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Stability Results and Parametric Delayed Mittag–Leffler Matrices in Symmetric Fuzzy–Random Spaces with Application

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Abstract: We introduce a matrix-valued fractional delay differential system in diverse cases and present Fox type stability results with applications of aggregated special functions. In addition we present an example showing the numerical solutions based on the second type Kudryashov method. Finally, via the method of variation of constants, and some properties of the parametric Mittag–Leffler matrices, we obtain both symmetric random and symmetric fuzzy finite-time stability results for the governing fractional delay model. A numerical example is considered to illustrate applicability of the study.

Keywords: stability; special functions; delayed Mittag–Leffler matrices; aggregation maps; symmetric fuzzy-random spaces

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1. Introduction

Generally fractional-order equations are considered as an extension of ODEs, and they have been used as more appropriate models of real-world issues in physics, engineering, finance, etc. [1,2]. The applications of fractional-order calculus have been growing, including petroleum engineering, viscoelastic mechanics, anomalous diffusion, multi-strain tuberculosis model, control system, and many other branches of engineering and physics [3,4]. A good collection of diverse fractional-order models used to mechanics, viscoelasticity, thermodiffusion, and thermodynamics is given in [5]. In different processes, like technical processes, chemical processes, economics, biosciences, a delay is observed. With the combination of both time delay and fractional derivative, the subject of fractional-order delay differential models is enjoying growing interest among scientists [6–11].

finite-time stability is a notion that was first presented in the 1950s. This notion differs from classical stability [12,13] in two significant ways. First, finite-time stability requires prescribed bounds on system variables. Second, it deals with systems whose operation is limited to a fixed finite interval of time. For systems that are known to operate only over a finite interval of time and whenever, from specific considerations, the systems' variables must lie within particular bounds, finite-time stability is the only meaningful description of stability [14–16].

Motivated by [17,18], we consider the Caputo fractional delay differential system below:

$$D_{0+}^{\mathfrak{P}}\mathcal{F}(\chi) = \Xi\,\mathcal{F}(\chi - t) + \Xi_1\,\mathcal{G}(\chi) + \Xi_2\,\mathcal{H}(\chi, \mathcal{F}(\chi)), \quad \chi \in \nu := [0, T],\ t > 0, \quad (1)$$

$$\mathcal{F}(\chi) = \mathcal{K}(\chi), \qquad -t \leq \chi \leq 0, \quad (2)$$

where $D^{\mathfrak{P}}$ is the Caputo derivative with $0 < \mathfrak{P} < 1$, t is a fixed delay time, $T = \kappa^* t$ for $\kappa^* \in \rho := \{1, 2, \cdots\}$.

Let $[.]_{n\times n}$ be a square matrix of $n \times n$, for every $n \in \mathbb{N}$. We shall investigate the cases below:

Case 1 : $\Xi_1 = [1]_{1\times 1}$, $\Xi = \Xi_2 = [0]_{1\times 1}$, and $\mathcal{F}, \mathcal{G} \in C(\nu, \mathbb{R})$.
Case 2 : $\Xi_1 = \Xi_2 = [0]_{n\times n}$, $\Xi \in \mathbb{R}^{n\times n}$, $\mathcal{K} \in C([-t, 0], \mathbb{R}^n)$, and $\mathcal{F} \in C([-t, T], \mathbb{R}^n)$.
Case 3 : $\Xi_1 = [1]_{n\times n}$, $\Xi_2 = [0]_{n\times n}$, $\Xi \in \mathbb{R}^{n\times n}$, $\mathcal{K} \in C([-t, 0], \mathbb{R}^n)$, $\mathcal{G} \in C(\nu, \mathbb{R}^n)$, and $\mathcal{F} \in C([-t, T], \mathbb{R}^n)$.
Case 4 : $\Xi_1 = [0]_{n\times n}$, $\Xi_2 = [1]_{n\times n}$, $\Xi \in \mathbb{R}^{n\times n}$, $\mathcal{K} \in C([-t, 0], \mathbb{R}^n)$, $\mathcal{H} \in C(\nu \times \mathbb{R}^n, \mathbb{R}^n)$, and $\mathcal{F} \in C([-t, T], \mathbb{R}^n)$.

In **Case (1)**, we study Fox type stability results with applications of aggregation maps and special functions. Finally, a numerical method is applied to find the approximate solutions. In **Case (2)**, via the method of variation of constants, and some properties of the delayed one parameter Mittag–Leffler matrix, we investigate the explicit formula of solutions. Thereafter, we propose symmetric random finite-time stability results. In **Case (3)**, by the method of variation of constants, and some properties of the delayed two parameter Mittag–Leffler matrix, we study the explicit formula of solutions. In **Case (4)**, through the delayed Mittag–Leffler matrices in one and two parameters, we prove symmetric fuzzy finite-time stability for the above fractional-order delay system. Next, a numerical example is considered to illustrate applicability of the study.

2. Preliminaries

2.1. Special Functions

2.1.1. Fox Type Functions

In this part, we present the Fox $\mathbb{H}$-function and its variations (see [19]).

The Fox $\mathbb{H}$-function introduced by Charles Fox (1961) is defined as follows:

$$ {}^{A}_{C}\mathbb{H}^{B}_{D}\left[\chi\middle|\begin{smallmatrix}(V_j,W_j)_{1,C}\\(N_j,M_j)_{1,D}\end{smallmatrix}\right] := \frac{1}{2\pi i}\int_{\mathscr{X}} \varpi(S)\chi^S dS, \tag{3}$$

where $i^2 = -1$, $\chi \in \mathbb{C}\backslash\{0\}$, $\chi^S = \exp(S[\log|\chi| + i\ \arg(\chi)])$, and

$$\varpi(S) := \frac{\prod_{j=1}^{A}\Gamma(N_j - M_jS)\prod_{j=1}^{B}\Gamma(1 - V_j + W_jS)}{\prod_{j=A+1}^{D}\Gamma(1 - N_j + M_jS)\prod_{j=B+1}^{C}\Gamma(V_j - W_jS)},$$

in which an empty product is interpreted as 1, and the integers A, B, C, D satisfy the inequalities $0 \le B \le C$ and $1 \le A \le D$. Let the coefficients

$$W_j > 0\ (j = 1, \ldots, C) \quad \text{and} \quad M_j > 0\ (j = 1, \ldots, D),$$

and the complex parameters

$$V_j\ (j = 1, \ldots, C) \quad \text{and} \quad N_j\ (j = 1, \ldots, D)$$

be constrained s.t. no poles of the integrand in (3) coincide, and $\mathscr{X}$ is a suitable contour of the Mellin–Barnes type (in the complex S-plane) which has one of the forms below:

- $\mathscr{X} = \mathscr{X}_{-\infty}$ is a left loop beginning at $-\infty$ and terminating at $-\infty$, enclosing all the poles of $\Gamma(S)$.
- $\mathscr{X} = \mathscr{X}_{+\infty}$ is a left loop beginning at $+\infty$ and terminating at $+\infty$, enclosing all the poles of $\Gamma(\underbrace{V_j}_{1\le j\le C} - S)$, situated in a horizontal strip beginning at the point $+\infty + i\lambda_1$ and ending at the point $+\infty + i\lambda_2$ with $-\infty < \lambda_1 < \lambda_2 < +\infty$, and $V_j \in \mathbb{C}$.
- $\mathscr{X} = \mathscr{X}_{i\lambda\infty}$ is a contour beginning at the point $\lambda - i\infty$ and ending at the point $\lambda + i\infty$, for every $\lambda \in \mathbb{R}$.

Plus, if,

$$R := \sum_{j=1}^{B} W_j - \sum_{j=B+1}^{C} W_j + \sum_{j=1}^{A} M_j - \sum_{j=A+1}^{D} M_j > 0,$$

then, the integral in (3) converges absolutely and defines the $\mathbb{H}$-function that is analytic in the sector:

$$|\arg(\chi)| < \frac{\pi}{2}R$$

and with the point $\chi = 0$ being tacitly excluded. Indeed, the $\mathbb{H}$-function makes sense and also presents an analytic function of χ when either

$$R_1 := \sum_{j=1}^{C} W_j - \sum_{j=1}^{D} M_j < 0 \quad \text{and} \quad 0 < |\chi| < \infty,$$

or

$$R_1 = 0 \quad \text{and} \quad 0 < |\chi| < R_2 := \prod_{j=1}^{C} W_j^{-W_j} \prod_{j=1}^{D} M_j^{M_j}.$$

We now propose the special cases of Fox's $\mathbb{H}$-function as follows:

- **Exponential function:**

$$_0\mathbb{H}_0[\chi] := \exp(\chi) = \sum_{j=0}^{\infty} \frac{\chi^j}{\Gamma(j+1)},$$

where $\chi \in \mathbb{C}$.

- **One parameter Mittag–Leffler function:**

$$_0\mathbb{H}_1[N_1;\chi] := \sum_{j=0}^{\infty} \frac{\chi^j}{\Gamma(1+N_1 j)},$$

where $\chi, N_1 \in \mathbb{C}$, and $\Re(N_1) > 0$.

- **Gauss Hypergeometric function:**

$$_2\mathbb{H}_1[V_1,V_2;N_1;\chi] = \sum_{j=0}^{\infty} \frac{(V_1)_j (V_2)_j}{(N_1)_j} \frac{\chi^j}{j!} = \frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)} \sum_{j=0}^{\infty} \frac{\Gamma(V_1+j)\Gamma(V_2+j)}{\Gamma(N_1+j)} \frac{\chi^j}{j!}.$$

Furthermore, this function can be represented in terms of the Mellin–Barnes integral of the form

$$_2\mathbb{H}_1[V_1,V_2;N_1;\chi] = \frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)} \frac{1}{2\pi i} \int_{\mathscr{X}} \frac{\Gamma(S)\Gamma(V_1-S)\Gamma(V_2-S)}{\Gamma(N_1-S)} (-\chi)^{-S} dS,$$

where $\chi, V_1, V_2, N_1 \in \mathbb{C}$, $N_1 \neq 0, -1, -2, -3, \dots$, and $\Re(V_1), \Re(V_2), \Re(N_1) > 0$.

- **Wright function:**

$$_1\mathbb{H}_1[V_1;N_1;\chi] := \sum_{j=0}^{\infty} \frac{\chi^j}{j!\Gamma(V_1 j + N_1)},$$

where $V_1, N_1, \chi \in \mathbb{C}$ and $\Re(V_1), \Re(N_1) > 0$.

- **Fox–Wright function:**

Consider the positive vectors $\mathbf{W} = (W_1, \ldots, W_C)$, $\mathbf{M} = (M_1, \ldots, M_D)$, and complex vectors $\mathbf{V} = (V_1, \ldots, V_C)$, and $\mathbf{N} = (N_1, \ldots, N_D)$. The Fox–Wright function is given by the series

$$ {}_C\mathbb{H}_D\left[\chi\middle|\begin{smallmatrix}(V_1,W_1),\ldots,(V_C,W_C)\\(N_1,M_1),\ldots,(N_D,M_D)\end{smallmatrix}\right] = {}_C\mathbb{H}_D\left[\chi\middle|\begin{smallmatrix}(\mathbf{V},\mathbf{W})\\(\mathbf{N},\mathbf{M})\end{smallmatrix}\right] = \sum_{n=0}^{\infty}\frac{\Gamma(\mathbf{W}n+\mathbf{V})}{\Gamma(\mathbf{M}n+\mathbf{N})}\frac{\chi^n}{n!}, \tag{4}$$

where

$$\Gamma(\mathbf{W}n+\mathbf{V}) = \prod_{j=1}^{C}\Gamma(W_j n + V_j),$$

and

$$\Gamma(\mathbf{M}n+\mathbf{N}) = \prod_{j=1}^{D}\Gamma(M_j n + N_j).$$

The series (4) has a nonzero radius of convergence if

$$R_1 := \sum_{j=1}^{D} M_j - \sum_{j=1}^{C} W_j \geq -1. \tag{5}$$

Plus, if $R_1 > -1$, then, the series converges for all finite values of χ, and if $R_1 = -1$, its radius of convergence equals

$$R_2 := \prod_{j=1}^{C} W_j^{-W_j} \prod_{j=1}^{D} M_j^{M_j}. \tag{6}$$

The convergence on the boundary $|\chi| = R_2$, however, depends on the value of

$$R_3 := \sum_{j=1}^{D} N_j - \sum_{k=1}^{C} V_k + \frac{C-D-1}{2}, \tag{7}$$

by noting that series (4) converges absolutely for $|\chi| = R_2$, if $\Re(R_3) > 0$.

- **Meijer $\mathbb{G}$-function:**

$$\begin{aligned} {}^{A}_{C}\mathbb{G}^{B}_{D}\left[\chi\middle|\begin{smallmatrix}V_1,\ldots,V_C\\N_1,\ldots,N_D\end{smallmatrix}\right] &\equiv {}^{A}_{C}\mathbb{H}^{B}_{D}\left[\chi\middle|\begin{smallmatrix}(V_1,1),\ldots,(V_C,1)\\(N_1,1),\ldots,(N_D,1)\end{smallmatrix}\right] \\ &= \frac{1}{2\pi i}\int_{\mathscr{X}} \varpi'(S)\chi^{-S}dS, \end{aligned} \tag{8}$$

where

$$\varpi'(S) := \frac{\prod_{j=1}^{A}\Gamma(N_j+S)\prod_{i=1}^{B}\Gamma(1-V_i-S)}{\prod_{i=B+1}^{C}\Gamma(V_i+S)\prod_{j=A+1}^{D}\Gamma(1-V_j-S)}, \tag{9}$$

and $\chi^{-S} = \exp(-S[\log|\chi| + i\ \arg(\chi)])$, $\chi \neq 0$ and $i^2 = -1$.

Note that an empty product in (9) is defined to be one, and the poles

$$N_{j\sigma} = -(N_j + \sigma), \quad j = 1, \ldots, A, \quad \sigma \in \mathbb{N}_0, \tag{10}$$

of the gamma functions $\Gamma(N_j + S)$ and the poles

$$V_{i\sigma'} = 1 - V_i + \sigma', \quad i = 1, \ldots, B, \quad \sigma' \in \mathbb{N}_0, \tag{11}$$

of the gamma functions $\Gamma(1 - V_i - S)$ do not coincide, that is

$$N_j + \sigma \neq V_i - \sigma' - 1, \quad i = 1, \dots, B, \quad j = 1, \dots, A, \quad \sigma, \sigma' \in \mathbb{N}_0. \tag{12}$$

Besides, $\mathscr{X}$ is one of the contours given above that separate all poles $N_{j\sigma}$ in (10) on the left from all poles $V_{i\sigma}$ in (11) on the right of $\mathscr{X}$.

- **$\mathbb{G}$-function:**

$${}_C\mathbb{H}_D[V_1, \dots, V_C; N_1, \dots, N_C; \chi] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{C} (V_i)_n}{\prod_{j=1}^{D} (N_j)_n} \frac{\chi^n}{n!}, \tag{13}$$

where $\chi \in \mathbb{C}$, $C, D \in \mathbb{N}_0$, and $V_i, N_j \in \mathbb{C}$, for $i = 1, \dots, C$ and $j = 1, \dots, D$.

For $\rho \in \mathbb{C}$, we define

$$\begin{aligned} &(\rho)_0 = 1, \quad \rho \neq 0, \\ &(\rho)_n = \rho(\rho + 1) \dots (\rho + n - 1), \quad n \in \mathbb{N}. \end{aligned}$$

If $V_j \neq -\sigma$, $j = 1, \dots, D$ and $\sigma \in \mathbb{N}_0$, then, this function (13) can be represented in terms of the Mellin–Barnes integral of the following form

$${}_C\mathbb{H}_D[V_1, \dots, V_C; N_1, \dots, N_D; \chi] = \frac{\prod_{j=1}^{D} \Gamma(N_j)}{\prod_{i=1}^{C} \Gamma(V_i)} \frac{1}{2\pi i} \int_{\mathscr{X}} \frac{\Gamma(S) \prod_{i=1}^{C} \Gamma(V_i - S)}{\prod_{j=1}^{D} \Gamma(N_j - S)} (-\chi)^{-S} dS, \quad \chi \neq 0,$$

where $N_j \neq 0, -1, -2, \dots, j = 1, \dots D$, $V_i \neq 0, -1, -2, \dots, i = 1, \dots C$, and with the special contour $\mathscr{X}$.

2.1.2. Mittag–Leffler Type Functions

We introduce a novel Mittag–Leffler function with m-parameters as follows [20]:

Suppose $(V, W)_C := [V_1, W_1; \dots; V_C, W_C]$, $(N, M)_D := [N_1, M_1; \dots; N_D, M_D]$, $C + D = m - 1$, and $m \in \mathbb{N}$. The m-parameter function of the Mittag–Leffler type is given by

$$\begin{aligned} \mathbb{M}^{V_1, W_1; \dots; V_C, W_C}_{\alpha, \tau; N_1, M_1; \dots; N_D, M_D}(\chi) &= \mathbb{M}^{(V,W)_C}_{\alpha, \tau; (N,M)_D} \\ &= \sum_{n=0}^{\infty} \frac{(V_1)_{W_1 n} \cdots (V_C)_{W_C n}}{\Gamma(\alpha n + \tau)(N_1)_{M_1 n} \cdots (N_D)_{M_D n}} \chi^n, \end{aligned}$$

where $\chi, \alpha, \tau, V_i, W_i, N_j, M_j \in \mathbb{C}$, with $\min\{\alpha, \tau, V_i, W_i, N_j, M_j\} > 0$, for every $i = 1, \cdots, C$ and $j = 1, \cdots, D$. Note that the generalized Pochhammer symbol $(A)_{Bn}$ is defined by

$$(A)_{Bn} = \frac{\Gamma(A + Bn)}{\Gamma(A)}.$$

We now introduce a family of parametric Mittag–Leffler type functions, as follows:

- **One-parameter function of the Mittag–Leffler type:**

$$\mathbb{M}_\alpha(\chi) = \sum_{j=0}^{\infty} \frac{\chi^j}{\Gamma(j\alpha + 1)},$$

where $\chi, \alpha \in \mathbb{C}$, and $\Re(\alpha) > 0$.

- **Two-parameter function of the Mittag–Leffler type:**

$$\mathbb{M}_{\alpha,\tau}(\chi) = \sum_{j=0}^{\infty} \frac{\chi^j}{\Gamma(j\alpha + \tau)},$$

where $\chi, \alpha, \tau \in \mathbb{C}$, and $\Re(\alpha), \Re(\tau) > 0$.

- **Three-parameter function of the Mittag–Leffler type:**

$$\mathbb{M}_{\alpha,\tau}^{V_1}(\chi) = \sum_{j=0}^{\infty} \frac{(V_1)_j \chi^j}{j!\Gamma(j\alpha + \tau)},$$

where the Pochhammer symbol $(V_1)_j$ defined by

$$(V_1)_j = V_1(V_1 + 1) \cdots (V_1 + j - 1), \quad (V_1)_0 = 1,$$

and $\chi, \alpha, \tau, V_1 \in \mathbb{C}$, and $\Re(\alpha), \Re(\tau) > 0$.

- **Four-parameter function of the Mittag–Leffler type** [21]:

$$\mathbb{M}_{\alpha,\tau}^{V_1,W_1}(\chi) = \sum_{j=0}^{\infty} \frac{(V_1)_{W_1 j}\, \chi^j}{j!\Gamma(j\alpha + \tau)},$$

where $\chi, \alpha, \tau, V_1, W_1 \in \mathbb{C}$, and $\min\{\Re(\alpha), \Re(\tau), \Re(V_1)\} > 0$.

- **Five-parameter function of the Mittag–Leffler type** [22]:

$$\mathbb{M}_{\alpha,\tau,N_1}^{V_1,W_1}(\chi) = \sum_{j=0}^{\infty} \frac{(V_1)_{W_1 j}}{\Gamma(\alpha j + \tau)(N_1)_j} \chi^j,$$

where $\min\{\Re(\alpha), \Re(\tau), \Re(N_1), \Re(V_1)\} > 0$, and $W_1 \in (0,1) \cup \mathbb{N}$.

