1, \xi_2 \in [0,1]$.*

Next, we revisit the definition of a fuzzy rectangular metric space, as originally presented in [15].

Definition 2. *Consider a nonempty set denoted as $\mathcal{F}$. We define the notion of a fuzzy rectangular metric space using the triplet notation $(\mathcal{F}, \mathcal{S}, *)$. Here, $*$ represents a continuous t-norm, and $\mathcal{S}$ is a fuzzy set defined over $\mathcal{F}^2 \times (0, +\infty)$. This definition adheres to the following conditions, valid for all $\eta, \xi, x, y \in \mathcal{F}$:*
(F1) $\mathcal{S}(\eta, \xi, t) = 0$,
(F2) $\mathcal{S}(\eta, \xi, t) = 1, \forall\, t > 0 \leftrightarrow \eta = \xi$;
(F3) $\mathcal{S}(\eta, \xi, t) = \mathcal{S}(\xi, \eta, t)$, *exhibiting symmetry in η and ξ, $\forall\, t > 0$;*

(F4) $\mathcal{S}(\eta,\xi,.) : (0,+\infty) \to [0,1]$ *is left continuous, with* $\lim_{t\to\infty} \mathcal{S}(\eta,\xi,t) = 1$*;*
(F5) $\mathcal{S}(\eta,\xi,(t+s+w)) \geq \mathcal{S}(\eta,x,t) * \mathcal{S}(x,y,s) * \mathcal{S}(y,\xi,w)$*.*

These conditions collectively delineate the characteristics of the fuzzy rectangular metric space, elucidating the behavior of the fuzzy set $\mathcal{S}$ *in relation to its parameters and the continuous t-norm operation* $*$*.*

A broader concept within the realm of fuzzy rectangular metric spaces is the fuzzy rectangular b-metric space, as introduced by Mehmood et al. [19].

Definition 3. *Consider a nonempty set* $\mathcal{F}$ *and a real number* $b \geq 1$*. Let* $*$ *denote a continuous t-norm. A fuzzy set* $\mathcal{S}$ *defined on* $\mathcal{F}^2 \times (0,+\infty)$ *is referred to as a fuzzy rectangular b-metric on* $\mathcal{F}$ *if it satisfies the following conditions for all* $\eta,\xi,x,y \in \mathcal{F}$ *and* $t,s,w > 0$*:*
(F1) $\mathcal{S}(\eta,\xi,t) = 0$*;*
(F2) $\mathcal{S}(\eta,\xi,t) = 1 \leftrightarrow \eta = \xi$*;*
(F3) $\mathcal{S}(\eta,\xi,t) = \mathcal{S}(\xi,\eta,t)$*, symmetric in* η *and* ξ *for all* $t > 0$*;*
(F4) $\mathcal{S}(\eta,\xi,b(t+s+w)) \geq \mathcal{S}(\eta,x,t) * \mathcal{S}(x,y,s) * \mathcal{S}(y,\xi,w)$*;*
(F5) $\mathcal{S}(\eta,\xi,.) : (0,+\infty) \to [0,1]$ *is left continuous, and* $\lim_{t\to\infty} \mathcal{S}(\eta,\xi,t) = 1$*.*
Then, $(\mathcal{F},\mathcal{S},*)$ *is termed a fuzzy rectangular b-metric space.*

The two functions P and Q are referred to as noncomparable functions, which implies that they cannot be meaningfully compared in the sense that neither of them is greater than or equal to the other function. Next, we introduce the concept of fuzzy double controlled metric spaces. For more information, please consult [21].

Definition 4. *Let* P *and* Q *be two noncomparable functions, both mapping from* $\mathcal{F}^2$ *to the interval* $[1,+\infty)$*. Here,* $\mathcal{F}$ *represents a nonempty set. Moreover, let the symbol* $*$ *represent a continuous t-norm operation. A fuzzy set denoted as* $\mathcal{S}$ *on the domain* $\mathcal{F}^2 \times (0,+\infty)$ *is termed a fuzzy double controlled metric on* $\mathcal{F}$ *if it adheres to the subsequent criteria for all* η*,* ξ*, and* ϖ *within the set* $\mathcal{F}$*:*
(FD1) $\mathcal{S}(\eta,\xi,t) > 0 \;\forall\, l\; t > 0$*;*
(FD2) $\mathcal{S}(\eta,\xi,t) = 1 \;\forall\, t > 0 \leftrightarrow \eta = \xi$*;*
(FD3) $\mathcal{S}(\eta,\xi,t) = \mathcal{S}(\xi,\eta,t)$*, symmetric in* η *and* ξ*, and* $\forall\, t > 0$*;*
(FD4) $\mathcal{S}(\eta,\varpi,t+s) \geq \mathcal{S}(\eta,\xi,\frac{t}{P(\eta,\xi)}) * \mathcal{S}(\xi,\varpi,\frac{s}{Q(\xi,\varpi)})$*, for all* $s,t > 0$*;*
(FD5) $\mathcal{S}(\eta,\xi,.) : (0,+\infty) \to [0,1]$ *is continuous .*

In that case, we refer to the triple $(\mathcal{F},\mathcal{S},*)$ *as a fuzzy double controlled metric space.*

In this context, we present the definition of a fuzzy triple controlled metric space as introduced in the work by Furqan et al. [22].

