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The λ-Fold Spectrum Problem for the Orientations of the Eight-Cycle

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Abstract: A D-decomposition of a graph (or digraph) G is a partition of the edge set (or arc set) of G into subsets, where each subset induces a copy of the fixed graph D. Graph decomposition finds motivation in numerous practical applications, particularly in the realm of symmetric graphs, where these decompositions illuminate intricate symmetrical patterns within the graph, aiding in various fields such as network design, and combinatorial mathematics, among various others. Of particular interest is the case where G is ${}^{\lambda}K_v^*$, the λ-fold complete symmetric digraph on v vertices, that is, the digraph with λ directed edges in each direction between each pair of vertices. For a given digraph D, the set of all values v for which ${}^{\lambda}K_v^*$ has a D-decomposition is called the λ-fold spectrum of D. An eight-cycle has 22 non-isomorphic orientations. The λ-fold spectrum problem has been solved for one of these oriented cycles. In this paper, we provide a complete solution to the λ-fold spectrum problem for each of the remaining 21 orientations.

Keywords: λ-fold spectrum problem; complete symmetric digraph; decompositions; orientations of an eight-cycle

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1. Introduction

Assuming m and n are positive integers with $m \leq n$, we use the notation $[m, n]$ to represent the set containing all integers between m and n inclusive. When referring to a graph (or digraph) G, we use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set (or arc set). Moreover, we will use ${}^{\lambda}G$ to represent the multigraph (or directed multigraph) with vertex set $V(G)$ and λ copies of each edge (or arc) in $E(G)$. For the sake of clarity, edges are denoted using curly brackets and arcs are denoted using parentheses. For a given simple undirected graph G, we use G^* to denote the symmetric digraph with vertex set $V(G^*) = V(G)$ and arc set $E(G^*) = \bigcup_{\{x,y\} \in E(G)} \{(x, y), (y, x)\}$. In essence, G^* represents the digraph obtained from G by replacing each of its edges with a pair of symmetric arcs. Hence, ${}^{\lambda}K_v^*$ and ${}^{\lambda}K_{x,y}^*$, respectively, are the λ-fold complete symmetric digraph with v vertices and λ-fold complete bipartite symmetric digraph with x and y vertices in the parts.

Let G be a directed multigraph. A *decomposition* of G is a set $\mathcal{D} = \{D_1, D_2, \ldots, D_t\}$ of subgraphs of G such that $E(D_1) \cup E(D_2) \cup \ldots \cup E(D_t) = E(G)$ and $E(D_i) \cap E(D_j) = \emptyset$ for $i \neq j$. A decomposition of a digraph G into subgraphs, each isomorphic to a given digraph D, is called a *D-decomposition* of G. A *(G, D)-design* refers to the same concept as a D-decomposition of G. The *spectrum of D* is the set of all v for which K_v^* has a D-decomposition. Similarly, the set of all v for which ${}^{\lambda}K_v^*$ has a D-decomposition is called the *λ-fold spectrum of D*.

Graphs, digraphs, and multigraphs are fundamental mathematical structures of great significance in various fields, including computer science, logistics, chemistry, and biology [1–4]. In particular, digraphs have a wide range of real-world applications such as social networks, communication networks, electrical circuit design, network flow analysis, and biological networks (see [5]). In this context, graph decomposition, an essential concept in both graph theory and combinatorial design theory, involves breaking down intricate structures into smaller, structured components. Many problems in these fields can be viewed in terms of decomposition of graphs into prescribed subgraphs. Beyond its theoretical significance, graph decomposition finds practical applications in diverse areas such as graph similarity and matching [6] and parallel computations on large graphs [7], among others.

For a decomposition of ${}^{\lambda}K_v^*$ into copies of a digraph D, it is necessary that the number of arcs in ${}^{\lambda}K_v^*$, namely, $\lambda v(v-1)$, be a multiple of the number of the arcs in D. Moreover, both $\gcd\{d^+(x) : x \in V(D)\}$ and $\gcd\{d^-(x) : x \in V(D)\}$ divide $\lambda(v-1)$, which is both indegree and outdegree of every vertex in ${}^{\lambda}K_v^*$, where $d^+(x)$ and $d^-(x)$ denote the outdegree and indegree of the vertex x, respectively. Thus, based on these discussions, we now have the following obvious necessary conditions for the existence of a decomposition of ${}^{\lambda}K_v^*$ into digraph D.

Lemma 1. *Let D be a digraph. The necessary conditions for the existence of a D-decomposition of ${}^{\lambda}K_v^*$ are*

(a) $|V(D)| \leq v$,
(b) $|E(D)|$ *divides* $\lambda v(v-1)$, *and*
(c) *both* $\gcd\{d^+(x) : x \in V(D)\}$ *and* $\gcd\{d^-(x) : x \in V(D)\}$ *divide* $\lambda(v-1)$.

The number of non-isomorphic orientations of a cycle of order n is denoted by $o(C_n)$ and is given as follows [8]:

$$o(C_n) = \frac{1}{2n}\sum_{d|n}\varphi(d)2^{n/d} + \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2^{n/2-2}, & \text{if } n \text{ is even} \end{cases} \quad (1)$$

where φ is the Euler's totient function. We can quickly verify by Equation (1) that $o(C_3) = 2$, $o(C_4) = o(C_5) = 4$, $o(C_6) = 9$, $o(C_7) = 10$, and $o(C_8) = 22$.

The spectrum problem for certain subgraphs of K_4^* (both bipartite and non-bipartite) has already been studied [9–13]. Two non-isomorphic orientations of C_3 are called cyclic and transitive orientations. If D is a cyclic orientation of C_3, then a (K_v^*, D)-design is commonly referred to as a Mendelsohn triple system. The spectrum for Mendelsohn triple systems was independently studied and settled by Mendelsohn [12] and Bermond [9]. A (K_v^*, D)-design with D being a transitive orientation of K_3 is referred to as a transitive triple system. Hung and Mendelsohn [11] found the spectrum for transitive triple systems. For all remaining simple connected subgraphs of K_3^*, the spectrum was found by Hartman and Mendelsohn in [14]. A four-cycle (referred to as a quadrilateral) can have precisely four distinct orientations. In [13], it was proven that if D is a cyclic orientation of a quadrilateral, then a (K_v^*, D)-design exists if and only if $v > 4$ and $v \equiv 0$ or $1 \pmod 4$. For the remaining three orientations of a four-cycle, the spectrum problem was settled in [10]. Alspach et al. [15] showed that any of the four orientations of a five-cycle (referred to as a pentagon) can decompose K_v^* if and only if $v \equiv 0$ or $1 \pmod 5$. It is shown in [16] that for given positive integers m and v such that $2 \leq m \leq v$, the digraph K_v^* can be decomposed into directed cycles (i.e., with all the edges being oriented in the same direction) of length m if and only if $(m, v) \notin \{(3,6), (4,4), (6,6)\}$ and the number of arcs in K_v^* is divisible by m. For all nine possible orientations of a six-cycle, the λ-fold spectrum problem was settled by Adams et al. [17]. Also, recently [18], the spectrum problem for all ten possible orientations of a seven-cycle (referred to as a heptagon) was completely settled.

Twenty-two non-isomorphic orientations of an eight-cycle exist. We denote these as $D_1, D_2, \ldots, D_{21}$ and D_{22}, as illustrated in Figure 1. The λ-fold spectrum problem was settled for the directed eight-cycle (D_1 in Figure 1) in [19]. The focus of this research is to settle this problem for the remaining twenty-one orientations. Let us now state the main result of this paper, which is proved in Section 4.

Theorem 1. *Let D be an orientation of an eight-cycle and let λ and v be positive integers such that $v \geq 8$. There exists a D-decomposition of ${}^{\lambda}K_v^*$ if and only if $\lambda v(v-1) \equiv 0 \pmod 8$ except for $D = D_{14}$ and $\lambda(v-1)$ is odd.*