2.1.3. Supertrigonometric and Superhyperbolic Mittag–Leffler Type Functions

In this part, let $\chi, N_1 \in \mathbb{C}$, and $\Re(N_1) > 0$. We shall consider the Supertrigonometric and Superhyperbolic Mittag–Leffler type functions in one parameter, as follows [23–27]:

- **Pre-supercosine-Mittag–Leffler-type function:**

$$precos_{N_1}(\chi) = \sum_{i=0}^{\infty} \frac{(-1)^j \chi^{2j}}{\Gamma((2j)N_1 + 1)}.$$

- **Pre-supersine-Mittag–Leffler-type function:**

$$presin_{N_1}(\chi) = \sum_{j=0}^{\infty} \frac{(-1)^i \chi^{2j+1}}{\Gamma((2j+1)N_1 + 1)}.$$

- **Pre-superhyperbolic supercosine-Mittag–Leffler-type function**

$$precosh_{N_1}(\chi) = \sum_{j=0}^{\infty} \frac{\chi^{2j}}{\Gamma((2j)N_1 + 1)}.$$

- **Pre-superhyperbolic supersine-Mittag–Leffler-type function**

$$presinh_{N_1}(\chi) = \sum_{j=0}^{\infty} \frac{\chi^{2j+1}}{\Gamma((2j+1)N_1 + 1)}.$$

2.1.4. Supertrigonometric and Superhyperbolic Gauss–Hypergeometric Type Functions

Let $\chi, N_1, V_1, V_2 \in \mathbb{C}$, and $\Re(N_1), \Re(V_1), \Re(V_2) > 0$. We shall consider the Supertrigonometric and Superhyperbolic Gauss–Hypergeometric type functions, as follows [28,29]:

- **Supercosine-Gauss–Hypergeometric type function:**

$$ {}_2supercos_1(V_1, V_2, N_1; \chi) = \sum_{j=0}^{\infty} \frac{(V_1)_{2j}(V_2)_{2j}}{(N_1)_{2j+1}} \frac{(-1)^j \chi^{2j}}{(2j)!}. $$

- **Supersine-Gauss–Hypergeometric type function:**

$$ {}_2supersin_1(V_1, V_2, N_1; \chi) = \sum_{j=0}^{\infty} \frac{(V_1)_{2j+1}(V_2)_{2j+1}}{(N_1)_{2j+1}} \frac{(-1)^j \chi^{2j+1}}{(2j+1)!}. $$

- **Superhyperbolic cosine-Gauss–Hypergeometric type function:**

$$ {}_2supercosh_1(V_1, V_2, N_1; \chi) = \sum_{j=0}^{\infty} \frac{(V_1)_{2j}(V_2)_{2j}}{(N_1)_{2j+1}} \frac{\chi^{2j}}{(2j)!}. $$

- **Superhyperbolic sine-Gauss–Hypergeometric type function:**

$$ {}_2supersinh_1(V_1, V_2, N_1; \chi) = \sum_{j=0}^{\infty} \frac{(V_1)_{2j+1}(V_2)_{2j+1}}{(N_1)_{2j+1}} \frac{\chi^{2j+1}}{(2j+1)!}. $$

2.2. Generalized Triangular Norms (GTNs)

Let $\epsilon := [0,1]$ and

$$ \mathrm{diag}M_n(\epsilon) := \left\{ \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ 0 & A_{22} & 0 & \cdots & 0 \\ 0 & 0 & A_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & A_{nn} \end{bmatrix} = \mathrm{diag}[A_{11}, \cdots, A_{nn}], \quad \underbrace{A_{ij}}_{\substack{1\le i\le n \\ 1\le j\le n}} \in \epsilon \right\}, $$

with the partial order relation below:

$$ \mathbf{A} := \mathrm{diag}[A_{11}, \cdots, A_{nn}], \quad \mathbf{B} := \mathrm{diag}[B_{11}, \cdots, B_{nn}] \in \mathrm{diag}M_n(\epsilon), $$

$$ \mathbf{A} \preceq \mathbf{B} \Longleftrightarrow \underbrace{A_{ij}}_{\substack{1\le i\le n \\ 1\le j\le n}} \le \underbrace{B_{ij}}_{\substack{1\le i\le n \\ 1\le j\le n}}, $$

and the bold symbols **0** and **1** defind by

$$ \mathbf{0} := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n} := \mathrm{diag}[0, \cdots, 0]_{n\times n}, $$

and

$$ \mathbf{1} := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n} := \mathrm{diag}[1, \cdots, 1]_{n\times n}. $$

Definition 1 ([18]). *A GTN on diagonal matrices is an operation* $\bigodot : (diagM_n(\epsilon))^2 \to diagM_n(\epsilon)$, *s.t. for every* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in diagM_n(\epsilon)$, *satisfies the following:*

(1) $\mathbf{A} \bigodot \mathbf{1} = \mathbf{A}$,
(2) $\mathbf{A} \bigodot \mathbf{B} = \mathbf{B} \bigodot \mathbf{A}$,
(3) $\mathbf{A} \bigodot (\mathbf{B} \bigodot \mathbf{C}) = (\mathbf{A} \bigodot \mathbf{B}) \bigodot \mathbf{C}$,
(4) $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{C} \preceq \mathbf{D} \Longrightarrow \mathbf{A} \bigodot \mathbf{C} \preceq \mathbf{B} \bigodot \mathbf{D}$.

For every sequences $\{\mathbf{A}_m\}$, $\{\mathbf{B}_m\}$ converging to $\mathbf{A}, \mathbf{B} \in \text{diag}M_n(\epsilon)$ respectively, if

$$\lim_{m \to \infty} (\mathbf{A}_m \bigodot \mathbf{B}_m) = \mathbf{A} \bigodot \mathbf{B},$$

then $\bigodot$ on diagonal matrices is continuous.

For instance, consider the continuous GTNs $\bigodot_P, \bigodot_M, \bigodot_L : (\text{diag}M_n(\epsilon))^2 \to \text{diag}M_n(\epsilon)$ defined as follows:

$$\begin{aligned} &\mathbf{A} \underset{P}{\bigodot} \mathbf{B} \\ &= \text{diag}[A_{11}, \cdots, A_{nn}] \underset{P}{\bigodot} \text{diag}[B_{11}, \cdots, B_{nn}] \\ &= \text{diag}[A_{11} \cdot B_{11}, \cdots, A_{nn} \cdot B_{nn}], \end{aligned}$$

$$\begin{aligned} &\mathbf{A} \underset{L}{\bigodot} \mathbf{B} \\ &= \text{diag}[A_{11}, \cdots, A_{nn}] \underset{L}{\bigodot} \text{diag}[B_{11}, \cdots, B_{nn}] \\ &= \text{diag}[\max\{A_{11} + B_{11} - 1, 0\}, \cdots, \max\{A_{nn} + B_{nn} - 1, 0\}], \end{aligned}$$

and

$$\begin{aligned} &\mathbf{A} \underset{M}{\bigodot} \mathbf{B} \\ &= \text{diag}[A_{11}, \cdots, A_{nn}] \underset{M}{\bigodot} \text{diag}[B_{11}, \cdots, B_{nn}] \\ &= \text{diag}[\min\{A_{11}, B_{11}\}, \cdots, \min\{A_{nn}, B_{nn}\}]. \end{aligned}$$

If, for every GTNs $\bigodot_1, \bigodot_2$, and every $\mathbf{A}, \mathbf{B} \in \text{diag}M_n(\epsilon)$,

$$\mathbf{A} \underset{1}{\bigodot} \mathbf{B} \preceq \mathbf{A} \underset{2}{\bigodot} \mathbf{B},$$

then, we say that $\bigodot_2$ is stronger than $\bigodot_1$, or, equivalently, $\bigodot_1$ is weaker than $\bigodot_2$.

In the above examples, $\bigodot_L$ and $\bigodot_M$ are weaker and stronger GTNs, respectively. In other words, we get the ordering below:

$$\underset{L}{\bigodot} \preceq \underset{P}{\bigodot} \preceq \underset{M}{\bigodot}.$$

Throughout the paper, we let $\bigodot := \bigodot_M$.

2.3. Symmetric Matrix Valued Fuzzy Normed Spaces

Let $\mathfrak{J}$ be a vector space and $\mathfrak{G}$ be a collection of all matrix valued fuzzy sets (in short, MVF sets), with the continuous increasing mappings $\Phi : \mathfrak{J} \times (0, +\infty) \to \text{diag}M_n(\epsilon)$, s.t. $\lim_{\phi \to \infty} \Phi(\chi, \phi) = \mathbf{1}$, for every $\chi \in \mathfrak{J}$.

In $\mathfrak{G}$, we define the ordering below:

$$\Phi \preceq \Phi' \iff \Phi(\chi, \phi) \preceq \Phi'(\chi, \phi),$$

for every $\phi > 0$ and $\chi \in \mathfrak{J}$.

Definition 2 ([18])**.** *Consider the continuous GTN $\odot$, the vector space $\mathfrak{J}$, and the MVF set $\Phi : \mathfrak{J} \times (0, +\infty) \to diagM_n(\epsilon)$. A symmetric matrix valued fuzzy normed space (shortly, SMVFNS) is a triple $(\mathfrak{J}, \Phi, \odot)$, s.t. for every $\chi, \chi' \in \mathfrak{J}$, and $\phi, \phi' > 0$, we have that*

(1) $\Phi(\chi, \phi) > \mathbf{0}$,
(2) $\Phi(\chi, \phi) = \mathbf{1}$, *iff* $\chi = 0$,
(3) $\Phi(\upsilon\chi, \phi) = \Phi(\chi, \frac{\phi}{|\upsilon|})$, *for every* $0 \neq \upsilon \in \mathbb{C}$,
(4) $\Phi(\chi + \chi', \phi + \phi') \succeq \Phi(\chi, \phi) \odot \Phi(\chi', \phi')$.

Note 1. *A symmetric matrix valued fuzzy Banach space (or SMVFBS) is a complete SMVFNS.*

In this paper we consider the minimum GTN.

Example 1. *We prove in the following four steps that the parametric Mittag–Leffler function below defines a symmetric fuzzy norm as follows:*

$$\mathbb{M}_{N_1}\left(-\frac{|\chi|}{\phi}\right) = \sum_{m=0}^{\infty} \frac{\left(-\frac{|\chi|}{\phi}\right)^m}{\Gamma(1 + N_1 m)},$$

for every $N_1 \in (0, 1)$, $\chi \in \mathfrak{J}$ and $\phi > 0$.

(1) If $0 < N_1 \leq 1$, then, $\mathbb{M}_{N_1}(0) = 1$ and $\lim_{\chi \to -\infty} \mathbb{M}_{N_1}(\chi) = 0$. Thus, $\mathbb{M}_{N_1}$ is an increasing mapping, for every $0 < N_1 \leq 1$, and also we have that $0 < \mathbb{M}_{N_1} \leq 1$.

(2) The equality $\mathbb{M}_{N_1}\left(-\frac{|\chi|}{\phi}\right) = 1$, clearly shows that $\chi = 0$, for every $\phi \in (0, \infty)$, and vice versa.

(3) For every $\chi \in \mathfrak{J}$, $\upsilon \in \mathbb{C}$ and $\phi > 0$, we get

$$\begin{aligned} \mathbb{M}_{N_1}\left(-\frac{|\upsilon\chi|}{\phi}\right) &= \sum_{m=0}^{\infty} \frac{\left(-\frac{|\upsilon\chi|}{\phi}\right)^m}{\Gamma(N_1 m + 1)} \\ &= \sum_{m=0}^{\infty} \frac{\left(-\frac{|\chi|}{\frac{\phi}{|\upsilon|}}\right)^m}{\Gamma(N_1 m + 1)} \\ &= \mathbb{M}_{N_1}\left(-\frac{|\chi|}{\frac{\phi}{|\upsilon|}}\right). \end{aligned}$$

(4) Let

$$\mathbb{M}_{N_1}\left(-\frac{|\chi|}{\phi}\right) \leq \mathbb{M}_{N_1}\left(-\frac{|\chi'|}{\phi'}\right).$$

Thus, for every $\chi, \chi' \in \mathfrak{J}$ and $\phi, \phi' > 0$, we have that

$$\frac{|\chi'|}{\phi'} \leq \frac{|\chi|}{\phi}.$$

If $\chi = \chi'$, we get $\phi \leq \phi'$. Thus, we have

$$
\begin{aligned}
& \frac{|\chi|}{\phi}+\frac{|\chi|}{\phi} \\
\geq \quad & \frac{|\chi|}{\phi}+\frac{|\chi'|}{\phi'} \\
\geq \quad & 2\frac{|\chi|}{\phi+\phi'}+2\frac{|\chi'|}{\phi+\phi'} \\
\geq \quad & 2\frac{|\chi+\chi'|}{\phi+\phi'},
\end{aligned}
$$

thus, $\frac{|\chi|}{\phi} \geq \frac{|\chi+\chi'|}{\phi+\phi'}$. *But* $-\frac{|\chi|}{\phi} \leq -\frac{|\chi+\chi'|}{\phi+\phi'}$, *and also*

$$
\sum_{m=0}^{\infty} \frac{\left(-\frac{|\chi|}{\phi}\right)^{m}}{\Gamma(1+N_1 m)} \leq \sum_{m=0}^{\infty} \frac{\left(-\frac{|\chi+\chi'|}{\phi+\phi'}\right)^{m}}{\Gamma(1+N_1 m)}, \tag{14}
$$

which implies that

$$
\mathbb{M}_{N_1}\left(-\frac{|\chi|}{\phi}\right) \leq \mathbb{M}_{N_1}\left(-\frac{|\chi+\chi'|}{\phi+\phi'}\right).
$$

Hence,

$$
\mathbb{M}_{N_1}\left(-\frac{|\chi+\chi'|}{\phi+\phi'}\right) \geq \min\left\{\mathbb{M}_{N_1}\left(-\frac{|\chi|}{\phi}\right), \mathbb{M}_{N_1}\left(-\frac{|\chi'|}{\phi'}\right)\right\},
$$

for every $\chi, \chi' \in \mathfrak{J}$ *and* $\phi, \phi' \in (0, \infty)$. *Thus,* $\mathbb{M}_{N_1}\left(-\frac{|\chi|}{\phi}\right)$ *is a symmtric fuzzy norm for every* $\chi \in \mathfrak{J}$, $\phi > 0$ *and* $0 < N_1 \leq 1$.

2.4. Symmetric Matrix Valued random Normed Spaces

Let $\mathfrak{E}$ be a collection of all matrix valued distribution functions (shortly, MVDFs) with the left–continuous and non–decreasing mappings $\Psi : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow \mathrm{diag} M_n(\epsilon)$, s.t., $\Psi(0) = 0$ and $\Psi(+\infty) = 1$. Assume that the subset $\mathfrak{E}^+ \subseteq \mathfrak{E}$ contains all functions $\Psi \in \mathfrak{E}$, s.t., the left limit of the function Ψ at the point $+\infty$ is $\mathbf{1}$.

In $\mathfrak{E}^+$, we define the ordering below:

$$
\Psi \preceq \Psi' \Longleftrightarrow \Psi(\psi) \preceq \Psi'(\psi),
$$

for every $\psi \in \mathbb{R}$. The maximal element for $\mathfrak{E}^+$ in the above order is the MVDF $\mathcal{E}_0(\phi)$, defined as

$$
\mathcal{E}_0(\phi) = \begin{cases} \mathbf{0}, & \psi \leq 0, \\ \mathbf{1}, & \psi > 0. \end{cases}
$$

Example 2. *The function* $\Psi(\psi)$ *given by*

$$
\Psi(\psi) = \begin{cases} \mathbf{0}_{2\times 2}, & \psi \leq 0, \\ diag[\exp(-|\psi|^{\frac{1}{2}}), 1 - \frac{1}{exp(\psi)}], & \psi > 0, \end{cases}
$$

is a MVDF. Note $\lim_{\psi \to +\infty} \Psi(\psi) = \mathbf{1}$, *and* $\Psi \in \mathfrak{E}^+$.

Definition 3 ([17])**.** *Consider the continuous GTN* $\odot$, *the vector space* $\mathfrak{J}$, *and the DF* $\Psi : \mathfrak{J} \to \mathfrak{E}^+$. *A symmetric matrix valued random normed space (shortly, SMVRNS) is a triple* $(\mathfrak{J}, \Psi, \odot)$, *s.t. for every* $\chi, \chi' \in \mathfrak{J}$, *and* $\psi > 0$, *we have that*

(1) $\Psi_\chi(\psi) = \mathcal{E}_0(\phi)$, *iff* $\chi = 0$,

(2) $\Psi_{v\chi}(\psi) = \Psi_\chi(\frac{\psi}{|v|})$, *for every* $0 \neq v \in \mathbb{C}$,

(3) $\Psi_{\chi+\chi'}(\psi + \psi') = \Psi_\chi(\psi) \odot \Psi_{\chi'}(\psi')$,

where Ψ_χ *denotes the value of* Ψ *at a point* $\chi \in \mathfrak{J}$.

Example 3. *We prove in the following steps that the increasing Hypergeometric function below defines a symmetric random norm as follows:*

$$ {}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi\|}{\psi}\right) = \sum_{k=0}^{\infty} \frac{(V_1)_k (V_2)_k}{(N_1)_k} \frac{\left(-\dfrac{\|\chi\|}{\psi}\right)^k}{k!}, $$

in which $V_1, V_2, N_1 \geq 0$, $\chi \in \mathfrak{J}$, *and* $\psi > 0$.

(1) *We can easily show that* ${}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi\|}{\psi}\right) = 1$, *for every* $\psi \in (0, +\infty)$, *iff* $\chi = 0$.

(2) *For every* $\chi \in \mathfrak{J}$, $v \in \mathbb{C}$ *and* $\psi > 0$, *we get*

$$ \begin{aligned} {}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|v\chi\|}{\psi}\right) &= \sum_{k=0}^{\infty} \frac{(V_1)_k (V_2)_k}{(N_1)_k} \frac{-\dfrac{\|v\chi\|^k}{\psi}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(V_1)_k (V_2)_k}{(N_1)_k} \frac{-\dfrac{\|\chi\|^k}{\frac{\psi}{|v|}}}{k!} \\ &= {}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi\|}{\frac{\psi}{|v|}}\right). \end{aligned} $$

(3) *Let* ${}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi\|}{\psi}\right) \leq {}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi'\|}{\psi'}\right)$. *Then, we have that* $\frac{\|\chi'\|}{\psi'} \leq \frac{\|\chi\|}{\psi}$, *for every* $\chi, \chi' \in \mathfrak{J}$ *and* $\psi, \psi' > 0$. *If* $\chi = \chi'$, *we have* $\psi \leq \psi'$. *Thus, we obtain*

$$ \begin{aligned} \frac{\|\chi\|}{\psi} + \frac{\|\chi\|}{\psi} &\geq \frac{\|\chi\|}{\psi} + \frac{\|\chi'\|}{\psi'} \\ &\geq 2\frac{\|\chi\|}{\psi + \psi'} + 2\frac{\| \rtimes' \|}{\psi + \psi'} \\ &\geq 2\frac{\|\chi + \chi'\|}{\psi + \psi'}, \end{aligned} $$

hence, $\dfrac{\|\chi\|}{\psi} \geq \dfrac{\|\chi + \chi'\|}{\psi + \psi'}$. *But* $-\dfrac{\|\chi\|}{\psi} \leq -\dfrac{\|\chi + \chi'\|}{\psi + \psi'}$, *and*

$$ \sum_{k=0}^{\infty} \frac{(V_1)_k (V_2)_k}{(N_1)_k} \frac{\left(-\dfrac{\|\chi\|}{\psi}\right)^k}{k!} \leq \sum_{k=0}^{\infty} \frac{(V_1)_k (V_2)_k}{(N_1)_k} \frac{\left(-\dfrac{\|\chi + \chi'\|}{\psi + \psi'}\right)^k}{k!}, \tag{15} $$

which implies that

$$ {}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi\|}{\psi}\right) \leq {}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi + \chi'\|}{\psi + \psi'}\right). $$

Thus, we conclude that

$$
{}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi+\chi'\|}{\psi+\psi'}\right) \geq
$$
$$
\min\left\{{}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi\|}{\psi}\right), {}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi'\|}{\psi'}\right)\right\},
$$

for every $\chi, \chi' \in \mathfrak{J}$ and $\psi, \psi' > 0$. Hence, ${}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{\|\chi\|}{\psi}\right)$ is a symmetric random norm, for every $\chi \in \mathfrak{J}$, and $\psi > 0$, where $(\xi, \|.\|)$ is a normed linear space.

2.5. Caputo Fractional Derivatives

The fractional integral of order $0 < \mathfrak{P} < 1$, for a function $\mathcal{F} : [0, +\infty) \longrightarrow \mathbb{R}$ can be written as follows

$$
I^{\mathfrak{P}}(\chi) = \frac{1}{\Gamma(\mathfrak{P})} \int_0^{\chi} (\chi - t)^{\mathfrak{P}-1} \mathcal{F}(t) dt,
$$

for every $\chi > 0$.

The Riemann–Liouville derivative of order $0 < \mathfrak{P} < 1$, for a function $\mathcal{F} : [0, +\infty) \longrightarrow \mathbb{R}$ is defined by

$$
{}^{RL}D_{0^+}^{\mathfrak{P}}(\chi) = \frac{1}{\Gamma(1-\mathfrak{P})} \frac{d}{d\chi} \int_0^{\chi} (\chi - t)^{-\mathfrak{P}} \mathcal{F}(t) dt,
$$

for every $\chi > 0$.

The Caputo derivative of order $0 < \mathfrak{P} < 1$, for a function $\mathcal{F} : [0, +\infty) \longrightarrow \mathbb{R}$ is given by

$$
D_{0^+}^{\mathfrak{P}}(\chi) = ({}^{RL}D_{0^+}^{\mathfrak{P}})(\chi) - \frac{\mathcal{F}(0)}{\Gamma(1-\mathfrak{P})} \chi^{-\mathfrak{P}},
$$

for every $\chi > 0$.

