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Inexact Iterates of Nonexpansive Mappings with Summable Errors in Metric Spaces with Graphs

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Abstract: In our joint work with Dan Butnariu (2006) we established the stability of the convergence of iterates of a nonexpansive mapping on a complete metric space in the presence of summable computational errors. In a recent paper of ours, we extended this result to inexact iterates of nonexpansive mappings on complete metric spaces with graphs under a certain assumption on the iterates. In the present paper we obtain an analogous result by removing that assumption on the iterates and replacing it with an additional assumption on the graph.

Keywords: complete metric space; fixed point; graph; inexact iterate; nonexpansive mapping

MSC: 47H09; 47H10; 54E50

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1. Introduction

During the last sixty years, many interesting developments have taken place in the fixed point theory of nonlinear mappings [1–13]. The origin of these investigations is Banach's classical work [14]. Since Banach's celebrated theorem, many important and interesting results have been obtained in this field, including results on common fixed point problems, feasibility, iterative projection algorithms, and variational inequalities. In addition to their own importance, these results have numerous useful applications in various areas of research [13,15–18].

In [2], the authors considered a nonexpansive mapping acting on a complete metric space under the assumption that every sequence of its iterates converges to a fixed point. It was shown there that every sequence of inexact iterates of such a mapping with summable errors converges to a fixed point as well. This result is a generalization of a classical result of Ostrowski [8], which was obtained for strict contractions. In [19], we established an analog of this result for nonexpansive mappings acting on complete metric spaces with graphs under a certain assumption on the iterates. This result is an extension of an analogous theorem which was proved in [20] for strict contractions. In this connection, note that the investigation of mappings in metric spaces with graphs is now of great research interest [5,9,10,21–24]. In the present work, we obtain an analog of the main result of [19] by removing the assumption on the iterates and replacing it with an additional assumption on the graph.

At this point, it is worth recalling that the study of the behavior of inexact iterates is very important, as computational errors are always produced in calculations. Therefore, this has been and continues to be an important topic in analysis, beginning with the seminal paper [8], though see [2,13] as well.

Let (X, ρ) be a complete metric space and let G be a graph. We assume that the set of vertices $V(G)$ of G is contained in the space X, its set of edges $E(G)$ is a closed subset of the space $X \times X$ endowed with the product metric, and that the following assumption holds.

(A1) For each pair of points $u, v \in X$, if $(u, v) \in E(G)$, then $(T(u), T(v)) \in E(G)$ and the inequality

$$\rho(T(u), T(v)) \leq \rho(u, v)$$

is true.

Assumption (A1) appears in [21] (see [22,23]).

For each $\xi \in X$ and every $E \subset X$, put

$$\rho(\xi, E) := \inf\{\rho(\xi, \eta) : \eta \in E\}.$$

For every $\xi \in X$ and every $\Delta > 0$, set

$$B(\xi, \Delta) := \{\eta \in X : \rho(\xi, \eta) \leq \Delta\}.$$

For each map $G : X \to X$, let $G^0\eta = \eta$ for all $\eta \in X$.

In [19], we established the following result.

Theorem 1. *Assume that a sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *satisfies*

$$(T(x_i), x_{i+1}) \in E(G), \ i = 0, 1, \ldots$$

and

$$\sum_{i=0}^{\infty} \rho(T(x_i), x_{i+1}) < \infty,$$

and that a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ *is given. Then, the following assertions hold.*

1. Assume that for each integer k, the sequence $\{T^j(x_{i_k})\}_{j=1}^{\infty}$ *converges. Then, there exists*

$$x_* = \lim_{i \to \infty} x_i,$$

$(x_*, T(x_*)) \in E(G)$*, and if T is continuous at the point* x_**, then* $T(x_*) = x_*$*.*

