symmetry

MDPI

Article

The Existence and Averaging Principle for a Class of Fractional Hadamard Itô–Doob Stochastic Integral Equations

Mohamed Rhaima [1], Lassaad Mchiri [2] and Abdellatif Ben Makhlouf [3,*]

[1] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[2] Panthéon-Assas University Paris II, 92 Rue d'Assas, 75006 Paris, France; lassaad.mchiri@u-paris2.fr
[3] Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Road of Soukra, Sfax 1171, Tunisia
* Correspondence: benmakhloufabdellatif@gmail.com

Abstract: In this paper, we investigate the existence and uniqueness properties pertaining to a class of fractional Hadamard Itô–Doob stochastic integral equations (FHIDSIE). Our study centers around the utilization of the Picard iteration technique (PIT), which not only establishes these fundamental properties but also unveils the remarkable averaging principle within FHIDSIE. To accomplish this, we harness powerful mathematical tools, including the Hölder and Gronwall inequalities.

Keywords: stochastic system; Hadamard fractional integral; Gronwall inequality

MSC: 34A08; 60H10; 34C29

check for updates

Citation: Rhaima, M.; Mchiri, L.; Ben Makhlouf, A. The Existence and Averaging Principle for a Class of Fractional Hadamard Itô–Doob Stochastic Integral Equations. *Symmetry* **2023**, *15*, 1910. https://doi.org/10.3390/sym15101910

Academic Editors: Junesang Choi and Ji-Huan He

Received: 25 August 2023
Revised: 10 October 2023
Accepted: 11 October 2023
Published: 12 October 2023

1. Introduction

The theory of fractional calculus is an ancient subject that appeared in the same century as classical calculus. The question of fractional derivatives was raised as early as 1695 by Leibniz in a letter to l'Hospital, but when Leibniz asked what the derivative of a function means when the order is $\frac{1}{2}$, Leibniz replied that this leads to a paradox. This letter to l'Hospital was considered the first incident of fractional calculus theory. Later, Euler's functions made the definition of fractional operators possible. However, a systematic development of the subject did not occur until the nineteenth century with Riemann, Liouville, Grünwald, and Letnikov (see [1–3]).

Fractional calculus has practical applications across a range of disciplines, including biology, machine learning, and physics. A notable example is found in the study of anomalous diffusion phenomena, where fractional variants of the diffusion equation serve as indispensable tools for investigation (as highlighted in references such as [4–6]). These applications underscore the versatility and potency of fractional calculus in unraveling intricate phenomena that conventional calculus may struggle to address effectively.

Caputo–Hadamard and Hadamard derivatives are crucial classes of fractional derivatives. The scientific community has extensively studied Hadamard fractional integral equations (HFIE); for further details, readers can consult [7–9]. The FHIDSIE represents a significant category within HFIE, drawing the attention of numerous researchers (see [10,11]). In [12], the authors analyzed the existence and uniqueness of solutions using PIT and the averaging principle of FHIDSIE.

Inspired by the analysis above, this paper investigates the averaging principle for FHIDSIE. In the literature, most published articles that study the averaging principle focus on ordinary stochastic differential equations. In this sense, this paper extends the results in [12] to the neutral delay FHIDSIE. Therefore, in this paper, our main aims are:

1. Analyze the existence and uniqueness of the solution of FHIDSIE by employing the PIT;

2. Investigate the averaging principle of FHIDSIE using the famous stochastic calculus techniques.

The structure of this article is summarized as follows. Section 2 is devoted to several definitions and classical results. In Section 3, we prove the existence and uniqueness result using the PIT. In Section 4, we present two theorems related to demonstrating the averaging principle of FHIDSIE. Section 5 is dedicated to showcasing an application related to the studied problem. Finally, we provide a conclusion in Section 6.

2. Basic Notions

Let $\mathcal{D} = \{\mathcal{N}, \mathfrak{Z}, (\mathfrak{Z}_{\varkappa})_{\varkappa\geq 1}, \mathfrak{P}\}$ be a complete probability space and $\mathcal{W}(\varkappa)$ be a standard Brownian motion with respect the space $\mathcal{D}$. Denote by $C([1-\phi,1];\mathbb{R}^n)$ the family of functions φ from $[1-\phi,1]$ to $\mathbb{R}^n$ that are right-continuous and have limits on the left. $C([1-\phi,1];\mathbb{R}^n)$ is equipped with the norm $\|\varphi\| = \sup_{1-\phi\leq r\leq 1}|\varphi(r)|$ and $|y| = \sqrt{y^T y}$ for any $y \in \mathbb{R}^n$. Denote by $L^2_{\mathfrak{F}_{\varkappa}}([1-\phi,1];\mathbb{R}^n)$, $\varkappa \geq 1$ the set of all $\mathfrak{F}_{\varkappa}$-measurable, $C([1-\phi,1];\mathbb{R}^n)$-valued random variables $\varphi = \{\varphi(\tau) : 1-\phi \leq \tau \leq 1\}$ satisfying $\mathbb{E}\sup_{1-\phi\leq\tau\leq 1}|\varphi(\tau)|^2 < \infty$.

Definition 1. *The Hadamard fractional integral of order ς for a function h is defined as*

$$I^{\varsigma}h(\vartheta) = \frac{1}{\Gamma(\varsigma)}\int_1^{\vartheta}\big(\log\frac{\vartheta}{\nu}\big)^{\varsigma-1}\frac{h(\nu)}{\nu}d\nu, \quad \varsigma > 0,$$

provided the integral exists.

For a more comprehensive and in-depth understanding of fractional calculus, we highly recommend consulting the authoritative work in [1,13,14], which offers a wealth of detailed insights and thorough explanations on this subject matter.

Let

$$\begin{aligned} H &: C([1-\phi,1],\mathbb{R}^n) \to \mathbb{R}^n, \\ b &: [1,T]\times C([1-\phi,1],\mathbb{R}^n) \to \mathbb{R}^n, \\ \sigma_1 &: [1,T]\times C([1-\phi,1],\mathbb{R}^n) \to \mathbb{R}^n, \\ \sigma_2 &: [1,T]\times C([1-\phi,1],\mathbb{R}^n) \to \mathbb{R}^n \end{aligned}$$

all be Borel measurable. Consider the following FHIDSIE:

$$\zeta(\varkappa) - H(\zeta_{\varkappa}) = \zeta(1) - H(\zeta_1) + \int_1^{\varkappa} b(s,\zeta_s)ds + \int_1^{\varkappa}\sigma_1(s,\zeta_s)d\mathcal{W}(s) + \iota\int_1^{\varkappa}\Big(\log\frac{\varkappa}{s}\Big)^{\iota-1}\frac{\sigma_2(s,\zeta_s)}{s}ds \tag{1}$$

with initial condition

$$\zeta_1 = \alpha = \{\alpha(\lambda); 1-\phi\leq\lambda\leq 1\} \in L^2_{\mathfrak{Z}_1}([1-\phi,1],\mathbb{R}^n), \tag{2}$$

$\frac{1}{2} < \iota < 1$ and $\phi > 0$.

Take the following hypotheses:

$\mathcal{H}_1$: There exists $\underline{K} > 0$ that satisfies

$$|b(\rho,\theta) - b(\rho,\widetilde{\theta})|^2 \vee |\sigma_1(\rho,\theta) - \sigma_1(\rho,\widetilde{\theta})|^2 \vee |\sigma_2(\rho,\alpha_1) - \sigma_2(\rho,\widetilde{\theta})|^2 \leq \underline{K}||\theta - \widetilde{\theta}||^2, \tag{3}$$

for all $(\rho,\theta,\widetilde{\theta}) \in [1,T]\times C([1-\phi,1],\mathbb{R}^n)\times C([1-\phi,1],\mathbb{R}^n)$.

$\mathcal{H}_2$: There exists $K > 0$ that satisfies

$$|b(\rho,\theta)|^2 \vee |\sigma_1(\rho,\theta)|^2 \vee |\sigma_2(\rho,\theta)|^2 \leq K(1 + ||\theta||^2), \tag{4}$$

for all $(\rho,\theta) \in [1,T]\times C([1-\phi,1],\mathbb{R}^n)$.