2.6. Delayed Parametric Mittag–Leffler Type Matrices

We first introduce parametric Mittag–Leffler matrices and then, we define delayed parametric Mittag–Leffler matrices and some of their properties.

Definition 4 ([17]). *The one parameter and two parameter Mittag–Leffler matrices with parameters $\mathfrak{P}_1, \mathfrak{P}_2 > 0$, and square matrix $[\chi]_{n\times n}$ are respectively defined by*

$$
\begin{aligned}
\mathbb{M}_{\mathfrak{P}_1}(\chi) &= \sum_{k=0}^{\infty} \frac{\chi^k}{\Gamma(\mathfrak{P}_1 k + 1)} \\
&= I_n + \frac{\chi}{\Gamma(1+\mathfrak{P}_1)} + \frac{\chi^2}{\Gamma(1+2\mathfrak{P}_1)} + \cdots,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{M}_{\mathfrak{P}_1,\mathfrak{P}_2}(\chi) &= \sum_{k=0}^{\infty} \frac{\chi^k}{\Gamma(\mathfrak{P}_1 k + \mathfrak{P}_2)} \\
&= I_n + \frac{\chi}{\Gamma(\mathfrak{P}_1+\mathfrak{P}_2)} + \frac{\chi^2}{\Gamma(2\mathfrak{P}_1+\mathfrak{P}_2)} + \cdots.
\end{aligned}
$$

Definition 5 ([30])**.** *Delayed one parameter and two parameter Mittag–Leffler matrices* $\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} : \mathbb{R} \to \mathbb{R}^{n^2}$ *and* $\mathbb{M}_{t,\mathcal{Z}}^{\Xi\chi^{\mathfrak{P}}} : \mathbb{R} \to \mathbb{R}^{n^2}$ *are respectively defined as follows:*

$$\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} = \begin{cases} \mathbf{0}, & -\infty < \chi < -t, \\ I, & -t \le \chi \le 0, \\ I + \Xi\frac{\chi^{\mathfrak{P}}}{\Gamma(1+\mathfrak{P})} + \cdots + \Xi^j \frac{(\chi-(j-1)t)^{j\mathfrak{P}}}{\Gamma(j\mathfrak{P}+1)}, & j \in \rho, \end{cases} \tag{16}$$

and

$$\mathbb{M}_{t,\mathcal{Z}}^{\Xi\chi^{\mathfrak{P}}} = \begin{cases} \mathbf{0}, & -\infty < \chi < -t, \\ I\frac{(t+\chi)^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}, & -t \le \chi \le 0, \\ I\frac{(t+\chi)^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} + \Xi\frac{\chi^{2\mathfrak{P}-1}}{\Gamma(\mathcal{Z}+\mathfrak{P})} + \cdots + \Xi^j \frac{(\chi-(j-1)t)^{(j+1)\mathfrak{P}-1}}{\Gamma(j\mathfrak{P}+\mathcal{Z})}, & (j-1)t < \chi \le jt,\ j \in \rho, \end{cases}$$

where I (or **1***) and* **0** *are identity and zero matrices.*

Lemma 1. *For every $\chi \in [(j-1)t, jt]$, with $j \in \rho$, $\mathfrak{P} > 0$, and $\phi, \psi > 0$, we have that*

$$\Phi(\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}, \phi) \succeq \Phi(\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}), \phi),$$

and

$$\Psi_{\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}}(\psi) \succeq \Psi_{\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}(\psi).$$

Proof. Making use of (16), we get

$$\begin{aligned} \Phi(\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}, \phi) &\succeq \Phi\left(I + |\Xi|\frac{\chi^{\mathfrak{P}}}{\Gamma(1+\mathfrak{P})} + \cdots + |\Xi^j|\frac{\chi^{j\mathfrak{P}}}{\Gamma(j\mathfrak{P}+1)}, \phi\right) \\ &\succeq \Phi\left(\sum_{j=0}^{\infty} \frac{(|\Xi|\chi^{\mathfrak{P}})^j}{\Gamma(j\mathfrak{P}+1)}, \phi\right) \\ &= \Phi(\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}), \phi). \end{aligned}$$

□

Lemma 2. *For $\chi \in [(j-1)t, jt]$, with $j \in \rho$, we get*

$$\begin{aligned} &\int_{(j-1)t}^{\chi} (\chi-\tau)^{-\mathfrak{P}} (\tau-(j-1)t)^{j\mathfrak{P}-1} d\tau \\ &= (\chi-(j-1)t)^{(j-1)\mathfrak{P}} \mathbb{B}[1-\mathfrak{P}, j\mathfrak{P}], \end{aligned}$$

where $\mathbb{B}[\alpha, \beta] = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$ is the Beta function.

Proof. Using integration by parts, we have

$$\begin{aligned} &\int_{(j-1)t}^{\chi} (\chi-\tau)^{-\mathfrak{P}} (\tau-(j-1)t)^{j\mathfrak{P}-1} d\tau \\ &= \int_0^{\chi-(j-1)t} (\chi-(j-1)t)^{-\mathfrak{P}} \left(1 - \frac{\mathcal{Y}}{\chi-(j-1)t}\right)^{-\mathfrak{P}} \mathcal{Y}^{j\mathfrak{P}-1} d\mathcal{Y} \\ &= (\chi-(j-1)t)^{(j-1)\mathfrak{P}} \mathbb{B}[1-\mathfrak{P}, j\mathfrak{P}]. \end{aligned}$$

□

Lemma 3. *For $\chi \in ((j-1)t, jt]$, $\mathcal{Y} \in [0, \tau]$ and fixed number $j \in \rho$, we get*

$$\int_{(j-1)t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} (\tau-(j-1)t-\mathcal{Y})^{j\mathfrak{P}-1} d\tau$$
$$= (\chi-(j-1)t-\mathcal{Y})^{(j-1)\mathfrak{P}} \mathbb{B}[1-\mathfrak{P}, j\mathfrak{P}].$$

Proof. From integration by parts, we have

$$\begin{aligned}
& \int_{(j-1)t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} (\tau-(j-1)t-\mathcal{Y})^{j\mathfrak{P}-1} d\tau \\
\overset{\tau-(j-1)t-\mathcal{Y}=M}{=\!=\!=} & \int_{0}^{\chi-(j-1)t-\mathcal{Y}} (\chi-(j-1)t-\mathcal{Y}-M)^{-\mathfrak{P}} M^{j\mathfrak{P}-1} dM \\
= & \int_{0}^{\chi-(j-1)t-\mathcal{Y}} (\chi-(j-1)t-\mathcal{Y})^{-\mathfrak{P}} \left(1-\frac{M}{\chi-(j-1)t-\mathcal{Y}}\right)^{-\mathfrak{P}} M^{j\mathfrak{P}-1} dM \\
\overset{N(\chi-(j-1)t-\mathcal{Y})=M}{=\!=\!=} & \int_{0}^{1} (\chi-(j-1)t-\mathcal{Y})^{(j-1)\mathfrak{P}} (1-N)^{-\mathfrak{P}} N^{j\mathfrak{P}-1} dN \\
= & (\chi-(j-1)t-\mathcal{Y})^{(j-1)\mathfrak{P}} \mathbb{B}[1-\mathfrak{P}, j\mathfrak{P}].
\end{aligned}$$

□

Lemma 4. *For $\chi \in ((j-1)t, jt]$, $\mathcal{Y} \in [0, \tau]$ and fixed number $j \in \rho$, we get*

$$\begin{aligned}
& \int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}} d\tau \\
&= \int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} d\tau + \int_{t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \Xi \frac{(\tau-t-\mathcal{Y})^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P})} d\tau \\
&\quad + \cdots + \int_{(j-1)t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \Xi^{j-1} \frac{(\tau-(j-1)t-\mathcal{Y})^{j\mathfrak{P}-1}}{\Gamma(j\mathfrak{P})} d\tau.
\end{aligned} \tag{17}$$

Proof. Applying mathematical induction, for every $\chi \in ((j-1)t, jt]$, $\mathcal{Y} \in [0, \tau]$ and fixed number $j \in \rho$, we get

(1) For $j = 1$, $\chi \in (0, t]$, applying $\mathbb{M}_{t,\mathfrak{P}}^{\Xi.\mathfrak{P}}$, we obtain

$$\int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}} d\tau = \int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} d\tau.$$

(2) For $j = 2$, $\chi \in (t, 2t]$, applying $\mathbb{M}_{t,\mathfrak{P}}^{\Xi.\mathfrak{P}}$, we obtain

$$\begin{aligned}
& \int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}} d\tau \\
&= \int_{\mathcal{Y}}^{t+\mathcal{Y}} (\chi-\tau)^{-\mathfrak{P}} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}} d\tau + \int_{t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}} d\tau \\
&= \int_{\mathcal{Y}}^{t+\mathcal{Y}} (\chi-\tau)^{-\mathfrak{P}} I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} d\tau + \int_{t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \left[I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} + \Xi \frac{(\tau-t-\mathcal{Y})^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P})} \right] d\tau \\
&= \int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} d\tau + \int_{t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \Xi \frac{(\tau-t-\mathcal{Y})^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P})} d\tau.
\end{aligned}$$

(3) For $j = E$, $\chi \in ((E-1)t, Et]$, and $E \in \rho$, we have that

$$\begin{aligned}
&\int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}} d\tau \\
&= \int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} d\tau + \int_{t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \Xi \frac{(\tau-t-\mathcal{Y})^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P})} d\tau \\
&\qquad + \cdots + \int_{(E-1)t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \Xi^{E-1} \frac{(\tau-(E-1)t-\mathcal{Y})^{E\mathfrak{P}-1}}{\Gamma(E\mathfrak{P})} d\tau.
\end{aligned}$$

For $j = E+1$, $\chi \in (Et, (E+1)t]$, the relation below holds:

$$\begin{aligned}
&\int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}} d\tau \\
&= \int_{\mathcal{Y}}^{t+\mathcal{Y}} (\chi-\tau)^{-\mathfrak{P}} I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} d\tau + \int_{t+\mathcal{Y}}^{2t+\mathcal{Y}} (\chi-\tau)^{-\mathfrak{P}} \left[I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} + \Xi \frac{(\tau-t-\mathcal{Y})^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P})} \right] d\tau \\
&\qquad + \ldots + \int_{Et+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \left[I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} + \ldots + \Xi^{E} \frac{(\tau-Et-\mathcal{Y})^{(E+1)\mathfrak{P}-1}}{\Gamma((E+1)\mathfrak{P})} \right] d\tau \\
&= \int_{\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} I \frac{(\tau-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} d\tau + \int_{t+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \Xi \frac{(\tau-t-\mathcal{Y})^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P})} d\tau \\
&\qquad + \ldots + \int_{Et+\mathcal{Y}}^{\chi} (\chi-\tau)^{-\mathfrak{P}} \Xi^{E} \frac{(\tau-Et-\mathcal{Y})^{(E+1)\mathfrak{P}-1}}{\Gamma((E+1)\mathfrak{P})} d\tau.
\end{aligned}$$

□

Lemma 5. *For every $\chi \in ((j-1)t, jt]$, with fixed number $j \in \rho$ and $\tau \in [0, \chi)$, we get the following items:*

(i) For every $\tau \in [0, \chi-(j-1)t)$, $j \in \rho$, and $\phi > 0$, we obtain

$$\Phi\left(\mathbb{M}_{t,\mathcal{Z}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}, \phi\right) \succeq \Phi\left(\sum_{n=1}^{j} |\Xi|^{n-1} \frac{(\chi-(n-1)t-\tau)^{n\mathfrak{P}-1}}{\Gamma((n-1)\mathfrak{P}+\mathcal{Z})}, \phi \right).$$

(ii) For every $\tau \in [\chi-(E-1)t, \chi-(E-2)t)$ with $E = 2, 3, \ldots, j$, and $\phi > 0$, we have

$$\Phi\left(\mathbb{M}_{t,\mathcal{Z}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}, \phi\right) \succeq \Phi\left(\sum_{n=2}^{E} |\Xi|^{n-2} \frac{(\chi-(n-2)t-\tau)^{(n-1)\mathfrak{P}-1}}{\Gamma((n-2)\mathfrak{P}+\mathcal{Z})}, \phi \right).$$

Proof. (i) For $\tau \in [0, \chi-(j-1)t)$, $j \in \rho$, and $\phi > 0$, we get

$$\begin{aligned}
& \Phi\left(\mathbb{M}_{t,\mathcal{Z}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}, \phi\right) \\
= \;& \Phi\left(I \frac{(\chi-\tau)^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})} + \Xi \frac{(\chi-t-\tau)^{2\mathfrak{P}-1}}{\Gamma(\mathfrak{P}+\mathcal{Z})} + \ldots + \Xi^{j-1} \frac{(\chi-(j-1)t-\tau)^{j\mathfrak{P}-1}}{\Gamma((j-1)\mathfrak{P}+\mathcal{Z})}, \phi \right) \\
\succeq \;& \Phi\left(\frac{(\chi-\tau)^{\mathfrak{P}-1}}{\Gamma(\mathcal{Z})} + |\Xi| \frac{(\chi-t-\tau)^{2\mathfrak{P}-1}}{\Gamma(\mathfrak{P}+\mathcal{Z})} + \cdots + |\Xi|^{j-1} \frac{(\chi-(j-1)t-\tau)^{j\mathfrak{P}-1}}{\Gamma((j-1)\mathfrak{P}+\mathcal{Z})}, \phi \right) \\
\succeq \;& \Phi\left(\frac{(\chi-\tau)^{\mathfrak{P}-1}}{\Gamma(\mathcal{Z})} + \sum_{n=2}^{j} |\Xi|^{n-1} \frac{(\chi-(n-1)t-\tau)^{n\mathfrak{P}-1}}{\Gamma((n-1)\mathfrak{P}+\mathcal{Z})}, \phi \right) \\
\succeq \;& \Phi\left(\sum_{n=1}^{j} |\Xi|^{n-1} \frac{(\chi-(n-1)t-\tau)^{n\mathfrak{P}-1}}{\Gamma((n-1)\mathfrak{P}+\mathcal{Z})}, \phi \right).
\end{aligned}$$

(ii) For every $\tau \in [\chi-(E-1)t, \chi-(E-2)t)$, $E = 2, 3, \cdots, j$, and $\phi > 0$, we get

$$\begin{aligned}
&\Phi\left(\mathbb{M}_{t,\mathcal{Z}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}, \phi\right) \\
\succeq\ &\Phi\left(I\frac{(\chi-\tau)^{\mathfrak{P}-1}}{\Gamma(\mathcal{Z})} + \Xi\frac{(\chi-t-\tau)^{2\mathfrak{P}-1}}{\Gamma(\mathfrak{P}+\mathcal{Z})} + \cdots + \Xi^{E-2}\frac{(\chi-(E-2)t-\tau)^{(E-1)\mathfrak{P}-1}}{\Gamma((E-2)\mathfrak{P}+\mathcal{Z})}, \phi\right) \\
\succeq\ &\Phi\left(\frac{(\chi-\tau)^{\mathfrak{P}-1}}{\Gamma(\mathcal{Z})} + |\Xi|\frac{(\chi-t-\tau)^{2\mathfrak{P}-1}}{\Gamma(\mathfrak{P}+\mathcal{Z})} + \cdots + |\Xi|^{E-2}\frac{(\chi-(E-2)t-\tau)^{(E-1)\mathfrak{P}-1}}{\Gamma((E-2)\mathfrak{P}+\mathcal{Z})}, \phi\right) \\
\succeq\ &\Phi\left(\frac{(\chi-\tau)^{\mathfrak{P}-1}}{\Gamma(\mathcal{Z})} + \sum_{n=3}^{E}|\Xi|^{n-2}\frac{(\chi-(n-2)t-\tau)^{(n-1)\mathfrak{P}-1}}{\Gamma((n-2)\mathfrak{P}+\mathcal{Z})}, \phi\right) \\
\succeq\ &\Phi\left(\sum_{n=2}^{E}|\Xi|^{n-2}\frac{(\chi-(n-2)t-\tau)^{(n-1)\mathfrak{P}-1}}{\Gamma((n-2)\mathfrak{P}+\mathcal{Z})}, \phi\right).
\end{aligned}$$

□

Lemma 6. *For every $\chi \in ((j-1)t, jt]$, with fixed number $j \in \rho$, and $\phi > 0$, we obtain*

$$\Phi\left(\int_{-t}^{0}\mathbb{M}_{t}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}'(\tau)d\tau, \phi\right) \succeq \Phi\left(\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})\int_{-t}^{0}\mathcal{K}'(\tau)d\tau, \phi\right).$$

Proof.

$$\begin{aligned}
&\Phi\left(\int_{-t}^{0}\mathbb{M}_{t}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}'(\tau)d\tau, \phi\right) \\
&= \Phi\left(\int_{-t}^{\chi-jt}\left[I + \Xi\frac{(\chi-t-\tau)^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)} + \cdots + \Xi^{j}\frac{(\chi-jt-\tau)^{j\mathfrak{P}}}{\Gamma(j\mathfrak{P}+1)}\right]\mathcal{K}'(\tau)d\tau\right. \\
&\qquad \left. + \int_{\chi-jt}^{0}\left[I + \Xi\frac{(\chi-t-\tau)^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)} + \cdots + \Xi^{j-1}\frac{(\chi-(j-1)t-\tau)^{(j-1)\mathfrak{P}}}{\Gamma((j-1)\mathfrak{P}+1)}\right]\mathcal{K}'(\tau)d\tau, \phi\right) \\
&\succeq \Phi\left(\left[\int_{-t}^{0}\left(1 + |\Xi|\frac{\chi^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)} + \ldots + |\Xi|^{j-1}\frac{\chi^{(j-1)\mathfrak{P}}}{\Gamma((j-1)\mathfrak{P}+1)}\right) + \int_{-t}^{\chi-jt}|\Xi|^{j}\frac{\chi^{j\mathfrak{P}}}{\Gamma(j\mathfrak{P}+1)}\right]\mathcal{K}'(\tau)d\tau, \phi\right) \\
&\succeq \Phi\left(\left[I + \frac{|\Xi|\chi^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)} + \ldots + \frac{|\Xi|^{j-1}\chi^{(j-1)\mathfrak{P}}}{\Gamma((j-1)\mathfrak{P}+1)}\right]\int_{-t}^{0}\mathcal{K}'(\tau)d\tau + \frac{|\Xi|^{j}\chi^{j\mathfrak{P}}}{\Gamma(j\mathfrak{P}+1)}\int_{-t}^{\chi-jt}\mathcal{K}'(\tau)d\tau, \phi\right) \\
&\succeq \Phi\left(\left[I + \frac{|\Xi|\chi^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)} + \ldots + \frac{|\Xi|^{j-1}\chi^{(j-1)\mathfrak{P}}}{\Gamma((j-1)\mathfrak{P}+1)}\right]\int_{-t}^{0}\mathcal{K}'(\tau)d\tau + \frac{|\Xi|^{j}\chi^{j\mathfrak{P}}}{\Gamma(j\mathfrak{P}+1)}\int_{-t}^{0}\mathcal{K}'(\tau)d\tau, \phi\right) \\
&\succeq \Phi\left(\left[I + \frac{|\Xi|\chi^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)} + \ldots + \frac{|\Xi|^{j-1}\chi^{(j-1)\mathfrak{P}}}{\Gamma((j-1)\mathfrak{P}+1)} + \frac{|\Xi|^{j}\chi^{j\mathfrak{P}}}{\Gamma(j\mathfrak{P}+1)}\right]\int_{-t}^{0}\mathcal{K}'(\tau)d\tau, \phi\right) \\
&\succeq \Phi\left(\sum_{j=0}^{\infty}\frac{(|\Xi|\chi^{\mathfrak{P}})^{j}}{\Gamma(j\mathfrak{P}+1)}\int_{-t}^{0}\mathcal{K}'(\tau)d\tau, \phi\right) \\
&\succeq \Phi\left(\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})\int_{-t}^{0}\mathcal{K}'(\tau)d\tau, \phi\right).
\end{aligned}$$

□

Lemma 7. *For every $\chi \in ((j-1)t, jt]$, $j \in \rho$, $E = 2, 3, \cdots, j$, $\mathfrak{P} \geq \frac{1}{2}$, $\phi > 0$, and $\mathcal{K} \in C(\nu, \mathbb{R}^n)$, we get*

$$\begin{aligned}
&\Phi\left(\int_{0}^{\chi-(j-1)t}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}(\tau)d\tau + \int_{\chi-(j-1)t}^{\chi}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}(\tau)d\tau, \phi\right) \\
&\succeq \Phi\left(\mathbb{M}_{\mathfrak{P},\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})\int_{0}^{\chi}(\chi-\tau)^{\mathfrak{P}-1}\mathcal{K}(\tau)d\tau, \phi\right).
\end{aligned}$$

Proof.