2. Assume that there exists a nonempty set F such that for each integer $k \geq 1$*,*

$$\lim_{j \to \infty} \rho(T^j(x_{i_k}), F) = 0.$$

Then,

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

3. Assume that for each integer $k \geq 1$ *there exists a nonempty compact set* $E_k \subset X$ *such that*

$$\lim_{j \to \infty} \rho(T^j(x_{i_k}), E_k) = 0.$$

Then, there exists a nonempty compact set $E \subset X$ *such that* $\lim_{i \to \infty} \rho(x_i, E) = 0$*.*

In our work, we obtain an extension of this result without assuming that

$$(T(x_i), x_{i+1}) \in E(G), \ i = 0, 1, \ldots.$$

Instead, we assume that a certain property of the graph is satisfied (see (A2) in the next section).

2. The Main Result

We use all the definitions and notations from Section 1 and assume that all the assumptions introduced in Section 1 hold. We then prove the following result.

Theorem 2. *Assume that a sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *satisfies*

$$\sum_{i=0}^{\infty} \rho(T(x_i), x_{i+1}) < \infty, \tag{1}$$

$c_0 \geq 1$, $r_0 \in (0,1]$ *and that the following assumption holds.*

(A2) for each integer $i \geq 0$ *and each point* $z \in B(x_i, r_0)$*, there exists*

$$\xi \in B(x_i, c_0\rho(z, x_i))$$

such that

$$(x_i, \xi),\ (z, \xi) \in E(G).$$

Let a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ *be given; then, the following assertions hold true:*

1. Assume that the sequence $\{T^j(x_{i_k})\}_{j=1}^{\infty}$ *converges for each integer k. Then, there exists*

$$x_* = \lim_{i\to\infty} x_i,$$

$(x_*, T(x_*)) \in E(G)$*, and if T is continuous at the point* x_**, then* $T(x_*) = x_*$.

2. Assume that there exists a nonempty set F such that, for each integer $k \geq 1$*,*

$$\lim_{j\to\infty} \rho(T^j(x_{i_k}), F) = 0.$$

Then,

$$\lim_{i\to\infty} \rho(x_i, F) = 0.$$

3. Assume that for each integer $k \geq 1$ *there exists a nonempty compact set* $E_k \subset X$ *such that*

$$\lim_{j\to\infty} \rho(T^j(x_{i_k}), E_k) = 0.$$

Then, there exists a nonempty compact set $E \subset X$ *such that* $\lim_{i\to\infty} \rho(x_i, E) = 0$.

Example 1. *Assume that* $(Z, \|\cdot\|)$ *is a Banach space ordered by a closed convex cone* Z_+ *(*$x \leq y$ *for* $x, y \in Z$ *if and only if* $y - x \in Z_+$*) such that*

$$Z_+ - Z_+ = Z.$$

Then, per the Krein–Shmulyan theorem [25], there exists $c_0 > 0$ *such that for each* $z \in Z$ *there exist points* $z_1, z_2 \in Z_+$ *such that*

$$z = z_1 - z_2,\ \|z_i\| \leq c_0\|z\|,\ i = 1, 2.$$

Let $V = X$ *be a nonempty closed subset of Z, with* $(x, y) \in E(G)$ *if and only if* $y \geq x$ *and* $\rho(x, y) = \|x - y\|$, $x, y \in X$.

Assume that $r_0 \in (0,1]$, $c_0 \geq 1$, $\{x_i\}_{i=0}^{\infty} \subset X$ *and that*

$$B(x_i, c_0 r_0) \subset X,\ i = 0, 1, \ldots.$$

Assume that $i \geq 0$ *is an integer and that* $z \in X$ *satisfies*

$$\|z - x_i\| \leq r_0.$$

Then, there exist $v_1, v_2 \in Z_+$ *such that*

$$z - x_2 = v_2 - v_1,\ \|v_j\| \leq c_0\|z - x_i\|,\ j = 1, 2.$$

We now have

$$z = x_i + v_2 - v_1 \leq x_i + v_2,$$

$$x_i \leq x_i + v_2.$$

Setting

$$\xi = x_i + v_2,$$

it is easy to see that

$$\rho(x_i, \xi) = \|v_2\| \le c_0\|z - x_i\| = \rho(z, x_i)c_0,$$

and that

$$\rho(x_i, \xi) = \|v_2\| \le c_0\|z - x_i\| \le c_0 r_0.$$

Thus, (A2) holds.