$\mathcal{H}_3$: Assume that $H(0) = 0$ and there is $\underline{\varrho} \in (0,1)$ such that

$$|H(\theta) - H(\widetilde{\theta})| \leq \underline{\varrho}||\theta - \widetilde{\theta}||, \tag{5}$$

for all $(\theta, \widetilde{\theta}) \in C([1-\phi, 1], \mathbb{R}^n) \times C([1-\phi, 1], \mathbb{R}^n)$.

3. Existence and Uniqueness of Solution

Theorem 1. *Under assumptions $\mathcal{H}_1$–$\mathcal{H}_3$, there exists a unique solution $\zeta(\varkappa)$ of system (1) that satisfies $\zeta(\varkappa) \in \mathcal{M}_2([1-\phi, T]; \mathbb{R}^n)$.*

Before demonstrating Theorem 1, we require the following lemma.

Lemma 1. *Assume that assumptions $\mathcal{H}_2$ and $\mathcal{H}_3$ hold. If $\zeta(\varkappa)$ is a solution to Equation (1) with initial condition (2), then*

$$\mathbb{E}\big(\sup_{1-\phi \leq \varkappa \leq T} |\zeta(\varkappa)|^2\big) \leq L_1, \tag{6}$$

where

$$L_1 = \big(1 + \frac{4 + \underline{\varrho}\sqrt{\underline{\varrho}}}{(1-\sqrt{\underline{\varrho}})(1-\underline{\varrho})}\mathbb{E}||\alpha||^2\big) \exp\left[\frac{3K(T-1)\big(T + 3 + \frac{3\iota^2 K(\log T)^{2\iota-1}}{2\iota-1}\big)}{(1-\underline{\varrho})(1-\sqrt{\underline{\varrho}})}\right].$$

Moreover, $\zeta(\varkappa) \in \mathcal{M}_2([1-\phi, T]; \mathbb{R}^n)$.

Proof. For any $n \in \mathbb{N}^*$, denote by $\tau_n = T \wedge \inf\{\varkappa \in [1, T]; ||\zeta_\varkappa|| \geq n\}$ a stopping time. It is obvious that $\tau_n \uparrow T$ a.s. Let $\zeta^n(\varkappa) = \zeta(\varkappa \wedge \tau_n)$, for $\varkappa \in [1-\phi, T]$. Thus, for $1 \leq \varkappa \leq T$,

$$\zeta^n(\varkappa) = H(\zeta^n_\varkappa) - H(\alpha) + O^n(\varkappa), \tag{7}$$

where

$$O^n(\varkappa) = \alpha(1) + \int_1^\varkappa b\big(s, \zeta^n_s\big)1_{[1,\tau_n]}(s)ds + \int_1^\varkappa \sigma_1\big(s, \zeta^n_s\big)1_{[1,\tau_n]}(s)d\mathcal{W}(s) \tag{8}$$
$$+\iota\int_1^\varkappa \left(\log\frac{\varkappa \wedge \tau_n}{s}\right)^{\iota-1}\frac{\sigma_2\big(s, \zeta^n_s\big)}{s}1_{[1,\tau_n]}(s)ds.$$

By Lemma 2.3 in [15] and assumption $\mathcal{H}_3$, we can derive that

$$\begin{aligned} |\zeta^n(\varkappa)|^2 &\leq \frac{1}{\underline{\varrho}}|H(\zeta^n_\varkappa) - H(\alpha)|^2 + \frac{1}{1-\underline{\varrho}}|O^n(\varkappa)|^2 \\ &\leq \underline{\varrho}||\zeta^n_\varkappa - \alpha||^2 + \frac{1}{1-\underline{\varrho}}|O^n(\varkappa)|^2 \\ &\leq \sqrt{\underline{\varrho}}||\zeta^n_\varkappa||^2 + \frac{\underline{\varrho}}{1-\sqrt{\underline{\varrho}}}||\alpha||^2 + \frac{1}{1-\underline{\varrho}}|O^n(\varkappa)|^2. \end{aligned} \tag{9}$$

Consequently,

$$\mathbb{E}\left(\sup_{1 \leq s \leq \varkappa}|\zeta^n(s)|^2\right) \leq \sqrt{\underline{\varrho}}\mathbb{E}\left(\sup_{1-\phi \leq s \leq \varkappa}|\zeta^n(s)|^2\right) + \frac{\underline{\varrho}}{1-\sqrt{\underline{\varrho}}}\mathbb{E}||\alpha||^2 + \frac{1}{1-\underline{\varrho}}\mathbb{E}\left(\sup_{1 \leq s \leq \varkappa}|O^n(s)|^2\right).$$

It is not hard to see that $\sup\limits_{1-\phi\leq s\leq\varkappa}|\zeta^n(s)|^2\leq||\alpha||^2+\sup\limits_{1\leq s\leq\varkappa}|\zeta^n(s)|^2$. Then,

$$\mathbb{E}\sup_{1-\phi\leq s\leq\varkappa}|\zeta^n(s)|^2\leq\mathbb{E}||\alpha||^2+\mathbb{E}\sup_{1\leq s\leq\varkappa}|\zeta^n(s)|^2.$$

Therefore,

$$\mathbb{E}\left(\sup_{1-\phi\leq s\leq\varkappa}|\zeta^n(s)|^2\right)\leq\sqrt{\underline{\varrho}}\mathbb{E}\left(\sup_{1-\phi\leq s\leq\varkappa}|\zeta^n(s)|^2\right)+\frac{1+\underline{\varrho}-\sqrt{\underline{\varrho}}}{1-\sqrt{\underline{\varrho}}}\mathbb{E}||\alpha||^2+\frac{1}{1-\underline{\varrho}}\mathbb{E}\left(\sup_{1\leq s\leq\varkappa}|O^n(s)|^2\right).$$

Hence,

$$\mathbb{E}\left(\sup_{1-\phi\leq s\leq\varkappa}|\zeta^n(s)|^2\right)\leq\frac{1+\underline{\varrho}-\sqrt{\underline{\varrho}}}{(1-\sqrt{\underline{\varrho}})^2}\mathbb{E}||\alpha||^2+\frac{1}{(1-\underline{\varrho})(1-\sqrt{\underline{\varrho}})}\mathbb{E}\left(\sup_{1\leq s\leq\varkappa}|O^n(s)|^2\right).\tag{10}$$

Using the Cauchy–Schwartz inequality and assumption $\mathcal{H}_2$, we obtain

$$\begin{aligned}|O^n(\varkappa)|^2 &\leq 3|\alpha(1)|^2+3(\varkappa-1)\int_1^\varkappa|b(s,\zeta_s^n)|^2ds\\ &+ 3\left|\int_1^\varkappa\sigma_1(s,\zeta_s^n)1_{[1,\tau_n]}(s)d\mathcal{W}(s)\right|^2\\ &+ 3\iota^2\left(\int_1^\varkappa\frac{1}{s^2}\left(\log\frac{\varkappa}{s}\right)^{2\iota-2}ds\right)\left(\int_1^\varkappa|\sigma_2(s,\zeta_s^n)|^2ds\right)\\ &\leq 3|\alpha(1)|^2+3K(\varkappa-1)\int_1^\varkappa(1+||\zeta_s^n||)^2ds\\ &+ 3\left|\int_1^\varkappa\sigma_1(s,\zeta_s^n)1_{[1,\tau_n]}(s)d\mathcal{W}(s)\right|^2\\ &+ \frac{3\iota^2K}{2\iota-1}(\log\varkappa)^{2\iota-1}\int_1^\varkappa(1+||\zeta_s^n||)^2ds.\end{aligned}\tag{11}$$

Therefore, applying Theorem 7.2 (p. 40 in [15]) and hypothesis $\mathcal{H}_2$, one can obtain

$$\begin{aligned}\mathbb{E}\Big(\sup_{1\leq s\leq\varkappa}|O^n(s)|^2\Big) &\leq 3\mathbb{E}||\alpha||^2+3K(T-1)\int_1^\varkappa(1+\mathbb{E}||\zeta_s^n||^2)ds\\ &+ 12\mathbb{E}\int_1^\varkappa|\sigma_1(s,\zeta_s^n)|^2 1_{[1,\tau_n]}(s)ds\\ &+ \frac{3\iota^2K}{2\iota-1}(\log T)^{2\iota-1}\int_1^\varkappa(1+\mathbb{E}||\zeta_s^n||^2)ds\\ &\leq 3\mathbb{E}||\alpha||^2+3K\Big(T+3+\frac{3\iota^2K(\log T)^{2\iota-1}}{2\iota-1}\Big)\int_1^\varkappa(1+\mathbb{E}||\zeta_s^n||^2)ds.\end{aligned}\tag{12}$$