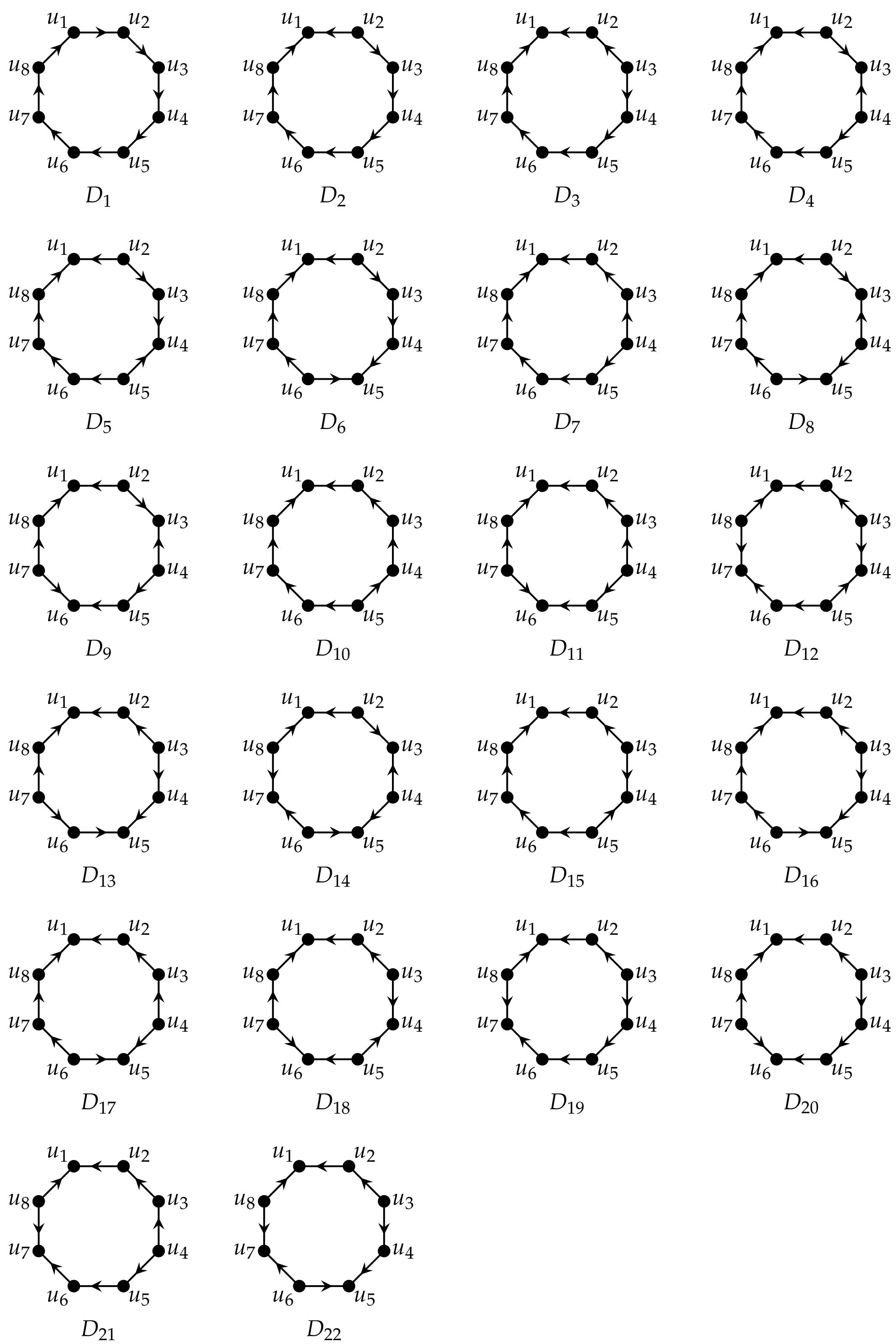

Figure 1. The twenty-two orientations of an eight-cycle. The λ-fold spectrum problem for the directed eight-cycle (D_1) has previously been settled.

An *antidirected cycle* of length n is obtained by orienting the edges of a cycle of length n such that no directed path of length 2 is induced. Thus, a cycle C_n is antidirected if any vertex of C_n has either indegree 2 or outdegree 2 in C_n, and antidirected cycles necessarily have an even number of arcs. In Figure 1, D_{14} is the antidirected eight-cycle. Thus, applying the necessary conditions provided in Lemma 1 to the 21 directed cycles under consideration, we obtain the following necessary conditions.

Lemma 2. *Let $D \in \{D_2, D_3, \ldots, D_{22}\} \backslash \{D_{14}\}$ and let λ and v be positive integers such that $v \geq 8$. There exists a D-decomposition of ${}^{\lambda}K_v^*$ only if $\lambda v(v-1) \equiv 0 \pmod 8$. Furthermore, there exists a D_{14}-decomposition of ${}^{\lambda}K_v^*$ only if $\lambda v(v-1) \equiv 0 \pmod 8$ and $\lambda(v-1) \equiv 0 \pmod 2$.*

Proof. In accordance with Lemma 1, for $D \in \{D_2, D_3, \ldots, D_{22}\}$, it is evident that $|E(D)| = 8$, which divides $\lambda v(v-1)$. In other words, $\lambda v(v-1) \equiv 0 \pmod 8$. Additionally, for $D \in \{D_2, D_3, \ldots, D_{22}\} \backslash \{D_{14}\}$, $\gcd\{d^+(x) : x \in V(D)\} = \gcd\{d^-(x) : x \in V(D)\} = 1$, which always divides $\lambda(v-1)$. Furthermore, $\gcd\{d^+(x) : x \in V(D_{14})\} = \gcd\{d^-(x) : x \in V(D_{14})\} = 2$. Thus, the result follows. □

It was shown by Bermond, Huang, and Sotteau [19] in 1978 that these necessary conditions are sufficient for the directed eight-cycle D_1.

Theorem 2 ([19])**.** *For integer $v \geq 8$, there exists a D_1-decomposition of ${}^{\lambda}K_v^*$ if and only if $\lambda v(v-1) \equiv 0 \pmod 8$.*

The rest of this paper is devoted to establishing the sufficiency of the necessary conditions given in Lemma 2. We achieve this by exhibiting constructions for the desired decompositions (see Section 3) using certain small examples (see Section 2). Henceforth, each of the graphs in Figure 1, with vertices labeled as in the figure, will be represented by $D_i[u_1, u_1, \ldots, u_8]$. For instance, $D_5[0, 1, 2, \ldots, 7]$ refers to the digraph with vertex set $\{0, 1, 2, \ldots, 7\}$ and arc set $\{(1,0), (1,2), (2,3), (4,3), (4,5), (5,6), (6,7), (7,0)\}$.

We denote the *reverse orientation* of D by $\text{Rev}(D)$, which is the digraph with vertex set $V(D)$ and arc set $\{(v,u) : (u,v) \in E(D)\}$. It is important to note that, if graph G has a D-decomposition, then its reverse graph ($\text{Rev}(G)$) must have a $\text{Rev}(D)$-decomposition. In this paper, we will make use of the following fact:

Fact 1. *Let D and G be digraphs. A (G, D)-design exists if and only if a $(\text{Rev}(G), \text{Rev}(D))$-design exists.*

Since ${}^{\lambda}K_v^*$ is its own reverse orientation, we observe that the λ-fold spectrum of D and $\text{Rev}(D)$ are equivalent. This fact leads to the following corollary:

Corollary 1. *Let D be a digraph. A D-decomposition of ${}^{\lambda}K_v^*$ exists if and only if a $\text{Rev}(D)$-decomposition of ${}^{\lambda}K_v^*$ exists.*

The following result of Sotteau proves the existence of $2m$-cycle decompositions of complete bipartite graphs for $m \geq 2$. We will use this result to obtain a decomposition of $K_{2x,2y}^*$ into several oriented eight-cycles.

Theorem 3 ([20])**.** *Let x, y, and m be positive integers such that $m \geq 2$. There exists a $2m$-cycle decomposition of $K_{2x,2y}$ if and only if $m \mid 2xy$ and $\min\{2x, 2y\} \geq m$.*

Note that 8 of the 22 oriented eight-cycles of interest in this paper occur in pairs with respect to their reverse orientations (see Figure 1), namely, $\text{Rev}(D_{15}) \cong D_{19}$, $\text{Rev}(D_{16}) \cong D_{20}$, $\text{Rev}(D_{17}) \cong D_{21}$, and $\text{Rev}(D_{18}) \cong D_{22}$. The other 14 orientations in this paper are reverse orientations of themselves, namely, $\text{Rev}(D_i) \cong D_i$ for all $i \in [1, 14]$.

Consider an orientation of an eight-cycle that is isomorphic to its own reverse, i.e., any D_i in Figure 1 such that $i \notin [15, 22]$. By definition of reverse orientation, the set $\{D_i, \mathrm{Rev}(D_i)\}$ is an obvious D_i-decomposition of C_8^* (the symmetric digraph with an eight-cycle as the underlying simple graph). By Fact 1, we obtain the following corollary from the case $m = 4$ in Theorem 3.

Corollary 2. *Let $D \in \{D_1, D_2, \ldots, D_{14}\}$. There exists a D-decomposition of $K^*_{2x,2y}$ if $xy \equiv 0 \pmod 2$ and $\min\{x, y\} \geq 2$.*

2. Examples of Small Designs

In this section, we focus on the designs of small orders that will establish the existence of necessary base cases. These designs in Examples 1 to 4 were obtained through a computer search by the first author. Additionally, these decompositions are extensively utilized in the general constructions discussed in Section 3.

Given a digraph represented by the notation $D[u_1, u_2, \ldots, u_8]$ and $i \in \mathbb{Z}_n$, we define $D[u_1, u_2, \ldots, u_8] + i = D[u_1 + i, u_2 + i, \ldots, u_8 + i]$, where all addition is performed modulo n. Similarly, if the vertices of D are ordered pairs in $\mathbb{Z}_m \times \mathbb{Z}_n$, then $D\big[(u_1, v_1), (u_2, v_2), \ldots, (u_8, v_8)\big] + (i, 0)$ means the digraph $D\big[(u_1 + i, v_1), (u_2 + i, v_2), \ldots, (u_8 + i, v_8)\big]$. We also adopt the convention that both $\infty + i$ and $\infty + (i, 0)$ result in simply ∞.