Definition 5. *Let* $P,Q,R : \mathcal{F}^2 \to [1,+\infty)$ *be three noncomparable functions defined on a nonempty set* $\mathcal{F}$*, and let* $*$ *be a continuous t-norm operation. A fuzzy set denoted as* $\mathcal{S}$ *defined on* $\mathcal{F}^2 \times (0,+\infty)$ *is termed a fuzzy triple controlled metric on the set* $\mathcal{F}$*. This definition is satisfied when the following conditions hold for all distinct* $\eta,\xi,\varpi,x \in \mathcal{F}$*:*
(FT1) $\mathcal{S}(\eta,\xi,t) > 0 \;\forall\, t > 0$*;*
(FT2) $\mathcal{S}(\eta,\xi,t) = 1 \;\forall\, t > 0 \leftrightarrow \eta = \xi$*;*
(FT3) $\mathcal{S}(\eta,\xi,t) = \mathcal{S}(\xi,\eta,t)$*, symmetric in* η *and* ξ*, and for all* $t > 0$*;*
(FT4) $\mathcal{S}(\eta,\xi,t+s+w) \geq \mathcal{S}(\eta,\varpi,\frac{t}{P(\eta,\varpi)}) * \mathcal{S}(\varpi,x,\frac{s}{Q(\varpi,x)}) * \mathcal{S}(x,\xi,\frac{w}{R(x,\xi)})$*, for all* $s,t,w > 0$*;*
(FT5) $\mathcal{S}(\eta,\xi,.) : (0,+\infty) \to [0,1]$ *is continuous.*

Therefore, we designate $(\mathcal{F},\mathcal{S},*)$ *as a fuzzy triple controlled metric space.*

Observe that throughout this article, fuzzy triple controlled metric spaces will be denoted as $\mathcal{FTCMS}$*.*

Remark 1. *The class of* $\mathcal{FTCMS}$ *encompasses a broader range of structures compared to the class of fuzzy rectangular b-metric spaces (this distinction becomes evident when we set* $P(\eta,\varpi) = Q(\varpi,x) = R(x,\xi) = b \geq 1$*). Furthermore, the class of fuzzy rectangular b-metric spaces is more* *extensive than the class of fuzzy rectangular metric spaces, with the latter being a specific case when $b = 1$. Refer to Figure 1 for a visual representation of the relationships among these different types of fuzzy metric spaces.*

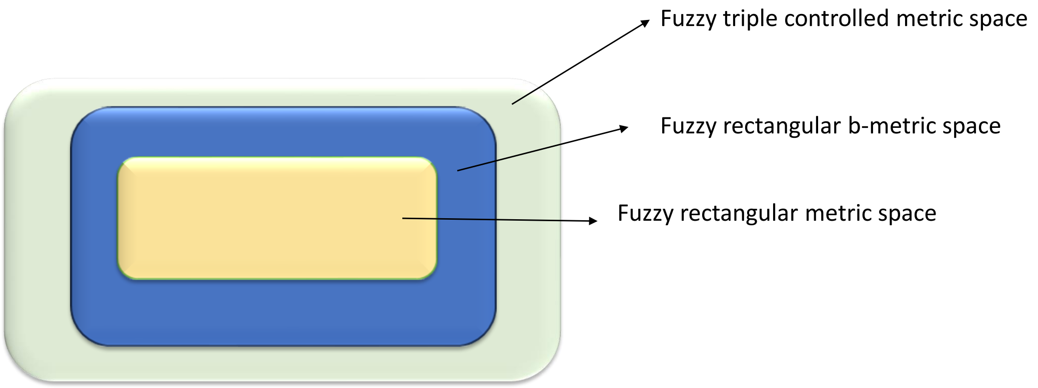

Figure 1. The diagram illustrates the interrelationships among various types of fuzzy metric spaces.

Our next example presents an $\mathcal{FTCMS}$ that is neither a fuzzy rectangular b-metric space nor a fuzzy rectangular metric space.

Example 1. *Let $\mathcal{F} = [0,1]$, and $P, Q, R : \mathcal{F}^2 \to [1, +\infty)$ are defined as follows: $P(\eta, \xi) = 2(\eta + \xi) + 1$, $Q(\eta, \xi) = 2(\eta^2 + \xi^2 + 1)$, and $R(\eta, \xi) = max\{\eta, \xi\} + 1$. Define*

$$\mathcal{S}(\eta, \xi, t) = \exp^{-\dfrac{(\eta - \xi)^2}{t}}, \eta, \xi \in \mathcal{F}, t > 0.$$

Then, $(\mathcal{F}, \mathcal{S}, *)$ is an $\mathcal{FTCMS}$ with product t-norm, i.e., $t_1 * t_2 = t_1 t_2$.
Axioms (FT1) to (FT3) and (FT5) are straightforward; we will only prove (FT4). Note that

$$\begin{aligned} \mathcal{S}(\eta, \xi, \frac{t}{P(\eta, \xi)}) &= \exp^{-\dfrac{(\eta-\xi)^2}{\frac{t}{P(\eta,\xi)}}} = \exp^{-\dfrac{(\eta-\xi)^2}{\frac{t}{2(\eta+\xi)+1}}} \\ &\leq \exp^{-\dfrac{(\eta-\xi)^2}{t}} \leq \exp^{-\dfrac{(\eta-\xi)^2}{t+s+w}}. \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{S}(\xi, x, \frac{s}{Q(\xi, x)}) &= \exp^{-\dfrac{(\xi-x)^2}{\frac{s}{Q(\xi,x)}}} = \exp^{-\dfrac{(\xi-x)^2}{\frac{s}{2(\xi^2+x^2+1)}}} \\ &\leq \exp^{-\dfrac{(\xi-x)^2}{s}} \leq \exp^{-\dfrac{(\xi-x)^2}{t+s+w}}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}(x, \varpi, \frac{w}{R(x, \varpi)}) &= \exp^{-\dfrac{(x-\varpi)^2}{\frac{w}{R(x,\varpi)}}} = \exp^{-\dfrac{(x-\varpi)^2}{\frac{w}{2(x^2+\varpi^2+1)}}} \\ &\leq \exp^{-\dfrac{(x-\varpi)^2}{s}} \leq \exp^{-\dfrac{(x-\varpi)^2}{t+s+w}}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathcal{S}(\eta,\varpi,t+s+w) &= \exp^{-\dfrac{(\eta-\varpi)^2}{t+s+w}} = \exp^{-\dfrac{(\eta-\xi+\xi-x+x-\varpi)^2}{t+s+w}}, \\ &\geq \exp^{-\frac{(\eta-\xi)^2}{t}} \exp^{-\frac{(\xi-x)^2}{s}} \exp^{-\frac{(x-\varpi)^2}{w}} \\ &\geq \mathcal{S}(\eta,\xi,\frac{t}{P(\eta,\xi)}) * \mathcal{S}(\xi,x,\frac{s}{Q(\xi,x)}) * \mathcal{S}(x,\varpi,\frac{w}{R(x,\varpi)}).\end{aligned}$$