Plugging (12) into (10), we can obtain

$$\mathbb{E}\left(\sup_{1-\phi\leq s\leq\varkappa}|\zeta^n(s)|^2\right)\leq\frac{4+\underline{\varrho}\sqrt{\underline{\varrho}}}{(1-\sqrt{\underline{\varrho}})(1-\underline{\varrho})}\mathbb{E}||\alpha||^2+\frac{3K\Big(T+3+\frac{3\iota^2K(\log T)^{2\iota-1}}{2\iota-1}\Big)}{(1-\underline{\varrho})(1-\sqrt{\underline{\varrho}})}\int_1^\varkappa(1+\mathbb{E}||\zeta_s^n||^2)ds.\tag{13}$$

Hence,

$$\begin{aligned}1+\mathbb{E}\Big(\sup_{1-\phi\leq s\leq\varkappa}|\zeta^n(s)|^2\Big) &\leq 1+\frac{4+\underline{\varrho}\sqrt{\underline{\varrho}}}{(1-\sqrt{\underline{\varrho}})(1-\underline{\varrho})}\mathbb{E}||\alpha||^2\\ &+ \frac{3K\Big(T+3+\frac{3\iota^2K(\log T)^{2\iota-1}}{2\iota-1}\Big)}{(1-\underline{\varrho})(1-\sqrt{\underline{\varrho}})}\int_1^\varkappa\left(1+\mathbb{E}\Big(\sup_{1-\phi\leq l\leq s}|\zeta^n(l)|^2\Big)\right)ds.\end{aligned}$$

By Gronwall's inequality, one has

$$1+\mathbb{E}\big(\sup_{1-\phi\le s\le\varkappa}|\zeta^n(s)|^2\big)\le\big(1+\frac{4+\underline{\varrho}\sqrt{\underline{\varrho}}}{(1-\sqrt{\underline{\varrho}})(1-\underline{\varrho})}\mathbb{E}||\alpha||^2\big)\exp\left[\frac{3K(T-1)\big(T+3+\frac{3\iota^2K(\log T)^{2\iota-1}}{2\iota-1}\big)}{(1-\underline{\varrho})(1-\sqrt{\underline{\varrho}})}\right].$$

Consequently,

$$\mathbb{E}\big(\sup_{1-\phi\le s\le\varkappa}|\zeta^n(s)|^2\big)\le\big(1+\frac{4+\underline{\varrho}\sqrt{\underline{\varrho}}}{(1-\sqrt{\underline{\varrho}})(1-\underline{\varrho})}\mathbb{E}||\alpha||^2\big)\exp\left[\frac{3K(T-1)\big(T+3+\frac{3\iota^2K(\log T)^{2\iota-1}}{2\iota-1}\big)}{(1-\underline{\varrho})(1-\sqrt{\underline{\varrho}})}\right].$$

Therefore, we can deduce the desired result when $n\longrightarrow\infty$. □

Proof of Theorem 1:

Proof. Uniqueness:

Consider $\zeta(\varkappa)$ and $\bar{\zeta}(\varkappa)$ to be two solutions to Equation (1). Using Lemma 1, $\zeta(\varkappa)$ and $\bar{\zeta}(\varkappa)$ belong to $\mathcal{M}_2\big([1-\phi,T];\mathbb{R}^n\big)$. Thus,

$$\zeta(\varkappa)-\bar{\zeta}(\varkappa)=H(\zeta_\varkappa)-H(\bar{\zeta}_\varkappa)+O(\varkappa), \tag{14}$$

where

$$O(\varkappa)=\int_1^\varkappa\Big(b\big(s,\zeta_s\big)-b\big(s,\bar{\zeta}_s\big)\Big)ds+\int_1^\varkappa\Big(\sigma_1\big(s,\zeta_s\big)-\sigma_1\big(s,\bar{\zeta}_s\big)\Big)d\mathcal{W}(s)$$
$$+\iota\int_1^\varkappa\Big(\log\frac{\varkappa}{s}\Big)^{\iota-1}\frac{\Big(\sigma_2\big(s,\zeta_s\big)-\sigma_2\big(s,\bar{\zeta}_s\big)\Big)}{s}ds.$$

By Lemma 2.3 in [15] and assumption $\mathcal{H}_3$, we can see that

$$|\zeta(\varkappa)-\bar{\zeta}(\varkappa)|^2\le\underline{\varrho}||\zeta_\varkappa-\bar{\zeta}_\varkappa||^2+\frac{1}{1-\underline{\varrho}}|O(\varkappa)|^2.$$

Hence,

$$\mathbb{E}\left(\sup_{1\le s\le\varkappa}|\zeta(s)-\bar{\zeta}(s)|^2\right)\le\underline{\varrho}\mathbb{E}\left(\sup_{1\le s\le\varkappa}|\zeta(s)-\bar{\zeta}(s)|^2\right)+\frac{1}{1-\underline{\varrho}}\mathbb{E}\left(\sup_{1\le s\le\varkappa}|O(s)|^2\right). \tag{15}$$

Using (15), it yields that

$$\mathbb{E}\left(\sup_{1\le s\le\varkappa}|\zeta(s)-\bar{\zeta}(s)|^2\right)\le\frac{1}{(1-\underline{\varrho})^2}\mathbb{E}\left(\sup_{1\le s\le\varkappa}|O(s)|^2\right).$$

Moreover, by Hölder's inequality, Theorem 7.2 (p. 40 in [15]), and $\mathcal{H}_1$, using a similar method as in the proof of Lemma 1, we obtain

$$\begin{aligned}\mathbb{E}\big(\sup_{1\le s\le\varkappa}|O(s)|^2\big)&\le 3\underline{K}\big(T+4+\frac{\iota^2(\log T)^{2\iota-1}}{2\iota-1}\big)\int_1^\varkappa\mathbb{E}||\zeta_s-\bar{\zeta}_s||^2ds\\&\le 3\underline{K}\big(T+4+\frac{\iota^2(\log T)^{2\iota-1}}{2\iota-1}\big)\int_1^\varkappa\mathbb{E}\big(\sup_{1\le w\le s}|\zeta(w)-\bar{\zeta}(w)|^2\big)ds.\end{aligned}$$

Therefore,

$$\mathbb{E}\left(\sup_{1\le s\le\varkappa}|\zeta(s)-\bar{\zeta}(s)|^2\right)\le\frac{3\underline{K}\big(T+4+\frac{\iota^2(\log T)^{2\iota-1}}{2\iota-1}\big)}{(1-\underline{\varrho})^2}\int_1^\varkappa\mathbb{E}\big(\sup_{1\le w\le s}|\zeta(w)-\bar{\zeta}(w)|^2\big)ds.$$

Using the Gronwall inequality, we can derive that: $\mathbb{E}\big(\sup_{1\leq\varkappa\leq T}|\zeta(\varkappa)-\bar{\zeta}(\varkappa)|^2\big) = 0$. Consequently, for all $1-\phi\leq\varkappa\leq T$, $\zeta(\varkappa)=\bar{\zeta}(\varkappa)$, a.s.
Existence: Let λ be sufficiently small such that

$$\vartheta = \underline{\varrho} + \frac{3\underline{K}\big(\lambda+5+\frac{\iota^2(\log(\lambda+1))^{2\iota-1}}{2\iota-1}\big)\lambda}{1-\underline{\varrho}} < 1, \tag{16}$$

and write $[1,T]=\bigcup_{k=1}^{r}[1+(k-1)\lambda, 1+k\lambda]\bigcup[1+r\lambda, T]$. We split the proof into two steps:
Step 1: Let $\zeta_1^0=\alpha$ and $\zeta^0(\varkappa)=\alpha(1)$ for $1\leq\varkappa\leq 1+\lambda$. For each $n=1,2,3,\ldots$, let $\zeta_1^n=\alpha$, and we define, using the Picard iterations:

$$\zeta^n(\varkappa)-H(\zeta_\varkappa^{n-1}) = \alpha(1)-H(\alpha)+\int_1^\varkappa b\big(s,\zeta_s^{n-1}\big)ds+\int_1^\varkappa \sigma_1\big(s,\zeta_s^{n-1}\big)d\mathcal{W}(s)+\iota\int_1^\varkappa\Big(\log\frac{\varkappa}{s}\Big)^{\iota-1}\frac{\sigma_2\big(s,\zeta_s^{n-1}\big)}{s}ds. \tag{17}$$