3. An Auxiliary Result

Lemma 1. *Assume that a sequence $\{x_i\}_{i=0}^{\infty}$ satisfies*

$$\sum_{i=0}^{\infty} \rho(T(x_i), x_{i+1}) < \infty$$

and that there exist numbers $c_0 \ge 1$ and $r_0 \in (0, 1]$ such that (A2) holds. Let $q \ge 0$ be an integer such that

$$\sum_{i=q}^{\infty} \rho(T(x_i), x_{i+1}) < r_0. \tag{2}$$

Then, for each integer $n \ge 1$,

$$\rho(x_{n+q}, T^n(x_q)) \le (2c_0 + 1) \sum_{i=0}^{n-1} \rho(T(x_{q+i}), x_{q+i+1}).$$

Proof. Setting

$$y_{q,0} = x_q, \tag{3}$$

$$y_{q+1,0} = T(y_{q,0}),\ y_{q+1,1} = x_{q+1}, \tag{4}$$

per (2)–(4) we have

$$\rho(y_{q+1,0}, x_{q+1}) \le \rho(T(x_q), x_{q+1}) < r_0. \tag{5}$$

Assumption (A2) and (5) imply that there exists $y_{q+1,1} \in X$ such that

$$(x_{q+1}, y_{q+1,1}) \in E(G),\ (y_{q+1,0}, y_{q+1,1}) \in E(G), \tag{6}$$

$$\rho(x_{q+1}, y_{q+1,1}) \le c_0\rho(x_{q+1}, y_{q+1,0}) \le c_0 r_0. \tag{7}$$

Per (3), (4), and (7), we have

$$\rho(y_{q+1,1}, y_{q+1,0}) \le \rho(x_{q+1}, y_{q+1,1}) + \rho(x_{q+1}, y_{q+1,0}),$$

$$\le (c_0 + 1)\rho(x_{q+1}, y_{q+1,0}) = (c_0 + 1)\rho(x_{q+1}, T(x_q)). \tag{8}$$

Setting

$$y_{q+1,2} = x_{q+1}, \tag{9}$$

assume that $n \ge 1$ is a natural number and that we have defined

$$y_{q+n,i} \in X,\ i = 0, \dots, 2n,$$

such that

$$y_{q+n,0} = T^n(x_q),\ y_{q+n,2n} = x_{q+n} \tag{10}$$

for each $i \in \{0, \dots, 2n - 1\}$ and at least one of the following relations holds:

$$(y_{q+n,i}, y_{q+n,i+1}) \in E(G),\ (y_{q+n,i+1}, y_{q+n,i}) \in E(G), \tag{11}$$

$$\rho(y_{q+n,2n}, y_{q+n,2n-1}) \le c_0\rho(x_{q+n}, T(x_{q+n-1})), \tag{12}$$

$$\rho(y_{q+n,2n-1}, y_{q+n,2n-2}) \le (c_0 + 1)\rho(x_{q+n}, T(x_{q+n-1})), \tag{13}$$

and

$$\rho(x_{q+n}, T^n(x_q)) \le \sum_{i=0}^{2n-1} \rho(y_{q+n,i}, y_{q+n,i+1}),$$

$$\le (2c_0 + 1) \sum_{i=0}^{n-1} (x_{q+i+1}, T(x_{q+i})). \tag{14}$$

In view of (2)–(4) and (6)–(9), our assumption holds for $n = 1$. We now define $y_{q+n+1,i} \in X$, $i = 0, \ldots, 2n+2$. For $i = 0, \ldots, 2n-1$, set