Thus, $(\mathcal{F},\mathcal{S},*)$ *is an* $\mathcal{FTCMS}$*, which is not a fuzzy rectangular b-metric space since* $P(\eta,\xi) = 2(\eta+\xi)+1 \neq Q(\eta,\xi) = 2(\eta^2+\xi^2+1) \neq R(\eta,\xi) = max\{\eta,\xi\}+1 \neq b \geq 1$*, and it is evident that this does not constitute a fuzzy rectangular metric space.*

Example 2. *Consider the set* $\mathcal{F} = [0,1]$*, and define the control functions* $P,Q,R : \mathcal{F}^2 \to [1,+\infty)$ *as follows:*
$P(\eta,\xi) = \eta+\xi+1,$
$Q(\eta,\xi) = \eta^2+\xi+1,$
$R(\eta,\xi) = \eta^2+\xi^2+1$*. We will now define the fuzzy set* $\mathcal{S}$ *using these functions as follows:*

$$\mathcal{S}(\eta,\xi,t) = exp^{\frac{-|\eta-\xi|}{t}},\ t>0. \tag{1}$$

It can be readily demonstrated that $(\mathcal{F},\mathcal{S},*)$ *constitutes an* $\mathcal{FTCMS}$*. We will specifically confirm the condition (FT4). It is worth noting that*

$$\mathcal{S}(\eta,\xi,\frac{t}{P(\eta,\xi)}) = exp^{\frac{\frac{-|\eta-\xi|}{t}}{P(\eta,\xi)}} = exp^{\frac{-P(\eta,\xi)|\eta-\xi|}{t}} \leq exp^{\frac{-|\eta-\xi|}{t}} \leq exp^{\frac{-|\eta-\xi|}{t+s+w}},\ s>0.$$

Similarly,

$$\mathcal{S}(\xi,x,\frac{s}{Q(\xi,x)}) = exp^{\frac{\frac{-|\xi-x|}{s}}{Q(\xi,x)}} = exp^{\frac{-Q(\xi,x)|\xi-x|}{s}} \leq exp^{\frac{-|\xi-x|}{s}} \leq exp^{\frac{-|\xi-x|}{t+s+w}},\ t>0.$$

and

$$\mathcal{S}(x,\varpi,\frac{w}{R(x,\varpi)}) = exp^{\frac{\frac{-|x-\varpi|}{w}}{R(x,\varpi)}} = exp^{\frac{-R(x,\varpi)|x-\varpi|}{w}} \leq exp^{\frac{-|x-\varpi|}{t}} \leq exp^{\frac{-|x-\varpi|}{t+s+w}},\ w>0.$$

Hence, for $t,s,w>0$*,*

$$\begin{aligned}\mathcal{S}(\eta,\varpi,t+s+w) &= exp^{\frac{-|\eta-\varpi|}{t+s+w}}. \\ &\geq exp^{\frac{-|\eta-\xi|}{t+s+w}} exp^{\frac{-|\xi-x|}{t+s+w}} exp^{\frac{-|x-\varpi|}{t+s+w}}. \\ &\geq M(\eta,\xi,\frac{t}{P(\eta,\xi)}) * M(\xi,x,\frac{s}{Q(\xi,x)}) * M(x,\varpi,\frac{w}{R(x,\varpi)}).\end{aligned}$$

Which implies that $(\mathcal{F},\mathcal{S},*)$ *is an* $\mathcal{FTCMS}$*.*

Here is another example of an $\mathcal{FTCMS}$, as detailed in [22].

Example 3. *Let $\mathcal{F} = \{1,2,3,4\}$, and $P, Q, R : \mathcal{F}^2 \to [1, +\infty)$ are three noncomparable functions defined as follows: $P(\eta, \xi) = 1 + \eta + \xi$, $Q(\eta, \xi) = \eta^2 + \xi + 1$, and $R(\eta, \xi) = \eta^2 + \xi^2 - 1$. Define*

$$\mathcal{S}(\eta, \xi, t) = \frac{min\{\eta, \xi\} + t}{max\{\eta, \xi\} + t}.$$

Then, $(\mathcal{F}, \mathcal{S}, *)$ forms an $\mathcal{FTCMS}$ employing the product t-norm, denoted as $t_1 * t_2 = t_1 t_2$. The specific values for P, Q, and R are as outlined below:

P	1	2	3	4
1	3	4	5	6
2	4	5	6	7
3	5	6	7	8
4	6	7	8	9,

and

Q	1	2	3	4
1	3	4	5	6
2	6	7	8	9
3	11	12	13	14
4	18	19	20	21

also,

R	1	2	3	4
1	1	4	9	16
2	4	7	12	19
3	9	12	17	24
4	16	19	24	31

Axioms (FT1) to (FT3) and (FT5) are straightforward; we will only prove (FT4).