It is not hard to see that $\zeta^n(\cdot)\in\mathcal{M}_2\big([1-\phi,1+\lambda];\mathbb{R}^n\big)$. Noting that for $1\leq\varkappa\leq 1+\lambda$,

$$\begin{aligned}\zeta^1(\varkappa)-\zeta^0(\varkappa) &= \zeta^1(\varkappa)-\alpha(1) = H(\zeta_\varkappa^0)-H(\alpha)+\int_1^\varkappa b\big(s,\zeta_s^0\big)ds \\ &+ \int_1^\varkappa\sigma_1\big(s,\zeta_s^0\big)d\mathcal{W}(s)+\iota\int_1^\varkappa\Big(\log\frac{\varkappa}{s}\Big)^{\iota-1}\frac{\sigma_2\big(s,\zeta_s^0\big)}{s}ds.\end{aligned}$$

Proceeding as in the proof of the uniqueness, we have

$$\begin{aligned}\mathbb{E}\bigg(\sup_{1\leq s\leq\varkappa}|\zeta^1(\varkappa)-\zeta^0(\varkappa)|^2\bigg) &\leq \underline{\varrho}\mathbb{E}\bigg(\sup_{1\leq s\leq\varkappa}||\zeta_\varkappa^0-\alpha||^2\bigg)+\frac{3K\big(T+4+\frac{\iota^2(\log T)^{2\iota-1}}{2\iota-1}\big)}{1-\underline{\varrho}}\mathbb{E}\int_1^\varkappa\Big(1+||\zeta_\varkappa^0||^2\Big)d\varkappa \\ &\leq C_1\end{aligned} \tag{18}$$

where

$$C_1 = 2\underline{\varrho}\mathbb{E}||\alpha||^2+\frac{3K\big(\lambda+5+\frac{\iota^2(\log(\lambda+1))^{2\iota-1}}{2\iota-1}\big)}{1-\underline{\varrho}}\Big(1+\mathbb{E}||\alpha||^2\Big)\lambda.$$

Noting that for $n\geq 1$ and $1\leq\varkappa\leq 1+\lambda$, we have

$$\begin{aligned}\zeta^{n+1}(\varkappa)-\zeta^n(\varkappa) &= H(\zeta_\varkappa^n)-H(\zeta_\varkappa^{n-1})+\int_1^\varkappa\Big[b\big(s,\zeta_s^n\big)-b\big(s,\zeta_s^{n-1}\big)\Big]ds \\ &+ \int_1^\varkappa\Big[\sigma_1\big(s,\zeta_s^n\big)-\sigma_1\big(s,\zeta_s^{n-1}\big)\Big]d\mathcal{W}(s) \\ &+ \iota\int_1^\varkappa\Big(\log\frac{\varkappa}{s}\Big)^{\iota-1}\frac{\big[\sigma_2\big(s,\zeta_s^n\big)-\sigma_2\big(s,\zeta_s^{n-1}\big)\big]}{s}ds.\end{aligned}$$

Proceeding as in the proof of the uniqueness and using (18), we can obtain

$$\begin{aligned}\mathbb{E}\bigg(\sup_{1\leq\varkappa\leq1+\lambda}|\zeta^{n+1}(\varkappa)-\zeta^n(\varkappa)|^2\bigg) &\leq \underline{\varrho}\mathbb{E}\bigg(\sup_{1\leq\varkappa\leq1+\lambda}|\zeta^n(\varkappa)-\zeta^{n-1}(\varkappa)|^2\bigg) \\ &+ \frac{3\underline{K}\big(\lambda+5+\frac{\iota^2(\log(\lambda+1))^{2\iota-1}}{2\iota-1}\big)}{1-\underline{\varrho}}\int_1^\varkappa\mathbb{E}\bigg(\sup_{1\leq s\leq\varkappa}|\zeta^n(s)-\zeta^{n-1}(s)|^2\bigg)d\varkappa \\ &\leq \vartheta\mathbb{E}\bigg(\sup_{1\leq\varkappa\leq1+\lambda}|\zeta^n(\varkappa)-\zeta^{n-1}(\varkappa)|^2\bigg) \\ &\leq \vartheta^n\mathbb{E}\bigg(\sup_{1\leq\varkappa\leq1+\lambda}|\zeta^1(\varkappa)-\zeta^0(\varkappa)|^2\bigg) \\ &\leq C_1\vartheta^n.\end{aligned} \tag{19}$$

By (16), we can prove from (19) that there exists a solution ${}^1\zeta$ to Equation (1) on $[1-\phi, 1+\lambda]$ in the same way as in the proof of Theorem 3.1 in [12].

Step 2: For $2 \leq k \leq r$ and $\varkappa \in [1+(k-1)\lambda, 1+k\lambda]$: let $\zeta^0(\varkappa) = {}^{k-1}\zeta(\varkappa)$ for $1-\phi \leq \varkappa \leq 1+(k-1)\lambda$ and $\zeta^0(\varkappa) = {}^{k-1}\zeta(1+(k-1)\lambda)$ for $1+(k-1)\lambda \leq \varkappa \leq 1+k\lambda$. For each $n = 1, 2, 3, \ldots$, let $\zeta^n(\varkappa) = {}^{k-1}\zeta(\varkappa)$ for $\varkappa \in [1-\phi, 1+(k-1)\lambda]$. We define the following Picard iterations:

$$\begin{aligned}
\zeta^n(\varkappa) - H(\zeta^{n-1}_\varkappa) &= \alpha(1) - H(\alpha) + \int_1^{1+(k-1)\lambda} b(s, {}^{k-1}\zeta_s)ds + \int_1^{1+(k-1)\lambda} \sigma_1(s, {}^{k-1}\zeta_s)d\mathcal{W}(s) \\
&+ \iota \int_1^{1+(k-1)\lambda} \left(\log \frac{\varkappa}{s}\right)^{\iota-1} \frac{\sigma_2(s, {}^{k-1}\zeta_s)}{s} ds \\
&+ \int_{1+(k-1)\lambda}^{\varkappa} b(s, \zeta^{n-1}_s)ds + \int_{1+(k-1)\lambda}^{\varkappa} \sigma_1(s, \zeta^{n-1}_s)d\mathcal{W}(s) \\
&+ \iota \int_{1+(k-1)\lambda}^{\varkappa} \left(\log \frac{\varkappa}{s}\right)^{\iota-1} \frac{\sigma_2(s, \zeta^{n-1}_s)}{s} ds.
\end{aligned} \tag{20}$$

Proceeding as in step 1, we obtain the existence of the solution ${}^k\zeta$ on $[1-\phi, 1+k\lambda]$. Repeating this on $[1+r\lambda, T]$, we can observe that there exists a solution to Equation (1) on the entire interval $[1-\phi, T]$, which complete the proof. □

4. Averaging Principle

Take the following standard form of Equation (1):

$$\begin{aligned}
\zeta_\epsilon(\varkappa) &= \zeta(1) - H(\zeta_1) + H(\zeta_{\epsilon,\varkappa}) + \epsilon \int_1^{\varkappa} b(s, \zeta_{\epsilon,s})ds + \sqrt{\epsilon} \int_1^{\varkappa} \sigma_1(s, \zeta_{\epsilon,s})d\mathcal{W}(s) \\
&+ \epsilon^\iota \iota \int_1^{\varkappa} \left(\log \frac{\varkappa}{s}\right)^{\iota-1} \frac{\sigma_2(s, \zeta_{\epsilon,s})}{s} ds,
\end{aligned} \tag{21}$$

where b, σ_1, and σ_2 satisfy $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\epsilon \in (0, \epsilon_0]$, with $\epsilon_0 \in (0, \frac{1}{2})$, is a fixed number. Using Theorem 1, Equation (21) has a unique global solution $\zeta_\epsilon(\varkappa)$, $\varkappa \in [1-\phi, T]$, for any $\epsilon \in (0, \epsilon_0]$.