2.1. Small Designs for $\lambda = 1$

Example 1. *Let the vertex set of K_8^* be $\mathbb{Z}_8$ and let*

$$\begin{aligned}
\mathcal{D}_5 = \{&D_5[0,1,7,5,4,3,6,2], D_5[0,6,7,1,5,3,2,4], D_5[1,0,5,6,4,7,2,3], D_5[2,1,5,0,3,7,6,4],\\
&D_5[3,7,0,6,2,5,4,1], D_5[4,0,2,1,6,3,5,7], D_5[7,0,3,4,1,6,5,2]\},\\
\mathcal{D}_{12} = \{&D_{12}[0,2,4,6,1,3,7,5], D_{12}[0,4,1,5,2,7,3,6], D_{12}[1,7,6,4,5,2,3,0], D_{12}[2,6,5,1,3,0,4,7],\\
&D_{12}[3,5,0,6,2,1,7,4], D_{12}[4,2,1,0,7,6,5,3], D_{12}[5,7,0,2,3,6,1,4]\},\\
\mathcal{D}_{13} = \{&D_{13}[0,6,3,4,7,1,5,2], D_{13}[0,4,6,2,7,3,1,5], D_{13}[2,7,0,5,6,1,4,3], D_{13}[4,1,2,5,3,6,7,0],\\
&D_{13}[5,3,7,4,2,1,0,6], D_{13}[5,4,2,3,0,1,6,7], D_{13}[6,2,0,3,1,7,5,4]\}.
\end{aligned}$$

Then, $\mathcal{D}_i$ forms a D_i-decomposition of K_8^* for $i \in \{5, 12, 13\}$.

Example 2. *Let the vertex set of K_8^* be $\mathbb{Z}_7 \cup \{\infty\}$ and let*

$$\begin{aligned}
\mathcal{D}_2 &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_2[0,1,3,6,4,5,2,\infty] + i\},\\
\mathcal{D}_3 &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_3[0,1,6,3,4,2,5,\infty] + i\},\\
\mathcal{D}_4 &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_4[0,1,6,2,3,5,\infty,4] + i\},\\
\mathcal{D}_6 &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_6[0,1,6,3,4,2,5,\infty] + i\},\\
\mathcal{D}_7 &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_7[0,1,3,6,2,4,5,\infty] + i\},\\
\mathcal{D}_8 &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_8[0,1,3,6,2,4,5,\infty] + i\},\\
\mathcal{D}_9 &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_9[0,1,2,5,3,6,4,\infty] + i\},\\
\mathcal{D}_{10} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{10}[0,1,5,3,2,6,4,\infty] + i\},\\
\mathcal{D}_{11} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{11}[0,1,6,3,4,2,5,\infty] + i\},\\
\mathcal{D}_{15} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{15}[0,1,3,6,2,4,5,\infty] + i\},\\
\mathcal{D}_{16} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{16}[0,1,3,4,6,2,5,\infty] + i\},\\
\mathcal{D}_{17} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{17}[0,1,6,2,3,5,\infty,4] + i\},\\
\mathcal{D}_{18} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{18}[0,1,5,3,2,6,4,\infty] + i\}.
\end{aligned}$$

Then, $\mathcal{D}_i$ forms a D_i-decomposition of K_8^* for $i \in [2, 18] \backslash \{5, 12, 13, 14\}$.

Example 3. *Let the vertex set of* K_9^* *be* $\mathbb{Z}_9$ *and let*

$$\begin{aligned}
\mathcal{D}_2 = \{&D_2[0,1,2,3,4,5,6,7], D_2[1,3,0,5,2,4,7,8], D_2[1,4,0,3,6,2,8,5], D_2[3,8,7,6,0,2,1,5],\\
&D_2[4,7,3,8,2,5,0,6], D_2[5,6,1,8,4,3,2,7], D_2[6,1,3,7,2,0,4,8], D_2[7,5,4,2,6,8,0,1],\\
&D_2[8,0,7,1,4,6,3,5]\},\\
\mathcal{D}_3 = \{&D_3[0,1,2,3,4,5,6,7], D_3[1,0,3,6,2,5,8,4], D_3[1,3,4,2,6,0,7,8], D_3[2,7,4,0,3,5,1,8],\\
&D_3[3,5,0,2,8,6,1,7], D_3[3,6,7,1,5,0,4,8], D_3[4,2,3,8,0,6,5,7], D_3[4,6,1,2,0,8,7,5],\\
&D_3[7,3,1,4,6,8,5,2]\},\\
\mathcal{D}_4 = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_4[0,1,3,2,7,4,8,6]+i\},\\
\mathcal{D}_5 = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_5[0,1,3,8,7,4,2,5]+i\},\\
\mathcal{D}_6 = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_6[0,1,3,8,5,4,2,6]+i\},\\
\mathcal{D}_7 = \{&D_7[0,1,2,3,4,5,6,7], D_7[0,2,1,3,5,4,6,8], D_7[1,8,3,7,4,2,6,0], D_7[2,7,5,6,3,0,4,8],\\
&D_7[3,0,4,1,5,2,7,8], D_7[3,2,6,0,8,7,5,1], D_7[6,3,5,2,8,4,1,7], D_7[6,8,5,0,2,4,7,1],\\
&D_7[7,3,4,6,1,8,5,0]\},\\
\mathcal{D}_8 = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_8[0,1,2,4,6,3,8,5]+i\},\\
\mathcal{D}_9 = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_9[0,1,2,5,3,8,4,6]+i\},\\
\mathcal{D}_{10} = \{&D_{10}[0,1,2,3,4,5,6,7], D_{10}[1,0,8,5,7,4,2,3], D_{10}[2,0,4,3,1,8,7,6], D_{10}[4,8,2,5,0,7,1,6],\\
&D_{10}[6,0,5,3,7,2,4,8], D_{10}[6,3,0,2,8,1,5,4], D_{10}[7,1,4,0,6,3,8,5], D_{10}[7,4,1,5,2,6,8,3],\\
&D_{10}[8,0,3,5,6,1,2,7]\},\\
\mathcal{D}_{11} = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_{11}[0,1,5,4,6,3,8,2]+i\},\\
\mathcal{D}_{12} = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_{12}[0,1,4,8,5,7,3,2]+i\},\\
\mathcal{D}_{13} = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_{13}[0,1,3,8,5,4,2,6]+i\},\\
\mathcal{D}_{14} = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_{14}[0,1,3,8,5,4,2,6]+i\},\\
\mathcal{D}_{15} = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_{15}[0,1,3,5,2,6,7,4]+i\},\\
\mathcal{D}_{16} = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_{16}[0,1,3,4,8,5,2,7]+i\},\\
\mathcal{D}_{17} = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_{17}[0,1,4,3,8,5,7,2]+i\},\\
\mathcal{D}_{18} = {}&\textstyle\bigcup_{i\in\mathbb{Z}_9}\{D_{18}[0,1,3,8,7,4,2,5]+i\}.
\end{aligned}$$

Then, $\mathcal{D}_i$ *forms a* D_i*-decomposition of* K_9^* *for* $i \in [2,18]$.

Example 4. *Let* $V(K_{4,4}^*) = \mathbb{Z}_8$ *with vertex partition* $\{0,1,2,3\} \cup \{4,5,6,7\}$ *and let*

$$\begin{aligned}
\mathcal{D}_{15} &= \{D_{15}[0,4,1,5,2,6,3,7], D_{15}[1,5,0,4,2,7,3,6], D_{15}[6,1,7,2,4,3,5,0], D_{15}[7,0,6,2,5,3,4,1]\},\\
\mathcal{D}_{16} &= \{D_{16}[0,4,1,5,2,6,3,7], D_{16}[1,5,0,4,2,7,3,6], D_{16}[6,0,5,3,4,2,7,1], D_{16}[7,1,4,3,5,2,6,0]\},\\
\mathcal{D}_{17} &= \{D_{17}[0,4,1,5,2,6,3,7], D_{17}[1,4,0,5,3,7,2,6], D_{17}[5,0,6,1,7,2,4,3], D_{17}[5,1,7,0,6,3,4,2]\},\\
\mathcal{D}_{18} &= \{D_{18}[0,4,1,5,2,6,3,7], D_{18}[2,4,3,5,0,6,1,7], D_{18}[4,0,5,1,7,3,6,2], D_{18}[7,0,6,1,4,3,5,2]\}.
\end{aligned}$$

Then, $\mathcal{D}_i$ *forms a* D_i*-decomposition of* $K_{4,4}^*$ *for* $i \in [15,18]$.