If $\eta = 1$ and $\xi = 2$, then either $x = 3$ and $y = 4$ or $x = 4$ and $y = 3$. We will consider the case where $x = 3$ and $y = 4$. The proof of the other cases is similar. Observe

$$\mathcal{S}(1, 2, t + s + w) = \frac{min\{1,2\} + t + s + w}{max\{1,2\} + t + s + w} = \frac{1 + t + s + w}{2 + t + s + w}. \tag{2}$$

Also, we have

$$\mathcal{S}(1, 3, \frac{t}{P(1,3)}) = \frac{min\{1,3\} + \frac{t}{P(1,3)}}{max\{1,3\} + \frac{t}{P(1,3)}} = \frac{5 + t}{15 + t},$$

$$\mathcal{S}(3, 4, \frac{s}{Q(3,4)}) = \frac{min\{3,4\} + \frac{s}{Q(3,4)}}{max\{3,4\} + \frac{s}{Q(3,4)}} = \frac{42 + s}{56 + s},$$

and

$$\mathcal{S}(4, 2, \frac{w}{R(4,2)}) = \frac{min\{4,2\} + \frac{w}{R(4,2)}}{max\{4,2\} + \frac{w}{R(4,2)}} = \frac{38 + w}{76 + w}. \tag{3}$$

Clearly, from Equations (2) and (3), we have

$$\mathcal{S}(1, 2, t + s + w) \geq \mathcal{S}(1, 3, \frac{t}{P(1,3)}) * \mathcal{S}(3, 4, \frac{s}{Q(3,4)}) * \mathcal{S}(4, 2, \frac{w}{R(4,2)}). \tag{4}$$

The remaining cases can be shown similarly; hence, $(\mathcal{F}, \mathcal{S}, *)$ is an $\mathcal{FTCMS}$.

Moving forward, we establish the definitions for open balls, sequence convergence, and the concept of Cauchy sequences within the framework of $\mathcal{FTCMS}$.

Definition 6. *Let $(\mathcal{F}, \mathcal{S}, *)$ be an $\mathcal{FTCMS}$. Then, the open ball $\mathcal{B}(\eta_0, r, t)$ with center η_0, radius $r \in (0, 1)$, and $t > 0$ is defined as follows:*

$$\mathcal{B}(\eta_0, r, t) = \{\xi \in \mathcal{F} : \mathcal{S}(\eta_0, \xi, t) > 1 - r\}.$$

Definition 7. *Let $(\mathcal{F}, \mathcal{S}, *)$ be an $\mathcal{FTCMS}$. Then, we define the following concepts:*

(1) *A sequence* $\{\eta_n\}$ *converges to* $\eta \in \mathcal{F}$ *if, for all* $t > 0$, *the limit as n approaches infinity of* $\mathcal{S}(\eta_n, \eta, t)$ *equals 1, expressed as:*

$$\lim_{n \to +\infty} \mathcal{S}(\eta_n, \eta, t) = 1.$$

(2) *A sequence* $\{\eta_n\}$ *is termed Cauchy if, for each* $m \in \mathbb{N}$ *and* $t > 0$, $\mathcal{S}(\eta_n, \eta_{n+m}, t) = 1$.

(3) *The* $\mathcal{FTCMS}$ $(\mathcal{F}, \mathcal{S}, *)$ *is designated as a complete* $\mathcal{FTCMS}$ *if every Cauchy sequence converges.*

Moving forward, we present a lemma that plays a crucial role in demonstrating our findings. For a more comprehensive explanation, please refer to [21].

Lemma 1. *Consider a Cauchy sequence* $\{\eta_n\}$ *in an* $\mathcal{FTCMS}$ $(\mathcal{F}, \mathcal{S}, *)$ *such that* $\eta_n \neq \eta_m$ *whenever* $n \neq m$, *and both m and n belong to the set of natural numbers* $\mathbb{N}$. *Then, the sequence* $\{\eta_n\}$ *can converge to a maximum of one limit point.*

Definition 8 ([31])**.** *Let* $(\mathcal{F}, \mathcal{S}, *)$ *denote an* $\mathcal{FTCMS}$. *The fuzzy triple controlled metric* $\mathcal{S}$ *is considered triangular if the following condition is satisfied:*

$$\left(\frac{1}{\mathcal{S}(\eta, \xi, t)} - 1\right) \leq \left(\frac{1}{\mathcal{S}(\eta, \varpi, t)} - 1\right) + \left(\frac{1}{\mathcal{S}(\xi, \varpi, t)} - 1\right). \tag{5}$$

This condition holds true for all $\eta, \xi, \varpi \in \mathcal{F}$ *and for all* $t > 0$.

3. The Main Results

Gopal and Vetro, in their work [30], introduced the notion of (α-ψ)-fuzzy contractive mappings in the context of fuzzy metric spaces. Inspired by their work, Azmi, as reported in [32], extended the notion of (α-ψ)-fuzzy contractive mappings to fuzzy double controlled metric spaces, subsequently establishing results related to fixed points.

In the upcoming sections, we will introduce two key notions: (α-ψ)-fuzzy contractive mappings and α-admissible mappings within the framework of $\mathcal{FTCMS}$.

Definition 9. *Consider the* $\mathcal{FTCMS}$ $(\mathcal{F}, \mathcal{S}, *)$. *We define a mapping* $T : \mathcal{F} \longrightarrow \mathcal{F}$ *as* α-*admissible if there exists a function* $\alpha : \mathcal{F}^2 \times (0, +\infty) \to [0, +\infty)$ *such that, for all* $t > 0$:

$$\text{For } \eta, \xi \in \mathcal{F}, \text{ if } \alpha(\eta, \xi, t) \geq 1, \text{then } \alpha(T\eta, T\xi, t) \geq 1. \tag{6}$$

Let Ψ represent the collection of all continuous functions (from the right) $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) < t$ for all $t > 0$.

Remark 2. *It is important to observe that for any function* $\psi \in \Psi$, *the limit* $\lim_{n \to +\infty} \psi^n(t) = 0$ *for all* $t > 0$, *where* ψ^n *denotes the n-th iteration of* ψ.

Definition 10. *Let* $(\mathcal{F}, \mathcal{S}, *)$ *represent an* $\mathcal{FTCMS}$. *We define the mapping* $T : \mathcal{F} \longrightarrow \mathcal{F}$ *to be an* (α-ψ)*-fuzzy contractive mapping if there exist two functions* $\psi \in \Psi$, *and* $\alpha : \mathcal{F}^2 \times (0, +\infty) \to [0, +\infty)$ *such that the following inequality holds for all* $\eta, \xi \in \mathcal{F}$ *and for all* $t > 0$:

$$\alpha(\eta, \xi, t)\left(\frac{1}{\mathcal{S}(T\eta, T\xi, t)} - 1\right) \leq \psi\left(\frac{1}{\mathcal{S}(\eta, \xi, t)} - 1\right). \tag{7}$$

We will now introduce our initial major result and proceed to prove it.