$$y_{q+n+1,i} = T(y_{q+n,i}). \tag{15}$$

Assumption (A1), (10), and (15) imply that for $i = 0, \ldots, 2n-1$ at least one of the following two inclusions holds:

$$(y_{q+n+1,i}, y_{q+n+1,i+1}) \in E(G),\ (y_{q+n+1,i+1}, y_{q+n+1,i}) \in E(G), \tag{16}$$

$$\rho(y_{q+n+1,i}, y_{q+n+1,i+1}) \le \rho(y_{q+n,i}, y_{q+n,i+1}), \tag{17}$$

$$y_{q+n+1,0} = T^{n+1}(x_q), \tag{18}$$

and

$$y_{q+n+1,2n} = T(x_{q+n}). \tag{19}$$

In view of (2),

$$\rho(T(x_{q+n}), x_{q+n+1}) < r_0. \tag{20}$$

Assumption (A2), (19), and (20) imply that there exists

$$y_{q+n+1,2n+1} \in X$$

such that

$$(y_{q+n+1,2n}, y_{q+n+1,2n+1}) \in E(G), \tag{21}$$

$$(T(x_{q+n}), y_{q+n+1,2n+1}) \in E(G), \tag{22}$$

$$(x_{q+n+1}, y_{q+n+1,2n+1}) \in E(G), \tag{23}$$

and

$$\rho(x_{q+n+1}, y_{q+n+1,2n+1}) \le c_0 \rho(x_{q+n+1}, T(x_{q+n})). \tag{24}$$

Setting

$$y_{q+n+1,2n+2} = x_{q+n+1}, \tag{25}$$

per (24) and (25) we have

$$\rho(y_{q+n+1,2n+2}, y_{q+n+1,2n+1}) \le c_0 \rho(x_{q+n+1}, T(x_{q+n})). \tag{26}$$

It follows from (19), (25), and (26) that

$$\rho(y_{q+n+1,2n+1}, y_{q+n+1,2n}),$$

$$\le \rho(y_{q+n+1,2n+1}, x_{q+n+1}) + \rho(x_{q+n+1}, T(x_{q+n})),$$

$$\le (c_0 + 1)\rho(x_{q+n+1}, T(x_{q+n})). \tag{27}$$

It now follows from (14), (18), and (25)–(27) that

$$\rho(T^{n+1}(x_q), x_{q+n+1}) = \rho(y_{q+n+1,0}, y_{q+n+1,2n+2}),$$

$$\le \sum_{i=0}^{2n+1} \rho(y_{q+n+1,i}, y_{q+n+1,i+1}),$$

$$= \sum_{i=0}^{2n-1} \rho(y_{q+n+1,i}, y_{q+n+1,i+1}),$$
$$+\rho(y_{q+n+1,2n}, y_{q+n+1,2n+1}) + \rho(y_{q+n+1,2n+1}, y_{q+n+1,2n+2}),$$
$$\leq \sum_{i=0}^{2n-1} \rho(y_{q+n,i}, y_{q+n,i+1}),$$
$$+(2c_0+1)\rho(T(x_{q+n}), x_{q+n+1}),$$
$$\leq (2c_0+1)\sum_{i=0}^{n} \rho(x_{q+i+1}, T(x_{q+i})). \tag{28}$$

Per (18), (23), and (25)–(28), the assumption made for n holds for $n+1$ as well. Thus, using induction, for each integer $n \geq 1$ we have defined $y_{q+n,i} \in X$, $i = 0, \ldots, 2n$, such that (10)–(14) hold. This completes the proof of Lemma 1. □

4. Proof of Theorem 2

Let

$$\epsilon \in (0, r_0/2). \tag{29}$$

Per (1), there exists an integer $k \geq 1$ such that

$$\sum_{j=i_k-1}^{\infty} \rho(x_{j+1}, T(x_j)) < (2c_0+1)^{-1}\epsilon/4. \tag{30}$$