Example 5. *Let* $V(K_{5,8}^*) = (\mathbb{Z}_5 \times \mathbb{Z}_2) \cup \{\infty_0, \infty_1, \infty_2\}$. *For brevity, we use* i_j *to denote the ordered pair* $(i,j) \in V(K_{5,8}^*)$ *and we (continue to) use the convention that* $\infty_k + i_0 = \infty_k$ *for each* $k \in \{0,1,2\}$. *Thus, the vertex bipartition of* $K_{5,8}^*$ *is* $\{a_0 : a \in \mathbb{Z}_5\} \cup (\{a_1 : a \in \mathbb{Z}_5\} \cup \{\infty_0, \infty_1, \infty_2\})$. *Let*

$$\begin{aligned}
\mathcal{D}_{15} &= \textstyle\bigcup_{i\in\mathbb{Z}_5}\{D_{15}[0_0,0_1,1_0,2_1,4_0,\infty_0,3_0,\infty_1]+i_0, D_{15}[0_1,0_0,1_1,2_0,4_1,1_0,\infty_2,3_0]+i_0\},\\
\mathcal{D}_{16} &= \textstyle\bigcup_{i\in\mathbb{Z}_5}\{D_{16}[0_0,0_1,2_0,\infty_0,3_0,1_1,4_0,\infty_1]+i_0, D_{16}[0_1,0_0,1_1,2_0,3_1,4_0,\infty_2,3_0]+i_0\},\\
\mathcal{D}_{17} &= \textstyle\bigcup_{i\in\mathbb{Z}_5}\{D_{17}[0_0,\infty_0,2_0,2_1,1_0,0_1,3_0,\infty_1]+i_0, D_{17}[0_1,0_0,\infty_2,4_0,1_1,3_0,4_1,1_0]+i_0\}.
\end{aligned}$$

Then, $\mathcal{D}_i$ *forms a* D_i*-decomposition of* $K_{5,8}^*$ *for* $i \in [15,17]$.

Example 6. *Let* $V(K^*_{6,8}) = \{a_b : a \in \mathbb{Z}_6 \text{ and } b \in \mathbb{Z}_2\} \cup \{\infty_0, \infty_1\}$ *with vertex partition* $\{a_0 : a \in \mathbb{Z}_6\} \cup (\{a_1 : a \in \mathbb{Z}_6\} \cup \{\infty_0, \infty_1\})$. *Let*

$$\mathcal{D}_{18} = \bigcup_{i\in\mathbb{Z}_6}\{D_{18}[0_1, 0_0, 1_1, 1_0, 3_1, 4_0, 2_1, 5_0] + i_0, D_{18}[0_0, \infty_0, 1_0, 3_1, 5_0, 2_1, 3_0, \infty_1] + i_0\}.$$

Then, $\mathcal{D}_{18}$ *forms a* D_{18}*-decomposition of* $K^*_{6,8}$.

Example 7. *Let* $V(K^*_{7,8}) = \{a_b : a \in \mathbb{Z}_7 \text{ and } b \in \mathbb{Z}_2\} \cup \{\infty\}$ *with vertex partition* $\{a_0 : a \in \mathbb{Z}_7\} \cup (\{a_1 : a \in \mathbb{Z}_7\} \cup \{\infty\})$. *Let*

$$\begin{aligned}
\mathcal{D}_{15} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{15}[0_0, 0_1, 1_0, 1_1, 3_0, 4_1, 6_0, 3_1] + i_0, D_{15}[0_1, 5_0, 6_1, 0_0, 4_1, 2_0, \infty, 4_0] + i_0\},\\
\mathcal{D}_{16} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{16}[0_0, 0_1, 4_0, 4_1, 3_0, 2_1, 6_0, \infty] + i_0, D_{16}[0_1, 2_0, 4_1, 0_0, 1_1, 4_0, 6_1, 1_0] + i_0\},\\
\mathcal{D}_{17} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{17}[0_0, 0_1, 5_0, 2_1, 4_0, 3_1, 2_0, \infty] + i_0, D_{17}[0_1, 0_0, 3_1, 4_0, 2_1, 5_0, 1_1, 6_0] + i_0\},\\
\mathcal{D}_{18} &= \textstyle\bigcup_{i\in\mathbb{Z}_7}\{D_{18}[0_1, 0_0, 1_1, 2_0, 4_1, 6_0, 3_1, 3_0] + i_0, D_{18}[0_0, 3_1, 1_0, 2_1, 3_0, 1_1, 5_0, \infty] + i_0\}.
\end{aligned}$$

Then, $\mathcal{D}_i$ *forms a* D_i*-decomposition of* $K^*_{7,8}$ *for* $i \in [15, 18]$.

2.2. Small Designs for $\lambda = 2$

Example 8. *Let the vertex set of* ${}^2K^*_8$ *be* $\mathbb{Z}_7 \cup \{\infty\}$ *and let*

$$\mathcal{D}_{14} = \bigcup_{i\in\mathbb{Z}_7}\{D_{14}[0, 1, 3, 6, 2, 4, 5, \infty] + i, D_{14}[0, 1, 3, 5, 6, 2, \infty, 4] + i\}.$$

Then, $\mathcal{D}_{14}$ *forms a* D_{14}*-decomposition of* ${}^2K^*_8$.

Example 9. *Let the vertex set of* ${}^2K^*_{12}$ *be* $\mathbb{Z}_{11} \cup \{\infty\}$ *and let*

$$\begin{aligned}
\mathcal{D}_2 &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_2[0, 3, 1, 5, 10, 9, 4, \infty] + i), D_2[0, 9, 1, 2, 3, 5, 8, 4] + i\},\\
\mathcal{D}_3 &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_3[0, 1, 3, 7, 2, 10, 4, \infty] + i), D_3[0, 10, 9, 1, 3, 5, 8, 4] + i\},\\
\mathcal{D}_4 &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_4[0, 1, 5, 7, 2, 10, 4, \infty] + i), D_4[0, 10, 2, 1, 3, 5, 8, 4] + i\},\\
\mathcal{D}_5 &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_5[0, 1, 5, 3, 8, 2, 10, \infty] + i), D_5[0, 10, 1, 3, 2, 5, 8, 4] + i\},\\
\mathcal{D}_6 &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_6[0, 1, 5, 2, 7, 9, 4, \infty] + i), D_6[0, 10, 1, 3, 6, 5, 8, 4] + i\},\\
\mathcal{D}_7 &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_7[0, 1, 3, 9, 2, 8, 5, \infty] + i), D_7[0, 4, 1, 9, 5, 6, 8, 10] + i\},\\
\mathcal{D}_8 &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_8[0, 1, 5, 7, 4, 9, 3, \infty] + i), D_8[0, 4, 5, 2, 9, 6, 8, 10] + i\},\\
\mathcal{D}_9 &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_9[0, 1, 5, 7, 4, 9, 3, \infty] + i), D_9[0, 4, 5, 2, 3, 10, 7, 9] + i\},\\
\mathcal{D}_{10} &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_{10}[0, 1, 3, 8, 4, 9, 6, \infty] + i), D_{10}[0, 4, 1, 10, 9, 5, 6, 8] + i\},\\
\mathcal{D}_{11} &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_{11}[0, 1, 3, 6, 10, 4, 9, \infty] + i), D_{11}[0, 4, 2, 1, 3, 10, 7, 8] + i\},\\
\mathcal{D}_{12} &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_{12}[0, 1, 3, 8, 2, \infty, 4, 7] + i), D_{12}[0, 4, 1, 3, 2, 9, 10, 8] + i\},\\
\mathcal{D}_{13} &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{D_{13}[0, 1, 3, 8, 5, 10, 6, \infty] + i, D_{13}[0, 4, 1, 2, 9, 7, 6, 8] + i\},\\
\mathcal{D}_{14} &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{D_{14}[0, 1, 2, 3, 4, 6, 8, 5] + i, D_{14}[0, 2, 4, 1, 5, 8, 3, \infty] + i,\\
&\qquad D_{14}[0, 3, 8, 1, 5, 9, \infty, 6] + i\},\\
\mathcal{D}_{15} &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_{15}[0, 1, 3, 7, 2, 8, 5, \infty] + i), D_{15}[0, 4, 1, 2, 10, 6, 7, 9] + i\},\\
\mathcal{D}_{16} &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_{16}[0, 1, 3, 7, 2, 5, 10, \infty] + i), D_{16}[0, 4, 1, 2, 9, 6, 8, 10] + i\},\\
\mathcal{D}_{17} &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_{17}[0, 1, 3, 6, 10, 4, 9, \infty] + i), D_{17}[0, 4, 1, 10, 6, 5, 7, 8] + i\},\\
\mathcal{D}_{18} &= \textstyle\bigcup_{i\in\mathbb{Z}_{11}}\{2(D_{18}[0, 1, 3, 7, 2, 10, 4, \infty] + i), D_{18}[0, 4, 1, 3, 2, 9, 7, 8] + i\}.
\end{aligned}$$

Then, $\mathcal{D}_i$ *forms a* D_i*-decomposition of* ${}^2K^*_{12}$ *for* $i \in [2, 18]$.