Theorem 1. *Consider a complete* $\mathcal{FTCMS}$ $(\mathcal{F}, \mathcal{S}, *)$ *with three noncomparable functions* $P, Q, R : \mathcal{F}^2 \to [1, 1/\tau)$, *where* $\tau \in (0, 1)$. *Let* $T : \mathcal{F} \to \mathcal{F}$ *be an* (α-ψ)*-fuzzy contractive mapping, with* $\psi \in \Psi$, *satisfying the following conditions:*

(1) *T possesses α-admissibility;*
(2) *There exists $\eta_0 \in \mathcal{F}$ such that $\alpha(\eta_0, T\eta_0, t) \geq 1$ for all $t > 0$;*
(3) *T exhibits continuity;*
(4) *For some $\eta_0 \in \mathcal{F}$, the sequence $\{\eta_n\}$ is defined as $\eta_n = T^n\eta_0$, and for any $\eta \in \mathcal{F}$, the following limits exist and are finite:*

$$\lim_{n\to+\infty} P(\eta_n, \eta), \quad \lim_{n\to+\infty} Q(\eta, \eta_n), \quad \textit{and} \quad \lim_{n\to+\infty} R(\eta, \eta_n). \tag{8}$$

Then, T possesses a fixed point, meaning there exists $\eta^* \in \mathcal{F}$ such that $T(\eta^*) = \eta^*$.

Proof. Choose an initial point $\eta_0 \in \mathcal{F}$ such that $\alpha(\eta_0, T\eta_0, t) \geq 1$ for all $t > 0$. We also have a sequence $\{\eta_n\}$ in $\mathcal{F}$ where $T^n\eta_0 = \eta_n$ for all $n \in \mathbb{N}$.

Observe that if $\eta_m = \eta_{m+1}$ for a specific $m \in \mathbb{N}$, it implies that $T^m\eta_0$ serves as a fixed point for the mapping T. Therefore, we can proceed without any loss of generality and assume that $\eta_n \neq \eta_{n+1}$ holds for all $n \in \mathbb{N}$.

Based on the provided assumptions, we observe that $\alpha(\eta_0, \eta_1, t) = \alpha(\eta_0, T\eta_0, t) \geq 1$. Since T is α-admissible, this implies $\alpha(T\eta_0, T\eta_1, t) = \alpha(\eta_1, \eta_2, t) \geq 1$. Using induction, we can readily infer that:

$$\alpha(\eta_n, \eta_{n+1}, t) \geq 1, \text{ holds for all } n \in \mathbb{N}, \text{and for all } t > 0. \tag{9}$$

Therefore, by employing Equations (7) and (9), we can deduce:

$$\begin{aligned}
\left(\frac{1}{\mathcal{S}(\eta_n, \eta_{n+1}, t)} - 1\right) &= \left(\frac{1}{\mathcal{S}(T\eta_{n-1}, T\eta_n, t)} - 1\right). \\
&\leq \alpha(\eta_{n-1}, \eta_n, t)\left(\frac{1}{\mathcal{S}(T\eta_{n-1}, T\eta_n, t)} - 1\right). \\
&\leq \psi\left(\frac{1}{\mathcal{S}(\eta_{n-1}, \eta_n, t)} - 1\right), \quad \text{repeating the process} \\
&\leq \psi\left(\psi\left(\frac{1}{\mathcal{S}(\eta_{n-2}, \eta_{n-1}, t)} - 1\right)\right) = \psi^2\left(\frac{1}{\mathcal{S}(\eta_{n-2}, \eta_{n-1}, t)} - 1\right). \\
&\leq \cdots \leq \psi^n\left(\frac{1}{\mathcal{S}(\eta_0, \eta_1, t)} - 1\right).
\end{aligned} \tag{10}$$

Taking the limit as n approaches infinity in Equation (10) and utilizing the fact that $\lim_{n\to+\infty} \psi^n(r) = 0$ with $r = \dfrac{1}{\mathcal{S}(\eta_0, \eta_1, t)} - 1$, we derive:

$$\lim_{n\to+\infty} \mathcal{S}(\eta_{n-1}, \eta_n, t) = 1 \text{ for all } t > 0. \tag{11}$$

For any pair of natural numbers n and m, where $n < m$, we have:

$$\begin{aligned}
\mathcal{S}(\eta_n, \eta_m, t) \geq{}& \mathcal{S}\Big(\eta_n, \eta_{n+1}, \frac{\frac{t}{3}}{P(\eta_n, \eta_{n+1})}\Big) * \mathcal{S}\Big(\eta_{n+1}, \eta_{n+2}, \frac{\frac{t}{3}}{Q(\eta_{n+1}, \eta_{n+2})}\Big) * \mathcal{S}\Big(\eta_{n+2}, \eta_m, \frac{\frac{t}{3}}{R(\eta_{n+2}, \eta_m)}\Big). \\
\geq{}& \mathcal{S}\Big(\eta_n, \eta_{n+1}, \frac{\frac{t}{3}}{P(\eta_n, \eta_{n+1})}\Big) * \mathcal{S}\Big(\eta_{n+1}, \eta_{n+2}, \frac{\frac{t}{3}}{Q(\eta_{n+1}, \eta_{n+2})}\Big) \\
&* \mathcal{S}\Big(\eta_{n+2}, \eta_{n+3}, \frac{\frac{t}{3^2}}{P(\eta_{n+2}, \eta_{n+3})R(\eta_{n+2}, \eta_{n+3})}\Big) * \mathcal{S}\Big(\eta_{n+3}, \eta_{n+4}, \frac{\frac{t}{3^2}}{Q(\eta_{n+3}, \eta_{n+4})R(\eta_{n+3}, \eta_{n+4})}\Big) \\
&* \mathcal{S}\Big(\eta_{n+4}, \eta_m, \frac{\frac{t}{3^2}}{R(\eta_{n+4}, \eta_m)R(\eta_{n+2}, \eta_m)}\Big).
\end{aligned}$$