Suppose that the following hypothesis holds true:

$\mathcal{H}_4$: Let the measurable functions $\bar{b}(\zeta) : C([1-\phi, 1], \mathbb{R}^n) \to \mathbb{R}^n$, $\bar{\sigma_1}(\zeta) : C([1-\phi, 1], \mathbb{R}^n) \to \mathbb{R}^n$ and $\bar{\sigma_2}(\zeta) : C([1-\phi, 1], \mathbb{R}^n) \to \mathbb{R}^n$ exist, satisfying $\mathcal{H}_1$ and $\mathcal{H}_2$. For any $(\zeta, T_1) \in C([1-\phi, 1], \mathbb{R}^n) \times [1, T]$, we have

1. $\displaystyle \frac{1}{T_1} \int_1^{T_1} |b(s, \zeta) - \bar{b}(\zeta)|^2 ds \leq \psi_1(T_1)(1 + ||\zeta||^2);$
2. $\displaystyle \frac{1}{T_1} \int_1^{T_1} |\sigma_1(s, \zeta) - \bar{\sigma_1}(\zeta)|^2 ds \leq \psi_2(T_1)(1 + ||\zeta||^2);$
3. $\displaystyle \frac{1}{T_1} \int_1^{T_1} |\sigma_2(s, \zeta) - \bar{\sigma_2}(\zeta)|^2 ds \leq \psi_3(T_1)(1 + ||\zeta||^2);$

with $\psi_j(T_1)$ as positive bounded functions satisfying $\lim_{x \longrightarrow \infty} \psi_j(x) = 0$ for $j = 1, 2, 3$.

Our objective is to establish that the solution $\zeta_\epsilon(\varkappa)$ can be approximated by the solution $y_\epsilon(\varkappa)$ of the next equation

$$\begin{aligned}
y_\epsilon(\varkappa) &= \zeta(1) - H(\zeta_1) + H(y_{\epsilon,\varkappa}) + \epsilon \int_1^{\varkappa} \bar{b}(y_{\epsilon,s})ds + \sqrt{\epsilon} \int_1^{\varkappa} \bar{\sigma}_1(y_{\epsilon,s})d\mathcal{W}(s) \\
&+ \epsilon^\iota \iota \int_1^{\varkappa} \left(\log \frac{\varkappa}{s}\right)^{\iota-1} \frac{\bar{\sigma}_2(y_{\epsilon,s})}{s} ds,
\end{aligned} \tag{22}$$

for $\varkappa \in [1, T]$.

Now, we will show the first main theorem in this section.

Theorem 2. *Assume that $\mathcal{H}_1$–$\mathcal{H}_4$ hold. For a given arbitrary small constant $\pi_1 > 0$, there are two constants $L > 0$ and $\epsilon_1 \in (0, \epsilon_0]$ that satisfy $\forall \epsilon \in (0, \epsilon_1]$,*

$$\mathbb{E}\big(\sup_{\varkappa \in [1, \frac{L}{\epsilon}]} |\zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa)|^2 \big) \leq \pi_1.$$

Proof. Using Lemma 2.3 in [15] and assumption $\mathcal{H}_3$, one can derive

$$\mathbb{E} \sup_{\varkappa \in [1,u]} |\zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa)|^2 \leq \frac{\mathbb{E} \sup_{\varkappa \in [1,u]} |\zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa) - (H(\zeta_{\epsilon,\varkappa}) - H(y_{\epsilon,\varkappa}))|^2}{(1 - \underline{\varrho})^2}. \tag{23}$$

Note that

$$\begin{aligned} \zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa) - (H(\zeta_{\epsilon,\varkappa}) - H(y_{\epsilon,\varkappa})) &= \epsilon \int_1^\varkappa \big[b\big(s, \zeta_{\epsilon,s}\big) - \bar{b}\big(y_{\epsilon,s}\big)\big] ds \\ &+ \sqrt{\epsilon} \int_1^\varkappa \big[\sigma_1\big(s, \zeta_{\epsilon,s}\big) - \bar{\sigma}_1\big(y_{\epsilon,s}\big)\big] d\mathcal{W}(s) \\ &+ \iota \epsilon^\iota \int_1^\varkappa \Big(\log \frac{\varkappa}{s}\Big)^{\iota - 1} \frac{\big[\sigma_2\big(s, \zeta_{\epsilon,s}\big) - \bar{\sigma}_2\big(y_{\epsilon,s}\big)\big]}{s} ds. \end{aligned} \tag{24}$$

According to the elementary inequality, one has

$$\begin{aligned} &\mathbb{E} \sup_{\varkappa \in [1,u]} |\zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa) - (H(\zeta_{\epsilon,\varkappa}) - H(y_{\epsilon,\varkappa}))|^2 \qquad (25) \\ &\leq \epsilon^2 \mathbb{E} \sup_{\varkappa \in [1,u]} | \int_1^\varkappa \big[b\big(s, \zeta_{\epsilon,s}\big) - \bar{b}\big(y_{\epsilon,s}\big)\big] ds|^2 \\ &+ 3\epsilon \mathbb{E} \sup_{\varkappa \in [1,u]} | \int_1^\varkappa \big[\sigma_1\big(s, \zeta_{\epsilon,s}\big) - \bar{\sigma}_1\big(y_{\epsilon,s}\big)\big] d\mathcal{W}(s)|^2 \\ &+ 3\iota^2 \epsilon^{2\iota} \mathbb{E} \sup_{\varkappa \in [1,u]} | \int_1^\varkappa \Big(\log \frac{\varkappa}{s}\Big)^{\iota - 1} \frac{\big[\sigma_2\big(s, \zeta_{\epsilon,s}\big) - \bar{\sigma}_2\big(y_{\epsilon,s}\big)\big]}{s} ds|^2 \\ &= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3, \qquad (26) \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}_1 &= 3\epsilon^2 \mathbb{E} \sup_{\varkappa \in [1,u]} | \int_1^\varkappa \big[b\big(s, \zeta_{\epsilon,s}\big) - \bar{b}\big(y_{\epsilon,s}\big)\big] ds|^2 \\ &\leq 6\epsilon^2 \mathbb{E} \sup_{\varkappa \in [1,u]} | \int_1^\varkappa \big[b\big(s, \zeta_{\epsilon,s}\big) - b\big(s, y_{\epsilon,s}\big)\big] ds|^2 \\ &+ 6\epsilon^2 \mathbb{E} \sup_{\varkappa \in [1,u]} | \int_1^\varkappa \big[b\big(s, y_{\epsilon,s}\big) - \bar{b}\big(y_{\epsilon,s}\big)\big] ds|^2 \\ &\leq 6\epsilon^2 \mathcal{J}_{11} + 6\epsilon^2 \mathcal{J}_{12}. \end{aligned} \tag{27}$$

By the Hölder inequality and $\mathcal{H}_1$, one has

$$\begin{aligned} \mathcal{J}_{11} &= \mathbb{E} \sup_{\varkappa \in [1,u]} | \int_1^\varkappa \big[b\big(s, \zeta_{\epsilon,s}\big) - b\big(s, y_{\epsilon,s}\big)\big] ds|^2 \\ &\leq (u-1) \mathbb{E} \int_1^u |b\big(s, \zeta_{\epsilon,s}\big) - b\big(s, y_{\epsilon,s}\big)|^2 ds \\ &\leq \underline{K} u \mathbb{E} \int_1^u ||\zeta_{\epsilon,s} - y_{\epsilon,s}||^2 ds \\ &\leq \underline{K} u \int_1^u \mathbb{E} \sup_{v \in [1,s]} |\zeta_\epsilon(v) - y_\epsilon(v)|^2 ds. \end{aligned} \tag{28}$$

By $\mathcal{H}_4$, one can obtain

$$\begin{aligned} \mathcal{J}_{12} &= \mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}[b(s,y_{\epsilon,s})-\bar{b}(y_{\epsilon,s})]ds|^2 \\ &\leq u\mathbb{E}\sup_{\varkappa\in[1,u]}\int_1^{\varkappa}|b(s,y_{\epsilon,s})-\bar{b}(y_{\epsilon,s})|^2ds \\ &\leq u^2\mathbb{E}\sup_{\varkappa\in[1,u]}\frac{1}{\varkappa}\int_1^{\varkappa}|b(s,y_{\epsilon,s})-\bar{b}(y_{\epsilon,s})|^2ds. \end{aligned} \tag{29}$$