$$\begin{aligned}
&\Phi\bigg(\int_0^{\chi-(j-1)t}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}(\tau)d\tau+\int_{\chi-(j-1)t}^{\chi}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}(\tau)d\tau,\phi\bigg)\\
&\succeq\Phi\bigg(\int_0^{\chi-(j-1)t}\bigg[\sum_{m=1}^{j}|\Xi|^{m-1}\frac{(\chi-(m-1)t-\tau)^{m\mathfrak{P}-1}}{\Gamma((m-1)\mathfrak{P}+\mathfrak{P})}\bigg]\mathcal{K}(\tau)d\tau\\
&\qquad+\sum_{E=2}^{j}\int_{\chi-(E-1)t}^{\chi-(E-2)t}\bigg[\sum_{m=2}^{E}|\Xi|^{m-2}\frac{(\chi-(m-2)t-\tau)^{(m-1)\mathfrak{P}-1}}{\Gamma((m-2)\mathfrak{P}+\mathfrak{P})}\bigg]\mathcal{K}(\tau)d\tau,\phi\bigg)\\
&\succeq\Phi\bigg(\int_0^{\chi-(j-1)t}\bigg[\sum_{m=1}^{j}|\Xi|^{m-1}\frac{(\chi-t)^{m\mathfrak{P}-1}}{\Gamma((m-1)\mathfrak{P}+\mathfrak{P})}\bigg]\mathcal{K}(\tau)d\tau\\
&\qquad+\int_{\chi-(j-1)t}^{\chi}\bigg[\sum_{m=2}^{j}|\Xi|^{m-2}\frac{(\chi-\tau)^{(m-1)\mathfrak{P}-1}}{\Gamma((m-2)\mathfrak{P}+\mathfrak{P})}\bigg]\mathcal{K}(\tau)d\tau,\phi\bigg)\\
&\succeq\Phi\bigg(\bigg[\int_0^{\chi}\sum_{m=2}^{j}|\Xi|^{m-2}\frac{(\chi-\tau)^{(m-1)\mathfrak{P}-1}}{\Gamma((m-2)\mathfrak{P}+\mathfrak{P})}+\int_0^{\chi-(j-1)t}|\Xi|^{j-1}\frac{(\chi-\tau)^{j\mathfrak{P}-1}}{\Gamma((j-1)\mathfrak{P}+\mathfrak{P})}\bigg]\mathcal{K}(\tau)d\tau,\phi\bigg)\\
&\succeq\Phi\bigg(\int_0^{\chi}(\chi-\tau)^{\mathfrak{P}-1}\mathbb{M}_{\mathfrak{P},\mathfrak{P}}(|\Xi|(\chi-\tau)^{\mathfrak{P}})\mathcal{K}(\tau)d\tau,\phi\bigg)\\
&\succeq\Phi\bigg(\mathbb{M}_{\mathfrak{P},\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})\int_0^{\chi}(\chi-\tau)^{\mathfrak{P}-1}\mathcal{K}(\tau)d\tau,\phi\bigg),
\end{aligned}$$

in which we apply the monotonic property of function $\Gamma(.)=.^{j\mathfrak{P}-1}$, for $\mathfrak{P}\geq\frac{1}{2}$ that infers $(\chi-(j-1)t-\tau)^{j\mathfrak{P}-1}\leq(\chi-\tau)^{j\mathfrak{P}-1}$, for every $\chi\in((j-1)t,jt]$. □

2.7. Multiple Aggregate Window Maps

Let $n\in\mathbb{N}$, $\mu=\mathrm{diag}[\mu_1,\cdots,\mu_n]$, and $\underbrace{\mu_i}_{1\leq i\leq n}\in\epsilon$. An n-ary aggregation map is a mapping $\mathtt{AG}^{(n)}:\mathrm{diag}M_n(\epsilon)\longrightarrow\epsilon$, s.t.

$$\inf_{\underbrace{\mu_i}_{1\leq i\leq n}\in\epsilon}\mathtt{AG}^{(n)}(\mu)=\inf\epsilon,$$

and

$$\sup_{\underbrace{\mu_i}_{1\leq i\leq n}\in\epsilon}\mathtt{AG}^{(n)}(\mu)=\sup\epsilon,$$

or, equivalently, $\mathtt{AG}^{(n)}(\mathbf{0})=0$ and $\mathtt{AG}^{(n)}(\mathbf{1})=1$.

Besides, for every $\mu,\mu'\in\mathrm{diag}M_n(\epsilon)$, if $\underbrace{\mu_i}_{1\leq i\leq n}\leq\underbrace{\mu_i'}_{1\leq i\leq n}$, then, $\mathtt{AG}^{(n)}(\mu)\leq\mathtt{AG}^{(n)}(\mu')$.

In case $n=1$, for every $\mu\in\epsilon$, we get $\mathtt{AG}^{(1)}(\mu)=\mu$.

Note that $n\in\mathbb{N}$ denotes the arity of the aggradation map. Also, the aggregation maps will simply be written $\mathtt{AG}$ instead of $\mathtt{AG}^{(n)}$.

Now, we present a small list of well-known aggregation maps $\underbrace{\mathtt{AG}_i}_{1\leq i\leq 8}:\mathrm{diag}M_n(\epsilon^n)\longrightarrow\epsilon$, as follows:

- Geometric mean functions:

$$\mathtt{AG}_1(\mu)=(\prod_{i=1}^{n}\mu_i)^{\frac{1}{n}},$$

- Arithmetric mean functions:
$$\mathtt{AG}_2(\mu) = \frac{1}{n}\sum_{i=1}^{n}\mu_i,$$
- Maximum functions:
$$\mathtt{AG}_3(\mu) = \max\{\mu_1, \cdots, \mu_n\},$$
- Minimum functions:
$$\mathtt{AG}_4(\mu) = \min\{\mu_1, \cdots, \mu_n\},$$
- Median of odd numbers:
$$\mathtt{AG}_5(\mathrm{diag}[\mu_1, \cdots, \mu_{2n-1}]) = \min_{\substack{N\subseteq[2n-1]\\ |N|=n}} \max_{i\in n} \mu_i,$$
- Median of even numbers:
$$\mathtt{AG}_6(\mathrm{diag}[\mu_1, \cdots, \mu_{2n}]) = \min_{\substack{N\subseteq[2n]\\ |N|=n}} \max_{i\in n} \mu_i,$$
- Sum functions:
$$\mathtt{AG}_7(\mu) = \sum_{i=1}^{n}\mu_i,$$
- Product functions:
$$\mathtt{AG}_8(\mu) = \prod_{i=1}^{n}\mu_i.$$

2.8. Second Type Kudryashov Method

Let us present the algorithm of the second type Kudryashov method, as follows:

(1) Consider the NPDE of the type:
$$\mathsf{N}(\chi, D_{t_2}^{\mathfrak{P}_1}\chi, D_{t_1}^{\mathfrak{P}_1}\chi, D_{t_1}^{\mathfrak{P}_2}\chi, D_{t_1}^{\mathfrak{P}_1}D_{t_1}^{\mathfrak{P}_2}\chi, D_{t_2}^{\mathfrak{P}_1}D_{t_2}^{\mathfrak{P}_2}\chi, \cdots) = 0, \quad 0 < \mathfrak{P}_1, \mathfrak{P}_2 \leqslant 1, \tag{18}$$
where $\chi = \chi(t_1, t_2)$.

(2) Transmute the NPDE (18) into an ODE via the transformations below
$$\eta = \frac{At_1^{A\mathfrak{P}_2}}{\Gamma(1+\mathfrak{P}_2)} + \frac{Bt_2^{B\mathfrak{P}_1}}{\Gamma(1+\mathfrak{P}_1)}, \quad \chi(t_1, t_2) = \chi(\eta), \tag{19}$$
for every constants A and B.

(3) Rewrite (18) as follows:
$$\tilde{\mathsf{N}}(\chi, \chi', \chi'', \chi''', \ldots) = 0, \tag{20}$$
where the $'$ denotes $\frac{d}{d\eta}$.

(4) Assume the general solution of (20) can be expressed by
$$\chi(\eta) = N_0 + N_1 Q(\eta) + N_2 Q^2(\eta) + \cdots + N_n Q^n(\eta), \tag{21}$$
where $\underbrace{N_i}_{1\leq i\leq n}$ are determined later, and $N \in \mathbb{N}$ can be computed via the homogeneous balance principle, and
$$Q(\eta) = \frac{1}{(\alpha+\beta)\cosh(\eta) + (\alpha-\beta)\sinh(\eta)}, \tag{22}$$

which satisfies

$$(Q'(\eta))^2 = Q^2(\eta)(1 - 4\alpha\beta Q^2(\eta)). \tag{23}$$

(5) Making use of (20)–(22), a system of algebraic type is gained, and by solving it, the general solutions are obtained.

3. Fox Type Stability of (1) for Case 1

Consider the following matrix valued Fox-type controller defined by

$$\begin{aligned}\mathfrak{Z}_1(\chi, \mathfrak{S}\phi) := \mathrm{diag}\Bigg[& {}_C^A\mathbb{H}_D^B\left(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi} \middle| \begin{matrix} (V_j, W_j)_{1,C} \\ (N_j, M_j)_{1,D} \end{matrix} \right), {}_0\mathbb{H}_0\left(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi} \right) \\ & {}_0\mathbb{H}_1\left(N_1; -\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi} \right), {}_2\mathbb{H}_1\left(V_1, V_2; N_1; -\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi} \right) \\ & {}_1\mathbb{H}_1\left(V_1; N_1; -\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi} \right), {}_C\mathbb{H}_D\left(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi} \middle| \begin{matrix} (V_1, W_1), \ldots, (V_C, W_C) \\ (N_1, M_1), \ldots, (N_D, M_D) \end{matrix} \right) \\ & {}_C^A\mathbb{H}_D^B\left(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi} \middle| \begin{matrix} (V_1, 1), \ldots, (V_C, 1) \\ (N_1, 1), \ldots, (N_D, 1) \end{matrix} \right), {}_C\mathbb{H}_D\left(V_1, \ldots, V_C; N_1, \ldots, N_C; -\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi} \right) \Bigg],\end{aligned} \tag{24}$$

where $\phi > 0$ and $\mathfrak{S} > 0$.

In view of (24), the plots of aggregation maps $\underbrace{\mathtt{AG}_i}_{1 \le i \le 8}[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)]$ are displayed separately in Figure 1. As you can see, the minimum aggregation map $\mathtt{AG}_4[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)]$, and the maximum aggregation map $\mathtt{AG}_3[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)]$, include the lowest and highest values respectively, and the rest of the aggregation maps $\underbrace{\mathtt{AG}_i}_{i=1,2,5,6,7,8}[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)]$, are placed between them. Thus, we conclude that the aggregate special controller $\mathtt{AG}_4[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)]$, can present a better approximation for (1) than the others.

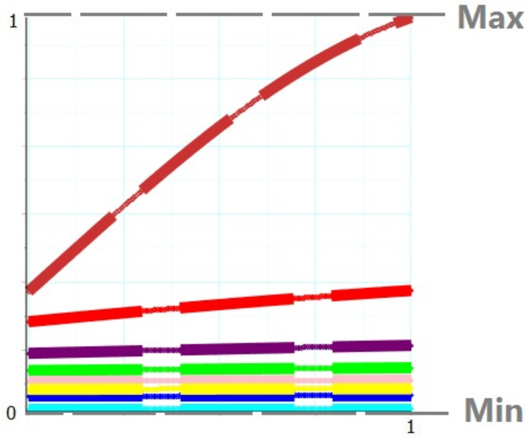

Figure 1. The plots of aggregation maps $\underbrace{\mathtt{AG}_i}_{1 \le i \le 8}$ on control function $\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)$. The minimum aggregation map $\mathtt{AG}_4$ and the maximum aggregation map $\mathtt{AG}_3$ are shown in cyan and brown colors, respectively, and the rest is between them.

Definition 6. *Taking into account* ***Case 1****, the fractional order Equation* (1) *has the Fox type stability with respect to* $\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)$ *given in* (24)*, if there exists an* $\ell > 0$*, s.t for every* $\mathfrak{S} > 0$*, and every solution* $\mathcal{F}$ *to*

$$\begin{aligned}&\Phi\Big(D^{\mathfrak{P}}_{0^+}\mathcal{F}(\chi)-\Xi_1\mathcal{G}(\chi),\phi\Big)\\ \succeq\ &\ diag\Big[\mathtt{AG}_4[\mathfrak{Z}_1(\chi,\mathfrak{S}\phi)],\ldots,\mathtt{AG}_4[\mathfrak{Z}_1(\chi,\mathfrak{S}\phi)]\Big],\end{aligned}\tag{25}$$

there exists a solution $\widetilde{\mathcal{F}}$ *to* (1), *with*

$$\begin{aligned}&\mathscr{N}\Big(\mathcal{F}(\chi)-\widetilde{\mathcal{F}}(\chi),\phi\Big)\\ &\succeq diag\Big[\mathtt{AG}_4[\mathfrak{Z}_1(\chi,\mathfrak{S}\ell\phi)],\ldots,\mathtt{AG}_4[\mathfrak{Z}_1(\chi,\mathfrak{S}\ell\phi)]\Big],\end{aligned}\tag{26}$$

in which $\phi>0$.

Now, consider the following matrix valued fuzzy controllers created by the Mittag–Leffler type functions, the one parameter Supertrigonometric and Superhyperbolic Mittag–Leffler type functions, and Supertrigonometric and Superhyperbolic Gauss–Hypergeometric type functions, as follows:

$$\begin{aligned}\mathfrak{Z}_2(\chi,\mathfrak{S}\phi):=\mathrm{diag}\Big[&\mathbb{M}_\alpha\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),\mathbb{M}_{\alpha,\tau}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),\mathbb{M}^{V_1}_{\alpha,\tau}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),\\ &\mathbb{M}^{V_1,W_1}_{\alpha,\tau}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),\mathbb{M}^{V_1,W_1}_{\alpha,\tau,N_1}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),\mathbb{M}^{V_1,W_1}_{\alpha,\tau,N_1,M_1}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),\\ &\mathbb{M}^{V_1,W_1,V_2}_{\alpha,\tau,N_1,M_1}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),\mathbb{M}^{V_1,W_1,V_2,W_2}_{\alpha,\tau,N_1,M_1}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big)\Big],\end{aligned}$$

$$\begin{aligned}\mathfrak{Z}_3(\chi,\mathfrak{S}\phi):=\mathrm{diag}\Big[&precos_{N_1}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),presin_{N_1}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),\\ &precosh_{N_1}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),presinh_{N_1}\Big(-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big)\Big],\end{aligned}$$

and

$$\begin{aligned}\mathfrak{Z}_4(\chi,\mathfrak{S}\phi):=\mathrm{diag}\Big[&{}_2supercos_1\Big(V_1,V_2,N_1;-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),{}_2supersin_1\Big(V_1,V_2,N_1;-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),\\ &{}_2supercosh_1\Big(V_1,V_2,N_1;-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big),{}_2supersinh_1\Big(V_1,V_2,N_1;-\frac{|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\Big)\Big].\end{aligned}$$

Likewise as above, we can see that the minimum aggregation maps $\mathtt{AG}_4[\underbrace{\mathfrak{Z}_i}_{i=2,3,4}(\chi,\mathfrak{S}\phi)]$, present a better approximation for (1), than $\underbrace{\mathtt{AG}_i}_{\substack{1\le i\le 8\\ i\ne 4}}$.

In summary, we have the following important theorem:

Theorem 1. *Consider the fractional-order differential Equation* (1) *and the inequality below*

$$\begin{aligned}&\Phi\Big(D^{\mathfrak{P}}_{0^+}\mathcal{F}(\chi)-\Xi_1\mathcal{G}(\chi),\phi\Big)\\ \succeq\ &\ diag\Big[\mathtt{AG}_4[\mathfrak{Z}_1(\chi,\mathfrak{S}\phi)],\ldots,\mathtt{AG}_4[\mathfrak{Z}_4(\chi,\mathfrak{S}\phi)]\Big]_{4\times 4},\end{aligned}\tag{27}$$

under the assumptions of ***Case 1****, with zero initial condition. Then,* (1) *is Fox-type stable with respect to* (24)*.*

Proof. It is easy to show that the unique solution of fractional Equation (1) is given by

$$\mathcal{F}(\chi) = \frac{1}{\Gamma(\mathfrak{P})} \int_0^{\chi} (\chi - S)^{\mathfrak{P}-1} \Xi_1 \mathcal{G}(S) dS. \tag{28}$$

Plus, if $\mathcal{F}$ is a solution of the inequality (27), then $\mathcal{F}$ is a solution of the following inequality, for every $\phi > 0$,

$$\begin{aligned} &\Phi\left(\mathcal{F}(\chi) - \frac{1}{\Gamma(\mathfrak{P})} \int_0^{\chi} (\chi - S)^{\mathfrak{P}-1} \Xi_1 \mathcal{G}(S) dS, \phi\right) \\ &\succeq \operatorname{diag}\Big[\mathtt{AG}_4[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)], \cdots, \mathtt{AG}_4[\mathfrak{Z}_4(\chi, \mathfrak{S}\phi)]\Big]_{4\times 4}. \end{aligned} \tag{29}$$

We prove the inequality (29) only for the special cases: one parameter of the Mittag–Leffler function and the Gauss Hypergeometric function, as follows:

$$\begin{aligned}
&\Phi\left(\mathcal{F}(\chi) - \frac{\Xi_1}{\Gamma(\mathfrak{P})} \int_0^{\chi} (\chi - S)^{\mathfrak{P}-1} \mathcal{G}(S) dS, \phi\right) \\
&\succeq \Phi\left(\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})} \int_0^{\chi} (\chi - S)^{\mathfrak{P}-1} {}_0\mathbb{H}_1[\mathfrak{P}; S^{\mathfrak{P}}] dS, \cdots, \frac{\mathfrak{S}}{\Gamma(\mathfrak{P})} \int_0^{\chi} (\chi - S)^{\mathfrak{P}-1} {}_0\mathbb{H}_1[\mathfrak{P}, S^{\mathfrak{P}}] dS, \phi\right) \\
&\succeq \Phi\left(\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})} \int_0^{\chi} (\chi - S)^{\mathfrak{P}-1} \sum_{k=0}^{\infty} \frac{S^{k\mathfrak{P}}}{\Gamma(k\mathfrak{P}+1)} dS, \cdots, \right. \\
&\qquad\qquad \left. \frac{\mathfrak{S}}{\Gamma(\mathfrak{P})} \int_0^{\chi} (\chi - S)^{\mathfrak{P}-1} \sum_{k=0}^{\infty} \frac{S^{k\mathfrak{P}}}{\Gamma(k\mathfrak{P}+1)} dS, \phi\right) \\
&\succeq \Phi\left(\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\mathfrak{P}+1)} \int_0^{\chi} (\chi - S)^{\mathfrak{P}-1} S^{k\mathfrak{P}} dS, \cdots, \right. \\
&\qquad\qquad \left. \frac{\mathfrak{S}}{\Gamma(\mathfrak{P})} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\mathfrak{P}+1)} \int_0^{\chi} (\chi - S)^{\mathfrak{P}-1} S^{k\mathfrak{P}} dS, \phi\right) \\
&\succeq \Phi\left(\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})} \sum_{k=0}^{\infty} \frac{\chi^{(k+1)\mathfrak{P}}}{\Gamma(k\mathfrak{P}+1)} \frac{\Gamma(\mathfrak{P})\Gamma(k\mathfrak{P}+1)}{\Gamma((k+1)\mathfrak{P}+1)}, \cdots, \right. \\
&\qquad\qquad \left. \frac{\mathfrak{S}}{\Gamma(\mathfrak{P})} \sum_{k=0}^{\infty} \frac{\chi^{(k+1)\mathfrak{P}}}{\Gamma(k\mathfrak{P}+1)} \frac{\Gamma(\mathfrak{P})\Gamma(k\mathfrak{P}+1)}{\Gamma((k+1)\mathfrak{P}+1)}, \phi\right) \\
&\succeq \Phi\left(\mathfrak{S} \sum_{k=0}^{\infty} \frac{\chi^{(k+1)\mathfrak{P}}}{\Gamma((k+1)\mathfrak{P}+1)}, \cdots, \mathfrak{S} \sum_{k=0}^{\infty} \frac{\chi^{(k+1)\mathfrak{P}}}{\Gamma((k+1)\mathfrak{P}+1)}, \phi\right) \\
&\succeq \Phi\left(\mathfrak{S} \sum_{k=0}^{\infty} \frac{\chi^{n\mathfrak{P}}}{\Gamma(n\mathfrak{P}+1)}, \cdots, \mathfrak{S} \sum_{k=0}^{\infty} \frac{\chi^{n\mathfrak{P}}}{\Gamma(n\mathfrak{P}+1)}, \phi\right) \\
&\succeq \operatorname{diag}\left[{}_0\mathbb{H}_1\left(\mathfrak{P}; \frac{-|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\right), \cdots, {}_0\mathbb{H}_1\left(\mathfrak{P}; \frac{-|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\right)\right],
\end{aligned}$$