$$\geq \mathcal{S}(\eta_n, \eta_{n+1}, \frac{\frac{t}{3}}{P(\eta_n, \eta_{n+1})}) * \mathcal{S}(\eta_{n+1}, \eta_{n+2}, \frac{\frac{t}{3}}{Q(\eta_{n+1}, \eta_{n+2})})$$
$$* \mathcal{S}(\eta_{n+2}, \eta_{n+3}, \frac{\frac{t}{3^2}}{P(\eta_{n+2}, \eta_{n+3})R(\eta_{n+2}, \eta_{n+3})}) * \mathcal{S}(\eta_{n+3}, \eta_{n+4}, \frac{\frac{t}{3^2}}{Q(\eta_{n+3}, \eta_{n+4})R(\eta_{n+3}, \eta_{n+4})})$$
$$\vdots$$
$$* \mathcal{S}(\eta_{m-2}, \eta_{m-1}, \frac{\frac{t}{3^m}}{Q(\eta_{m-2}, \eta_{m-1})R(\eta_{m-2}, \eta_m) \cdots R(\eta_{n+2}, \eta_m)})$$
$$* \mathcal{S}(\eta_{m-1}, \eta_m, \frac{\frac{t}{3^m}}{R(\eta_{m-1}, \eta_m)R(\eta_{m-2}, \eta_m) \cdots R(\eta_{n+2}, \eta_m)}).$$

By taking the limit as n approaches infinity in the above inequality and applying Equations (11) and (8), we obtain:

$$\lim_{n\to+\infty} \mathcal{S}(\eta_n, \eta_m, t) \geq 1 * 1 * 1 * \cdots * 1 = 1. \tag{12}$$

Hence, $\lim_{n\to+\infty} \mathcal{S}(\eta_n, \eta_m, t) = 1$; this suggests that the sequence $\{\eta_n\}$ is a Cauchy sequence within the space $\mathcal{F}$, and since $\mathcal{F}$ is a complete $\mathcal{FTCMS}$, there exists an element $\eta^* \in \mathcal{F}$ such that η_n converges to η^*, i.e.,

$$\lim_{n\to+\infty} \mathcal{S}(\eta_n, \eta^*, t) = 1. \tag{13}$$

The continuity of T implies that $\lim_{n\to+\infty} \mathcal{S}(T(\eta_n), T(\eta^*), t) = 1$ for all $t > 0$. Consequently, we have:

$$\lim_{n\to+\infty} \mathcal{S}(\eta_{n+1}, T(\eta^*), t) = \lim_{n\to+\infty} \mathcal{S}(T(\eta_n), T(\eta^*), t) = 1, \text{ for all } t > 0. \tag{14}$$

This leads to the conclusion that $\eta_n \to T(\eta^*)$, and utilizing Lemma 1, we conclude that $T(\eta^*) = \eta^*$, establishing that η^* is a fixed point of T.
□

As a specific instance, when we set $P(\eta, \xi) = Q(\eta, \xi) = R(\eta, \xi) = b$, Theorem 1 offers a demonstration for the situation of a complete fuzzy rectangular b-metric space, as illustrated in the following corollary.

Corollary 1. *Consider a complete fuzzy rectangular b-metric space denoted as $(\mathcal{F}, \mathcal{S}, *)$, and let the mapping $T : \mathcal{F} \to \mathcal{F}$ be characterized as an (α-ψ)-fuzzy contractive mapping, where ψ belongs to the set Ψ. This mapping satisfies the following conditions:*

(1) T possesses α-admissibility.
(2) There exists an element η_0 in $\mathcal{F}$ such that $\alpha(\eta_0, T\eta_0, t) \geq 1 \; \forall \; t > 0$.
(3) T exhibits continuity.

Then, T possesses a fixed point, which means that there exists an element $\eta^* \in \mathcal{F}$ satisfying $T(\eta^*) = \eta^*$.

Proof. By substituting $P(\eta, \xi) = Q(\eta, \xi) = R(\eta, \xi) = b$ into Theorem 1 and following the same steps outlined in the proof, it can be concluded that T possesses a fixed point, as it satisfies all the conditions stipulated in Theorem 1. □

Corollary 2. *Consider a complete fuzzy rectangular metric space denoted as $(\mathcal{F}, \mathcal{S}, *)$. Within this space, there exists a mapping $T : \mathcal{F} \to \mathcal{F}$ that can be characterized as an (α-ψ)-fuzzy contractive mapping, with ψ belonging to the set Ψ. These mappings adhere to the following conditions:*

(1) T possesses α-admissibility.
(2) There exists an element η_0 in $\mathcal{F}$ such that $\alpha(\eta_0, T\eta_0, t) \geq 1 \; \forall \; t > 0$.
(3) T exhibits continuity.

Consequently, T possesses a fixed point, meaning that there exists an element $\eta^* \in \mathcal{F}$ such that $T(\eta^*) = \eta^*$.

Proof. By setting $P(\eta, \xi) = Q(\eta, \xi) = R(\eta, \xi) = 1$ in Theorem 1 and revisiting the proof, we can conclude that T possesses a fixed point, as it satisfies all the conditions stipulated in Theorem 1. $\square$

In the upcoming theorem, we substitute the continuity assumptions of T in Theorem 1 with an alternative regularity condition.