Example 10. *Let the vertex set of* $^2K^*_{13}$ *be* $\mathbb{Z}_{13}$ *and let*

$$\begin{aligned}
\mathcal{D}_2 &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_2[0,8,1,2,4,3,12,9]+i), D_2[0,10,4,2,5,3,11,6]+i\},\\
\mathcal{D}_3 &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_3[0,1,8,4,5,2,6,11]+i), D_3[0,5,10,4,7,1,12,2]+i\},\\
\mathcal{D}_4 &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_4[0,1,2,9,5,7,4,8]+i), D_4[0,5,3,8,2,9,7,10]+i\},\\
\mathcal{D}_5 &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_5[0,1,2,4,11,3,12,9]+i), D_5[0,5,3,6,11,1,8,2]+i\},\\
\mathcal{D}_6 &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_6[0,1,2,6,3,10,12,4]+i), D_6[0,5,3,1,4,9,12,6]+i\},\\
\mathcal{D}_7 &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_7[0,1,4,8,9,2,6,11]+i), D_7[0,2,4,10,5,8,3,6]+i\},\\
\mathcal{D}_8 &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_8[0,1,2,5,7,11,3,9]+i), D_8[0,2,5,7,10,3,11,6]+i\},\\
\mathcal{D}_9 &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_9[0,1,2,5,7,12,3,9]+i), D_9[0,2,5,11,6,1,3,10]+i\},\\
\mathcal{D}_{10} &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{10}[0,1,4,12,8,9,2,11]+i), D_{10}[0,2,7,4,10,5,3,6]+i\},\\
\mathcal{D}_{11} &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{11}[0,1,4,12,5,6,2,11]+i), D_{11}[0,2,7,4,11,6,12,10]+i\},\\
\mathcal{D}_{12} &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{12}[0,1,4,6,2,3,12,7]+i), D_{12}[0,2,7,1,11,6,4,10]+i\},\\
\mathcal{D}_{13} &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{13}[0,1,5,2,3,10,6,8]+i), D_{13}[0,2,7,1,9,6,12,10]+i\},\\
\mathcal{D}_{14} &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{14}[0,1,2,5,7,3,9,4]+i), D_{14}[0,2,5,10,8,1,9,6]+i\},\\
\mathcal{D}_{15} &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{15}[0,1,4,5,9,2,6,8]+i), D_{15}[0,2,4,12,5,8,3,6]+i\},\\
\mathcal{D}_{16} &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{16}[0,1,4,5,7,11,2,8]+i), D_{16}[0,2,4,7,10,3,11,5]+i\},\\
\mathcal{D}_{17} &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{17}[0,1,8,4,5,9,6,11]+i), D_{17}[0,2,7,4,12,5,3,6]+i\},\\
\mathcal{D}_{18} &= \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{18}[0,1,4,6,2,3,11,7]+i), D_{18}[0,2,7,1,11,6,12,10]+i\}.
\end{aligned}$$

Then, $\mathcal{D}_i$ *forms a* D_i*-decomposition of* $^2K^*_{13}$ *for* $i \in [2,18]$.

2.3. Small Designs for $\lambda = 4$

Example 11. *Let the vertex set of* $^4K^*_{10}$ *be* $\mathbb{Z}_9 \cup \{\infty\}$ *and let*

$$\begin{aligned}
\mathcal{D}_2 &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_2[0,2,1,4,5,7,3,\infty]+i), 2(D_2[0,4,1,3,6,7,2,\infty]+i), D_2[0,2,1,5,4,8,6,3]+i\},\\
\mathcal{D}_3 &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_3[0,1,3,4,6,2,5,\infty]+i), 2(D_3[0,3,1,6,7,2,5,\infty]+i), D_3[0,1,2,6,4,8,5,3]+i\},\\
\mathcal{D}_4 &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_4[0,1,2,4,7,3,5,\infty]+i), 2(D_4[0,3,4,2,7,1,5,\infty]+i), D_4[0,1,5,6,3,7,4,2]+i\},\\
\mathcal{D}_5 &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_5[0,1,2,5,3,8,6,\infty]+i), 2(D_5[0,3,4,6,1,5,8,\infty]+i), D_5[0,1,7,5,6,4,8,3]+i\},\\
\mathcal{D}_6 &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_6[0,1,5,3,4,2,7,\infty]+i), 2(D_6[0,3,4,6,1,5,8,\infty]+i), D_6[0,1,4,2,5,7,6,3]+i\},\\
\mathcal{D}_7 &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_7[0,1,3,7,8,2,4,\infty]+i), 2(D_7[0,3,2,7,1,6,8,\infty]+i), D_7[0,1,6,2,8,7,5,3]+i\},\\
\mathcal{D}_8 &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_8[0,1,2,4,6,3,8,\infty]+i), 2(D_8[0,3,4,2,5,1,6,\infty]+i), D_8[0,1,8,4,2,7,6,3]+i\},\\
\mathcal{D}_9 &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_9[0,1,2,6,4,7,5,\infty]+i), 2(D_9[0,3,4,1,5,7,2,\infty]+i), D_9[0,3,1,6,5,4,7,2]+i\},\\
\mathcal{D}_{10} &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_{10}[0,1,3,2,8,4,6,\infty]+i), 2(D_{10}[0,3,1,5,2,6,7,\infty]+i), D_{10}[0,1,3,6,2,8,7,5]+i\},\\
\mathcal{D}_{11} &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_{11}[0,1,5,3,4,2,8,\infty]+i), 2(D_{11}[0,3,1,6,2,5,4,\infty]+i), D_{11}[0,1,3,8,5,4,7,2]+i\},\\
\mathcal{D}_{12} &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_{12}[0,1,3,5,2,\infty,4,8]+i), 2(D_{12}[0,3,1,2,6,\infty,8,5]+i), D_{12}[0,1,4,2,7,6,3,5]+i\},\\
\mathcal{D}_{13} &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_{13}[0,1,3,5,8,7,2,\infty]+i), 2(D_{13}[0,3,1,4,8,7,2,\infty]+i), D_{13}[0,1,3,2,8,4,7,5]+i\},\\
\mathcal{D}_{14} &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_{14}[0,1,2,4,6,3,8,\infty]+i), 2(D_{14}[0,3,4,1,6,2,\infty,5]+i), D_{14}[0,1,3,6,5,7,4,2]+i\},\\
\mathcal{D}_{15} &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_{15}[0,1,3,5,2,7,8,\infty]+i), 2(D_{15}[0,3,1,5,2,7,8,\infty]+i), D_{15}[0,1,6,3,8,5,4,2]+i\},\\
\mathcal{D}_{16} &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_{16}[0,1,3,6,7,2,4,\infty]+i), 2(D_{16}[0,3,1,2,7,4,8,\infty]+i), D_{16}[0,1,6,4,3,8,5,2]+i\},\\
\mathcal{D}_{17} &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_{17}[0,1,3,2,7,4,6,\infty]+i), 2(D_{17}[0,3,1,5,8,7,2,\infty]+i), D_{17}[0,1,4,6,3,8,7,2]+i\},\\
\mathcal{D}_{18} &= \bigcup_{i\in\mathbb{Z}_9}\{2(D_{18}[0,1,3,5,2,7,6,\infty]+i), 2(D_{18}[0,3,1,5,2,7,6,\infty]+i), D_{18}[0,1,4,2,7,6,8,3]+i\}.
\end{aligned}$$

Then, $\mathcal{D}_i$ *forms a* D_i*-decomposition of* $^4K^*_{10}$ *for* $i \in [2,18]$.

Example 12. *Let the vertex set of $^4K^*_{11}$ be $\mathbb{Z}_{11}$ and let*

$$\mathcal{D}_2 = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_2[0,1,2,4,7,3,8,5]+i), D_2[0,1,3,7,4,8,6,2]+i, D_2[0,7,1,2,6,4,10,8]+i\},$$
$$\mathcal{D}_3 = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_3[0,1,6,2,3,5,10,8]+i), D_3[0,3,2,1,5,9,6,4]+i, D_3[0,5,2,4,1,6,3,7]+i\},$$
$$\mathcal{D}_4 = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_4[0,1,2,7,3,5,8,6]+i), D_4[0,3,2,1,5,9,6,4]+i, D_4[0,5,2,10,1,6,3,7]+i\},$$
$$\mathcal{D}_5 = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_5[0,1,2,4,9,3,10,8]+i), D_5[0,3,2,6,5,1,9,7]+i, D_5[0,5,2,4,1,6,3,7]+i\},$$
$$\mathcal{D}_6 = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_6[0,1,2,4,9,3,10,8]+i), D_6[0,3,1,5,9,8,7,4]+i, D_6[0,5,2,4,1,9,3,7]+i\},$$
$$\mathcal{D}_7 = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_7[0,1,7,6,2,4,10,8]+i), D_7[0,1,3,2,6,10,7,4]+i), D_7[0,3,6,2,4,7,1,5]+i\},$$
$$\mathcal{D}_8 = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_8[0,1,3,2,9,4,7,5]+i), D_8[0,1,5,4,8,10,6,3]+i, D_8[0,3,5,1,4,7,2,6]+i\},$$
$$\mathcal{D}_9 = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_9[0,1,3,2,9,4,10,8]+i), D_9[0,2,1,5,6,3,10,7]+i, D_9[0,3,5,1,4,10,2,6]+i\},$$
$$\mathcal{D}_{10} = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_{10}[0,1,3,7,6,8,2,5]+i), D_{10}[0,1,3,6,2,9,10,7]+i, D_{10}[0,3,1,6,2,5,10,7]+i\},$$
$$\mathcal{D}_{11} = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_{11}[0,1,3,7,2,5,4,6]+i), D_{11}[0,1,4,6,2,3,10,7]+i, D_{11}[0,3,1,8,2,5,10,7]+i\},$$
$$\mathcal{D}_{12} = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_{12}[0,1,3,4,9,5,8,6]+i), D_{12}[0,1,4,8,10,6,7,3]+i, D_{12}[0,3,1,4,9,2,10,6]+i\},$$
$$\mathcal{D}_{13} = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_{13}[0,1,3,4,7,5,10,6]+i), D_{13}[0,1,4,8,5,7,3,10]+i, D_{13}[0,3,1,4,8,2,9,6]+i\},$$
$$\mathcal{D}_{14} = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_{14}[0,1,2,4,6,3,10,5]+i), D_{14}[0,1,2,9,6,8,4,7]+i, D_{14}[0,3,5,1,9,4,10,7]+i\},$$
$$\mathcal{D}_{15} = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_{15}[0,1,7,3,2,4,10,8]+i), D_{15}[0,2,1,5,6,3,7,4]+i, D_{15}[0,3,6,1,8,2,4,7]+i\},$$
$$\mathcal{D}_{16} = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_{16}[0,1,7,2,4,3,10,8]+i), D_{16}[0,2,1,5,9,10,6,3]+i, D_{16}[0,3,6,8,2,9,1,5]+i\},$$
$$\mathcal{D}_{17} = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_{17}[0,1,3,2,5,9,4,6]+i), D_{17}[0,1,3,6,2,9,10,7]+i, D_{17}[0,3,1,6,9,5,2,7]+i\},$$
$$\mathcal{D}_{18} = \bigcup_{i\in\mathbb{Z}_{11}}\{3(D_{18}[0,1,3,4,2,5,10,6]+i), D_{18}[0,1,4,8,10,6,2,3]+i, D_{18}[0,3,1,4,9,6,2,7]+i\}.$$