and

$$
\begin{aligned}
&\Phi\left(\mathcal{F}(\chi)-\frac{\Xi_1}{\Gamma(\mathfrak{P})}\int_0^\chi(\chi-S)^{\mathfrak{P}-1}\mathcal{G}(S)dS,\phi\right)\\
&\succeq\Phi\left(\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})}\int_0^\chi(\chi-S)^{\mathfrak{P}-1}{}_2\mathbb{H}_1[V_1,V_2;N_1;S^{\mathfrak{P}}]dS,\cdots,\right.\\
&\qquad\qquad\left.\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})}\int_0^\chi(\chi-S)^{\mathfrak{P}-1}{}_2\mathbb{H}_1[V_1,V_2;N_1;S^{\mathfrak{P}}]dS,\phi\right)\\
&\succeq\Phi\left(\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})}\int_0^\chi(\chi-S)^{\mathfrak{P}-1}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{k=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{S^{\mathfrak{P}k}}{k!}dS,\right.\\
&\qquad\left.\cdots,\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})}\int_0^\chi(\chi-S)^{\mathfrak{P}-1}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{k=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{S^{\mathfrak{P}k}}{k!}dS,\phi\right)\\
&=\Phi\left(\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{k=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{1}{k!}\int_0^\chi(\chi-S)^{\mathfrak{P}-1}S^{\mathfrak{P}k}dS,\right.\\
&\qquad\left.\cdots,\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{k=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{1}{k!}\int_0^\chi(\chi-S)^{\mathfrak{P}-1}S^{\mathfrak{P}k}dS,\phi\right)\\
&=\Phi\left(\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{k=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{\chi^{(k+1)\mathfrak{P}}}{k!}\frac{\Gamma(k\mathfrak{P}+1)\Gamma(\mathfrak{P})}{\Gamma((k+1)\mathfrak{P}+1)},\right.\\
&\qquad\left.\cdots,\frac{\mathfrak{S}}{\Gamma(\mathfrak{P})}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{k=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{\chi^{(k+1)\mathfrak{P}}}{k!}\frac{\Gamma(k\mathfrak{P}+1)\Gamma(\mathfrak{P})}{\Gamma((k+1)\mathfrak{P}+1)},\phi\right)\\
&\succeq\Phi\left(\mathfrak{S}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{k=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{\chi^{k\mathfrak{P}}}{k!}\frac{\Gamma(k\mathfrak{P}+1)}{\Gamma((k+1)\mathfrak{P}+1)},\right.\\
&\qquad\left.\cdots,\mathfrak{S}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{k=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{\chi^{k\mathfrak{P}}}{k!}\frac{\Gamma(k\mathfrak{P}+1)}{\Gamma((k+1)\mathfrak{P}+1)},\phi\right)\\
&\succeq\Phi\left(\mathfrak{S}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{K=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{\chi^{k\mathfrak{P}}}{k!}\frac{\Gamma(k\mathfrak{P}+1)}{\Gamma(k\mathfrak{P}+1)},\right.\\
&\qquad\left.\cdots,\mathfrak{S}\frac{\Gamma(N_1)}{\Gamma(V_1)\Gamma(V_2)}\sum_{K=0}^{\infty}\frac{\Gamma(V_1+k)\Gamma(V_2+k)}{\Gamma(N_1+k)}\frac{\chi^{k\mathfrak{P}}}{k!}\frac{\Gamma(k\mathfrak{P}+1)}{\Gamma(k\mathfrak{P}+1)},\phi\right)\\
&=\mathrm{diag}\left[{}_2\mathbb{H}_1\left(V_1,V_2;N_1;\frac{-|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\right),\cdots,{}_2\mathbb{H}_1\left(V_1,V_2;N_1;\frac{-|\chi|^{\mathfrak{P}}}{\mathfrak{S}\phi}\right)\right].
\end{aligned}
$$

A function $\widetilde{\mathcal{F}}$ is a solution of (25), iff there exists a function $\mathcal{P}\in C(\nu,\mathbb{R})$ (which depends on $\widetilde{\mathcal{F}}$), s.t.

$$\Phi(\mathcal{P}(\chi),\phi)\succeq\mathrm{diag}[\mathtt{AG}_4[\mathfrak{Z}_1(\chi,\mathfrak{S}\phi)],\cdots,\mathtt{AG}_4[\mathfrak{Z}_4(\chi,\mathfrak{S}\phi)],$$

and

$$D_{0^+}^{\mathfrak{P}}\widetilde{\mathcal{F}}(\chi)=\Xi_1\mathcal{G}(\chi)+\mathcal{P}(\chi). \tag{30}$$

Thus, $\widetilde{\mathcal{F}}$ is a solution of the inequality below

$$
\begin{aligned}
&\Phi\left(\widetilde{\mathcal{F}}(\chi)-\frac{1}{\Gamma(\mathfrak{P})}\int_0^\chi(\chi-S)^{\mathfrak{P}-1}\Xi_1\mathcal{G}(S)dS,\phi\right)\\
\succeq\ &\mathrm{diag}[\mathtt{AG}_4[\mathfrak{Z}_1(\chi,\mathfrak{S}\phi)],\cdots,\mathtt{AG}_4[\mathfrak{Z}_4(\chi,\mathfrak{S}\phi)].
\end{aligned} \tag{31}
$$

Making use of (28), we have that

$$\widetilde{\mathcal{F}}(\chi)=\frac{1}{\Gamma(\mathfrak{P})}\int_0^\chi(\chi-S)^{\mathfrak{P}-1}[\Xi_1\mathcal{G}(S)+\mathcal{P}(S)]dS.$$

Then, we get

$$\Phi\left(\widetilde{\mathcal{F}}(\chi) - \frac{1}{\Gamma(\mathfrak{P})}\int_0^{\chi}(\chi - S)^{\mathfrak{P}-1}\Xi_1\mathcal{G}(S)dS, \phi\right) \tag{32}$$
$$\succeq \Phi\left(\frac{1}{\Gamma(\mathfrak{P})}\int_0^{\chi}(\chi - S)^{\mathfrak{P}-1}\mathcal{P}(S)dS, \phi\right)$$
$$\succeq \mathrm{diag}[\mathtt{AG}_4[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)], \cdots, \mathtt{AG}_4[\mathfrak{Z}_4(\chi, \mathfrak{S}\phi)].$$

□

The plots of aggregation map $\mathtt{AG}_4$ on control special functions $\underbrace{\mathfrak{Z}_i}_{i=1,2,3,4}(\chi, \mathfrak{S}\phi)$, are displayed in Figure 2. As you can observe, the graph of the aggregate window function $\mathtt{AG}_4[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)]$, can present the best estimation among the rest of the drawn control functions. Considering relation (32) and the above, we have that

$$\mathrm{diag}[\mathtt{AG}_4[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)], \cdots, \mathtt{AG}_4[\mathfrak{Z}_4(\chi, \mathfrak{S}\phi)]$$
$$\succeq \mathrm{diag}[\mathtt{AG}_4[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)], \cdots, \mathtt{AG}_4[\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)].$$

Thus, fractional order equation (1) is Fox-type stable with respect to $\mathfrak{Z}_1(\chi, \mathfrak{S}\phi)$.

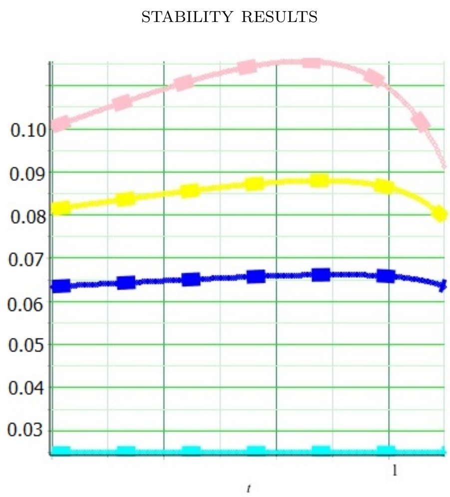

Figure 2. The plots of the aggregate window functions $\mathtt{AG}_4[\underbrace{\mathfrak{Z}_i}_{i=1,2,3,4}(\chi, \mathfrak{S}\phi)]$, in colors: cyan ($\mathfrak{Z}_1$), blue ($\mathfrak{Z}_2$), yellow ($\mathfrak{Z}_3$), and pink ($\mathfrak{Z}_4$).

3.1. Fractional-Order Harry Dym Equation

Consider (1), when $\Xi_1 = [1]_{1\times1}$, $\Xi = \Xi_2 = [0]_{1\times1}$, $\chi := (\chi_1, \chi_2)$, $\mathcal{G}(\chi_1, \chi_2) := \mathcal{F}^3(\chi_1, \chi_2)\mathcal{F}_{\chi_1\chi_1\chi_1}(\chi_1, \chi_2)$, and $\mathcal{F}$ is a function with continuous second derivative.

Putting the above in (1), we get the following nonlinear time fractional Harry Dym equation defined by

$$\begin{aligned} D^{\mathfrak{P}}_{\chi_2}\mathcal{F}(\chi_1, \chi_2) &= \mathcal{F}^3(\chi_1, \chi_2)\mathcal{F}_{\chi_1\chi_1\chi_1}(\chi_1, \chi_2), \quad 0 < \mathfrak{P} \le 1, \\ \mathcal{F}(\chi_1, 0) &= \left(a - \frac{3\sqrt{b}}{2}\chi_1\right)^{\frac{2}{3}}. \end{aligned} \tag{33}$$

Consider the following transformations that represent novel dependent variables below

$$\mathcal{Y}_1 = \int_{-\infty}^{\chi_1} \frac{dS}{\mathcal{F}(S, \chi_2)}, \tag{34}$$
$$\mathcal{Y}_2 = -\frac{\chi_2^{\mathfrak{P}}}{\Gamma(1+\mathfrak{P})},$$
$$\mathcal{Q}(\mathcal{Y}_1, \mathcal{Y}_2) = \mathcal{F}\Big(\chi_1(\mathcal{Y}_1, \mathcal{Y}_2), \chi_2(\mathcal{Y}_1, \mathcal{Y}_2)\Big),$$

in which $\chi_1 = \chi_1(\mathcal{Y}_1, \mathcal{Y}_2)$, and $\chi_2 = \chi_2(\mathcal{Y}_1, \mathcal{Y}_2)$.

Note that $\mathcal{F}(\chi_1, \chi_2)$ and its spatial derivative tend to zero as $|\chi_1| \to \infty$. Then, we have that

$$\begin{aligned}\frac{\partial^{\mathfrak{P}}}{\partial \chi_2^{\mathfrak{P}}} &= \frac{\partial}{\partial \mathcal{Y}_1}\frac{\partial^{\mathfrak{P}} \mathcal{Y}_1}{\partial \chi_2^{\mathfrak{P}}} + \frac{\partial}{\partial \mathcal{Y}_2}\frac{\partial^{\mathfrak{P}} \mathcal{Y}_2}{\partial \chi_2^{\mathfrak{P}}} \\ &= -\frac{\partial}{\partial \mathcal{Y}_2} - \left(\frac{\mathcal{Q}\mathcal{Q}_{\mathcal{Y}_1\mathcal{Y}_1} - \frac{3}{2}\mathcal{Q}_{\mathcal{Y}_1}^2}{\mathcal{Q}^2}\right)\frac{\partial}{\partial \mathcal{Y}_1},\end{aligned}$$

and

$$\frac{\partial}{\partial \chi_1} = \frac{1}{\mathcal{Q}(\mathcal{Y}_1, \mathcal{Y}_2)}\frac{\partial}{\partial \mathcal{Y}_1}.$$

Thus, (33) can be expressed as follows:

$$\mathcal{Q}_{\mathcal{Y}_2} + \frac{\mathcal{Q}_{\mathcal{Y}_1\mathcal{Y}_1\mathcal{Y}_1}\mathcal{Q}^2 - 3\mathcal{Q}_{\mathcal{Y}_1\mathcal{Y}_1}\mathcal{Q}_{\mathcal{Y}_1}\mathcal{Q} + \frac{3}{2}\mathcal{Q}_{\mathcal{Y}_1}^3}{\mathcal{Q}^2} = 0. \tag{35}$$

This time, apply the transformation below

$$\mathcal{L}(\mathcal{Y}_1, \mathcal{Y}_2) = \frac{\mathcal{Q}_{\mathcal{Y}_1}}{\mathcal{Q}}. \tag{36}$$

Based on (35) and (37), we obtain the Korteweg–De Vries equation defined by

$$\mathcal{L}_{\mathcal{Y}_2} - \frac{3}{2}\mathcal{L}^2\mathcal{L}_{\mathcal{Y}_1} + \mathcal{L}_{\mathcal{Y}_1\mathcal{Y}_1\mathcal{Y}_1} = 0. \tag{37}$$

Define the following new variable with constant c,

$$\mathcal{L}(\mathcal{Y}_1, \mathcal{Y}_2) := \mathcal{L}(\mathcal{S}), \tag{38}$$
$$\mathcal{S} := \mathcal{S}(\mathcal{Y}_1, \mathcal{Y}_2) = \mathcal{Y}_1 - c\mathcal{Y}_2.$$

Inserting (38) in (37), we obtain the ODE below

$$-c\mathcal{L}' - \frac{3}{2}\mathcal{L}^2\mathcal{L}' + \mathcal{L}''' = 0. \tag{39}$$

Integration of (39) yields

$$-c\mathcal{L} - \frac{1}{2}\mathcal{L}^3 + \mathcal{L}'' = 0. \tag{40}$$

Application of the Second Type Kudryashov Method

Balancing the highest order derivative $\mathcal{L}''$ in (40) with the nonlinear term $\mathcal{L}^3$, we obtain $N = 1$.

Let the solution of (40) be given by

$$\mathcal{L}(\mathcal{S}) = N_0 + N_1 Q(\mathcal{S}), \tag{41}$$

where N_0, N_1 are fixed.

Based on (40) and (41), as well as (23), we obtain a system of algebraic equations, as follows:

$$\begin{aligned}
&cN_1 - \frac{1}{2}N_0 = 0,\\
&-cN_1 + N_1 - \frac{3}{2}N_0^2 N_1 = 0,\\
&\frac{-3}{2}N_0 N_1^2 = 0,\\
&-8N_1\alpha\beta - \frac{1}{2}N_1^3 = 0.
\end{aligned}$$

Solving the above system, we get

$$c = 1, \quad N_0 = 0, \quad N_1 = \pm 4\sqrt{-\alpha\beta}. \tag{42}$$

From the above results, the following solution is derived as

$$\mathcal{F}(\chi_1, \chi_2) = \pm 4\sqrt{-\alpha\beta}\frac{1}{(\beta+\alpha)\cosh(\chi_1 - \frac{\chi_2^{\mathfrak{P}}}{\Gamma(1+\mathfrak{P})}) + (\beta-\alpha)\sinh(\chi_1 - \frac{\chi_2^{\mathfrak{P}}}{\Gamma(1+\mathfrak{P})})}. \tag{43}$$

Figures 3–8 display The 2-d with the plots of the imaginary and real parts of (43), for diverse values of $\mathfrak{P}$.

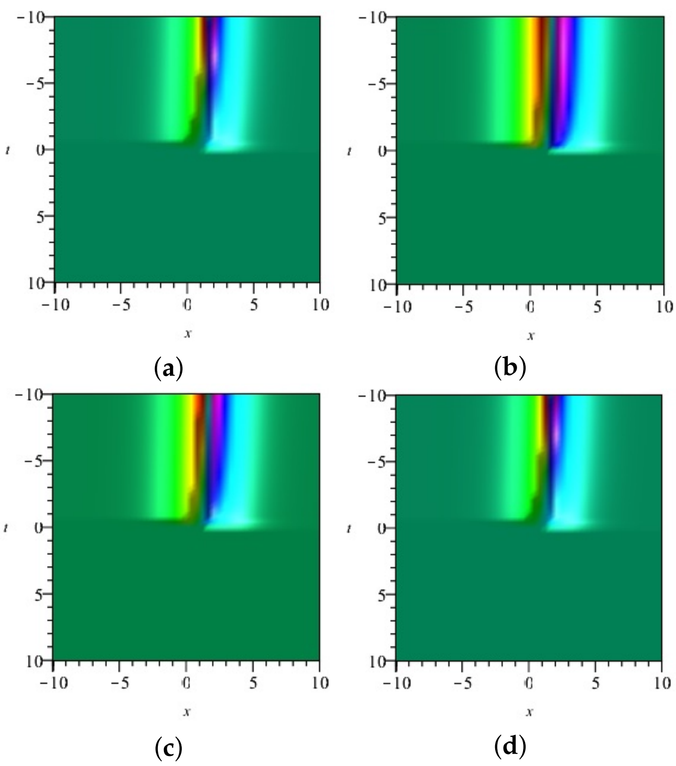

Figure 3. (**a**–**d**) The 2D with the diagrams of the real part of (43), in the z-axis orientation, for $\mathfrak{P} = 0.10, 0.15, 0.20, 0.25$.

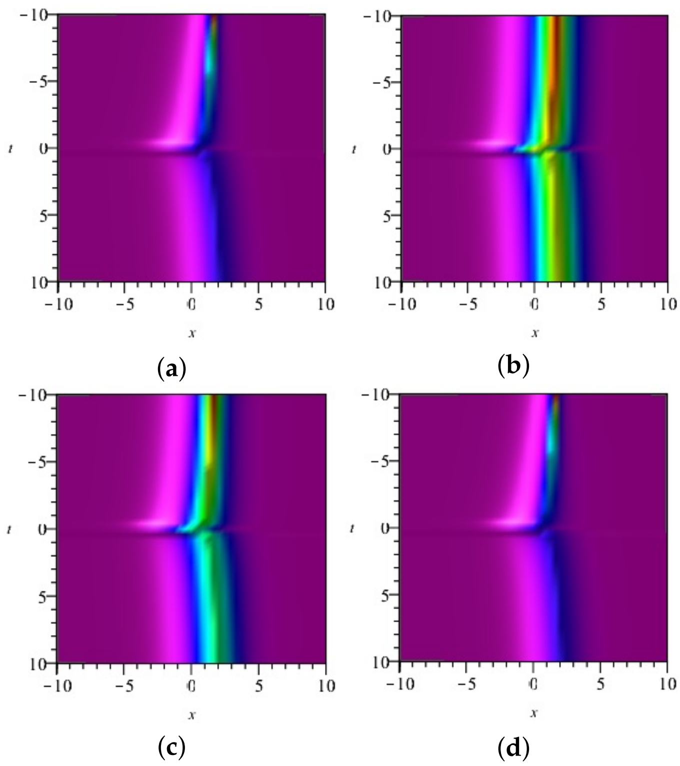

Figure 4. (**a**–**d**) The 2D with the diagrams of the imaginary part of (43), in the z-axis orientation, for $\mathfrak{P} = 0.10, 0.15, 0.20, 0.25$.

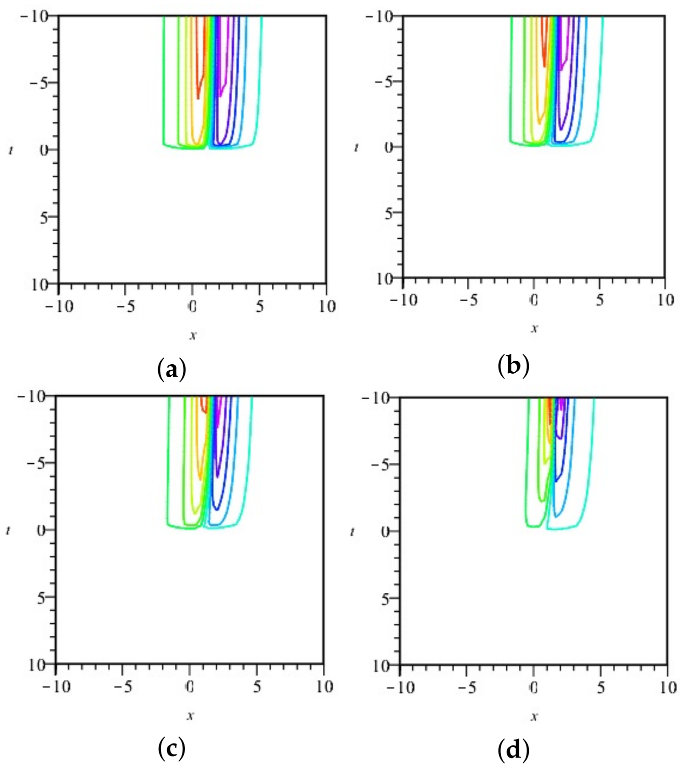

Figure 5. (**a**–**d**) The contour plots of the real part of (43), in the z-axis orientation, for $\mathfrak{P} = 0.10, 0.15, 0.20, 0.25$.