Theorem 2. *Consider a complete $\mathcal{FTCMS}$ denoted as $(\mathcal{F}, \mathcal{S}, *)$. Within this space, there are three noncomparable functions, namely, $P, Q, R : \mathcal{F}^2 \to [1, 1/\tau)$, where $\tau \in (0, 1)$. Furthermore, let $T : \mathcal{F} \to \mathcal{F}$ be an (α-ψ)-fuzzy contractive mapping, with ψ belonging to the set Ψ. These mappings adhere to the following conditions:*

(1) T possesses α-admissibility.
(2) There exists an element η_0 in $\mathcal{F}$ such that $\alpha(\eta_0, T\eta_0, t) \geq 1 \; \forall \; t > 0$.
(3) If $\{\eta_n\}$ is any sequence in $\mathcal{F}$ such that $\alpha(\eta_n, \eta_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $\eta_n \to \eta$ as n tends to $+\infty$, then $\alpha(\eta_n, \eta, t) \geq 1 \; \forall \; n \in \mathbb{N}$.
(4) If the sequence $\{\eta_n\}$ is defined as $\eta_n = T^n \eta_0$ for some $\eta_0 \in \mathcal{F}$, then for any $\eta \in \mathcal{F}$, the following limits exists and are finite;

$$\lim_{n \to +\infty} P(\eta_n, \eta), \; \lim_{n \to +\infty} Q(\eta, \eta_n) \; \textit{and} \; \lim_{n \to +\infty} R(\eta, \eta_n) \tag{15}$$

As a result, T admits a fixed point, signifying the existence of an element $\eta^* \in \mathcal{F}$ such that $T(\eta^*) = \eta^*$.

Proof. By following the proof outlined in Theorem 1, we can establish that the sequence $\{\eta_n\}$ is a Cauchy sequence within the context of a complete $\mathcal{FTCMS}$ denoted as $(\mathcal{F}, \mathcal{S}, *)$. This, in turn, implies the existence of an element $\eta^* \in \mathcal{F}$ such that $\eta_n \to \eta^*$ as n tends to positive infinity. Therefore, based on hypothesis (3), we can deduce the following:

$$\alpha(\eta_n, \eta^*, t) \geq 1 \; \text{for all} \; n \in \mathbb{N} \; \text{and} \forall \; t > 0. \tag{16}$$

Utilizing the triangular property of $\mathcal{S}$ and combining Equations (16) and (7), we can derive:

$$\begin{aligned}
\left(\frac{1}{\mathcal{S}(T\eta^*, \eta^*, t)} - 1\right) &\leq \left(\frac{1}{\mathcal{S}(T\eta^*, T\eta_n, t)} - 1\right) + \left(\frac{1}{\mathcal{S}(\eta_{n+1}, \eta^*, t)} - 1\right). \\
&\leq \alpha(\eta_n, \eta^*, t)\left(\frac{1}{\mathcal{S}(T\eta_n, T\eta^*, t)} - 1\right) + \left(\frac{1}{\mathcal{S}(\eta_{n+1}, \eta^*, t)} - 1\right). \\
&\leq \psi\left(\frac{1}{\mathcal{S}(\eta_n, \eta^*, t)} - 1\right) + \left(\frac{1}{\mathcal{S}(\eta_{n+1}, \eta^*, t)} - 1\right), \; \text{since} \, \psi(r) < r. \\
&< \left(\frac{1}{\mathcal{S}(\eta_n, \eta^*, t)} - 1\right) + \left(\frac{1}{\mathcal{S}(\eta_{n+1}, \eta^*, t)} - 1\right).
\end{aligned} \tag{17}$$

As we allow n to approach positive infinity in Equation (17), we acquire:

$$\lim_{n \to +\infty} \mathcal{S}(T\eta^*, \eta^*, t) = 1 \; \text{for all} \, t > 0, \tag{18}$$

that is, $T(\eta^*) = \eta^*$, demonstrating that T possesses a fixed point. $\square$

4. Application

Consider the space $\mathcal{F} = C([0, 1], \mathbb{R})$, which represents the set of all continuous real-valued functions defined on the interval $[0, 1]$. We establish three noncomparable functions, denoted as $P, Q, R : \mathcal{F}^2 \to [1, +\infty)$, in the following manner: $P(\eta, \xi) = \eta + \xi + 1$, $Q(\eta, \xi) =$

$\eta^2 + \xi + 1$, and $R(\eta, \xi) = \eta^2 + \xi^2 + 1$. The fuzzy metric $\mathcal{S}$ is defined over the set $\mathcal{F}$ as follows:

$$\mathcal{S}(\eta, \xi, t) = e^{-sup_{s\in[0,1]} \frac{|\eta(s)-\xi(s)|}{t}}, \text{ where } \eta, \xi \in \mathcal{F},\ t > 0. \tag{19}$$

Therefore, $(\mathcal{F}, \mathcal{S}, *)$ constitutes a complete $\mathcal{FTCMS}$.

Theorem 3. *Consider $(\mathcal{F}, \mathcal{S}, *)$ to be a complete $\mathcal{FTCMS}$ as previously defined. We introduce an integral operator $T : \mathcal{F} \longrightarrow \mathcal{F}$ defined by the following expression:*

$$T\eta(s) = p(s) + \int_0^s \mathcal{G}(s, x, \eta(x))dx. \tag{20}$$

Here, p and $\eta \in \mathcal{F}$, and $\mathcal{G}(s, x, \eta(x)) : [0,1]^2 \longrightarrow \mathbb{R}$ is a continuous function. Furthermore, $\mathcal{G}$ satisfies the following conditions:

$$|\mathcal{G}(s, x, \eta(x)) - \mathcal{G}(s, x, \xi(x))| \le h(s, x)|\eta(x) - \xi(x)|,\ \eta, \xi \in \mathcal{F}$$

for some function $h : [0,1]^2 \to [0, +\infty)$ such that $h \in L^1([0,1], \mathbb{R})$, and it satisfies the following:

- $0 < \sup_{s\in[0,1]} \int_0^s h(s, x)dx \le k < 1$, *for some $k \in (0, 1)$;*
- $e^{-\sup_{x\in[0,1]} \frac{k|\eta(x)-\xi(x)|}{t}} \ge 2\, e^{-\sup_{x\in[0,1]} \frac{|\eta(x)-\xi(x)|}{t}}$.