Lemma 1 and inequality (30) imply that for each integer $n \geq 1$, we have

$$\rho(x_{n+i_k}, T^n(x_{i_k})) \leq (2c_0+1) \sum_{j=i_k}^{i_k+n-1} \rho(T(x_j), x_{j+1}) < \epsilon/4. \tag{31}$$

We first prove Assertion 1. There exists

$$y_k = \lim_{j\to\infty} T^j(x_{i_k}).$$

Per (31), for all sufficiently large natural numbers n,

$$\rho(x_{n+i_k}, y_k) \leq \rho(y_k, T^n(x_{i_k})) + \rho(T^n(x_{i_k}), x_{i_k+n}) < \epsilon.$$

Because ϵ is an arbitrary sufficiently small positive number, we conclude that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and that there exists the limit

$$x_* = \lim_{n\to\infty} x_n.$$

It follows from (1) that

$$x_* = \lim_{i\to\infty} T(x_i).$$

Because the set $E(G)$ is closed, Assumption (A2) implies that

$$(x_*, x_*) \in E(G).$$

If the mapping T is continuous at the point x_*, then we have $T(x_*) = x_*$. Assertion 1 is proved.

Next, we prove Assertion 2. Based on our assumptions,

$$\lim_{j\to\infty} \rho(T^j(x_{i_k}), F) = 0.$$

When combined with (31), this implies that for each sufficiently large natural number n we have

$$\rho(x_{n+i_k}, F) \leq \rho(x_{n+i_k}, T^n(x_{i_k})) + \rho(T^n(x_{i_k}), F) < \epsilon.$$

Because ϵ is an arbitrary sufficiently small positive number, we can conclude that

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

Assertion 2 is proved.

Finally, we prove Assertion 3. There exists a compact set $E_0 \subset X$ such that

$$\lim_{n \to \infty} \rho(T^n(x_k), E_0) = 0.$$

Per (31), for each sufficiently large natural number n we have

$$\rho(x_{n+i_k}, E_0) \leq \rho(x_{n+i_k}, T^n(x_{i_k})) + \rho(T^n(x_{i_k}), E_0) < \epsilon.$$

Thus, we have shown that there exists a compact set E_0 such that

$$\rho(x_{n+i_k}, E_0) < \epsilon$$

for every sufficiently large natural number n. We may assume that E_0 is finite. Because ϵ is any element of the interval $(0, r_0/2)$, this implies that each subsequence of $\{x_i\}_{i=0}^{\infty}$ has a convergent subsequence. Denoting the set of all limit points of the sequence $\{x_i\}_{i=0}^{\infty}$ by E, it is not difficult to see that E is compact and that

$$\lim_{i \to \infty} \rho(x_i, E) = 0.$$

This completes the proof of Assertion 3 and of Theorem 2 itself.

5. Conclusions

In the present paper, we have shown that if all exact orbits of a nonexpansive self-mapping of a complete metric space with a graph converge, then this convergence property holds for all its inexact orbits with summable errors as well. This is an analog of the result of [2] for inexact iterates of nonexpansive mappings defined on complete metric spaces. In this connection, we recall that the study of the behavior of inexact iterates is very important, as computational errors are always present in calculations. Therefore, this is a rapidly growing area of research, starting with the seminal paper [8] (although see [2,13] and references mentioned therein as well). Our results show that if all exact iterates converge, then inexact iterates with summable errors converge as well.

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References

1. Betiuk-Pilarska, A.; Benavides, T.D. Fixed points for nonexpansive mappings and generalized nonexpansive mappings on Banach lattices. *Pure Appl. Func. Anal.* **2016**, *1*, 343–359.