As in the proof of Theorem 1 in [16], one can derive

$$\mathcal{J}_{12}\leq u^2(\sup_{\varkappa\in[1,u]}\psi_1(\varkappa))(1+\mathbb{E}\sup_{\varkappa\in[1-\phi,u]}|y_\epsilon(\varkappa)|^2). \tag{30}$$

Therefore,

$$\begin{aligned} \mathcal{J}_1 &\leq 6\epsilon^2\underline{K}u\int_1^u\mathbb{E}\sup_{v\in[1,s]}|\zeta_\epsilon(v)-y_\epsilon(v)|^2ds \\ &+ 6\epsilon^2u^2(\sup_{\varkappa\in[1,u]}\psi_1(\varkappa))(1+\mathbb{E}\sup_{\varkappa\in[1-\phi,u]}|y_\epsilon(\varkappa)|^2). \end{aligned} \tag{31}$$

$$\begin{aligned} \mathcal{J}_2 &= 3\epsilon\mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}[\sigma_1(s,\zeta_{\epsilon,s})-\bar{\sigma}_1(y_{\epsilon,s})]d\mathcal{W}(s)|^2 \\ &= 3\epsilon\mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}[(\sigma_1(s,\zeta_{\epsilon,s})-\sigma_1(s,y_{\epsilon,s}))+(\sigma_1(s,y_{\epsilon,s})-\bar{\sigma}_1(y_{\epsilon,s}))]d\mathcal{W}(s)|^2 \\ &\leq 6\epsilon\mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}[\sigma_1(s,\zeta_{\epsilon,s})-\sigma_1(s,y_{\epsilon,s})]d\mathcal{W}(s)|^2 \\ &+ 6\epsilon\mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}[\sigma_1(s,y_{\epsilon,s})-\bar{\sigma}_1(y_{\epsilon,s})]d\mathcal{W}(s)|^2 \\ &\leq 6\epsilon\mathcal{J}_{21}+6\epsilon\mathcal{J}_{22}. \end{aligned} \tag{32}$$

By employing Theorem 7.2 (p. 40 in [15]) and $\mathcal{H}_1$, one can obtain

$$\begin{aligned} \mathcal{J}_{21} &= \mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}[\sigma_1(s,\zeta_{\epsilon,s})-\sigma_1(s,y_{\epsilon,s})]d\mathcal{W}(s)|^2 \\ &\leq 4\mathbb{E}\int_1^u|\sigma_1(s,\zeta_{\epsilon,s})-\sigma_1(s,y_{\epsilon,s})|^2ds \\ &\leq 4\underline{K}\mathbb{E}\int_1^u||\zeta_{\epsilon,s}-y_{\epsilon,s}||^2ds \\ &\leq 4\underline{K}\int_1^u\mathbb{E}(\sup_{v\in[1,s]}|\zeta_\epsilon(v)-y_\epsilon(v)|^2)ds. \end{aligned} \tag{33}$$

Using $\mathcal{H}_4$, one can derive

$$\begin{aligned} \mathcal{J}_{22} &= \mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}[\sigma_1(s,y_{\epsilon,s})-\bar{\sigma}_1(y_{\epsilon,s})]d\mathcal{W}(s)|^2 \\ &\leq 4\mathbb{E}\int_1^u|\sigma_1(s,y_{\epsilon,s})-\bar{\sigma}_1(y_{\epsilon,s})|^2ds \\ &\leq 4u\mathbb{E}\frac{1}{u}\int_1^u|\sigma_1(s,y_{\epsilon,s})-\bar{\sigma}_1(y_{\epsilon,s})|^2ds \\ &\leq 4u(\sup_{\varkappa\in[1,u]}\psi_2(\varkappa))(1+\mathbb{E}\sup_{\varkappa\in[1-\phi,u]}|y_\epsilon(\varkappa)|^2). \end{aligned} \tag{34}$$

Consequently,

$$\begin{aligned}\mathcal{J}_2 &\leq 24\epsilon\underline{K}\int_1^u \mathbb{E}(\sup_{v\in[1,s]}|\zeta_\epsilon(v)-y_\epsilon(v)|^2)ds \\ &+ 24\epsilon u(\sup_{\varkappa\in[1,u]}\psi_2(\varkappa))(1+\mathbb{E}\sup_{\varkappa\in[1-\phi,u]}|y_\epsilon(\varkappa)|^2). \end{aligned} \tag{35}$$

Moreover,

$$\begin{aligned}\mathcal{J}_3 &= 3\iota^2\epsilon^{2\iota}\mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}\left(\log\frac{\varkappa}{s}\right)^{\iota-1}\frac{[\sigma_2(s,\zeta_{\epsilon,s})-\bar{\sigma}_2(y_{\epsilon,s})]}{s}ds|^2 \\ &\leq 6\iota^2\epsilon^{2\iota}\mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}\left(\log\frac{\varkappa}{s}\right)^{\iota-1}\frac{[\sigma_2(s,\zeta_{\epsilon,s})-\sigma_2(s,y_{\epsilon,s})]}{s}ds|^2 \\ &+ 6\iota^2\epsilon^{2\iota}\mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}\left(\log\frac{\varkappa}{s}\right)^{\iota-1}\frac{[\sigma_2(s,y_{\epsilon,s})-\bar{\sigma}_2(y_{\epsilon,s})]}{s}ds|^2 \\ &\leq 6\iota^2\epsilon^{2\iota}\mathcal{J}_{31}+6\iota^2\epsilon^{2\iota}\mathcal{J}_{32}. \end{aligned} \tag{36}$$

By Hölder's inequality and $\mathcal{H}_1$, one can derive

$$\begin{aligned}\mathcal{J}_{31} &= \mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}\left(\log\frac{\varkappa}{s}\right)^{\iota-1}\frac{[\sigma_2(s,\zeta_{\epsilon,s})-\sigma_2(s,y_{\epsilon,s})]}{s}ds|^2 \\ &\leq \sup_{\varkappa\in[1,u]}\left(\int_1^{\varkappa}\left(\log\frac{\varkappa}{s}\right)^{2\iota-2}\frac{1}{s}ds\right)\mathbb{E}\int_1^u|\sigma_2(s,\zeta_{\epsilon,s})-\sigma_2(s,y_{\epsilon,s})|^2ds \\ &\leq \frac{1}{2\iota-1}(\log u)^{2\iota-1}\underline{K}\mathbb{E}\int_1^u||\zeta_{\epsilon,s}-y_{\epsilon,s}||^2ds \\ &\leq \frac{1}{2\iota-1}(\log u)^{2\iota-1}\underline{K}\int_1^u\mathbb{E}(\sup_{v\in[1,s]}|\zeta_\epsilon(v)-y_\epsilon(v)|^2)ds. \end{aligned} \tag{37}$$

Using Hölder's inequality and $\mathcal{H}_4$, one has

$$\begin{aligned}\mathcal{J}_{32} &= \mathbb{E}\sup_{\varkappa\in[1,u]}|\int_1^{\varkappa}\left(\log\frac{\varkappa}{s}\right)^{\iota-1}\frac{[\sigma_2(s,y_{\epsilon,s})-\bar{\sigma}_2(y_{\epsilon,s})]}{s}ds|^2 \\ &\leq \frac{u}{2\iota-1}(\log u)^{2\iota-1}\mathbb{E}\frac{1}{u}\int_1^u|\sigma_2(s,y_{\epsilon,s})-\bar{\sigma}_2(y_{\epsilon,s})|^2ds \\ &\leq \frac{u}{2\iota-1}(\log u)^{2\iota-1}(\psi_3(u))(1+\mathbb{E}\sup_{\varkappa\in[1-\phi,u]}|y_\epsilon(\varkappa)|^2). \end{aligned} \tag{38}$$

Therefore,

$$\begin{aligned}\mathcal{J}_3 &\leq \frac{6\iota^2\epsilon^{2\iota}}{2\iota-1}(\log u)^{2\iota-1}\underline{K}\int_1^u\mathbb{E}(\sup_{v\in[1,s]}|\zeta_\epsilon(v)-y_\epsilon(v)|^2)ds \\ &+ \frac{6\iota^2\epsilon^{2\iota}u}{2\iota-1}(\log u)^{2\iota-1}(\psi_3(u))(1+\mathbb{E}\sup_{\varkappa\in[1-\phi,u]}|y_\epsilon(\varkappa)|^2). \end{aligned} \tag{39}$$