  Then, the integral Equation (20) has a solution.

Proof. Let us initially define the function $\alpha : \mathcal{F}^2 \times (0, +\infty) \to [0, +\infty)$ as follows:

$$\alpha(\eta, \xi, t) = \begin{cases} 1/4 & \text{if } \eta = 0, \text{or } \xi = 0. \\ 1/2 & \text{if } \eta = \xi. \\ 1 & \text{otherwise}. \end{cases} \tag{21}$$

Also, let $\psi(r) = r/2$. For $\eta, \xi \in \mathcal{F}$, we examine the fuzzy metric:

$$\begin{aligned} \mathcal{S}(T\eta, T\xi, t) &= e^{-sup_{s\in[0,1]} \frac{|T\eta(s) - T\xi(s)|}{t}}. \\ &\ge e^{-sup_{s\in[0,1]} \frac{\int_0^s |\mathcal{G}(s,x,\eta(x)) - \mathcal{G}(s,x,\xi(x))|dx}{t}}. \\ &\ge e^{-sup_{s\in[0,1]} \frac{\int_0^s h(s,x)|\eta(x) - \xi(x)|dx}{t}}. \\ &\ge e^{-sup_{x\in[0,1]} \frac{|\eta(x) - \xi(x)| sup_s \int_0^s h(s,x)dx}{t}}. \\ &\ge e^{-sup_{x\in[0,1]} \frac{k|\eta(x) - \xi(x)|}{t}} \\ &\ge 2e^{-sup_{x\in[0,1]} \frac{|\eta(x) - \xi(x)|}{t}} = 2(\mathcal{S}(\eta, \xi, t)). \end{aligned} \tag{22}$$

Therefore, in order to demonstrate that T is an $(\alpha$-$\psi)$-fuzzy contractive mapping, it is necessary to establish the validity of Equation (7). Thus, for any $\eta, \xi \in \mathcal{F}$, we can infer from Equations (21) and (22) that:

$$\begin{aligned}\alpha(\eta,\xi,t)\left(\frac{1}{\mathcal{S}(T\eta,T\xi,t)}-1\right) &\leq \frac{1}{\mathcal{S}(T\eta,T\xi,t)}-1. \\ &\leq \frac{1}{2\mathcal{S}(\eta,\xi,t)}-1 \leq \frac{1}{2}\left(\frac{1}{\mathcal{S}(\eta,\xi,t)}-1\right). \\ &= \psi\left(\frac{1}{\mathcal{S}(\eta,\xi,t)}-1\right).\end{aligned} \tag{23}$$

In conclusion, we established that the operator T possesses a fixed point, denoted as $\eta^* \in C([0,1],\mathbb{R})$, which serves as a solution to integral Equation (20). Therefore, all the conditions of Theorem 1 have been satisfied. □

Next, we present an example of Theorem 3.

Example 4 ([20])**.** *Let $(\mathcal{F},\mathcal{S},*)$ be a complete $\mathcal{FTCMS}$ as defined above. Consider the differential equation*

$$\eta''(s)-\eta(s)=cos(s),\ \ \eta(0)=0,\ \ \eta'(0)=0,$$

this produces the following integral equation

$$\eta(s)=1-cos(s)-\int_0^s(s-x)\eta(x)dx, \tag{24}$$

where $\mathcal{G}(s,x,\eta(x))=(s-x)\eta(x)$. Observe that

$$\begin{aligned}|\mathcal{G}(s,x,\eta(x))-\mathcal{G}(s,x,\xi(x))| &= |(s-x)\eta(x)-(s-x)\xi(x)|. \\ &= |(s-x)||\eta(x)-\xi(x)| = h(s,x)|\eta(x)-\xi(x)|,\end{aligned} \tag{25}$$

with $h(s,x)=|(s-x)|$; hence, $0<\sup_{s\in[0,1]}\int_0^s h(s,x)dx \leq k < 1$. Note that following the steps in Equation (22) we have

$$\begin{aligned}\mathcal{S}(T\eta,T\xi,t) &\geq e^{-sup_{s\in[0,1]}\frac{\int_0^s|\mathcal{G}(s,x,\eta(x))-\mathcal{G}(s,x,\xi(x))|dx}{t}}. \\ &\geq e^{-sup_{x\in[0,1]}\frac{|\eta(x)-\xi(x)|sup_s\int_0^s|s-x|dx}{t}}. \\ &\geq e^{-sup_{x\in[0,1]}\frac{k|\eta(x)-\xi(x)|}{t}} = 2(\mathcal{S}(\eta,\xi,t)).\end{aligned}$$

With α and ψ as defined in Theorem 3. Following the steps in Equation (23), we conclude that the integral Equation (24) has a solution.

5. Conclusions

This article has focused on the concept of fuzzy triple controlled metric spaces. We have established fixed point results within these spaces by employing $(\alpha$-$\psi)$-fuzzy contractive mappings. Additionally, we have provided several examples and demonstrated the application of our findings in the context of the existence of a solution to an integral equation. We have also given an example of a solution to an integral equation.

It is worth mentioning that in 1999, Molodstov introduced the theory of soft sets, a mathematical concept that deals with uncertainties [11]. Subsequently, Das and Samanta introduced the concepts of soft metric spaces, making significant contributions to this area [23–25]. Beaula and Raja introduced the concept of fuzzy soft metric spaces, paving the way for fixed point theory in the context of soft fuzzy metric spaces [26]. Related works can be found in references [34–36]. More recently, the work of Sonam et al. presented fixed point results in soft fuzzy metric spaces as well as fixed point results in soft rectangular*b*-metric spaces [27,28].

For potential future research, we suggest exploring the concept of soft double and triple controlled fuzzy metric spaces and investigating fixed point results within these spaces.

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