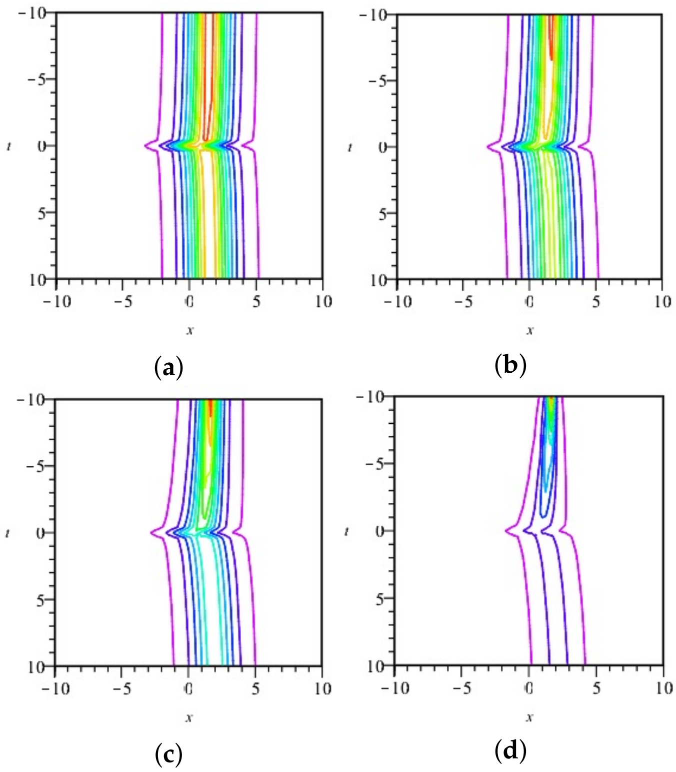

Figure 6. (**a**–**d**) The contour plots of the imaginary part of (43), in the z-axis orientation, for $\mathfrak{P} = 0.10, 0.15, 0.20, 0.25$.

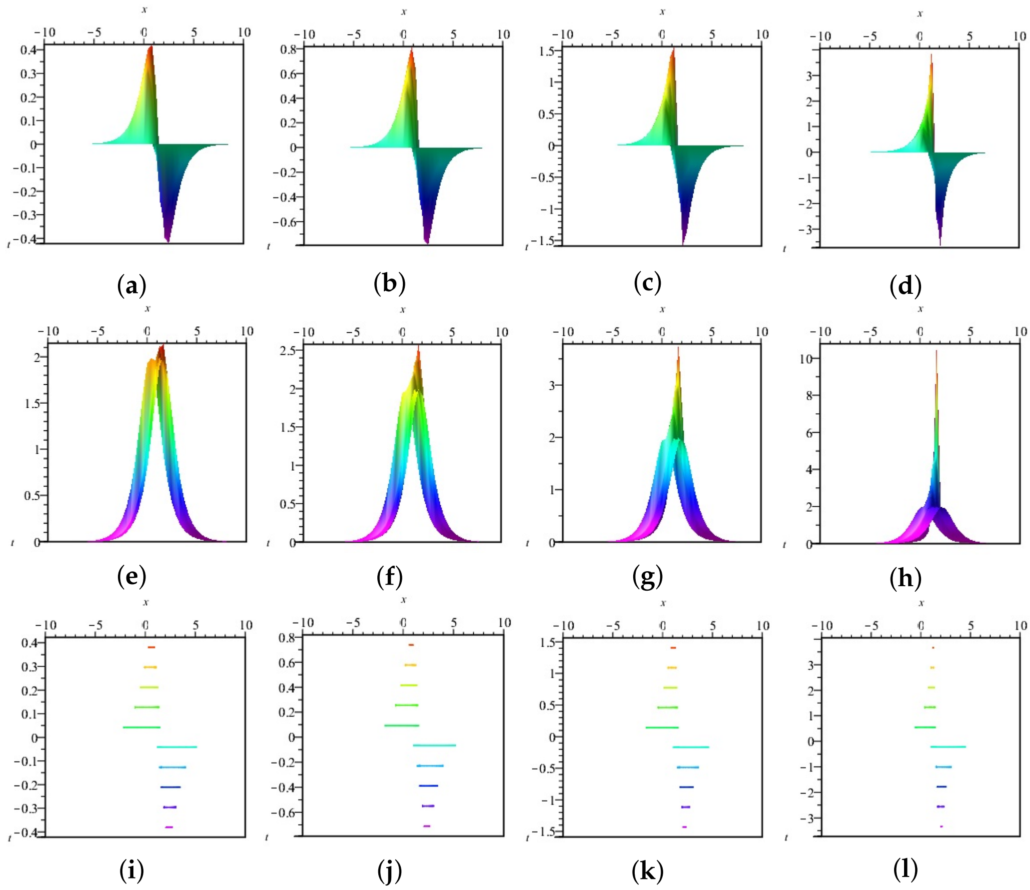

Figure 7. *Cont.*

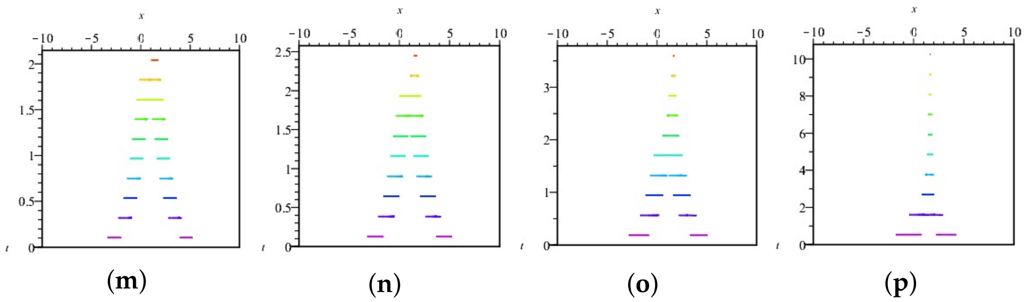

Figure 7. The 2D with the diagrams of the real part of (43), in the x-axis orientation (**a**–**d**,**i**–**l**) and the y-axis orientation (**e**–**h**,**m**–**p**), for $\mathfrak{P} = 0.10, 0.15, 0.20, 0.25$.

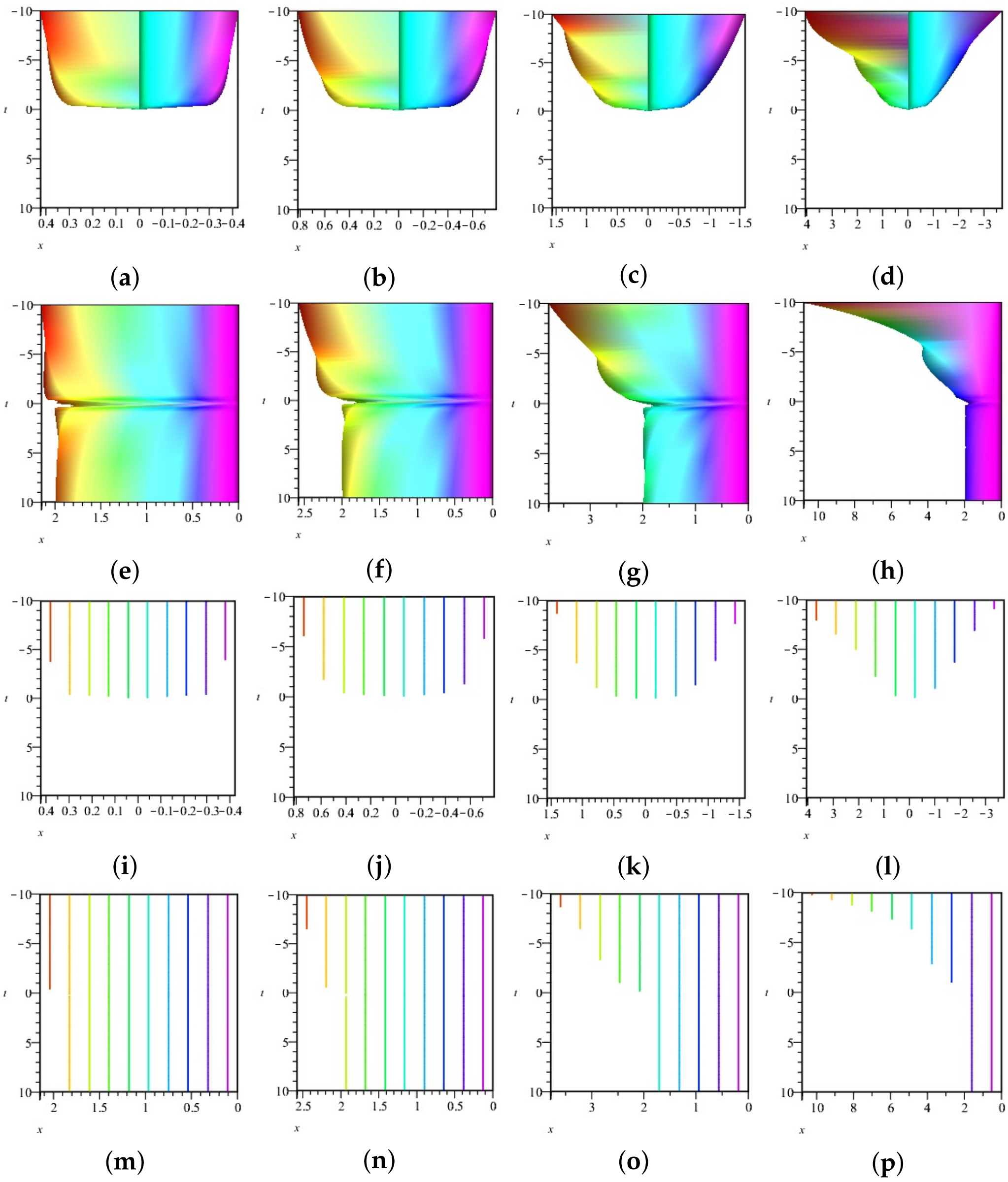

Figure 8. The 2D with the diagrams of the imaginary part of (43), in the x-axis orientation (**a**–**d**,**i**–**l**) and the y-axis orientation (**e**–**h**,**m**–**p**), for $\mathfrak{P} = 0.10, 0.15, 0.20, 0.25$.

4. Symmetric Random Finite-Time Stability of (1) for Case 2

4.1. Explicit Formula of Solutions

Making use of [8,10], we propose the proof of the theorem below:

Theorem 2. *(i) For the delayed Mittag–Leffler matrix $\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} : \mathbb{R} \to \mathbb{R}^n$, we have that*

$$(D_{0^+}^{\mathfrak{P}} \mathbb{M}_t^{\Xi\tau^{\mathfrak{P}}})(\chi) = \Xi \mathbb{M}_t^{\Xi(\chi-t)^{\mathfrak{P}}}, \tag{44}$$

i.e, $\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}$ is a solution of

$$(D_{0^+}^{\mathfrak{P}} \mathcal{F})(\chi) = \Xi \mathcal{F}(\chi - t),$$

with initial condition $\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} = I$, for every $\chi \in [-t, 0]$.

(ii) The solution $\mathcal{F} \in C([-t, T], \mathbb{R}^n)$ of (1), has the following form

$$\mathcal{F}(\chi) = \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} \mathcal{K}(-t) + \int_{-t}^{0} \mathbb{M}_t^{\Xi(\chi-t-\mathcal{Y})^{\mathfrak{P}}} \mathcal{K}'(\mathcal{Y}) d\mathcal{Y}.$$

Proof. (i) For every $\chi \in (-\infty, -t]$, $\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} = \mathbb{M}_t^{\Xi(\chi-t)^{\mathfrak{P}}} = \mathbf{0}$. Thus, (44) holds. For every $\chi \in [-t, 0]$, $\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} = I$ and $\mathbb{M}_t^{\Xi(\chi-t)^{\mathfrak{P}}} = \mathbf{0}$. Notice that $D_{0^+}^{\mathfrak{P}} I = \mathbf{0} = \Xi\mathbf{0}$. Then, (44) also holds. For every $\chi \in [(\kappa-1)t, \kappa t]$, with $\kappa \in \rho$, we have the following items:

(i.a) For $\kappa = 1$, and $\chi \in [0, t]$, we get

$$\begin{aligned}
\mathcal{F}(\chi) &= \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} = I + \frac{\Xi\chi^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)}, \\
\mathcal{F}'(\chi) &= \frac{\mathfrak{P}\Xi\chi^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P}+1)}.
\end{aligned} \tag{45}$$

Now, applying the fractional-order derivative in the Caputo sense, by (45) and Lemma 2, we get

$$\begin{aligned}
(D_{0^+}^{\mathfrak{P}} \mathbb{M}_t^{\Xi\tau^{\mathfrak{P}}})(\chi) &= \frac{\mathfrak{P}\Xi}{\Gamma(\mathfrak{P}+1)\Gamma(1-\mathfrak{P})} \int_0^{\chi} (\chi-\tau)^{-\mathfrak{P}} \tau^{\mathfrak{P}-1} d\tau \\
&= \frac{\mathfrak{P}\Xi\Gamma(1-\mathfrak{P})\Gamma(\mathfrak{P})}{\Gamma(\mathfrak{P}+1)\Gamma(1-\mathfrak{P})} \\
&= \Xi.
\end{aligned} \tag{46}$$

(i.b) For $\kappa = 2$, and $\chi \in [t, 2t]$, we obtain

$$\begin{aligned}
\mathcal{F}(\chi) &= \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} = I + \frac{\Xi\chi^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)} + \frac{\Xi^2(\chi-t)^{2\mathfrak{P}}}{\Gamma(2\mathfrak{P}+1)}, \\
\mathcal{F}'(\chi) &= \frac{\mathfrak{P}\Xi\chi^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P}+1)} + \frac{2\mathfrak{P}\Xi^2(\chi-t)^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P}+1)}.
\end{aligned} \tag{47}$$

Now, applying the fractional-order derivative in the Caputo sense by (46) and (47) and Lemma 2, we have

$$\begin{aligned}
(D_{0^+}^{\mathfrak{P}} \mathbb{M}_t^{\Xi\tau^{\mathfrak{P}}})(\chi) &= \Xi + \frac{2\mathfrak{P}\Xi^2}{\Gamma(1-\mathfrak{P})\Gamma(2\mathfrak{P}+1)} \int_t^{\chi} (\chi-\tau)^{-\mathfrak{P}} (\tau-t)^{2\mathfrak{P}-1} d\tau \\
&= \Xi + \frac{2\mathfrak{P}\Xi^2(\chi-t)^{\mathfrak{P}}}{\Gamma(1-\mathfrak{P})} \frac{\Gamma(1-\mathfrak{P})\Gamma(2\mathfrak{P})}{\Gamma(\mathfrak{P}+1)\Gamma(2\mathfrak{P}+1)} \\
&= \Xi + \Xi^2 \frac{(\chi-t)^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)}.
\end{aligned}$$

(i.c) For $\kappa = j$, $\chi \in [(j-1)t, jt]$, with $j \in \rho$, we have

$$(D_{0^+}^{\mathfrak{P}}\mathbb{M}_t^{\Xi\tau^{\mathfrak{P}}}) = \Xi + \frac{\Xi^2(\chi-t)^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)} + \frac{\Xi^3(\chi-2t)^{2\mathfrak{P}}}{\Gamma(2\mathfrak{P}+1)} + \cdots + \frac{\Xi^j(\chi-(j-1)t)^{(j-1)\mathfrak{P}}}{\Gamma((j-1)\mathfrak{P}+1)}.$$

For $\kappa = j+1$, $\chi \in [jt,(j+1)t]$, through elementary computation, we obtain

$$\mathcal{F}'(\chi) = \frac{\mathfrak{P}\Xi\chi^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P}+1)} + \frac{2\mathfrak{P}\Xi^2(\chi-t)^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P}+1)} + \ldots + \frac{(j+1)\mathfrak{P}\Xi^{j+1}(\chi-jt)^{(j+1)\mathfrak{P}-1}}{\Gamma((j+1)\mathfrak{P}+1)}. \quad (48)$$

Now, applying the fractional-order derivative in the Caputo sense by (49) and Lemma 2, we get

$$\begin{aligned}
&(D_{0^+}^{\mathfrak{P}}\mathbb{M}_t^{\Xi\tau^{\mathfrak{P}}})(\chi) \\
= \;& \frac{\mathfrak{P}\Xi}{\Gamma(1-\mathfrak{P})}\int_0^{\chi}(\chi-\tau)^{-\mathfrak{P}}\frac{\tau^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P}+1)}d\tau + \frac{2\mathfrak{P}\Xi^2}{\Gamma(1-\mathfrak{P})}\int_t^{\chi}(\chi-\tau)^{-\mathfrak{P}}\frac{(\tau-t)^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P}+1)}d\tau \\
&+ \cdots + \frac{(j+1)\mathfrak{P}\Xi^{j+1}}{\Gamma(1-\mathfrak{P})}\int_{jt}^{\chi}(\chi-\tau)^{-\mathfrak{P}}\frac{(\tau-jt)^{(j+1)\mathfrak{P}-1}}{\Gamma((j+1)\mathfrak{P}+1)}d\tau \\
= \;& \Xi + \frac{\Xi^2(\chi-t)^{\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)} + \frac{\Xi^3(\chi-2t)^{2\mathfrak{P}}}{\Gamma(2\mathfrak{P}+1)} + \cdots + \frac{\Xi^{j+1}(\chi-jt)^{j\mathfrak{P}}}{\Gamma(j\mathfrak{P}+1)},
\end{aligned}$$

thus, (44) holds, for every $\chi \in [(j-1)t, jt]$, with $j \in \rho$.

(ii) Suppose matrix $\Theta_0(\chi) = \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}$ satisfies (i) of Theorem 2, and every solution of (1) satisfies the initial condition $\mathcal{F}(\chi) = \mathcal{K}(\chi)$, for every $\chi \in [-t, 0]$. Then

$$\mathcal{F}(\chi) = \Theta_0(\chi)\varepsilon + \int_{-t}^{0}\Theta_0(\chi-t-\mathcal{Y})\mathfrak{C}(\mathcal{Y})d\mathcal{Y}, \quad (49)$$

where ε is a constant vector, and $\mathfrak{C}$ is a vector of a continuously differentiable function. Based on $\Theta_0(\chi)$ being a solution of (1), thus, for arbitrary ε and $\mathfrak{C}(.)$, (49) is also a solution of (1). Then, we claim ε and $\mathfrak{C}(.)$ satisfy the initial condition $\mathcal{F}(\chi) = \mathcal{K}(\chi)$, for every $\chi \in [-t, 0]$.

Considering $\chi = -t$, and from (16), we get $\Theta_0(-t) = I$, $\Theta_0(-2t-\mathcal{Y}) = \mathbf{0}$, $\mathcal{Y} \in [-t, 0]$ and $\Theta_0(-2t-\mathcal{Y}) = I$, with $\mathcal{Y} = -t$. Therefore, $\mathcal{F}(-t) = \mathcal{K}(-t) = \varepsilon$, and (49) takes the following form

$$\mathcal{F}(\chi) = \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}\mathcal{K}(-t) + \int_{-t}^{0}\mathbb{M}_t^{\Xi(\chi-t-\mathcal{Y})^{\mathfrak{P}}}\mathfrak{C}(\mathcal{Y})d\mathcal{Y}.$$

For $\chi \in [-t, 0]$, we have the following two cases:

(ii.a) For every $\mathcal{Y} \in [-t, \chi]$, so $-t \le \chi - t - \mathcal{Y} \le \chi$, the delayed Mittag–Leffler matrix is equivalent to $\mathbb{M}_t^{\Xi(\chi-t-\mathcal{Y})^{\mathfrak{P}}} = I$.

(ii.b) For every $\mathcal{Y} \in [\chi, 0]$, so $\chi - t \le \chi - t - \mathcal{Y} \le -t$, the delayed Mittag–Leffler matrix is equivalent to

$$\mathbb{M}_t^{\Xi(\chi-t-\mathcal{Y})^{\mathfrak{P}}} = \begin{cases} \mathbf{0}, & \mathcal{Y} \in (\chi, 0], \\ I, & \mathcal{Y} = \chi. \end{cases}$$

Then, for every $\chi \in [-t, 0]$, we get

$$\mathcal{K}(\chi) = \mathcal{K}(-t) + \int_{-t}^{\chi}\mathfrak{C}(\mathcal{Y})d\mathcal{Y}. \quad (50)$$

Taking the derivative in (50), we have $\mathfrak{C}(\chi) = \mathcal{K}'(\chi)$. $\square$

4.2. Symmetric Random Stability Results

Definition 7. *Fractional-order system (1) is random finite-time stable w.r.t $\{0, \nu, t, \alpha_1, \alpha_2\}$, iff $\Psi_{\mathcal{K}(\chi)}(\psi) \succ \Psi_{\alpha_1}(\psi)$, implies that $\Psi_{\mathcal{F}(\chi)}(\psi) \succ \Psi_{\alpha_2}(\psi)$, for every $\chi \in \nu$, and $\psi > 0$, with the initial time of observation $\mathcal{K}(\chi)$, $\chi \in [-t, 0]$, and every $\alpha_1, \alpha_2 \in \mathbb{R}^+$ with $\alpha_1 < \alpha_2$.*

Theorem 3. *(i) Let $\gamma := \int_{-t}^{0} \mathcal{K}(\mathcal{Y}) d\mathcal{Y} < \infty$. If*

$$2\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}) \min\{\alpha_1, \gamma\} < \alpha_2, \tag{51}$$

for every $\chi \in \nu$, then, (1) is random finite-time stable, w.r.t. $\{0, \nu, t, \alpha_1, \alpha_2\}$.