Then, $\mathcal{D}_i$ forms a D_i-decomposition of $^4K^*_{11}$ for $i \in [2,18]$.

Example 13. *Let the vertex set of $^4K^*_{14}$ be $\mathbb{Z}_{13} \cup \{\infty\}$ and let*

$$\mathcal{D}_2 = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_2[0,1,2,4,7,11,3,\infty]+i), 2(D_2[0,6,1,7,2,9,5,3]+i), D_2[0,3,1,7,4,10,6,2]+i\},$$
$$\mathcal{D}_3 = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_3[0,1,9,10,12,2,6,\infty]+i), 2(D_3[0,5,12,6,3,1,7,4]+i), D_3[0,2,4,11,7,1,9,5]+i\},$$
$$\mathcal{D}_4 = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_4[0,1,2,10,12,3,6,\infty]+i), 2(D_4[0,5,1,8,2,12,9,7]+i), D_4[0,2,9,11,6,1,8,4]+i\},$$
$$\mathcal{D}_5 = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_5[0,1,2,4,12,3,6,\infty]+i), 2(D_5[0,5,1,8,2,12,9,7]+i), D_5[0,2,9,5,7,3,11,6]+i\},$$
$$\mathcal{D}_6 = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_6[0,1,2,4,7,3,8,\infty]+i), 2(D_6[0,5,1,7,4,11,8,2]+i), D_6[0,2,9,3,12,1,10,5]+i\},$$
$$\mathcal{D}_7 = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_7[0,1,9,5,6,8,11,\infty]+i), 2(D_7[0,2,5,12,6,1,7,3]+i), D_7[0,5,11,4,12,8,6,2]+i\},$$
$$\mathcal{D}_8 = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_8[0,1,2,10,12,8,11,\infty]+i), 2(D_8[0,2,8,1,7,11,6,3]+i), D_8[0,5,1,7,2,8,6,4]+i\},$$
$$\mathcal{D}_9 = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_9[0,1,2,11,3,6,4,\infty]+i), 2(D_9[0,2,8,1,9,6,10,3]+i), D_9[0,5,1,7,3,11,4,2]+i\},$$
$$\mathcal{D}_{10} = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_{10}[0,1,9,5,3,4,7,\infty]+i), 2(D_{10}[0,2,7,1,6,3,10,4]+i), D_{10}[0,2,5,9,3,1,7,4]+i\},$$
$$\mathcal{D}_{11} = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_{11}[0,1,9,6,8,12,11,\infty]+i), 2(D_{11}[0,2,6,12,7,1,8,3]+i), D_{11}[0,2,5,9,6,4,11,7]+i\},$$
$$\mathcal{D}_{12} = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_{12}[0,1,9,10,7,\infty,2,11]+i), 2(D_{12}[0,2,6,1,7,4,11,5]+i), D_{12}[0,2,5,1,8,6,3,7]+i\},$$
$$\mathcal{D}_{13} = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_{13}[0,1,9,10,12,8,5,\infty]+i), 2(D_{13}[0,2,7,1,10,4,9,3]+i), D_{13}[0,2,5,1,12,3,10,7]+i\},$$
$$\mathcal{D}_{14} = \bigcup_{i\in\mathbb{Z}_{13}}\{2(D_{14}[0,1,2,10,12,8,1,\infty]+i), 2(D_{14}[0,1,3,2,7,4,\infty,9]+i), 2(D_{14}[0,2,8,1,11,3,12,5]+i),$$
$$D_{14}[0,2,8,11,7,1,12,3]+i\},$$
$$\mathcal{D}_{15} = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_{15}[0,1,9,10,6,8,11,\infty]+i), 2(D_{15}[0,2,5,1,8,3,9,6]+i), D_{15}[0,5,11,6,12,8,4,2]+i\},$$
$$\mathcal{D}_{16} = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_{16}[0,1,9,10,12,8,11,\infty]+i), 2(D_{16}[0,2,5,1,9,3,10,7]+i), D_{16}[0,5,11,7,2,8,6,4]+i\},$$
$$\mathcal{D}_{17} = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_{17}[0,1,9,5,6,4,7,\infty]+i), 2(D_{17}[0,2,5,9,3,6,1,7]+i), D_{17}[0,2,7,11,6,12,10,4]+i\},$$
$$\mathcal{D}_{18} = \bigcup_{i\in\mathbb{Z}_{13}}\{4(D_{18}[0,1,9,10,6,8,5,\infty]+i), 2(D_{18}[0,2,5,11,1,8,12,7]+i), D_{18}[0,2,6,1,7,5,10,4]+i\}.$$

Then, $\mathcal{D}_i$ forms a D_i-decomposition of $^4K^*_{14}$ for $i \in [2,18]$.

Example 14. *Let the vertex set of* ${}^4K_{15}^*$ *be* $\mathbb{Z}_{15}$ *and let*

$\mathcal{D}_2 = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_2[0,6,1,5,4,9,2,8]+i), 2(D_2[0,2,3,5,1,13,9,12]+i), D_2[0,2,3,1,13,10,12,14]+i\}$,
$\mathcal{D}_3 = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_3[0,1,6,10,2,8,13,7]+i), 2(D_3[0,3,2,4,7,5,1,12]+i), D_3[0,3,2,4,1,14,12,13]+i\}$,
$\mathcal{D}_4 = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_4[0,1,5,10,2,7,13,6]+i), 2(D_4[0,3,5,4,2,13,1,12]+i), D_4[0,3,5,4,1,14,12,13]+i\}$,
$\mathcal{D}_5 = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_5[0,1,5,12,2,7,13,6]+i), 2(D_5[0,3,1,12,11,7,10,13]+i), D_5[0,3,4,2,1,14,11,13]+i\}$,
$\mathcal{D}_6 = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_6[0,1,5,10,2,7,13,6]+i), 2(D_6[0,3,1,4,7,6,2,13]+i), D_6[0,3,1,13,11,10,12,14]+i\}$,
$\mathcal{D}_7 = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_7[0,3,11,7,1,6,12,5]+i), 2(D_7[0,1,3,7,6,8,9,12]+i), D_7[0,2,1,3,5,6,8,4]+i\}$,
$\mathcal{D}_8 = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_8[0,3,7,1,9,4,11,5]+i), 2(D_8[0,1,2,4,3,7,9,12]+i), D_8[0,2,3,5,7,6,8,4]+i\}$,
$\mathcal{D}_9 = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_9[0,3,7,1,8,2,12,5]+i), 2(D_9[0,1,2,4,3,5,9,12]+i), D_9[0,2,3,5,1,12,11,13]+i\}$,
$\mathcal{D}_{10} = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_{10}[0,1,7,12,6,10,3,8]+i), 2(D_{10}[0,3,1,5,2,13,11,12]+i), D_{10}[0,3,5,4,2,14,12,13]+i\}$,
$\mathcal{D}_{11} = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_{11}[0,3,8,1,5,11,2,7]+i), 2(D_{11}[0,1,2,4,5,6,10,12]+i), D_{11}[0,2,6,3,1,12,10,13]+i\}$,
$\mathcal{D}_{12} = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_{12}[0,3,8,1,7,11,2,10]+i), 2(D_{12}[0,1,2,3,5,6,8,12]+i), D_{12}[0,2,6,4,1,3,14,12]+i\}$,
$\mathcal{D}_{13} = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_{13}[0,3,8,1,5,11,4,9]+i), 2(D_{13}[0,1,2,3,4,6,10,12]+i), D_{13}[0,2,6,8,4,1,14,12]+i\}$,
$\mathcal{D}_{14} = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_{14}[0,3,7,12,2,8,1,9]+i), 2(D_{14}[0,1,2,3,4,6,8,12]+i), D_{14}[0,2,4,8,6,3,14,12]+i\}$,
$\mathcal{D}_{15} = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_{15}[0,6,1,3,10,14,2,8]+i), 2(D_{15}[0,1,2,12,11,7,5,3]+i), D_{15}[0,3,6,2,7,8,4,5]+i\}$,
$\mathcal{D}_{16} = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_{16}[0,1,10,2,7,3,11,5]+i), 2(D_{16}[0,2,5,1,12,9,10,13]+i), D_{16}[0,2,4,1,13,11,12,14]+i\}$,
$\mathcal{D}_{17} = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_{17}[0,3,8,1,5,11,2,7]+i), 2(D_{17}[0,1,2,4,5,9,10,12]+i), D_{17}[0,2,6,3,1,14,10,12]+i\}$,
$\mathcal{D}_{18} = \bigcup_{i\in\mathbb{Z}_{15}} \{4(D_{18}[0,3,8,1,7,11,4,9]+i), 2(D_{18}[0,1,2,3,5,6,10,12]+i), D_{18}[0,2,6,4,1,12,10,13]+i\}$.