Plugging (31)–(39) into (25), using Lemma 1 and (23), one can derive that

$$\mathbb{E}\sup_{\varkappa\in[1,u]}|\zeta_\epsilon(\varkappa)-y_\epsilon(\varkappa)|^2$$

$$
\begin{aligned}
&\leq \frac{1}{(1-\underline{\varrho})^2} 6\epsilon \underline{K}\left(\epsilon u + 4 + \frac{\iota^2 \epsilon^{2\iota-1} u}{2\iota - 1}(\log u)^{2\iota-1}\right) \int_1^u \mathbb{E}\big(\sup_{v\in[1,s]} |\zeta_\epsilon(v) - y_\epsilon(v)|^2\big) ds \\
&+ 6\epsilon u\Big(\epsilon u\big(\sup_{l\in[1,u]} \psi_1(l)\big) + 4\big(\sup_{l\in[1,u]} \psi_2(l)\big) + \frac{\iota^2\epsilon^{2\iota-1}u}{2\iota-1}(\log u)^{2\iota-1}\psi_3(u)\Big) \\
&\times \Big(1 + \mathbb{E}\sup_{\varkappa\in[1-\phi,u]} |y_\epsilon(\varkappa)|^2\Big) \times \frac{1}{(1-\underline{\varrho})^2} \\
&\leq \epsilon L_3 + \epsilon L_2 \int_1^u \mathbb{E}\big(\sup_{v\in[1,s]} |\zeta_\epsilon(v) - y_\epsilon(v)|^2\big) ds,
\end{aligned} \tag{40}
$$

with

$$
L_2 = \frac{6\underline{K}\left(\epsilon T + 4 + \frac{\iota^2\epsilon^{2\iota-1}u}{2\iota-1}(\log T)^{2\iota-1}\right)}{(1-\underline{\varrho})^2},
$$

and

$$
L_3 = \frac{6T\Big(\epsilon T \sup_{l\in[1,T]}\psi_1(l) + 4\sup_{l\in[1,T]}\psi_2(l) + \frac{\iota^2\epsilon^{2\iota-1}}{2\iota-1}(\log T)^{2\iota-1}\sup_{l\in[1,T]}\psi_3(l)\Big)(1+L_1)}{(1-\underline{\varrho})^2}. \tag{41}
$$

By Gronwall's inequality, one can obtain

$$
\mathbb{E}\Big(\sup_{\varkappa\in[1,u]} |\zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa)|^2\Big) \leq \epsilon L_3 e^{\epsilon L_2 u}.
$$

Therefore, given any number $\pi_1 > 0$, there are two constants $L > 0$ and $\epsilon_1 \in (0, \epsilon_0]$ that satisfy, for every $\epsilon \in (0, \epsilon_1]$,

$$
\mathbb{E}\Big(\sup_{\varkappa\in[1,\frac{L}{\epsilon}]} |\zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa)|^2\Big) \leq \pi_1.
$$

□

Next, we show the convergence in probability between $\zeta_\epsilon(\varkappa)$ and $y_\epsilon(\varkappa)$.

Theorem 3. *Suppose that the FHIDSIE (21) and (22) both satisfy $\mathcal{H}_1$–$\mathcal{H}_4$. For a given arbitrary small constant $\pi_1 > 0$, there are two constants $L > 0$ and $\epsilon_1 \in (0, \epsilon_0]$ that satisfy $\forall \epsilon \in (0, \epsilon_1]$,*

$$
\lim_{\epsilon\to 0} \mathbb{P}\big(\sup_{\varkappa\in[1,\frac{L}{\epsilon}]} |\zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa)| > \pi_1\big) = 0. \tag{42}
$$

Proof. Given any constant $\pi_1 > 0$, using Theorem 4.1 and the inequality of Chebyshev, we can obtain

$$
\begin{aligned}
\mathbb{P}\big(\sup_{\varkappa\in[1,\frac{L}{\epsilon}]} |\zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa)| > \pi_1\big) &\leq \frac{1}{\pi_1^2}\mathbb{E}\big(\sup_{\varkappa\in[1,\frac{L}{\epsilon}]} |\zeta_\epsilon(\varkappa) - y_\epsilon(\varkappa)|^2\big) \\
&\leq \frac{1}{\pi_1^2}\epsilon L_3 e^{\epsilon L_2 T}.
\end{aligned} \tag{43}
$$

Letting $\epsilon \to 0$, (42) holds. □

5. Application

In this section, we will present an application that illustrates the key findings outlined in the preceding section.

Consider the following perturbed Malthusian FHIDSIE model of population growth (see [15,17]):

$$\begin{aligned} \zeta_\epsilon(\varkappa) &= \zeta(1) - H(\zeta_1) + H(\zeta_{\epsilon,\varkappa}) + \epsilon\int_1^\varkappa b(s,\zeta_{\epsilon,s})ds + \sqrt{\epsilon}\int_1^\varkappa \sigma_1(s,\zeta_{\epsilon,s})d\mathcal{W}(s) \\ &+ \epsilon^\iota\iota\int_1^\varkappa \left(\log\frac{\varkappa}{s}\right)^{\iota-1}\frac{\sigma_2(s,\zeta_{\epsilon,s})}{s}ds, \end{aligned} \tag{44}$$

with initial condition

$$\zeta_1 = \alpha = \{\alpha(\lambda); 1-\phi \le \lambda \le 1\} \in L^2_{\mathfrak{I}_1}([1-\phi,1],\mathbb{R}^n), \tag{45}$$

where

$$H(\zeta_{\epsilon,\varkappa}) = 0.1\zeta_\epsilon(-\phi), \quad b(s,\zeta_{\epsilon,s}) = c_1\zeta_\epsilon(-\phi), \quad \sigma_1(s,\zeta_{\epsilon,s}) = c_2\zeta_\epsilon(-\phi), \quad \sigma_2(s,\zeta_{\epsilon,s}) = c_3\cos^2(s)\zeta_\epsilon(-\phi).$$

The assumptions $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ are satisfied for $\underline{K} = K = c_1^2 \vee c_2^2 \vee c_3^2$ and $\underline{\varrho} = 0.1$. We obtain the corresponding averaged equation

$$\begin{aligned} y_\epsilon(\varkappa) &= \zeta(1) - H(\zeta_1) + H(y_{\epsilon,\varkappa}) + \epsilon\int_1^\varkappa \bar{b}(y_{\epsilon,s})ds + \sqrt{\epsilon}\int_1^\varkappa \bar{\sigma}_1(y_{\epsilon,s})d\mathcal{W}(s) \\ &+ \epsilon^\iota\iota\int_1^\varkappa \left(\log\frac{\varkappa}{s}\right)^{\iota-1}\frac{\bar{\sigma}_2(y_{\epsilon,s})}{s}ds, \end{aligned} \tag{46}$$

where

$$\bar{b}(y_{\epsilon,s}) = c_1 y_\epsilon(-\phi), \qquad \bar{\sigma}_1(y_{\epsilon,s}) = c_2 y_\epsilon(-\phi), \qquad \bar{\sigma}_2(y_{\epsilon,s}) = 0.5c_3 y_\epsilon(-\phi).$$

Now, we present a numerical simulation of Equations (44) and (46), initialized with the condition $\alpha(\varkappa) = \exp(-\varkappa^2/2)$, while employing the selected parameters $\phi = 0.5$ and $\epsilon = 0.04$. Subsequently, in Figures 1 and 2, we offer a comparative analysis between the precise solution ζ_ϵ for Equation (44) and the averaged solution y_ϵ for Equation (46), each evaluated for two distinct values of ι.