(ii) Let κ and $\mathfrak{P}$ be constants, s.t., $\mathfrak{P} \le \frac{1}{\kappa}$, for every $\kappa \in \rho$. If

$$2\alpha_1 \min\left\{\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}), \mathbb{M}_{\mathfrak{P}}(|\Xi|t^{\mathfrak{P}})\right\} < \alpha_2, \tag{52}$$

for every $\chi \in \nu$, then, (1) is random finite-time stable, w.r.t. $\{0, \nu, t, \alpha_1, \alpha_2\}$.

Proof. (i) In view of item (ii) of Theorem 2, the solution of fractional system (1) has the form below:

$$\mathcal{F}(\chi) = \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}} \mathcal{K}(-t) + \int_{-t}^{0} \mathbb{M}_t^{\Xi(\chi-t-\mathcal{Y})^{\mathfrak{P}}} \mathcal{K}'(\mathcal{Y}) d\mathcal{Y}. \tag{53}$$

According to Lemma 1, and (51), we get

$$\begin{aligned}
\Psi_{\mathcal{F}(\chi)}(\psi) &\succeq \Psi_{\mathcal{K}(-t)\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}}\left(\frac{\psi}{2}\right) \bigodot \Psi_{\int_{-t}^{0} \mathcal{K}'(\mathcal{Y})\mathbb{M}_t^{\Xi(\chi-t-\mathcal{Y})^{\mathfrak{P}}} d\mathcal{Y}}\left(\frac{\psi}{2}\right) \\
&\succeq \Psi_{\mathcal{K}(-t)}\left(\frac{\psi}{2\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right) \bigodot \Psi_{\int_{-t}^{0} \mathcal{K}'(\mathcal{Y})d\mathcal{Y}}\left(\frac{\psi}{2\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right) \\
&\succeq \Psi_{\alpha_1}\left(\frac{\psi}{2\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right) \bigodot \Psi_{\gamma}\left(\frac{\psi}{2\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right) \\
&\succeq \Psi_{2\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}) \min\{\alpha_1, \gamma\}}(\psi) \\
&\succ \Psi_{\alpha_2}(\psi) \\
&= \Psi\left(\frac{\psi}{|\alpha_2|}\right).
\end{aligned}$$

for every $\chi \in \nu$, and by noting that $\mathbb{M}_t^{\Xi(\chi-t-\mathcal{Y})^{\mathfrak{P}}} \le \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}$.

(ii) From integration by parts, (53) has the following form:

$$\mathcal{F}(\chi) = \mathbb{M}_t^{\Xi(\chi-t)^{\mathfrak{P}}} \mathcal{K}(0) + \int_{-t}^{0} \sum_{i=1}^{\kappa} \frac{i\mathfrak{P}\Xi^i(\chi - it - \mathcal{Y})^{i\mathfrak{P}-1}}{\Gamma(i\mathfrak{P}+1)} \mathcal{K}(\mathcal{Y}) d\mathcal{Y}, \tag{54}$$

by $\mathbb{M}_t^{\Xi(\chi-t-\mathcal{Y})^{\mathfrak{P}}} = \sum_{i=0}^{\kappa} \Xi^i \frac{(\chi-it-\mathcal{Y})^{i\mathfrak{P}}}{\Gamma(i\mathfrak{P}+1)}$ and $\frac{d(\mathbb{M}_t^{\Xi(\chi-t-\mathcal{Y})})^{\mathfrak{P}}}{d\mathcal{Y}} = -\sum_{i=1}^{\kappa} \frac{i\mathfrak{P}\Xi^i(\chi-it-\mathcal{Y})^{i\mathfrak{P}-1}}{\Gamma(i\mathfrak{P}+1)}$.

Making use of Lemma 1 and (52), we obtain

$$
\begin{aligned}
\Psi_{\mathcal{F}(\chi)}(\psi) &\succeq \Psi_{\mathcal{K}(0)\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}+\sum_{i=1}^{\kappa}\int_{-t}^{0}\frac{i\mathfrak{P}\Xi^i(\chi-it-\mathcal{Y})^{i\mathfrak{P}-1}}{\Gamma(i\mathfrak{P}+1)}d\mathcal{Y}\mathcal{K}(\chi)}(\psi) \\
&\succeq \Psi_{\mathcal{K}(0)\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}}\left(\frac{\psi}{2}\right)\bigodot\Psi_{\sum_{i=1}^{\kappa}\int_{-t}^{0}\frac{i\mathfrak{P}\Xi^i(\chi-it-\mathcal{Y})^{i\mathfrak{P}-1}}{\Gamma(i\mathfrak{P}+1)}d\mathcal{Y}\mathcal{K}(\chi)}\left(\frac{\psi}{2}\right) \\
&\succeq \Psi_{\mathcal{K}(0)\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}}\left(\frac{\psi}{2}\right)\bigodot\Psi_{\sum_{i=1}^{\kappa}\frac{\Xi^i}{\Gamma(i\mathfrak{P}+1)}[((\chi-(i-1)t)^{i\mathfrak{P}}-(\chi-it)^{i\mathfrak{P}}]\mathcal{K}(\chi)}\left(\frac{\psi}{2}\right) \\
&\succeq \Psi_{\mathcal{K}(0)\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}}\left(\frac{\psi}{2}\right)\bigodot\Psi_{\mathcal{K}(\chi)\sum_{i=1}^{\kappa}\frac{\Xi^i}{\Gamma(i\mathfrak{P}+1)}t^{i\mathfrak{P}}}\left(\frac{\psi}{2}\right) \\
&\succeq \Psi_{\mathcal{K}(0)\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}}\left(\frac{\psi}{2}\right)\bigodot\Psi_{\mathcal{K}(\chi)\sum_{i=1}^{\infty}\frac{\Xi^i}{\Gamma(i\mathfrak{P}+1)}t^{i\mathfrak{P}}}\left(\frac{\psi}{2}\right) \\
&\succeq \Psi_{\alpha_1}\left(\frac{\psi}{2\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right)\bigodot\Psi_{\alpha_1}\left(\frac{\psi}{2\mathbb{M}_{\mathfrak{P}}(|\Xi|t^{\mathfrak{P}})}\right) \\
&\succeq \Psi_{2\alpha_1\min\left\{\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\mathbb{M}_{\mathfrak{P}}(|\Xi|t^{\mathfrak{P}})\right\}}(\psi) \\
&\succeq \Psi_{\alpha_2}(\psi) \\
&= \Psi\left(\frac{\psi}{|\alpha_2|}\right),
\end{aligned}
$$

for every $\chi \in \nu$, in which we use the relation $\alpha^a - \beta^a \le (\alpha-\beta)^a$, for $\alpha > \beta > 0$, and $a \in (0,1]$. □

5. Representation of Solutions to (1) for Case 3

Making use of [8,10], we present the proof of the theorem below:

Theorem 4. *Every solution $\widetilde{\mathcal{F}} \in C([-t,T],\mathbb{R}^n)$ of (1), with the initial condition $\mathcal{F}(\chi) = 0$, for every $\chi \in [-t,0]$, has the following form*

$$\widetilde{\mathcal{F}}(\chi) = \int_0^{\chi}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{F}(\tau)d\tau, \quad \chi > 0.$$

Proof. Making use of the variation of constants method, every solution of non-homogeneous system $\widetilde{\mathcal{F}}(\chi)$ has the form below:

$$\widetilde{\mathcal{F}}(\chi) = \int_0^{\chi}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}A(\tau)d\tau, \quad \chi > 0. \tag{55}$$

in which $A(\tau)$ is a vector function for every $\tau \in [0,\chi]$, and $\widetilde{\mathcal{F}}(0) = 0$.

Applying the fractional-order derivative in the Caputo sense on both sides of (55), we get the items below:

(i) For every $\chi \in (0,t]$, based on (1), we obtain

$$
\begin{aligned}
(D_{0^+}^{\mathfrak{P}}\widetilde{\mathcal{F}})(\chi) &= \Xi\widetilde{\mathcal{F}}(\chi-t)+\mathcal{G}(\chi) \\
&= \Xi\int_0^{\chi-t}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-2t-\tau)^{\mathfrak{P}}}A(\tau)d\tau+\mathcal{G}(\chi) \\
&= \mathcal{G}(\chi).
\end{aligned}
$$

Here, notice that $\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-2t-.)^{\mathfrak{P}}} = \mathbf{0}$.

In view of Lemma 3 and the definition of the Caputo fractional derivative, we get

$$\begin{aligned}
(D_{0^+}^{\mathfrak{P}}\widetilde{\mathcal{F}})(\chi) &= ({}^{RL}D_{0^+}^{\mathfrak{P}}\widetilde{\mathcal{F}})(\chi) \\
&= \frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi}(\chi-\tau)^{-\mathfrak{P}}\left[\int_0^{\tau}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}}A(\mathcal{Y})d\mathcal{Y}\right]d\tau \\
&= \frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi}A(\mathcal{Y})\int_{\mathcal{Y}}^{\chi}(\chi-\tau)^{-\mathfrak{P}}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}}d\mathcal{Y}d\tau \\
&= \frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi}A(\mathcal{Y})\left[\int_{\mathcal{Y}}^{\chi}(\chi-\tau)^{-\mathfrak{P}}I\frac{(\chi-\tau)^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}d\mathcal{Y}\right]d\tau \\
&= \frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi}\frac{\mathbb{B}[1-\mathfrak{P},\mathfrak{P}]}{\Gamma(\mathfrak{P})}A(\mathcal{Y})d\mathcal{Y} \\
&= A(\chi).
\end{aligned}$$

Thus, we have $A(\chi)=\mathcal{G}(\chi)$.

(ii) For every $\chi\in(jt,(j+1)t]$, with $j\in\rho$, based on (1), we get

$$\begin{aligned}
(D_{0^+}^{\mathfrak{P}}\widetilde{\mathcal{F}})(\chi) &= \Xi\widetilde{\mathcal{F}}(\chi-t)+\mathcal{G}(\chi) \\
&= \Xi\int_0^{\chi-t}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-2t-\tau)^{\mathfrak{P}}}A(\tau)d\tau+\mathcal{G}(\chi) \\
&= \Xi\left[\int_0^{\chi-t}\frac{(\chi-t-\tau)^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}A(\tau)d\tau+\int_0^{\chi-2t}\Xi\frac{(\chi-2t-\tau)^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P})}A(\tau)d\tau\right. \\
&\qquad\left.+\ldots+\int_0^{\chi-jt}\Xi^{j-1}\frac{(\chi-jt-\tau)^{j\mathfrak{P}-1}}{\Gamma(j\mathfrak{P})}A(\tau)d\tau\right]+\mathcal{G}(\chi).
\end{aligned}$$

By the definition of the Caputo fractional derivative, we obtain

$$\begin{aligned}
(D_{0^+}^{\mathfrak{P}}\widetilde{\mathcal{F}})(\chi) &= ({}^{RL}D_{0^+}^{\mathfrak{P}}\widetilde{\mathcal{F}})(\chi) \\
&= \frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi}(\chi-\tau)^{-\mathfrak{P}}\left[\int_0^{\tau}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}}A(\mathcal{Y})d\mathcal{Y}\right]d\tau \\
&= \frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi}\int_0^{\tau}(\chi-\tau)^{-\mathfrak{P}}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}}A(\mathcal{Y})d\mathcal{Y}d\tau \\
&= \frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi}A(\mathcal{Y})\left[\int_{\mathcal{Y}}^{\chi}(\chi-\tau)^{-\mathfrak{P}}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}}d\tau\right]d\mathcal{Y}.
\end{aligned}$$

In view of Lemmas 3 and 4, we have that

$$\begin{aligned}
(D_{0^+}^{\mathfrak{P}}\widetilde{\mathcal{F}})(\chi) &= ({}^{RL}D_{0^+}^{\mathfrak{P}}\widetilde{\mathcal{F}})(\chi) \\
&= \frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi}A(\mathcal{Y})\left[\int_{\mathcal{Y}}^{\chi}(\chi-\tau)^{-\mathfrak{P}}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\tau-t-\mathcal{Y})^{\mathfrak{P}}}d\tau\right]d\mathcal{Y} \\
&= \frac{d}{d\chi}\int_0^{\chi}A(\mathcal{Y})d\mathcal{Y}+\frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi-t}A(\mathcal{Y})\left[\int_{t+\mathcal{Y}}^{\chi}(\chi-\tau)^{-\mathfrak{P}}\Xi\frac{(\tau-t-\mathcal{Y})^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P})}d\tau\right]d\mathcal{Y} \\
&\qquad+\cdots+\frac{1}{\Gamma(1-\mathfrak{P})}\frac{d}{d\chi}\int_0^{\chi-jt}A(\mathcal{Y})\left[\int_{jt+\mathcal{Y}}^{\chi}(\chi-\tau)^{-\mathfrak{P}}\Xi^{j}\frac{(\tau-jt-\mathcal{Y})^{(j+1)\mathfrak{P}-1}}{\Gamma((j+1)\mathfrak{P})}d\tau\right]d\mathcal{Y} \\
&= A(\chi)+\Xi\int_0^{\chi-t}\frac{(\chi-t-\mathcal{Y})^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}A(\mathcal{Y})d\mathcal{Y}+\Xi^2\int_0^{\chi-2t}\frac{(\chi-2t-\mathcal{Y})^{2\mathfrak{P}-1}}{\Gamma(2\mathfrak{P})}A(\mathcal{Y})d\mathcal{Y} \\
&\qquad+\cdots+\Xi^{j}\int_0^{\chi-jt}\frac{(\chi-jt-\mathcal{Y})^{j\mathfrak{P}-1}}{\Gamma(j\mathfrak{P})}A(\mathcal{Y})d\mathcal{Y} \\
&= \Xi\int_0^{\chi-t}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-2t-\tau)^{\mathfrak{P}}}A(\tau)d\tau+\mathcal{G}(\chi).
\end{aligned}$$

Thus, $A(\chi)=\mathcal{G}(\chi)$.

□

It is straightforward that every solution $\mathcal{F}$ of (1) for **Case 3** has the form $\mathcal{F}(\chi) = \mathcal{F}_0(\chi) + \widetilde{\mathcal{F}}(\chi)$, in which $\mathcal{F}_0(\chi)$ is a solution of (1) for **Case 2**, with the initial condition $\mathcal{F}(\chi) = \mathcal{K}(\chi)$, for every $\chi \in [-t, 0]$, and $\widetilde{\mathcal{F}}(\chi)$ is a solution of (1) for **Case 3**, with $\mathcal{F}(0) = 0$. Considering the above descriptions, we will present the formula of solutions to (1) for **Case 3**, in the theorem below:

Theorem 5. *Every solution $\mathcal{F} \in C([-t, T], \mathbb{R}^n)$ of (1) for **Case 3** is given by*

$$\mathcal{F}(\chi) = \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}\mathcal{K}(-t) + \int_{-t}^{0} \mathbb{M}_t^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}'(\tau)d\tau + \int_0^{\chi} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{G}(\tau)d\tau.$$

6. Fuzzy Finite-Time Stability of (1) for Case 4

Definition 8. *The function $\mathcal{F} \in C([-t, T], \mathbb{R}^n)$ is a solution of (1) for **Case 4**, if,*

$$\mathcal{F}(\chi) = \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}\mathcal{K}(-t) + \int_{-t}^{0} \mathbb{M}_t^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}'(\tau)d\tau + \int_0^{\chi} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{H}(\tau, \mathcal{F}(\tau))d\tau. \quad (56)$$

Let us consider the assumptions below:

(Δ_1) The contractive mapping $\mathcal{H} \in C(\nu, \mathbb{R}^n)$ has the contraction property w.r.t the second component with positive Lipschitz constant a, i.e., $\Phi(\mathcal{H}(\chi, \mathcal{Y}) - \mathcal{H}(\chi, \mathcal{Z}), \phi) \succeq \Phi(\mathcal{Y} - \mathcal{Z}, \frac{\phi}{a})$.

(Δ_2) $b := a[\sum_{m=1}^{\kappa} \frac{|\Xi|^{m-1}}{\Gamma(m\mathfrak{P}+1)}(T-(m-1)t)^{m\mathfrak{P}}] < I$, for every fixed number $\kappa \in \rho$.

(Δ_3) There is a $\zeta_1(.) \in C(\nu, \mathbb{R}_+^n)$, s.t. $\Phi(\mathcal{H}(\chi, \mathcal{Y}), \phi) \succeq \Phi(\zeta_1(\chi), \phi)$, for every $\chi \in \nu$ and $\mathcal{Y} \in \mathbb{R}^n$.

(Δ_4) There is a $\zeta_2(.) \in L^q(\nu, \mathbb{R}_+^n)$, with $\frac{1}{q} + \frac{1}{p} = 1$ and $p > 1$, s.t., $\Phi(\mathcal{H}(\chi, \mathcal{Y}), \phi) \succeq \Phi(\zeta_2(\chi), \phi)$, for every $\chi \in \nu$ and $\mathcal{Y} \in \mathbb{R}^n$, and $\xi(\chi) := (\int_0^{\chi} \zeta_2(\tau)^q d\tau)^{\frac{1}{q}} < \infty$.

Let $c := \int_{-t}^{0} |\mathcal{K}'(\tau)|d\tau$.

Theorem 6. *Let Δ_1 and Δ_2 hold. Then, (1) has a unique solution $\mathcal{F} \in C([-t, T], \mathbb{R}^n)$.*

Proof. Consider the operator $\mathscr{L} : C([-t, T], \mathbb{R}^n) \to C([-t, T], \mathbb{R}^n)$ defined by

$$(\mathscr{L}\mathcal{F})(\chi) = \mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}\mathcal{K}(-t) + \int_{-t}^{0} \mathbb{M}_t^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}'(\tau)d\tau + \int_0^{\chi} \mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{H}(\tau, \mathcal{F}(\tau))d\tau. \quad (57)$$

The function $\mathscr{L}$ is well-defined because of Δ_1. We prove $\mathscr{L}$ is a contraction mapping. Applying Lemma 5, for every $\mathcal{F}, \mathcal{I} \in C([-t, T], \mathbb{R}^n)$, and $\phi > 0$, we have that

$$\Phi((\mathscr{L}\mathcal{F})(\chi) - (\mathscr{L}\mathcal{I})(\chi), \phi) \succeq \Phi\Big((\mathcal{F}(\chi) - \mathcal{I}(\chi)) \cdot a[\sum_{m=1}^{\kappa} \frac{|\Xi|^{m-1}}{\Gamma(m\mathfrak{P}+1)}(\chi-(m-1)t)^{m\mathfrak{P}}], \phi\Big),$$

which infers that

$$\Phi((\mathscr{L}\mathcal{F})(\chi) - (\mathscr{L}\mathcal{I})(\chi), \phi) \succeq \Phi(b[\mathcal{F}(\chi) - \mathcal{I}(\chi)], \phi),$$

Through (Δ_2), one can use contraction mapping principle to complete the proof. □

Definition 9. *Suppose $\mathcal{F}$ is a solution of (1). Fractional order Equations (1) and (2) is fuzzy finite-time stable w.r.t $\{0, \nu, t, \alpha_1, \alpha_2\}$, iff, $\Phi(\mathcal{K}(\chi), \phi) \succ \Phi(\alpha_1, \phi), \chi \in [-t, 0]$, infers that $\Phi(\mathcal{F}(\chi), \phi) \succ \Phi(\alpha_2, \phi), \chi \in \nu$, in which $\mathcal{K}(\chi)$, $-t \le \chi \le 0$ is the initial time, and α_1, α_2 are positive, with $\alpha_1 < \alpha_2$, and $\phi > 0$.*

Theorem 7. *(i) Assume the assumptions $(\Delta_1), (\Delta_2)$, and (Δ_3) hold. For the fixed $\kappa \in \rho$, if, for every $\chi \in \nu$,*

$$3\min\left\{\min\{\alpha_1,c\}\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\inf_{\chi\in\nu}\zeta_1(\chi)\left[\sum_{j=1}^{\kappa}\frac{|\Xi|^{j-1}}{\Gamma(j\mathfrak{P}+1)}(\chi-(j-1)t)^{j\mathfrak{P}}\right]\right\}<\alpha_2,\quad \chi\in\nu, \tag{58}$$

then the fractional order system (1)–(2) is fuzzy finite-time stable w.r.t $\{0,\nu,t,\alpha_1,\alpha_2\}$.

(ii) Assume the assumptions (Δ_1), (Δ_2), *and* (Δ_4) *hold and* $\mathfrak{P}>1-\frac{1}{q}$ $(p>1)$. *For the fixed* $\kappa\in\rho$, *if,*

$$3\min\left\{\min\{\alpha_1,c\}\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\xi(\chi)\sum_{j=1}^{\kappa}\left[\frac{|\Xi|^{j-1}}{\Gamma(j\mathfrak{P})}\frac{(\chi-(j-1)t)^{j\mathfrak{P}-1+\frac{1}{p}}}{(pi\mathfrak{P}-p+1)\frac{1}{p}}\right]\right\}<\alpha_2,\quad \chi\in\nu, \tag{59}$$

then the fractional order system (1)–(2) is fuzzy finite-time stable w.r.t $\{0,\nu,t,\alpha_1,\alpha_2\}$.