Then, $\mathcal{D}_i$ *forms a* D_i*-decomposition of* ${}^4K_{15}^*$ *for* $i \in [2,18]$.

3. General Constructions

The *union* $G \cup H$ of two edge-disjoint graphs (or digraphs) G and H has as vertex set and edge (or arc) set the unions of the vertex sets and edge (or arc) sets, respectively, of G and H. Moreover, given a positive integer α, we will denote the edge-disjoint union of α copies of G by αG, which are not necessarily vertex-disjoint. If G and H are vertex-disjoint, then we will denote the *join* of G and H by $G \vee H$, which has vertex set $V(G) \cup V(H)$ and edge (or arc) set $E(G) \cup E(H) \cup \{\{u,v\} : u \in V(G), v \in V(H)\}$. To illustrate the different types of notation described here, consider that K_{17} can be viewed as $(K_8 \cup K_8) \vee K_1 \cup K_{8,8} = K_9 \cup K_9 \cup K_{8,8}$. Note that the join precedes the union in the order of operations.

We begin by establishing a lemma concerning the decompositions of $K_{8,8}^*$, $K_{8,10}^*$, $K_{8,12}^*$, and $K_{8,14}^*$.

Lemma 3. *For each* $i \in [2,18]$ *and each* $y \in \{8,10,12,14\}$*, there exists* D_i*-decomposition of* $K_{8,y}^*$.

Proof. If $D \in \{D_2, D_3, \ldots, D_{14}\}$, then the result follows from Corollary 2.

For $i \in [15,18]$, a D_i-decomposition of $K_{4,4}^*$ and hence of $K_{8,8}^*$ and $K_{8,12}^*$ exists (since $K_{8,8}^* = 4K_{4,4}^*$ and $K_{8,12}^* = 6K_{4,4}^*$), as given in Example 4.

For $i \in [15,17]$, we have the required D_i-decomposition of $K_{8,10}^*$ since $K_{8,10}^* = 2K_{5,8}^*$, and $K_{5,8}^*$ has a D_i-decomposition by Example 5. In the case of $i = 18$, note that $K_{8,10}^* = K_{4,8}^* \cup K_{6,8}^* = 2K_{4,4}^* \cup K_{6,8}^*$, and the result follows from the existence of D_{18}-decompositions of $K_{4,4}^*$ and $K_{6,8}^*$, where the latter decomposition follows from Example 6.

Finally, a D_i-decomposition of $K_{7,8}^*$ and hence of $K_{8,14}^*$ exists (since $K_{8,14}^* = 2K_{7,8}^*$), as given in Example 7 for $i \in [15,18]$. □

In the subsequent lemmata, we present our constructions for decomposing ${}^\lambda K_v^*$, which cover values of v working modulo 8. Afterward, we summarize the main result in Theorem 1.

Lemma 4. *Let* λ *and* v *be positive integers such that* $v \equiv 0 \pmod 8$ *and* $v \geq 8$*. If* $D \in \{D_2, D_3, \ldots, D_{18}\} \backslash \{D_{14}\}$*, then there exists a* D*-decomposition of* ${}^\lambda K_v^*$*. Furthermore, if* λ *is even, then there exists a* D_{14}*-decomposition of* ${}^\lambda K_v^*$.

Proof. Let x be a nonnegative integer and $v = 8x$. If $v = 8$ and $i \in [2, 18] \backslash \{14\}$, then the result can be obtained from λ copies of a D_i-decomposition of K_8^* found in Examples 1 and 2. If $v = 8$, λ is even, and $D = D_{14}$, then the result follows from $\lambda/2$ copies of a D_{14}-decomposition of ${}^2K_8^*$ found in Example 8. Therefore, we may assume $x \geq 2$ and λ is even whenever $D = D_{14}$.

We note that K_{8x} can be represented as $xK_8 \cup \binom{x}{2}K_{8,8}$. Thus, ${}^\lambda K_{8x}^* = x\left({}^\lambda K_8^*\right) \cup \binom{x}{2}\left({}^\lambda K_{8,8}^*\right)$, and the result follows from the existence of D_i-decompositions of ${}^\lambda K_8^*$ and ${}^\lambda K_{8,8}^*$ for $i \in [2, 18]$, where the latter decomposition follows from λ copies of a D_i-decomposition of $K_{8,8}^*$ by Lemma 3. $\square$

Lemma 5. *Let λ and v be positive integers such that $v \equiv 1 \pmod 8$ and $v \geq 9$. If $D \in \{D_2, D_3, \ldots, D_{18}\}$, then there exists a D-decomposition of ${}^\lambda K_v^*$.*

Proof. When v is 9, the result follows from λ copies of a D_i-decomposition of K_9^* for each $i \in [2, 18]$ (see Example 3). Henceforth, during the remaining part of the proof, we let $v = 8x + 1$ for some integer $x \geq 2$.

Now, let $i \in [2, 18]$. Note that K_{8x+1} can be represented as $(xK_8) \vee K_1 \cup \binom{x}{2}K_{8,8} = xK_9 \cup \binom{x}{2}K_{8,8}$. Thus, ${}^\lambda K_{8x+1}^* = x\left({}^\lambda K_9^*\right) \cup \binom{x}{2}\left({}^\lambda K_{8,8}^*\right)$, and the result follows from the existence of D_i-decompositions of ${}^\lambda K_9^*$ and ${}^\lambda K_{8,8}^*$, where the latter decomposition follows from λ copies of a D_i-decomposition of $K_{8,8}^*$ (see Lemma 3). $\square$

Lemma 6. *Let λ and v be positive integers such that $\lambda \equiv 0 \pmod 4$, $v \equiv 2 \pmod 8$ and $v \geq 10$. If $D \in \{D_2, D_3, \ldots, D_{18}\}$, then there exists a D-decomposition of ${}^\lambda K_v^*$.*

Proof. Let x be a nonnegative integer and let $v = 8x + 2$. By Example 11, there exists a D_i-decomposition of ${}^4K_{10}^*$ for each $i \in [2, 18]$. Therefore, if $x = 1$, then the result follows from $\lambda/4$ copies of a D_i-decomposition of ${}^4K_{10}^*$. Hence, we may assume $x \geq 2$.

We note that $K_{8x+2} = (x-1)K_8 \cup K_{10} \cup (x-1)K_{8,10} \cup \binom{x-1}{2}K_{8,8}$. Thus, ${}^\lambda K_{8x+2}^*$ can be represented as $(x-1)\left({}^\lambda K_8^*\right) \cup {}^\lambda K_{10}^* \cup (x-1)\left({}^\lambda K_{8,10}^*\right) \cup \binom{x-1}{2}\left({}^\lambda K_{8,8}^*\right)$. It is shown in the Examples 1 and 2 that K_8^*, and hence ${}^\lambda K_8^*$, admits a D_i-decomposition for $i \in [2, 18]$. Moreover, D_i-decompositions of ${}^\lambda K_{8,8}^*$ and ${}^\lambda K_{8,10}^*$ follow from λ copies of a D_i-decomposition of $K_{8,8}^*$ and $K_{8,10}^*$ (see Lemma 3). Now, the result follows. $\square$

Lemma 7. *Let λ and v be positive integers such that $\lambda \equiv 0 \pmod 4$, $v \equiv 3 \pmod 8$ and $v \geq 11$. If $D \in \{D_2, D_3, \ldots, D_{18}\}$, then there exists a D-decomposition of ${}^\lambda K_v^*$.*

Proof. Let x be a nonnegative integer and let $v = 8x + 3$. By Example 12, there exists a D_i-decomposition of ${}^4K_{11}^*$ for each $i \in [2, 18]$. Therefore, if $x = 1$, then the result follows from $\lambda/4$ copies of a D_i-decomposition of ${}^4K_{11}^*$. Hence, we may assume $x \geq 2$.