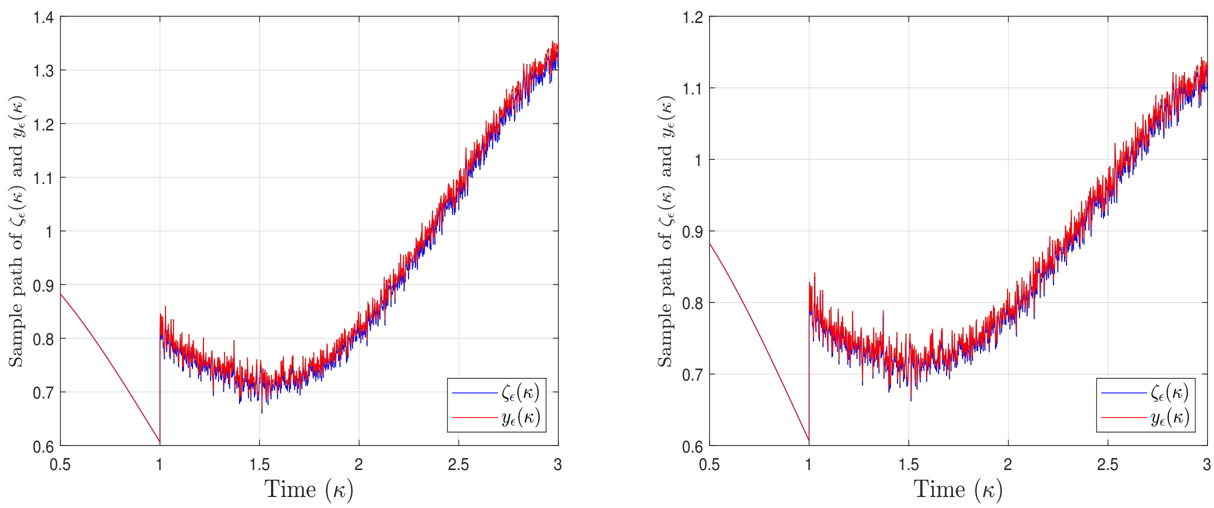

Figure 1. Comparison of the exact solution (blue) and the approximate solution (red), with $\iota = 0.75$ **(left)**, $\iota = 0.95$ **(right)**, $c_1 = -2$, $c_2 = -1$, $c_3 = 2$, and $\alpha(\varkappa) = \exp(-\varkappa^2/2)$, on the interval $[0.5, 3]$.

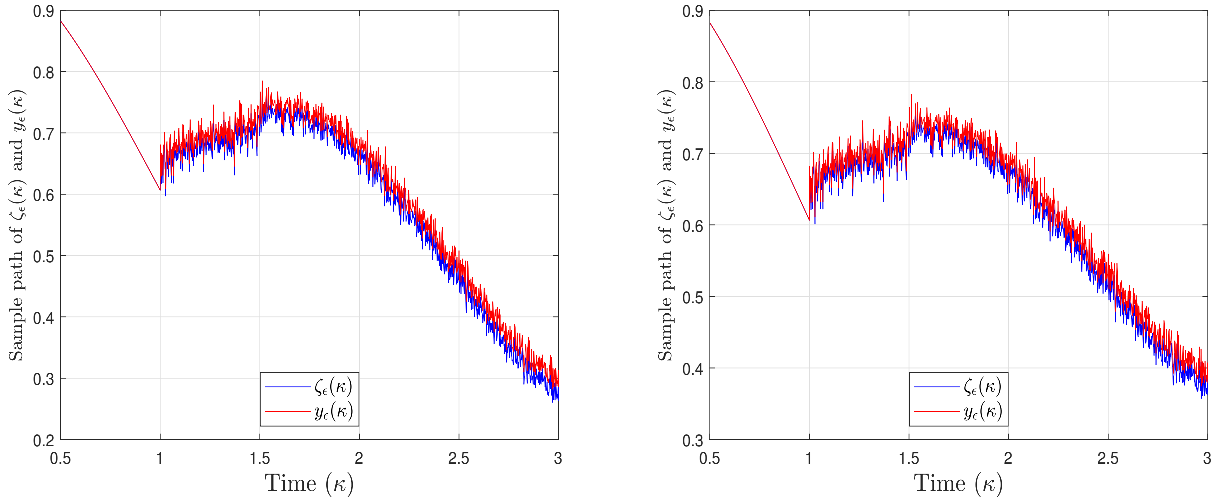

Figure 2. Comparison of the exact solution (blue) and the approximate solution (red), with $\iota = 0.5$ (**left**), $\iota = 0.65$ (**right**), $c_1 = 3$, $c_2 = 1$, $c_3 = -1$, and $\alpha(\varkappa) = \exp(-\varkappa^2/2)$, on the interval $[0.5, 3]$.

6. Conclusions

In this work, we delve into the intricacies surrounding the existence and uniqueness of FHIDSIE with an order denoted by ι, where ι belongs to the interval $(\frac{1}{2}, 1)$. Our investigation in this paper harnesses the power of the PIT method to unravel these fundamental aspects of FHIDSIE. Moreover, we establish the profound averaging principle associated with FHIDSIE, leveraging moment inequalities to provide a rigorous and comprehensive proof.

Author Contributions: Conceptualization, L.M.; methodology, M.R.; writing—original draft preparation, A.B.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research is funded by "Researchers Supporting Project number (RSPD2023R683), King Saud University, Riyadh, Saudi Arabia".

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors extend their appreciation to King Saud University in Riyadh, Saudi Arabia for funding this research work through Researchers Supporting Project number (RSPD2023R683).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
2. Podlubny, I. *Fractional Differential Equations*; Academic Press: Cambridge, MA, USA, 1999.
3. Ceng, L.C.; Cho, S.Y. On approximate controllability for systems of fractional evolution hemivariational inequalities with Riemann-Liouville fractional derivatives. *J. Nonlinear Var. Anal.* **2022**, *6*, 421–438.
4. Atanackovic, T.M.; Pilipovic, S.; Stankovic, B.; Zorica, D. *Fractional Calculus with Applications in Mechanics*; Wiley-ISTE: Indianapolis, IN, USA, 2014.
5. Baleanu, D.; Machado, J.A.; Luo, A.C. *Fractional Dynamics and Control*; Springer Science and Business Media: New York, NY, USA, 2011.
6. Pedjeu, J.C.; Ladde, G.S. Stochastic fractional differential equations: Modeling, method and analysis. *Chaos Solitons Fractals* **2012**, *45*, 279–293. [CrossRef]
7. Abbas, S.; Benchohra, M.; Lazreg, J.E.; Zhou, Y. A survey on Hadamard and Hilfer fractional differential equations: Analysis and stability. *Chaos Solitons Fractals* **2017**, *102*, 47–71. [CrossRef]
8. Almeida, R. Caputo-Hadamard fractional derivatives of variable order. *Numer. Funct. Anal. Optim.* **2017**, *38*, 1–19. [CrossRef]
9. Makhlouf, A.B.; Mchiri, L. Some results on the study of Caputo-Hadamard fractional stochastic differential equations. *Chaos Solitons Fractals* **2022**, *155*, 111757. [CrossRef]

10. Abouagwa, M.; Liu, J.; Li, J.; Caratheodory approximations and stability of solutions to non-Lipschitz stochastic fractional differential equations of Itô-Doob type. *Appl. Math. Comput.* **2018**, *329*, 143–153. [CrossRef]
11. Wang, W.; Guo, Z. Optimal index and averaging principle for Itô–Doob stochastic fractional differential equations. *Stoch. Dyn.* **2022**, *22*, 2250018. [CrossRef]
12. Ben Makhlouf, A.; Mchiri, L.; Arfaoui, H.; Dhahri, S.; El-Hady, E.S.; Cherif, B. Hadamard Itô-Doob Stochastic Fractional Order Systems. *Discret. Contin. Dyn. Syst.-S* **2023**, *16*, 2060–2074. [CrossRef]
13. Hattaf, K. A New Class of Generalized Fractal and Fractal-Fractional Derivatives with Non-Singular Kernels. *Fractal Fract.* **2023**, *7*, 395. [CrossRef]
14. Tatom, F.B. The Relationship between fractional calculus and fractals. *Fractals* **1995**, *3*, 217–229. [CrossRef]
15. Mao, X. *Stochastic Differential Equations and Applications*; Ellis Horwood: Chichester, UK, 1997.
16. Stoyanov, I.M.; Bainov, D.D. The averaging method for a class of stochastic differential equations. *Ukr. Math. J.* **1974**, *26*, 186–194. [CrossRef]
17. Lewis, E.R. Delay-line models of polpulation growth. *Ecology* **1972**, *53*, 797–807. [CrossRef]