(iii) Assume the assumptions (Δ_1), (Δ_2), *and* (Δ_3) *hold and* $\mathfrak{P}>\frac{1}{2}$. *For the fixed* $\kappa\in\rho$, *if,*

$$3\min\left\{\min\{\alpha_1,c\}\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\inf_{\chi\in\nu}\zeta_1(\chi)}{\mathfrak{P}}\chi^{\mathfrak{P}}\mathbb{M}_{\mathfrak{P},\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})\right\}<\alpha_2,\quad \chi\in\nu, \tag{60}$$

then the fractional order system (1) and (2) is fuzzy finite-time stable w.r.t $\{0,\nu,t,\alpha_1,\alpha_2\}$.

Proof. (i) Through the assumptions (Δ_1) and (Δ_2) and Theorem 6, we have that (1) has a unique solution $\mathcal{F}\in C([-t,T],\mathbb{R}^n)$. Making use of Lemmas 1, 5 and 6 and (56), we get

$$\begin{aligned}
&\Phi(\mathcal{F}(\chi),\phi)\\
\succeq\ &\Phi\left(\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}\mathcal{K}(-t)+\int_{-t}^{0}\mathbb{M}_t^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}'(\tau)d\tau+\int_0^{\chi}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\zeta_1(\tau)d\tau,\phi\right)\\
\succeq\ &\Phi\left(\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}\mathcal{K}(-t),\frac{\phi}{3}\right)\odot\Phi\left(\int_{-t}^{0}\mathbb{M}_t^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}'(\tau)d\tau,\frac{\phi}{3}\right)\odot\Phi\left(\int_0^{\chi}\mathbb{M}_{t,\mathfrak{P}}^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\zeta_1(\tau)d\tau,\frac{\phi}{3}\right)\\
\succeq\ &\Phi\left(\mathcal{K}(-t),\frac{\phi}{3\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right)\odot\Phi\left(\int_{-t}^{0}\mathcal{K}'(\tau)d\tau,\frac{\phi}{3\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right)\\
&\quad\odot\Phi\left(\inf_{\chi\in\nu}\zeta_1(\chi)\left[\int_0^{\chi-(\kappa-1)t}\sum_{j=1}^{\kappa}|\Xi|^{j-1}\frac{(\chi-(j-1)t-\tau)^{j\mathfrak{P}}-1}{\Gamma(j-1)\mathfrak{P}+\mathfrak{P}}d\tau\right.\right.\\
&\qquad\left.\left.+\sum_{i=2}^{\kappa}\int_{\chi-(i-1)t}^{\chi-(i-2)t}\left(\sum_{j=2}^{i}|\Xi|^{j-2}\frac{(\chi-(j-2)t-\tau)^{(j-1)\mathfrak{P}-1}}{\Gamma((j-2)\mathfrak{P}+\mathfrak{P})}\right)d\tau\right],\frac{\phi}{3}\right)\\
\succeq\ &\Phi\left(\alpha_1,\frac{\phi}{3\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right)\odot\Phi\left(c,\frac{\phi}{3\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right)\\
&\quad\odot\Phi\left(\inf_{\chi\in\nu}\zeta_1(\chi)\left[\sum_{j=1}^{\kappa}\frac{|\Xi|^{j-1}}{\Gamma(j\mathfrak{P}+1)}\left((\chi-(j-1)t)^{i\mathfrak{P}}-((\kappa-j)t)^{j\mathfrak{P}}\right)\right.\right.\\
&\qquad\left.\left.+\sum_{i=2}^{\kappa}\left[\sum_{j=2}^{i}\frac{|\Xi|^{j-2}}{\Gamma((j-1)\mathfrak{P}+1)}\left(((i-j+1)t)^{\mathfrak{P}(j-1)}-((i-j)t)^{\mathfrak{P}(j-1)}\right)\right],\frac{\phi}{3}\right)\right.\\
\succeq\ &\Phi\left(\alpha_1\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\odot\Phi\left(c\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\\
&\qquad\odot\Phi\left(\inf_{\chi\in\nu}\zeta_1(\chi)\left[\sum_{j=1}^{\kappa}\frac{|\Xi|^{j-1}}{\Gamma(j\mathfrak{P}+1)}(\chi-(j-1)t)^{j\mathfrak{P}}\right],\frac{\phi}{3}\right)\\
\succeq\ &\Phi\left(3\min\left\{\alpha_1\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),c\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\inf_{\chi\in\nu}\zeta_1(\chi)\left[\sum_{j=1}^{\kappa}\frac{|\Xi|^{j-1}}{\Gamma(j\mathfrak{P}+1)}(\chi-(j-1)t)^{j\mathfrak{P}}\right]\right\},\phi\right)\\
\succeq\ &\Phi(\alpha_2,\phi).
\end{aligned}$$

(ii) In view of Lemma 1, 5 and 6, and (56) and (59), we get

$$
\begin{aligned}
&\Phi(\mathcal{F}(\chi),\phi)\\
\succeq\;&\Phi\left(\mathbb{M}_t^{\Xi\chi^{\mathfrak{P}}}\mathcal{K}(-t),\frac{\phi}{3}\right)\bigodot\Phi\left(\int_{-t}^{0}\mathbb{M}_t^{\Xi(\chi-t-\tau)^{\mathfrak{P}}}\mathcal{K}'(\tau)d\tau,\frac{\phi}{3}\right)\\
&\bigodot\Phi\left(\int_0^{\chi}\frac{(\chi-\tau)^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}\zeta_2(\tau)d\tau+\cdots+\int_0^{\chi-(\kappa-1)t}|\Xi|^{\kappa-1}\frac{(\chi-(\kappa-1)t-\tau)^{\kappa\mathfrak{P}-1}}{\Gamma(\kappa\mathfrak{P})}\zeta_2(\tau)d\tau,\frac{\phi}{3}\right)\\
\succeq\;&\Phi\left(\mathcal{K}(-t),\frac{\phi}{3\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right)\bigodot\Phi\left(\int_{-t}^{0}\mathcal{K}'(\tau)d\tau,\frac{\phi}{3\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right)\\
&\bigodot\Phi\left(\sum_{j=1}^{\kappa}\frac{|\Xi|^{j-1}}{\Gamma(j\mathfrak{P})}\int_0^{\chi-(j-1)t}(\chi-(j-1)t-\tau)^{j\mathfrak{P}-1}\zeta_2(\tau)d\tau,\frac{\phi}{3}\right)\\
\succeq\;&\Phi\left(\alpha_1,\frac{\phi}{3\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right)\bigodot\Phi\left(c,\frac{\phi}{3\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})}\right)\\
&\bigodot\Phi\left(\sum_{j=1}^{\kappa}\frac{|\Xi|^{j-1}}{\Gamma(j\mathfrak{P})}\left(\int_0^{\chi-(j-1)t}(\chi-(j-1)t-\tau)^{p(j\mathfrak{P}-1)}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\chi-(j-1)t}\zeta_2(\tau)^q d\tau\right)^{\frac{1}{q}},\frac{\phi}{3}\right)\\
\succeq\;&\Phi\left(\alpha_1\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\bigodot\Phi\left(c\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\\
&\bigodot\Phi\left(\sum_{j=1}^{\kappa}\frac{|\Xi|^{j-1}}{\Gamma(j\mathfrak{P})}\left(\int_0^{\chi-(j-1)t}(\chi-(j-1)t-\tau)^{p(j\mathfrak{P}-1)}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\chi}\zeta_2(\tau)^q d\tau\right)^{\frac{1}{q}},\frac{\phi}{3}\right)\\
\succeq\;&\Phi\left(3\min\left\{\alpha_1\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),c\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\xi(\chi)\sum_{j=1}^{\kappa}\left(\frac{|\Xi|^{j-1}}{\Gamma(j\mathfrak{P})}\frac{(\chi-(j-1)t)^{j\mathfrak{P}-1+\frac{1}{p}}}{(pi\mathfrak{P}-p+1)^{\frac{1}{p}}}\right)\right\},\phi\right)\\
\succeq\;&\Phi(\alpha_2,\phi).
\end{aligned}
$$

(iii) In view of Lemma 1 and 5–7, and (56) and (60), we get

$$
\begin{aligned}
&\Phi(\mathcal{F}(\chi),\phi)\\
\succeq\;&\Phi\left(\alpha_1\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\bigodot\Phi\left(c\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\\
&\bigodot\Phi\left(\mathbb{M}_{\mathfrak{P},\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})\int_0^{\chi}(\chi-\tau)^{\mathfrak{P}-1}\zeta_1(\tau)d\tau,\frac{\phi}{3}\right)\\
\succeq\;&\Phi\left(\alpha_1\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\bigodot\Phi\left(c\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\\
&\bigodot\Phi\left(\inf_{\chi\in\nu}\zeta_1(\chi)\mathbb{M}_{\mathfrak{P},\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})\int_0^{\chi}(\chi-\tau)^{\mathfrak{P}-1}d\tau,\frac{\phi}{3}\right)\\
\succeq\;&\Phi\left(\alpha_1\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\bigodot\Phi\left(c\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\\
&\bigodot\Phi\left(\frac{\inf_{\chi\in\nu}\zeta_1(\chi)}{\mathfrak{P}}\chi^{\mathfrak{P}}\mathbb{M}_{\mathfrak{P},\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\phi}{3}\right)\\
\succeq\;&\Phi\left(3\min\left\{\alpha_1\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),c\mathbb{M}_{\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}}),\frac{\inf_{\chi\in\nu}\zeta_1(\chi)}{\mathfrak{P}}\chi^{\mathfrak{P}}\mathbb{M}_{\mathfrak{P},\mathfrak{P}}(|\Xi|\chi^{\mathfrak{P}})\right\},\phi\right)\\
\succeq\;&\Phi(\alpha_2,\phi).
\end{aligned}
$$

□

Example

Consider the fractional-order system below

$$D_{0^+}^{0.5}\mathcal{F}(\chi) = \Xi\mathcal{F}(\chi - 0.2) + \mathcal{H}(\chi, \mathcal{F}(\chi)), \quad \chi \in [0, 0.6], \tag{61}$$

$$\mathcal{K}(\chi) = \left(\frac{\chi}{2}, \frac{\chi}{4}, \frac{\chi}{6}, \frac{\chi}{8}, \frac{\chi}{10}\right), \quad \chi \in [-0.2, 0], \tag{62}$$

where $\Xi = \mathrm{diag}[0.1, 0.2, 0.3, 0.4, 0.5]$, $\mathcal{F}(\chi) = (\mathcal{F}_1(\chi), \mathcal{F}_2(\chi), \mathcal{F}_3(\chi), \mathcal{F}_4(\chi), \ \mathcal{F}_5(\chi))^\top$, and $\mathcal{H}(\chi, \mathcal{F}(\chi)) = \Big(\frac{\chi^2}{2}\ln\sqrt{|\mathcal{F}_1(\chi)|}, \frac{\chi^2}{2}\sin^2(\mathcal{F}_2(\chi)), \frac{\chi^2}{2}\cos(\mathcal{F}_3(\frac{\chi}{2})), \frac{\chi^2}{2}\mathrm{arccot}(\mathcal{F}_4(\chi)), \frac{\chi^2}{2}\frac{|\mathcal{F}_5(\chi)|}{1+|\mathcal{F}_5(\chi)|}\Big)^\top$.

The solution of fractional system (61) and (62) has the following form:

$$\mathcal{F}(\chi) = \mathbb{M}_{0.2}^{\Xi\chi^{0.5}}\mathcal{K}(-0.2) + \int_{-0.2}^{0}\mathbb{M}_{0.2}^{\Xi(\chi-0.2-\tau)^{0.5}}\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\right)^\top d\tau + \int_0 \chi\mathbb{M}_{0.2,0.5}^{\Xi(\chi-0.2-\tau)^{0.5}}\mathcal{H}(\chi, \mathcal{F}(\chi))d\tau.$$

Now, we have that

$$\begin{aligned}
& \Phi\left(\ln\sqrt{|\mathcal{F}_1(\chi)|} - \ln\sqrt{|\widetilde{\mathcal{F}}_1(\chi)|}, \phi\right) \\
= \ & \Phi\left(\ln\frac{\sqrt{|\mathcal{F}_1(\chi)|}}{\sqrt{|\widetilde{\mathcal{F}}_1(\chi)|}}, \phi\right) \\
\succeq \ & \Phi\left(\ln\left(1 + \left(\frac{\sqrt{|\mathcal{F}_1(\chi)|}}{\sqrt{|\widetilde{\mathcal{F}}_1(\chi)|}} - 1\right)\right), \phi\right) \\
\succeq \ & \Phi\left(\frac{\sqrt{|\mathcal{F}_1(\chi)|}}{\sqrt{|\widetilde{\mathcal{F}}_1(\chi)|}} - 1, \phi\right) \\
\succeq \ & \Phi\left(\frac{\sqrt{|\mathcal{F}_1(\chi)|} - \sqrt{|\widetilde{\mathcal{F}}_1(\chi)|}}{\sqrt{\min(|\widetilde{\mathcal{F}}_1(\chi)|)}}, \phi\right) \\
\succeq \ & \Phi\left(\sqrt{|\mathcal{F}_1(\chi)|} - \sqrt{|\widetilde{\mathcal{F}}_1(\chi)|}, \phi\right) \\
\succeq \ & \Phi\left(\frac{|\mathcal{F}_1(\chi)| - |\widetilde{\mathcal{F}}_1(\chi)|}{\sqrt{|\mathcal{F}_1(\chi)|} + \sqrt{|\widetilde{\mathcal{F}}_1(\chi)|}}, \phi\right) \\
\succeq \ & \Phi\left(\frac{|\mathcal{F}_1(\chi) - \widetilde{\mathcal{F}}_1(\chi)|}{\sqrt{\max(|\mathcal{F}_1(\chi)|, |\widetilde{\mathcal{F}}_1(\chi)|)}}, \phi\right) \\
\succeq \ & \Phi\left(\mathcal{F}_1(\chi) - \widetilde{\mathcal{F}}_1(\chi), \frac{\phi}{M_1}\right),
\end{aligned}$$

$$\begin{aligned}
& \Phi\left(\sin^2(\mathcal{F}_2(\chi)) - \sin^2(\widetilde{\mathcal{F}}_2(\chi)), \phi\right) \\
= \ & \Phi\left([\sin(\mathcal{F}_2(\chi)) - \sin(\widetilde{\mathcal{F}}_2(\chi))][\sin(\mathcal{F}_2(\chi)) + \sin(\widetilde{\mathcal{F}}_2(\chi))], \phi\right) \\
\succeq \ & \Phi\left(\mathcal{F}_2(\chi) - \widetilde{\mathcal{F}}_2(\chi)[2\max\{\sin(\mathcal{F}_2(\chi)), \sin(\widetilde{\mathcal{F}}_2(\chi))\}], \phi\right) \\
\succeq \ & \Phi\left(\mathcal{F}_2(\chi) - \widetilde{\mathcal{F}}_2(\chi), \frac{\phi}{M_2}\right),
\end{aligned}$$

$$
\begin{aligned}
& \Phi\bigg(\cos(\mathcal{F}_3(\frac{\chi}{2})) - \cos(\widetilde{\mathcal{F}_3}(\frac{\chi}{2})), \phi\bigg) \\
= \;& \Phi\Bigg(2\sin\left(\frac{\mathcal{F}_3(\frac{\chi}{2}) + \widetilde{\mathcal{F}_3}(\frac{\chi}{2})}{2}\right)\sin\left(\frac{\mathcal{F}_3(\frac{\chi}{2}) - \widetilde{\mathcal{F}_3}(\frac{\chi}{2})}{2}\right), \phi\Bigg) \\
\succeq \;& \Phi\Bigg(\sin\left(\frac{\mathcal{F}_3(\frac{\chi}{2}) - \widetilde{\mathcal{F}_3}(\frac{\chi}{2})}{2}\right), \frac{\phi}{2\left|\sin\left(\frac{\mathcal{F}_3(\frac{\chi}{2}) + \widetilde{\mathcal{F}_3}(\frac{\chi}{2})}{2}\right)\right|}\Bigg) \\
\succeq \;& \Phi\Bigg(\frac{\mathcal{F}_3(\frac{\chi}{2}) - \widetilde{\mathcal{F}_3}(\frac{\chi}{2})}{2}, \frac{\phi}{2\left|\sin\left(\frac{\mathcal{F}_3(\frac{\chi}{2}) + \widetilde{\mathcal{F}_3}(\frac{\chi}{2})}{2}\right)\right|}\Bigg) \\
\succeq \;& \Phi\bigg(\mathcal{F}_3(\chi) - \widetilde{\mathcal{F}_3}(\chi), \frac{\phi}{M_3}\bigg),
\end{aligned}
$$

$$
\begin{aligned}
& \Phi\bigg(\operatorname{arccot}(\mathcal{F}_4(\chi)) - \operatorname{arccot}(\widetilde{\mathcal{F}_4}(\chi)), \phi\bigg) \\
\succeq \;& \Phi\bigg((\operatorname{arccot}(M))'[\mathcal{F}_4(\chi)) - \widetilde{\mathcal{F}_4}(\chi)], \phi\bigg) \\
\succeq \;& \Phi\bigg(\frac{-1}{1+M^2}[\mathcal{F}_4(\chi)) - \widetilde{\mathcal{F}_4}(\chi)], \phi\bigg) \\
\succeq \;& \Phi\bigg(\mathcal{F}_4(\chi) - \widetilde{\mathcal{F}_4}(\chi), \frac{\phi}{M_4}\bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
& \Phi\bigg(\frac{|\mathcal{F}_5(\chi)|}{1+|\mathcal{F}_5(\chi)|} - \frac{|\widetilde{\mathcal{F}_5}(\chi)|}{1+|\widetilde{\mathcal{F}_5}(\chi)|}, \phi\bigg) \\
\succeq \;& \Phi\bigg(\mathcal{F}_5(\chi) - \widetilde{\mathcal{F}_5}(\chi), \frac{\phi}{M_5}\bigg),
\end{aligned}
$$

where $\phi > 0$, $M_i > 0$, $i = 1,2,3,4,5$ and $\mathcal{F}_4 \leq M \leq \widetilde{\mathcal{F}_4}$.

Making use of the above inequalities, we get

$$
\begin{aligned}
& \Phi\bigg(\mathcal{H}(\chi, \mathcal{F}_i(\chi)) - \mathcal{H}(\chi, \widetilde{\mathcal{F}_i}(\chi)), \phi\bigg) \\
\succeq \;& \Phi\bigg(\chi^2[\mathcal{F}_i(\chi) - \widetilde{\mathcal{F}_i}(\chi)], \frac{\phi}{M_i}\bigg), \quad i = 1,2,3,4,5.
\end{aligned}
$$

Here, we let $\zeta_1(\chi) = \zeta_2(\chi) = (\chi^2, \chi^2, \chi^2, \chi^2, \chi^2)^\top$.

Making use of the arithmetic mean aggregation map $\mathtt{AG}_2$ and the maximum aggregation map $\mathtt{AG}_3$, we calculate the numerical results of finite-time stability for fractional-order system (61) and (62) (see Table 1). In view of the required conditions in cases (i), (ii), (iii) of Theorem 7, we obtain the relative optimal thresholds $\mathtt{AG}_3(\alpha_2) = 0.70$, and $\mathtt{AG}_2(\alpha_2) = 0.04$.

Table 1. Stability results of Example 1 .

Theorem 7	(i)	(ii)	(iii)
$\mathfrak{P}$	0.50	0.50	0.50
T	0.60	0.60	0.60
t	0.20	0.20	0.20
$\mathtt{AG}_3\ (\mathcal{K})$	0.10	0.10	0.10
$\mathtt{AG}_3\ (\alpha_1)$	0.10	0.10	0.10
$\mathtt{AG}_3\ (\mathcal{F})$	0.7539	0.6992	0.9094
$\mathtt{AG}_3\ (\alpha_2)$	0.76	0.70	0.91
$\mathtt{AG}_2\ (\mathcal{K})$	0.02	0.02	0.02
$\mathtt{AG}_2\ (\alpha_1)$	0.02	0.02	0.02
$\mathtt{AG}_2\ (\mathcal{F})$	0.0746	0.0398	0.0971
$\mathtt{AG}_2\ (\alpha_2)$	0.08	0.04	0.10

7. Conclusions

The main target of this paper is to provide a new interpretation of Ulam type stability with the application of classical, well-known special functions and aggregation maps. This new notion of stability not only covers the previous notions but also considers the optimization of the problem. This stability allows us to get the best approximation error estimates for different fractional-order systems. In addition, we will be able to obtain maximal stability with minimal error which leads to calculate the optimal solution.

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