2. Butnariu, D.; Reich, S.; Zaslavski, A.J. Convergence to fixed points of inexact orbits of Bregman-monotone and of nonexpansive operators in Banach spaces. In *Fixed Point Theory and Its Applications*; Yokohama Publishers: Yokohama, Japan, 2006; pp. 11–32.
3. de Blasi, F.S.; Myjak, J. Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach. *C. R. Acad. Sci. Paris* **1976**, *283*, 185–187.
4. Goebel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory*; Cambridge University Press: Cambridge, UK, 1990.
5. Jachymski, J. Extensions of the Dugundji-Granas and Nadler's theorems on the continuity of fixed points. *Pure Appl. Funct. Anal.* **2017**, *2*, 657–666.
6. Kirk, W.A. Contraction mappings and extensions. In *Handbook of Metric Fixed Point Theory*; Kluwer: Dordrecht, The Netherlands, 2001; pp. 1–34.
7. Kubota, R.; Takahashi, W.; Takeuchi, Y. Extensions of Browder's demiclosedness principle and Reich's lemma and their applications. *Pure Appl. Func. Anal.* **2016**, *1*, 63–84.
8. Ostrowski, A.M. The round-off stability of iterations. *Z. Angew. Math. Mech.* **1967**, *47*, 77–81. [CrossRef]
9. Petruşel, A.; Petruşel, G.; Yao, J.-C. Multi-valued graph contraction principle with applications. *Optimization* **2020**, *69*, 1541–1556. [CrossRef]
10. Petruşel, A.; Petruşel, G.; Yao, J.-C. Graph contractions in vector-valued metric spaces and applications. *Optimization* **2021**, *70*, 763–775. [CrossRef]
11. Rakotch, E. A note on contractive mappings. *Proc. Amer. Math. Soc.* **1962**, *13*, 459–465. [CrossRef]
12. Reich, S.; Zaslavski, A.J. Genericity in nonlinear analysis. In *Developments in Mathematics*; Springer: New York, NY, USA, 2014; Volume 34.
13. Zaslavski, A.J. Approximate solutions of common fixed point problems. In *Springer Optimization and Its Applications*; Springer: Cham, Switzerland, 2016.
14. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
15. Censor, Y.; Zaknoon, M. Algorithms and convergence results of projection methods for inconsistent feasibility problems: A review. *Pure Appl. Func. Anal.* **2018**, *3*, 565–586.
16. Gibali, A. A new split inverse problem and an application to least intensity feasible solutions. *Pure Appl. Funct. Anal.* **2017**, *2*, 243–258.
17. Takahashi, W. The split common fixed point problem and the shrinking projection method for new nonlinear mappings in two Banach spaces. *Pure Appl. Funct. Anal.* **2017**, *2*, 685–699.
18. Takahashi, W. A general iterative method for split common fixed point problems in Hilbert spaces and applications. *Pure Appl. Funct. Anal.* **2018**, *3*, 349–369.
19. Reich, S.; Zaslavski, A.J. Convergence of inexact iterates of nonexpansive mappings with summable errors in metric spaces with graphs. 2023, *submitted*.
20. Reich, S.; Zaslavski, A.J. Convergence of inexact iterates of strict contractions in metric spaces with graphs. *J. Appl. Numer. Optim.* **2022**, *4*, 215–220.
21. Aleomraninejad, S.M.A.; Rezapour, S.; Shahzad, N. Some fixed point results on a metric space with a graph. *Topol. Appl.* **2012**, *159*, 659–663. [CrossRef]
22. Bojor, F. Fixed point of ϕ-contraction in metric spaces endowed with a graph. *An. Univ. Craiova Ser. Mat. Inform.* **2010**, *37*, 85–92.
23. Bojor, F. Fixed point theorems for Reich type contractions on metric spaces with a graph. *Nonlinear Anal.* **2012**, *75*, 3895–3901. [CrossRef]
24. Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Amer. Math. Soc.* **2008**, *136*, 1359–1373. [CrossRef]
25. Vulikh, B.Z. Introduction to the Theory of Cones in Normed Spaces; *Kalinin: Kalinin State Univ* , Russia, 1977. (In Russian)