We note that $K_{8x+3} = ((x-1)K_8 \cup K_{10}) \vee K_1 \cup (x-1)K_{8,10} \cup \binom{x-1}{2}K_{8,8}$. Thus, ${}^\lambda K_{8x+2}^*$ can be represented as $(x-1)\left({}^\lambda K_9^*\right) \cup {}^\lambda K_{11}^* \cup (x-1)\left({}^\lambda K_{8,10}^*\right) \cup \binom{x-1}{2}\left({}^\lambda K_{8,8}^*\right)$. It is shown in Example 3 that K_9^*, and hence ${}^\lambda K_9^*$, admits a D_i-decomposition for $i \in [2, 18]$. Then the proof proceeds similarly to that of Lemma 6. $\square$

Lemma 8. *Let λ and v be positive integers such that $\lambda \equiv 0 \pmod 2$, $v \equiv 4 \pmod 8$ and $v \geq 12$. If $D \in \{D_2, D_3, \ldots, D_{18}\}$, then there exists a D-decomposition of ${}^\lambda K_v^*$.*

Proof. Let x be a nonnegative integer and let $v = 8x + 4$. By Example 9, there exists a D_i-decomposition of ${}^2K_{12}^*$ for each $i \in [2, 18]$. Therefore, if $x = 1$, then the result follows from $\lambda/2$ copies of a D_i-decomposition of ${}^2K_{12}^*$. Hence, we may assume $x \geq 2$.

We note that $K_{8x+4} = (x-1)K_8 \cup K_{12} \cup (x-1)K_{8,12} \cup \binom{x-1}{2}K_{8,8}$. Thus, ${}^\lambda K_{8x+4}^*$ can be represented as $(x-1)\left({}^\lambda K_8^*\right) \cup {}^\lambda K_{12}^* \cup (x-1)\left({}^\lambda K_{8,12}^*\right) \cup \binom{x-1}{2}\left({}^\lambda K_{8,8}^*\right)$. It is shown in Examples

1 and 2 that K_8^*, and hence ${}^{\lambda}K_8^*$, admits a D_i-decomposition for $i \in [2,18]$. Furthermore, D_i-decompositions of ${}^{\lambda}K_{8,8}^*$ and ${}^{\lambda}K_{8,12}^*$ follow from λ copies of a D_i-decomposition of $K_{8,8}^*$ and $K_{8,10}^*$ (see Lemma 3). Now, the result follows. □

Lemma 9. *Let λ and v be positive integers such that $\lambda \equiv 0 \pmod 2$, $v \equiv 5 \pmod 8$ and $v \geq 13$. If $D \in \{D_2, D_3, \ldots, D_{18}\}$, then there exists a D-decomposition of ${}^{\lambda}K_v^*$.*

Proof. Let x be a nonnegative integer and let $v = 8x + 5$. By Example 10, there exists a D_i-decomposition of ${}^{2}K_{13}^*$ for each $i \in [2,18]$. Therefore, if $x = 1$, then the result follows from $\lambda/2$ copies of a D_i-decomposition of ${}^{2}K_{13}^*$. Hence, we may assume $x \geq 2$.

We note that $K_{8x+5} = ((x-1)K_8 \cup K_{12}) \vee K_1 \cup (x-1)K_{8,12} \cup \binom{x-1}{2}K_{8,8}$. Thus, ${}^{\lambda}K_{8x+5}^*$ can be represented as $(x-1)\left({}^{\lambda}K_9^*\right) \cup {}^{\lambda}K_{13}^* \cup (x-1)\left({}^{\lambda}K_{8,12}^*\right) \cup \binom{x-1}{2}\left({}^{\lambda}K_{8,8}^*\right)$. It is shown in Example 3 that K_9^*, and hence ${}^{\lambda}K_9^*$, admits a D_i-decomposition for $i \in [2,18]$. Then the proof proceeds similarly to that of Lemma 8. □

Lemma 10. *Let λ and v be positive integers such that $\lambda \equiv 0 \pmod 4$, $v \equiv 6 \pmod 8$ and $v \geq 14$. If $D \in \{D_2, D_3, \ldots, D_{18}\}$, then there exists a D-decomposition of ${}^{\lambda}K_v^*$.*

Proof. Let x be a nonnegative integer and let $v = 8x + 6$. By Example 13, there exists a D_i-decomposition of ${}^{4}K_{14}^*$ for each $i \in [2,18]$. Therefore, if $x = 1$, then the result follows from $\lambda/4$ copies of a D_i-decomposition of ${}^{4}K_{14}^*$. Hence, we may assume $x \geq 2$.

We note that $K_{8x+6} = (x-1)K_8 \cup K_{14} \cup (x-1)K_{8,14} \cup \binom{x-1}{2}K_{8,8}$. Thus, ${}^{\lambda}K_{8x+6}^*$ can be represented as $(x-1)\left({}^{\lambda}K_8^*\right) \cup {}^{\lambda}K_{14}^* \cup (x-1)\left({}^{\lambda}K_{8,14}^*\right) \cup \binom{x-1}{2}\left({}^{\lambda}K_{8,8}^*\right)$. It is shown in Examples 1 and 2 that K_8^*, and hence ${}^{\lambda}K_8^*$, admits a D_i-decomposition for $i \in [2,18]$. Moreover, D_i-decompositions of ${}^{\lambda}K_{8,8}^*$ and ${}^{\lambda}K_{8,14}^*$ follow from λ copies of a D_i-decomposition of $K_{8,8}^*$ and $K_{8,14}^*$ (see Lemma 3). Now, the result follows. □

Lemma 11. *Let λ and v be positive integers such that $\lambda \equiv 0 \pmod 4$, $v \equiv 7 \pmod 8$ and $v \geq 15$. If $D \in \{D_2, D_3, \ldots, D_{18}\}$, then there exists a D-decomposition of ${}^{\lambda}K_v^*$.*

Proof. Let x be a nonnegative integer and let $v = 8x + 7$. By Example 14, there exists a D_i-decomposition of ${}^{4}K_{15}^*$ for each $i \in [2,18]$. Therefore, if $x = 1$, then the result follows from $\lambda/2$ copies of a D_i-decomposition of ${}^{4}K_{15}^*$. Hence, we may assume $x \geq 2$.

We note that $K_{8x+7} = ((x-1)K_8 \cup K_{14}) \vee K_1 \cup (x-1)K_{8,14} \cup \binom{x-1}{2}K_{8,8}$. Thus, ${}^{\lambda}K_{8x+7}^*$ can be represented as $(x-1)\left({}^{\lambda}K_9^*\right) \cup {}^{\lambda}K_{15}^* \cup (x-1)\left({}^{\lambda}K_{8,14}^*\right) \cup \binom{x-1}{2}\left({}^{\lambda}K_{8,8}^*\right)$. It is shown in Example 3 that K_9^*, and hence ${}^{\lambda}K_9^*$, admits a D_i-decomposition for $i \in [2,18]$. Then the proof proceeds similarly to that of Lemma 10. □

4. Proof of Main Result

Combining the previous results from Lemmata 4 through 11, it is now possible to obtain the proof of Theorem 1, which gives a complete solution to the λ-fold spectrum problem for all possible orientations of the eight-cycle.

Proof of Theorem 1. The necessity follows from Lemma 2. Now, we prove the sufficiency. Let $v \geq 8$ and $D \in \{D_1, D_2, \ldots, D_{18}\}$. If $D = D_1$, sufficiency is guaranteed by Theorem 2. When $v \equiv 0 \pmod 8$, and λ is even for $D = D_{14}$, then sufficiency follows from Lemma 4. In the case $v \equiv 1 \pmod 8$, the result follows from Lemma 5. If $\lambda \equiv 0 \pmod 4$ and $v \equiv 2, 3, 6$, or $7 \pmod 8$, the results follow from, respectively, Lemmata 6, 7, 10, and 11. Furthermore, we have the required decompositions when $\lambda \equiv 0 \pmod 2$ and $v \equiv 4$ or $5 \pmod 8$ by Lemma 8 and Lemma 9, respectively.

For $i \in [15,18]$, the existence of a $({}^{\lambda}K_v^*, D_i)$-design is equivalent to the existence of a $({}^{\lambda}K_v^*, D_{i+4})$-design, given that D_i is the reverse orientation of D_{i+4}. Thus, we now have the sufficient conditions that allow ${}^{\lambda}K_v^*$ to be decomposed into oriented eight-cycles. □

When $\lambda = 1$, the spectrum problem for oriented cycles with orders $3, 4, 5, 6$, and 7 has already been addressed in the literature [9–13,15,17,18]. With the proof Theorem 1, the spectrum problem for oriented eight-cycles has now been solved. Furthermore, until now, C_6 was the only cycle for which the λ-fold spectrum problem had been solved (in [17]) for all its orientations. However, with the proof of the theorem above, the λ-fold spectrum problem has been solved for all oriented eight-cycles.

5. Conclusions

The main focus of this article was on the λ-fold spectrum problem concerning non-isomorphic orientations of an eight-cycle. The necessary conditions for such a decomposition are that $\lambda v(v-1) \equiv 0 \pmod 8$ and $\lambda(v-1)$ is even in the case of an antidirected cycle. Out of the twenty-two oriented eight-cycles, the problem has been settled for only one of them [19]. For all the remaining orientations, we have shown that these necessary conditions are also sufficient.

In future research, our initial focus will be on investigating the λ-fold spectrum problem for oriented heptagons. Subsequently, we aim to address the spectrum problem for antidirected cycles of arbitrary orders. Additionally, exploring a more comprehensive version of the spectrum problem concerning directed cycles of any order is of particular interest.

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